Network Systems, Kuramoto Oscillators, and Synchronous Power Flows

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Saber Jafarpour
UCSB

Elizabeth Huang
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Kevin D. Smith
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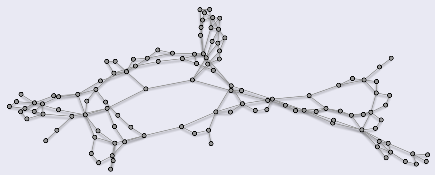
Outline

1. **Linear Network Systems**
   F. Bullo. *Lectures on Network Systems.*
   Kindle Direct Publishing, 1.4 edition, July 2020.
   With contributions by J. Cortés, F. Dörfler, and S. Martínez.
   URL: `http://motion.me.ucsb.edu/book-lns`
2. Kuramoto Synchronization and Synchronous Power Flows

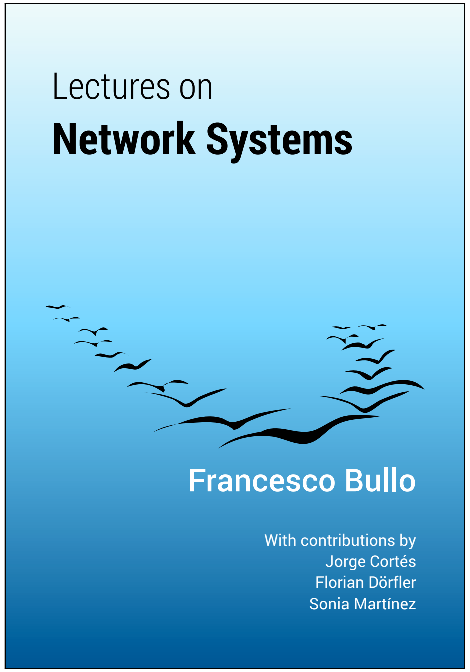
Network systems



network structure $\Longleftrightarrow$ **network function**

Model	Dynamics	Function	Structure
averaging system (Abelson '64)	$\dot{x} = -Lx$ Laplacian matrix	consensus	$\exists$ globally reach node
network flow (Noy Meir '73)	$\dot{x} = -(L^\top + D)x + u$ compartmental matrix	stationary distribution	outflow-connected

New text "Lectures on Network Systems"



Lectures on Network Systems, Francesco Bullo, KDP, 1.4 edition, 2020, ISBN 978-1-986425-64-3

1. Self-Published and Print-on-Demand at:
   https://www.amazon.com/dp/1986425649
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   http://motion.me.ucsb.edu/book-lns:
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   For instructors: slides and solution manual
3. incorporates lessons from 2 decades of research:
   robotic multi-agent, social networks, power grids

version 1.4
332 pages
171 exercises, 220 pages solution manual
5.8K downloads Jun 2016 - Oct 2020
46 instructors in 17 countries

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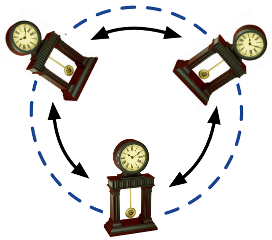
   URL: https://arxiv.org/pdf/1901.11189.pdf

Today: Sync & Multi-Stability in Coupled Oscillators

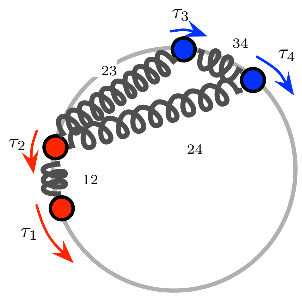
Kuramoto model

- n **oscillators** with angle $\theta_i \in \mathbb{S}^1$
- **non-identical** natural frequencies $\omega_i \in \mathbb{R}^1$
- **coupling** with strength $a_{ij} = a_{ji}$

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



Model #1: Spring network analog and applications



Coupled swing equations
Euler-Lagrange eq for spring network on ring:

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = \tau_i - \sum_j k_{ij} \sin(\theta_i - \theta_j)$$

Kuramoto coupled oscillators

$$\dot{\theta}_i = \omega_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

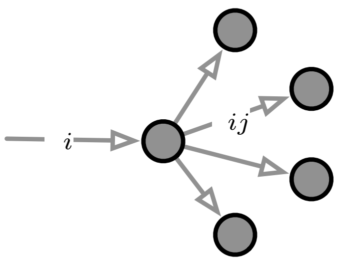
Kuramoto equilibrium equation

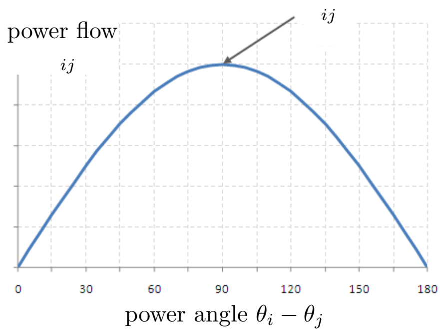
$$0 = \omega_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

Model #2: Active Power Flow Problem

AC, Kirchhoff and Ohm, quasi-sync, lossless lines, constant voltages.
supply/demand p_i, max power coeff a_{ij}, voltage phase θ_i

$$p_i = \sum_{j=1}^{n} f_{ij}, \qquad f_{ij} = a_{ij}\sin(\theta_i - \theta_j)$$



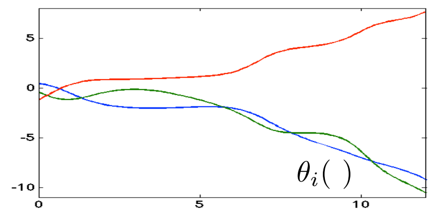


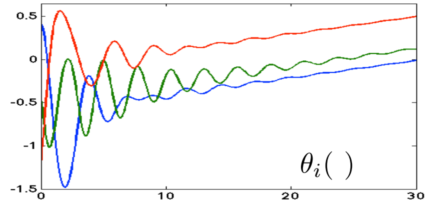
Given: network parameters & topology, load & generation profile,

Phenomenon #1: Transition from incoherence to sync

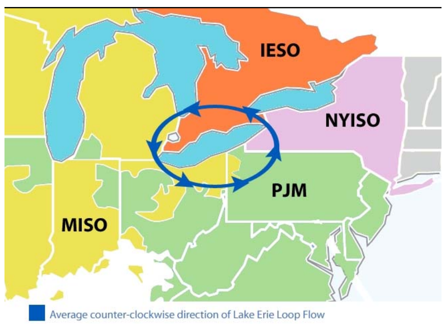
Function = synchronization

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)$$

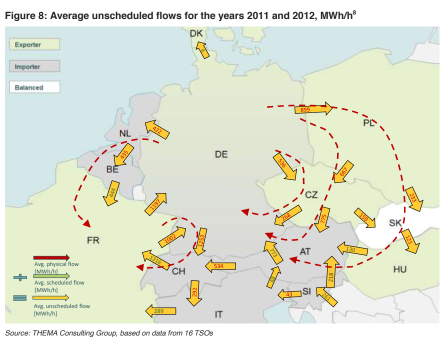




Phenomenon #2: Multi-stability in power flows



New York Independent System Operator, **Lake Erie Loop Flow Mitigation**, Technical Report, 2008



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Primer on algebraic graph theory (slide 1/2)

Incidence matrix: $n \times m$ matrix B s.t. $(B^\top p_{\text{actv}})_{(ij)} = p_i - p_j$

Weight matrix: $m \times m$ diagonal matrix $\mathcal{A}$

Laplacian stiffness: $L = B\mathcal{A}B^\top \geq 0$

Algebraic connectivity:

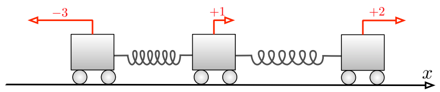
$$\lambda_2(L) = \text{second smallest eig of } L = \text{notion of connectivity and coupling}$$

Linearization of Kuramoto equilibrium equation:

$$p_{\text{actv}} = B\mathcal{A}\sin(B^\top\theta) \quad \Longrightarrow \quad p_{\text{actv}} \approx B\mathcal{A}(B^\top\theta) = L\,\theta$$

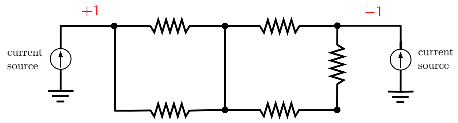
Laplacian linear balance equation

Primer on algebraic graph theory (slide 2/2)



(a) spring network

$L_{\text{stiffness}}\, x = f_{\text{load}}$



(b) resistive circuit

$L_{\text{conductance}}\, v = c_{\text{injected}}$

Laplacian linear balance equation: $p_{\text{actv}} = L\,\theta$

$\sum_i p_i = 0 \quad \Longleftrightarrow \quad$ equilibrium exists : $\theta = L^\dagger p_{\text{actv}}$

pairwise displacements : $B^\top\theta = B^\top L^\dagger p_{\text{actv}}$

From Old to New Tests

Question: Given balanced p_{actv}, do angles exist satisfying

$$p_{\text{actv}} = B\mathcal{A}\sin(B^\top\theta)$$

Old Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\|B^\top p_{\text{actv}}\|_2 < \lambda_2(L)$	for unweighted graphs	(Old 2-norm T)
$\|B^\top L^\dagger p_{\text{actv}}\|_\infty < 1$	for trees, complete	(Old ∞-norm T)

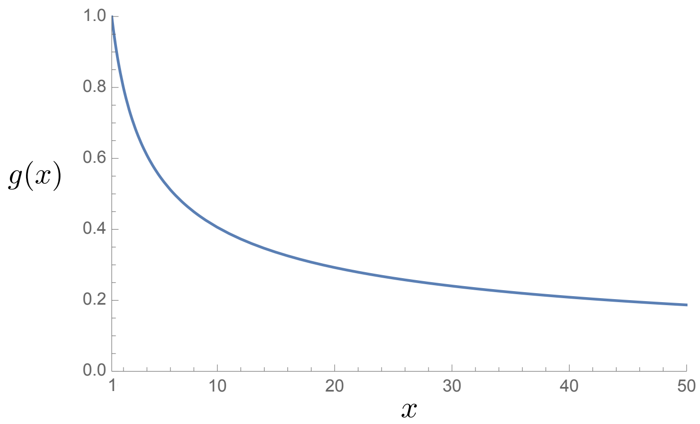
New Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\|B^\top L^\dagger p_{\text{actv}}\|_2 < 1$	for unweighted graphs	(New 2-norm T)
$\|B^\top L^\dagger p_{\text{actv}}\|_\infty < g(\|\mathcal{P}\|_\infty)$	for all graphs	(New ∞-norm T)

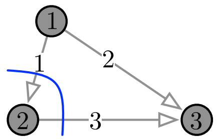
where g is monotonically decreasing

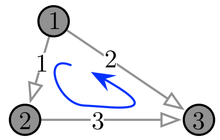
$$g : [1,\infty) \to [0,1]$$

$$x \mapsto \frac{y(x)+\sin(y(x))}{2} - x\,\frac{y(x)-\sin(y(x))}{2}\Big|_{y(x) = \arccos(\frac{x-1}{x+1})}$$



and where $\mathcal{P}$ is a projection matrix





$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\text{Im}(B^\top)}_{\substack{\text{cutset space}\\ \text{flow vectors}}} \oplus \underbrace{\text{Ker}(B\mathcal{A})}_{\substack{\text{weighted cycle space}\\ \text{cycle vectors}}}$$

$\mathcal{P} = B^\top L^\dagger B\mathcal{A}$ = oblique projection onto $\text{Im}(B^\top)$ parallel to $\text{Ker}(B\mathcal{A})$

1. if G unweighted, then $\mathcal{P}$ is orthogonal and $\|\mathcal{P}\|_2 = 1$
2. if G acyclic, then $\mathcal{P} = I_m$ and $\|\mathcal{P}\|_p = 1$
3. if G uniform complete or ring, then $\|\mathcal{P}\|_\infty = 2(n-1)/n \leq 2$

New Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\|B^\top L^\dagger p_{\text{actv}}\|_2 < 1$ for unweighted graphs (New 2-norm T)

$\|B^\top L^\dagger p_{\text{actv}}\|_\infty < g(\|\mathcal{P}\|_\infty)$ for all graphs (New ∞-norm T)

Unifying theorem with a family of tests

Equilibrium angles (neighbors within γ arc) exist if, in some p-norm,

$\|B^\top L^\dagger p_{\text{actv}}\|_p \leq \gamma \alpha_p(\gamma)$ for all graphs (New α_p T)

where nonconvex optimization problem:

$\alpha_p(\gamma) :=$ min amplification factor of $\mathcal{P}\,\text{diag}[\text{sinc}(x)]$

Comparison of sufficient and approximate sync tests

Any test predicts max transmittable power (before bifurcation).
Compare with numerically computed.
Test = linear inequalities vs iterative nonlinear solver

Test Case	ratio of test prediction to numerical computation			
	old 2-norm	new ∞-norm	$g(\|\mathcal{P}\|_\infty) \approx 1$ approximate	α_∞ test *fmincon*
IEEE 9	16.54 %	73.74 %	92.13 %	85.06 %†
IEEE 14	8.33 %	59.42 %	83.09 %	81.32 %†
IEEE RTS 24	3.86 %	53.44 %	89.48 %	89.48 %†
IEEE 30	2.70 %	55.70 %	85.54 %	85.54 %†
IEEE 118	0.29 %	43.70 %	85.95 %	—*
IEEE 300	0.20 %	40.33 %	99.80 %	—*
Polish 2383	0.11 %	29.08 %	82.85 %	—*

† *fmincon* with 100 randomized initial conditions
* *fmincon* does not converge

Summary: Kuramoto equilibrium and active power flow

Given topology (incidence B), admittances (Laplacian L), injections p_{actv},

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Q1: $\exists$ a **stable operating point** (with pairwise angles $\leq \gamma$)?
Q2: what is the **network capacity** to transmit active power?
Q3: how to quantify **robustness** as distance from loss of feasibility?

Equilibrium angles exist if, in some p-norm,

$\|B^\top L^\dagger p_{\text{actv}}\|_p \leq \gamma \alpha_p(\gamma)$ for all graphs (New α_p T)

For $p = \infty$, after bounding,

$\|B^\top L^\dagger p_{\text{actv}}\|_\infty \leq g(\|\mathcal{P}\|_\infty)$ (New ∞-norm T)

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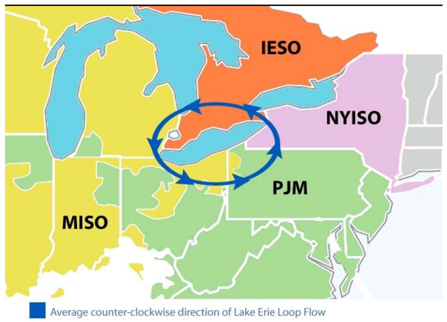
Phenomenon #2: Multiple power flows

Theoretical observation: multiple solutions exist

Practical observations:
- sometimes undesirable power flows around loops
- sometimes sizable difference between predicted and actual power flows



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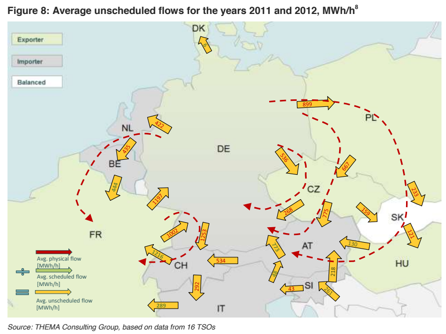

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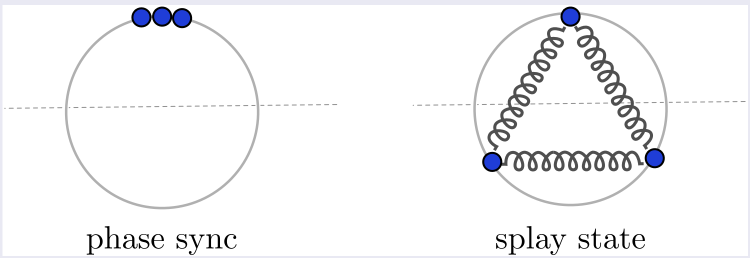
Lack of uniqueness and winding solutions

Given topology (incidence B), admittances (Laplacian L), injections p_{actv},

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

1. is solution unique?
2. how to localize/classify solutions?

triangle graph, homogeneous weights ($a_{ij} = 1$), $p_{\text{actv}} = 0$

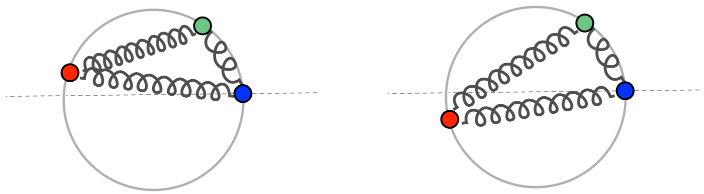


Winding number of n angles

Given undirected graph with a cycle $\sigma = (1, \dots, n_\sigma)$ and orientation

1. **winding number of $\theta \in \mathbb{T}^n$ along σ** is:

$$w_\sigma(\theta) = \frac{1}{2\pi} \sum_{i=1}^{n_\sigma} d_{\text{cc}}(\theta_i, \theta_{i+1})$$



$(\theta) = 0$ $\qquad$ $(\theta) = \pm 1$

2. given basis $\sigma_1, \dots, \sigma_r$ for cycles, **winding vector of θ** is

$$w(\theta) = (w_{\sigma_1}(\theta), \dots, w_{\sigma_r}(\theta))$$

"Kirckhoff Angle Law" and partition of the n-torus

Theorem: Kirchhoff angle law on $\mathbb{T}^n$

$$w_\sigma(\theta) = 0, \pm 1, \dots, \pm \lfloor n_\sigma/2 \rfloor$$
$$\implies w(\theta) \text{ is piecewise constant}$$
$$\implies w(\theta) \text{ takes value in a finite set}$$

Theorem: Winding partition

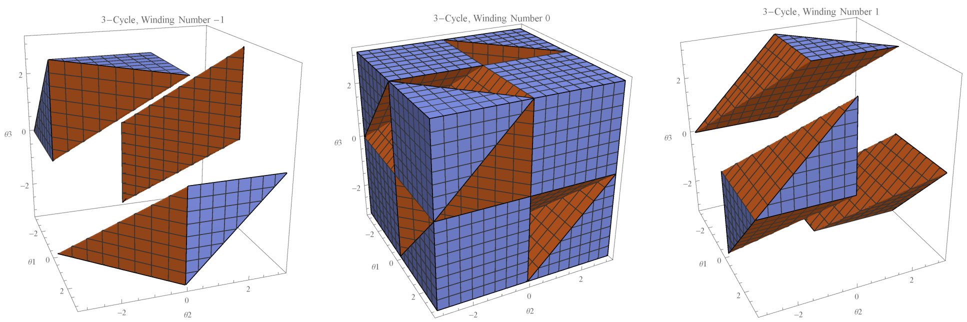
For each possible winding vector u, define

$$\text{WindingCell}(u) := \{\theta \in \mathbb{T}^n \mid w(\theta) = u\}$$

Then

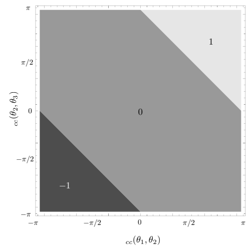
$$\mathbb{T}^n = \cup_u \text{WindingCell}(u)$$

Winding partition of triangle graph



$w = -1$ $w = 0$ $w = +1$

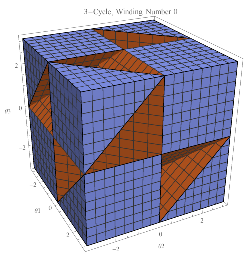
- each winding cell is connected
- each winding cell is invariant under rotation
- bijection:
  reduced winding cell $\longleftrightarrow$ open convex polytope



The Kuramoto model and the winding partition

Given topology (incidence B), admittances (Laplacian L), injections p_{actv},

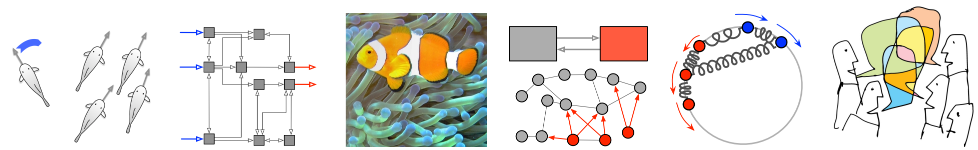
$$\dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$



Theorem: At-most-uniqueness and extensions

1. each WindingCell has at-most-1 equilibrium (n within $\pi/2$ arc)
2. loop winding number $\rightsquigarrow$ loop power flow
3. test for existence + uniqueness in WindingCell(u)

Summary and Future Work



Review

1. a rather comprehensive theory of linear network systems
2. synchronization and multistability for Kuramoto

Future research

1. emerging contractivity theory for network systems
2. learning and control of infrastructure networks
3. **outreach/collaboration opportunities for our community with** sociologists, biologists, economists, physicists ...