Numerical computation of contraction metrics

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Outline

Outline of the talk:

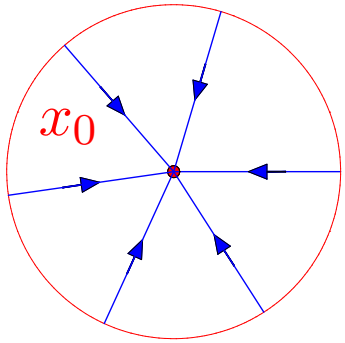
1. Contraction metrics
2. Numerical approximation by solving a PDE with RBF
3. Numerical approximation by integration and quadrature
4. Verification by CPA interpolation
5. Examples
6. Conclusions

Review on contraction analysis and computation of contraction metrics
Peter Giesl, Sigurdur Hafstein and Christoph Kawan
Journal of Computational Dynamics, 10(1), pp. 1-47, 2023.

Contraction metric: Basin of Attraction

System of autonomous ordinary differential equations

$$(1) \qquad \begin{cases} \dot{x} & = & f(x) \\ x(0) & = & \xi \end{cases}$$



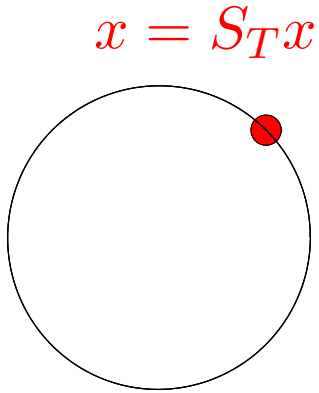
$x \in \mathbb{R}^n$, $f \in C^s(\mathbb{R}^n, \mathbb{R}^n)$ where $s \geq 1$, $n \in \mathbb{N}$.
Flow $S_t\xi := x(t)$, solution of (1)

Equilibrium

- equilibrium x_0 ($f(x_0) = 0$),
- basin of attraction $A(x_0) := \{\xi \in \mathbb{R}^n \mid S_t\xi \overset{t\to\infty}{\longrightarrow} x_0\}$

Periodic orbit

- periodic orbit $\Omega = \{S_t x \mid t \in [0, T)\}$ with $x = S_T x$
- basin of attraction $A(\Omega) = \{\xi \in \mathbb{R}^n \mid \mathrm{dist}(S_t\xi, \Omega) \overset{t\to\infty}{\longrightarrow} 0\}$



Riemannian metric

Definition (Riemannian metric)

A matrix-valued function $M \in C^1(\mathbb{R}^n, \mathbb{S}^{n\times n})$ (symmetric $n \times n$ matrices) is called Riemannian metric if $M(x) \succ 0$ (positive definite) for each $x \in \mathbb{R}^n$.

Note: $\langle v, w\rangle_{M(x)} := v^T M(x) w$ defines a point-dependent scalar product for $v, w \in \mathbb{R}^n$. $M(x) = I$ gives Euclidean metric.

Definition (Orbital derivative)

For $V \in C^1(\mathbb{R}^n, \mathbb{R})$, the orbital derivative, the derivative along solutions of $\dot{x} = f(x)$, is

$$V'(x) = \nabla V(x) \cdot f(x) = \frac{d}{dt} V(x(t))\big|_{t=0}.$$

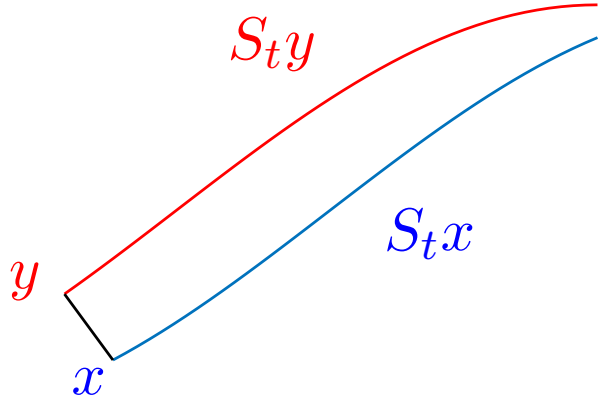
For $M \in C^1(\mathbb{R}^n, \mathbb{S}^{n\times n})$, the orbital derivative is defined component-wise

$$(M'(x))_{ij} = (M_{ij})'(x)$$

Idea of contraction criterion

Idea

- Solutions $S_t x$ and $S_t y$, y near x
- Time-dependent distance (squared)



$$d^2(t) := (S_t y - S_t x)^T M(S_t x)(S_t y - S_t x)$$

- Derivative, denoting $v = S_t y - S_t x$: exponential decay of $d(t)$

$$\begin{aligned}
\frac{d}{dt} d^2(t) \quad &\approx \quad (S_t y - S_t x)^T M(S_t x) Df(S_t x)(S_t y - S_t x) \\
&\quad +(S_t y - S_t x)^T M'(S_t x)(S_t y - S_t x) \\
&\quad +(S_t y - S_t x)^T Df(S_t x)^T M(S_t x)(S_t y - S_t x) \\
&= \quad \underbrace{v^T [M(S_t x) Df(S_t x) + M'(S_t x) + Df(S_t x)^T M(S_t x)] v}_{=:2\mathcal{L}_M(S_t x; v)} \\
&\leq \quad -2\nu\, v^T M(S_t x) v \;=\; -2\nu\, d^2(t)
\end{aligned}$$

Contraction Metric: equilibrium point

Theorem

- $\varnothing \neq K \subset \mathbb{R}^n$ *is forward invariant, compact and connected*
- *Riemannian metric* $M \in C^1(\mathbb{R}^n, \mathbb{S}^{n\times n})$
- $\mathcal{L}_M(x) \leq -\nu < 0$ *for all* $x \in K$ *where*

$$\begin{aligned} \mathcal{L}_M(x) &:= \max_{v^T M(x) v = 1} \mathcal{L}_M(x; v), \quad \mathcal{L}_M(x; v) = \frac{1}{2} v^T L_M(x) v \\ LM(x) &:= M(x) Df(x) + Df(x)^T M(x) + M'(x) \end{aligned}$$

Essentially $M(x) \succ 0$ *and* $LM(x) \prec 0$ *for all* $x \in K$

Then

- *Exists exactly one equilibrium* $x_0 \in K$*; it is exponentially stable*
- $-\nu$ *is upper bound of real parts of eigenvalues of* $Df(x_0)$
- $K \subset A(x_0)$ *(basin of attraction)*

Contraction Metric: periodic orbit

- $\varnothing \neq K \subset \mathbb{R}^n$ is forward invariant, compact and connected, $f(x) \neq 0$ for $x \in K$
- Riemannian metric $M \in C^1(\mathbb{R}^n, \mathbb{S}^{n\times n})$
- $\mathcal{L}_M(x) \leq -\nu < 0$ for all $x \in K$ where

$$\begin{aligned}
\mathcal{L}_M(x) &:= \max_{v^T M(x) v = 1, v^T f(x) = 0} \mathcal{L}_M(x; v), \quad \mathcal{L}_M(x; v) = \frac{1}{2} v^T L_M(x) v \\
V(x) &= Df(x) - \frac{f(x) f(x)^T (Df(x) + Df(x)^T)}{\|f(x)\|_2^2} \\
LM(x) &= M(x) V(x) + V(x)^T M(x) + M'(x)
\end{aligned}$$

Then

- Exists exactly one periodic orbit $\Omega \subset K$; it is exponentially stable
- $K \subset A(\Omega)$ (basin of attraction)

Contraction Metric: periodic orbit

Assume Ω is an exponentially stable periodic orbit of the system $\dot{x} = f(x)$ and $C \in C^s(A(\Omega), \mathbb{S}^{n\times n})$, $v^T C(x) v > 0$ for all $v \neq 0$ with $v^T f(x) = 0$, for all $x \in A(\Omega)$. Let $x_0 \in A(\Omega)$ and fix $c_0 > 0$.

Then the unique solution $M \in C^{s-1}(A(\Omega), \mathbb{S}^{n\times n})$ to the PDE

$$LM(x) = -C(x), \quad f(x_0)^T M(x_0) f(x_0) = c_0 \|f(x_0)\|_2^2$$

is a (periodic orbit) contraction metric for the system $\dot{x} = f(x)$.

Idea: use this PDE to compute a (periodic orbit) contraction metric

1. solve the PDE numerically to get $S \in C^2(A(\Omega), \mathbb{S}^{n\times n})$, $LS(x) \approx -C(x)$.
2. verify the conditions $S(x) \succ 0$ and $LS(x) \prec 0$ for $x \in A(\Omega)$.

Contraction Metric: periodic orbit

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WELL ...

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2. triangulate compact subset $K \subset A(\Omega)$ (subdivide into simplices), set $P(x_i) = S(x_i)$ at the vertices $x_i \in \mathcal{V}$ of the simplices, and interpolate P to a CPA (continuous, piecewise affine) function $P \in C(K, \mathbb{S}^{n\times n})$

(VP1) verify $P(x_i) \succ 0$ for all $x_i \in \mathcal{V}$
(VP2) and $LP(x_i) \prec 0$ for all $x_i \in \mathcal{V}$

Solve the PDE numerically: Optimal recovery

Generalised interpolation in Hilbert space H

- $\lambda_1, \ldots, \lambda_N \in H^*$ linearly independent functionals
- $r_1, \ldots, r_N \in \mathbb{R}$ given

Optimal recovery: $S \in H$ such that

$$\|S\|_H = \min\{\|S\|_H \,:\, \lambda_i(S) = r_i \text{ for } i = 1, 2, \ldots, N\}$$

- Solution $S = \sum_{j=1}^N \beta_j v_j$, v_j Riesz representer of λ_j, $\beta_j \in \mathbb{R}$
- Interpolation condition $\lambda_i(S) = \sum_{j=1}^N \beta_j \lambda_i(v_j) = r_i$ for $i = 1, \ldots, N$: system of linear equations

Optimal recovery of matrix-valued functions

Generalised interpolation in Hilbert space $H = H^\sigma(\Omega; \mathbb{S}^{n\times n})$, which is a Reproducing Kernel Hilbert Space: special form of Riesz representer

Functionals

- $L\colon H^\sigma(\Omega; \mathbb{S}^{n\times n}) \to H^{\sigma-1}(\Omega; \mathbb{S}^{n\times n})$ differential operator: $LM(x) = V(x)^T M(x) + M(x)V(x) + M'(x)$
- $X = \{x_1, \ldots, x_N\} \subset \Omega$ collocation points
- $\lambda_k^{(i,j)}(M) = e_i^T LM(x_k) e_j$, $1 \le i \le j \le n$, $k = 1, \ldots, N$ linearly independent functionals
- solve system of $N\frac{n(n+1)}{2}$ linear equations

Summary of construction

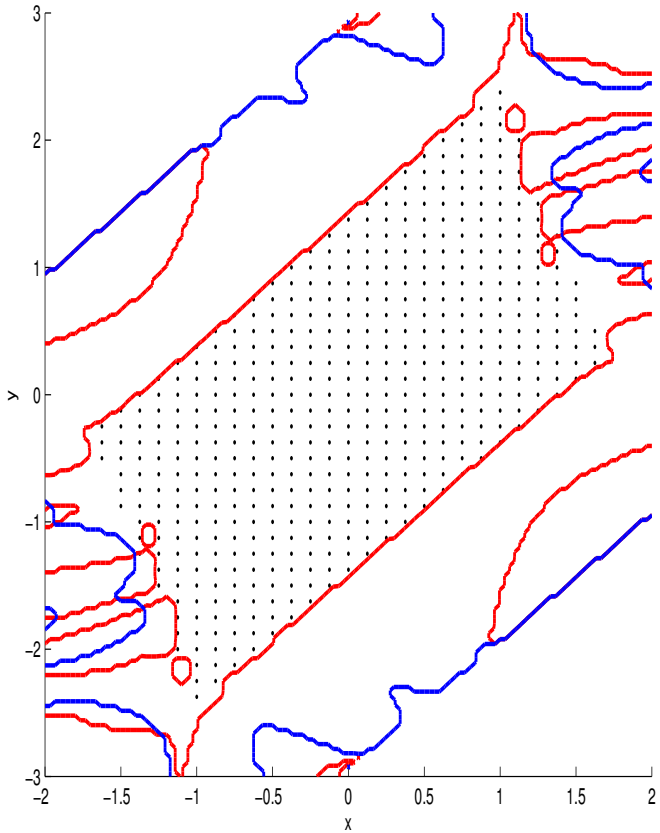
- Fix $\mathbb{S}^{n\times n} \ni C \succ 0$ (constant)
- Linear differential operator L,

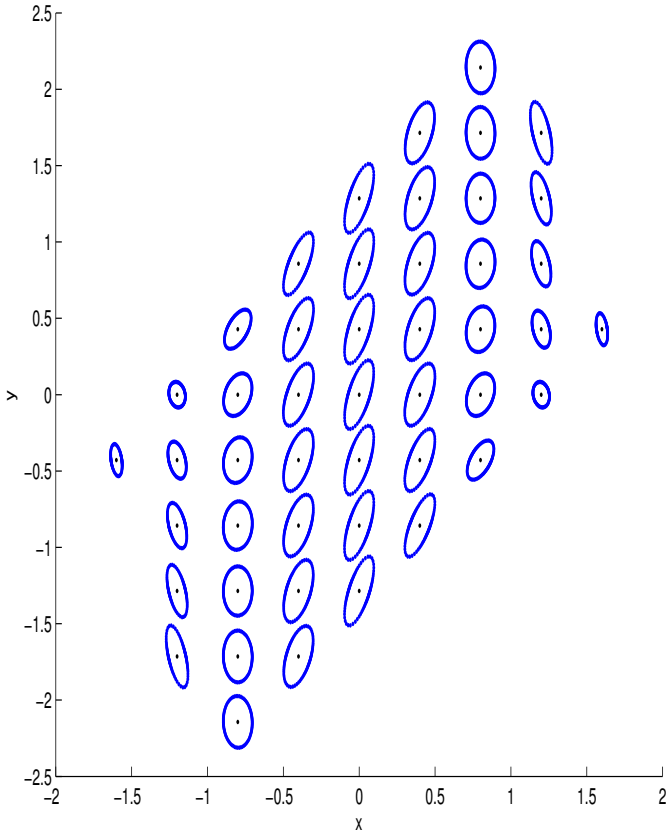
$$LM(x) := V(x)^T M(x) + M(x)V(x) + M'(x)$$

- Approximate matrix-valued solution M of $LM(x) = -C$ by S
- Solve linear system of equations for coefficients; matrix equation satisfied at given collocation points
- $S(x)$ positive definite and $LS(x)$ negative definite if collocation points are dense enough

Van der Pol with reversed time

$$\begin{aligned}\dot{x} &= -y \\ \dot{y} &= x - 3(1-x^2)y\end{aligned}$$





Left: Collocation points; $LS(x,y) \prec 0$ (red) and $S(x,y) \succ 0$ (blue)
Right: Curve of equal distance with respect to metric $S(x,y)$

Contraction Metric: CPA interpolation

Idea: use this PDE to compute a (periodic orbit) contraction metric

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2. triangulate compact subset $K \subset A(\Omega)$ (subdivide into simplices), set $P(x_i) = S(x_i)$ at the vertices $x_i \in \mathcal{V}$ of the simplices, and interpolate P to a CPA (continuous, piecewise affine) function $P \in C(K, \mathbb{S}^{n\times n})$

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$\widetilde{L}_T$ is a discrete version of L as in the CPA method assuring $LP(x) \prec 0$ for all $x \in T$. Somewhat involved for the periodic orbit case. P truly is a contraction metric, not an approximation!

Further, a forward invariant set is computed by generalized interpolation of

$$\nabla V(x) \cdot \frac{f(x)}{\sqrt{\delta^2 + \|f(x)\|_2^2}} = -1, \quad \delta^2 = 10^{-8}$$

Example 1

We consider the following system

$$\begin{cases} \dot{x} &= x(1-x^2-y^2)-y \\ \dot{y} &= y(1-x^2-y^2)+x \end{cases}$$

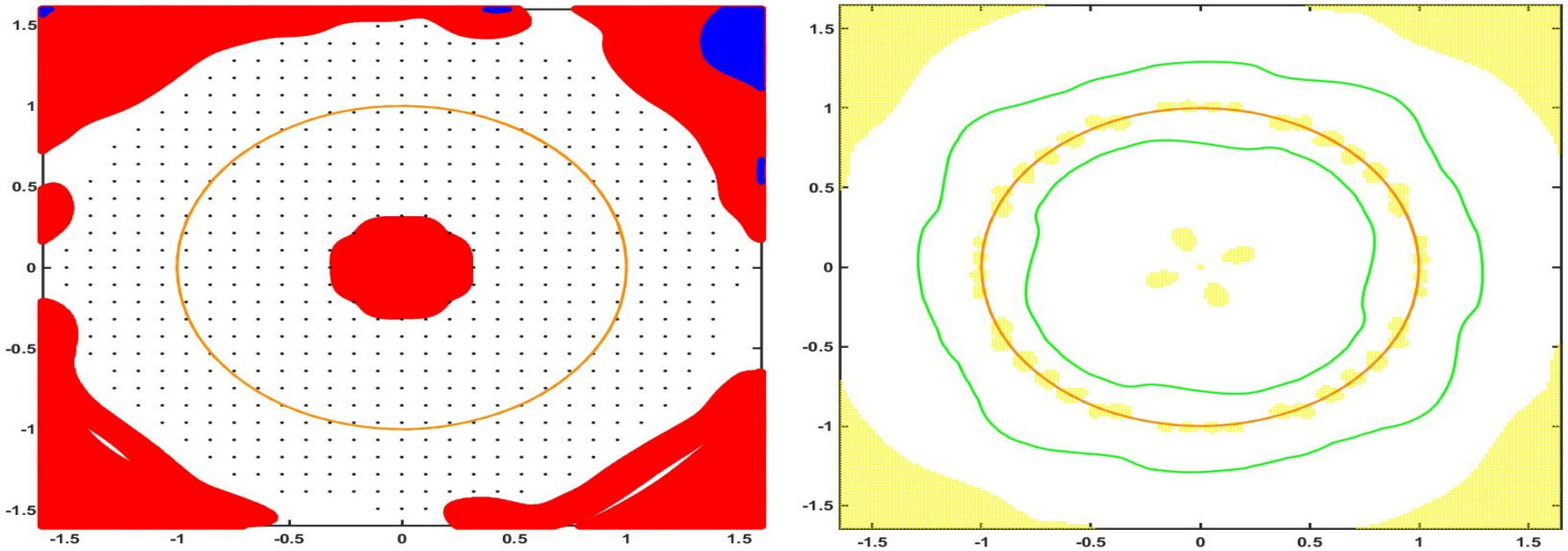

Figure: $P(x) \succ 0$ is not fulfilled in the blue area. $LP(x) \prec 0$ is not fulfilled in the red area. The orange curve is the periodic orbit. The Lyapunov-like function is not decreasing in the yellow area. The green curves are the level set of the Lyapunov-like function and form the boundary of a positively invariant set.

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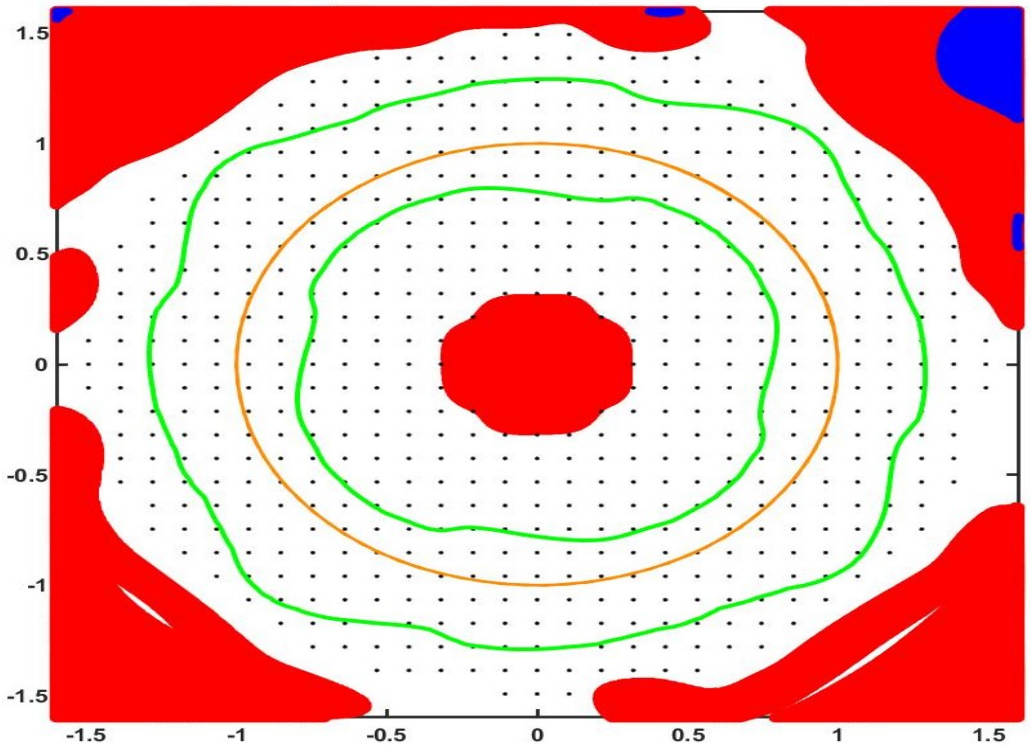

Figure: The positively invariant set (bounded by the green curves) is a subset of the basin of attraction of a unique periodic orbit within it.

Example 1 with perturbation $\varepsilon = 0.2$

Perturbed system

$$\begin{cases} \dot{x} &= (x+\varepsilon)(1-x^2-y^2)-(y+\varepsilon) \\ \dot{y} &= (y+\varepsilon)(1-x^2-y^2)+(x+\varepsilon) \end{cases}$$

same metric and Lyapunov-like function as for the unperturbed system

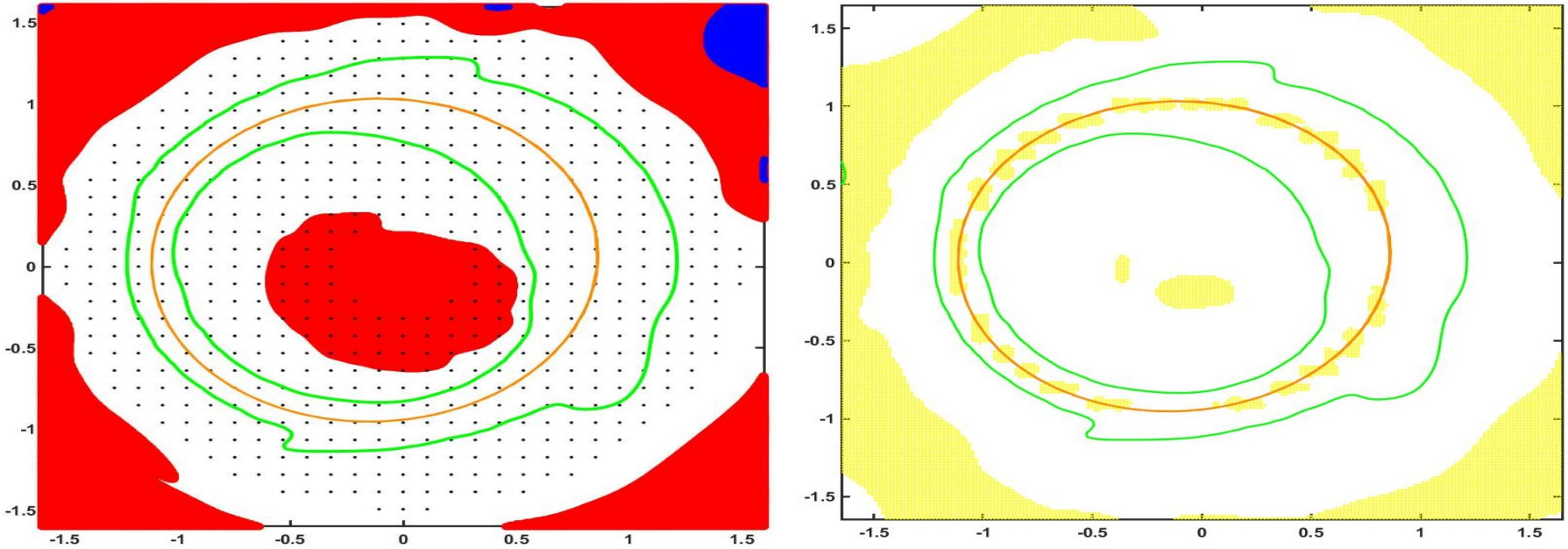

Figure: The positively invariant set (bounded by the green curves) is a subset of the basin of attraction of a unique periodic orbit within it.

Example 2

We consider the following system

$$\begin{cases} \dot{x} &= x(1-x^2-y^2)-y+0.1yz \\ \dot{y} &= y(1-x^2-y^2)+x \\ \dot{z} &= -z+xy \end{cases}$$

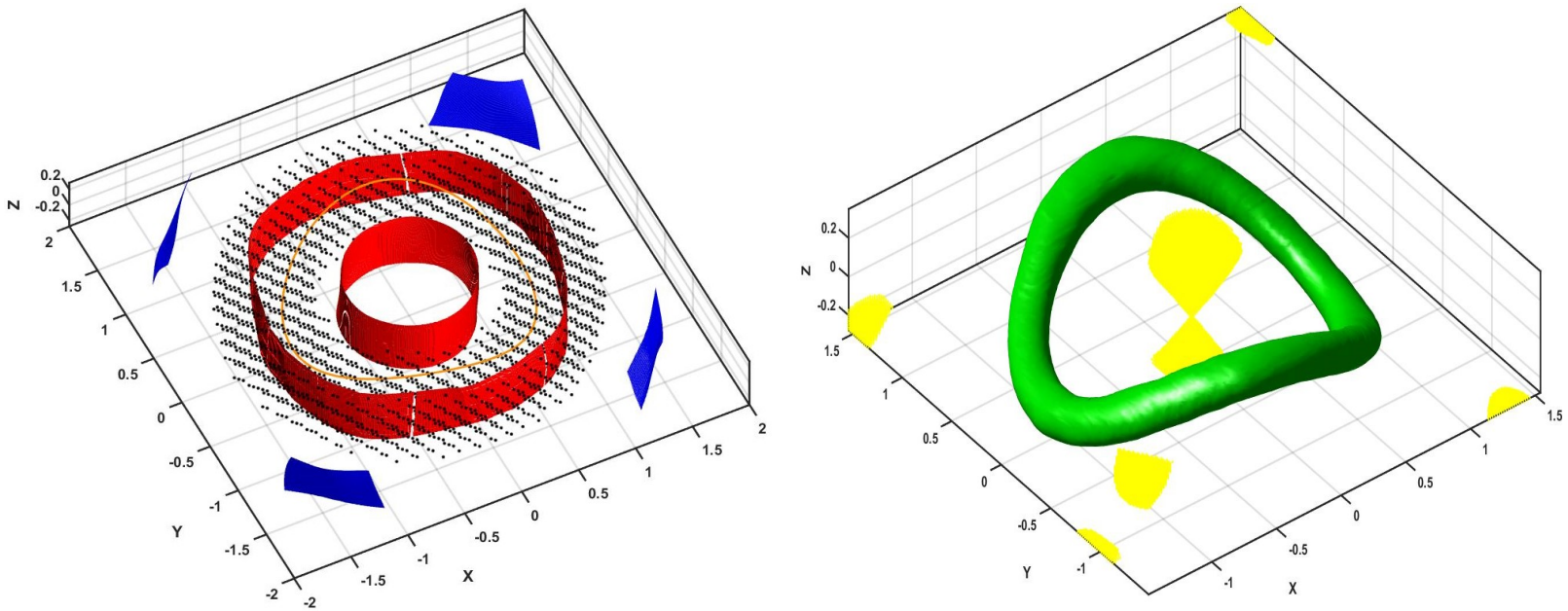

Figure: Periodic orbit (orange). $P(x) \succ 0$ not fulfilled (blue). $LP(x) \prec 0$ is not fulfilled (red). Lyapunov-like function not decreasing (yellow). Level set of the Lyapunov-like function (green).

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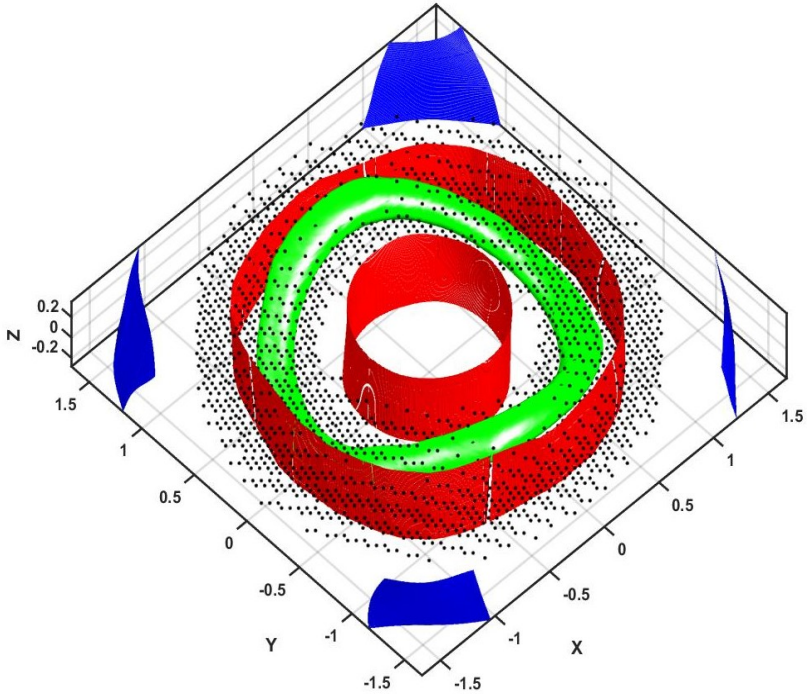

Figure: $P(x) \succ 0$ not fulfilled (blue). $LP(x) \prec 0$ is not fulfilled (red). Level set of the Lyapunov-like function (green). Must be an exponentially stable periodic orbit in the green tube!

Example 2 with perturbation $\varepsilon = 0.1$

Perturbed system

$$\begin{cases} \dot{x} &= x(1-x^2-y^2) - y + 0.1(y+\varepsilon)z \\ \dot{y} &= y(1-x^2-y^2) + x \\ \dot{z} &= -z + x(y+\varepsilon) \end{cases}$$

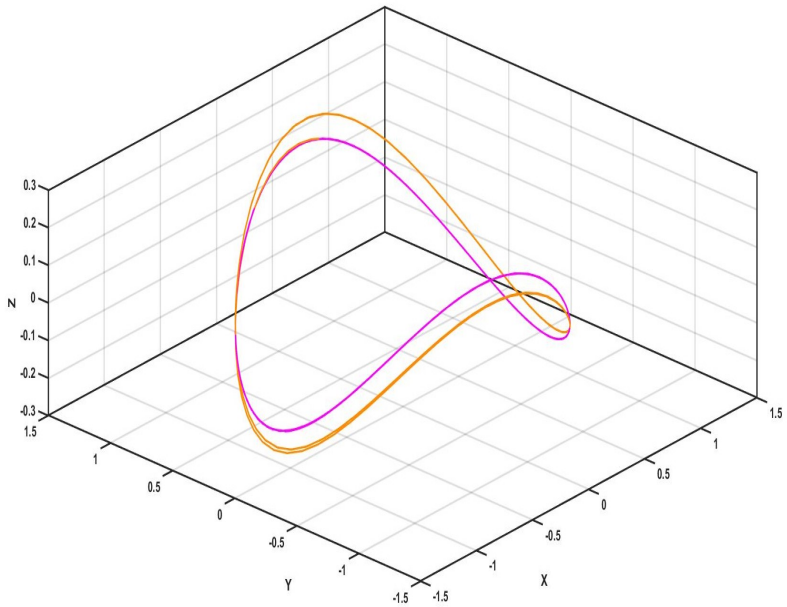

Figure: The periodic orbit for the original system $\varepsilon = 0$ (magenta); periodic orbit for the perturbed system $\varepsilon = 0.1$ (orange)

Example 2 with perturbation $\varepsilon = 0.1$

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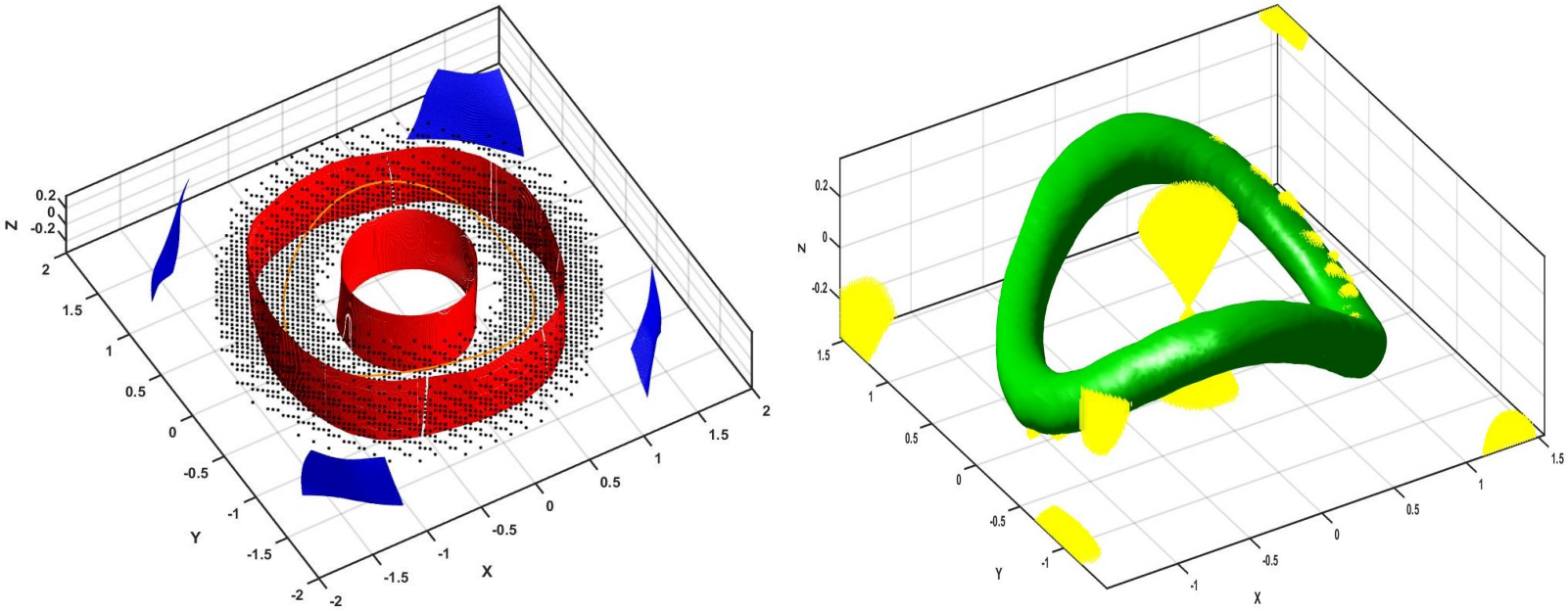
same metric as for the unperturbed system, but new Lyapunov-like function



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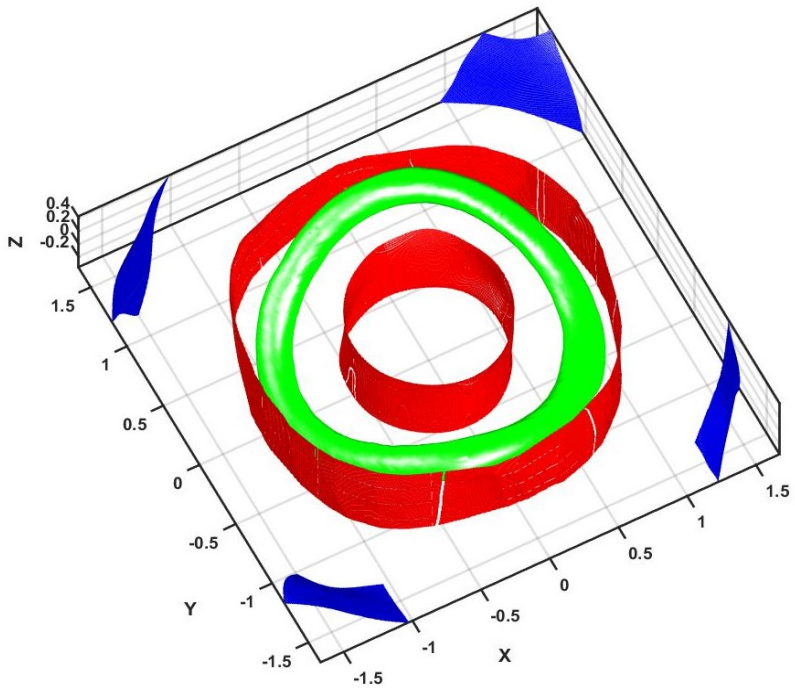

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Contraction Metric: CPA interpolation

Idea: use this PDE to compute a (periodic orbit) contraction metric

1. solve the PDE numerically to get $S \in C^2(A(\Omega), \mathbb{S}^{n\times n})$, $LS(x) \approx -C(x)$.
2. triangulate compact subset $K \subset A(\Omega)$ (subdivide into simplices), set $P(x_i) = S(x_i)$ at the vertices $x_i \in \mathcal{V}$ of the simplices, and interpolate P to a CPA (continuous, piecewise affine) function $P \in C(K, \mathbb{S}^{n\times n})$

(VP1) verify $P(x_i) \succ 0$ for all $x_i \in \mathcal{V}$

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Integration and Quadrature

Assume x_0 is an exponentially stable equilibrium of the system $\dot{x} = f(x)$ and $C \in C^s(A(x_0), \mathbb{S}^{n\times n})$, $v^T C(x) v > 0$ for all $v \neq 0$ and all $x \in A(x_0)$.

Then with $\dot{\phi}(t,x) = f(\phi(t,x)),\ \phi(0,x) = x$ and $\dot{\psi}(t,x) = Df(\phi(t,x))\psi(t,x),\ \psi(0,x) = I$:

$$M(x) = \int_0^\infty \psi(\tau,x)^T C(\phi(\tau,x))\psi(\tau,x)\, d\tau,$$

is a contraction metric for the system $\dot{x} = f(x)$.

Idea: use this solution formula to approximate a contraction metric

1. fix a time-horizon $T > 0$ and initial-values x_i, $i = 1, 2, \ldots, N$.
2. integrate $\dot{x} = f(x)$ numerically to obtain $\widetilde{\phi}(\cdot, x_i) \approx \phi(\cdot, x_i)$ on $[0, T]$
3. integrate $\dot{Y} = Df(\widetilde{\phi}(t, x_i))Y$, $Y(0) = I$, numerically to get $\widetilde{\psi}(\cdot, x_i) \approx \psi(\cdot, x_i)$ on $[0, T]$
4. compute $\widetilde{M}(x_i) := \int_0^T \widetilde{\psi}(\tau, x_i)^T C(\widetilde{\phi}(\tau, x_i))\widetilde{\psi}(\tau, x_i)\, d\tau.$

Contraction Metric: CPA interpolation

1. Integration and quadrature to obtain $\widetilde{M}(x_i)$
2. triangulate compact subset $K \subset A(\Omega)$ (subdivide into simplices), set $P(x_i) = \widetilde{M}(x_i)$ at the vertices $x_i \in \mathcal{V}$ of the simplices, and interpolate P to a CPA (continuous, piecewise affine) function $P \in C(K, \mathbb{S}^{n\times n})$

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$$\nabla V(x) \cdot \frac{f(x)}{\sqrt{\delta^2 + \|f(x)\|_2^2}} = -1, \quad \delta^2 = 10^{-8}$$

Example 3: Integration and Quadrature

$$\begin{cases} \dot{x} &= x(x^2+y^2-1) - y(z^2+1) \\ \dot{y} &= y(x^2+y^2-1) + x(z^2+1) \\ \dot{z} &= 10z(z^2-1) \end{cases}$$

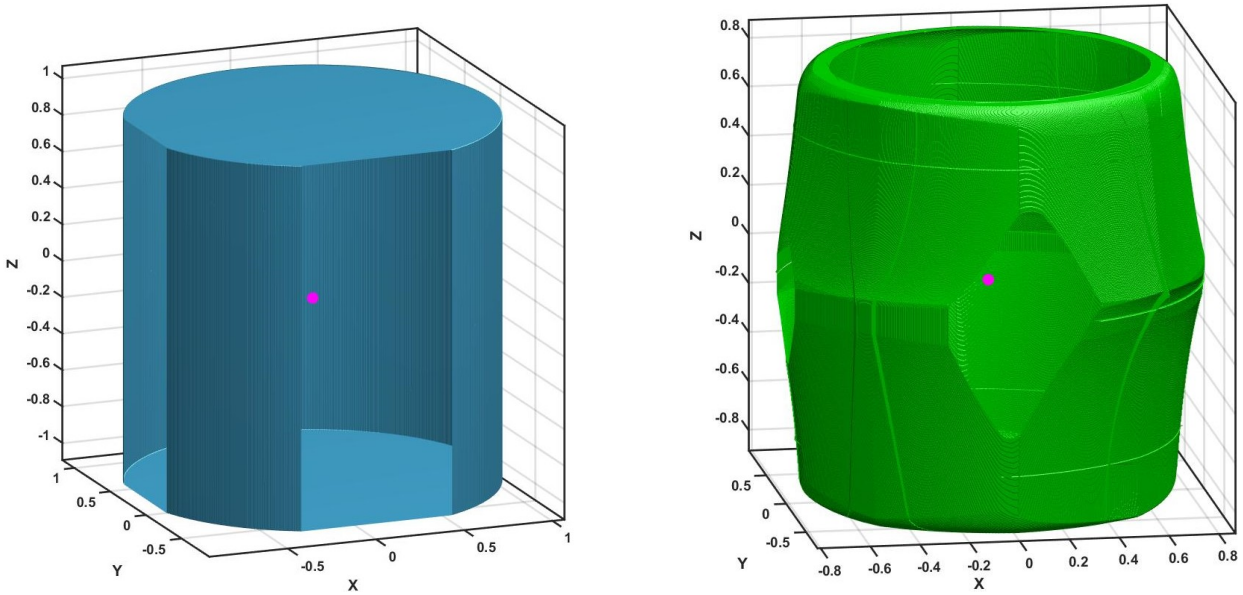

Figure: The blue surface (left) is the boundary between the area where the verification condition (VP1) is satisfied and where it is not satisfied, while the green surface (right) describes the verification condition (VP2) suggested by the Numerical Integration-CPA method. Hence, P is a contraction metric within the intersection of the areas bounded

Example 3: Integration and Quadrature

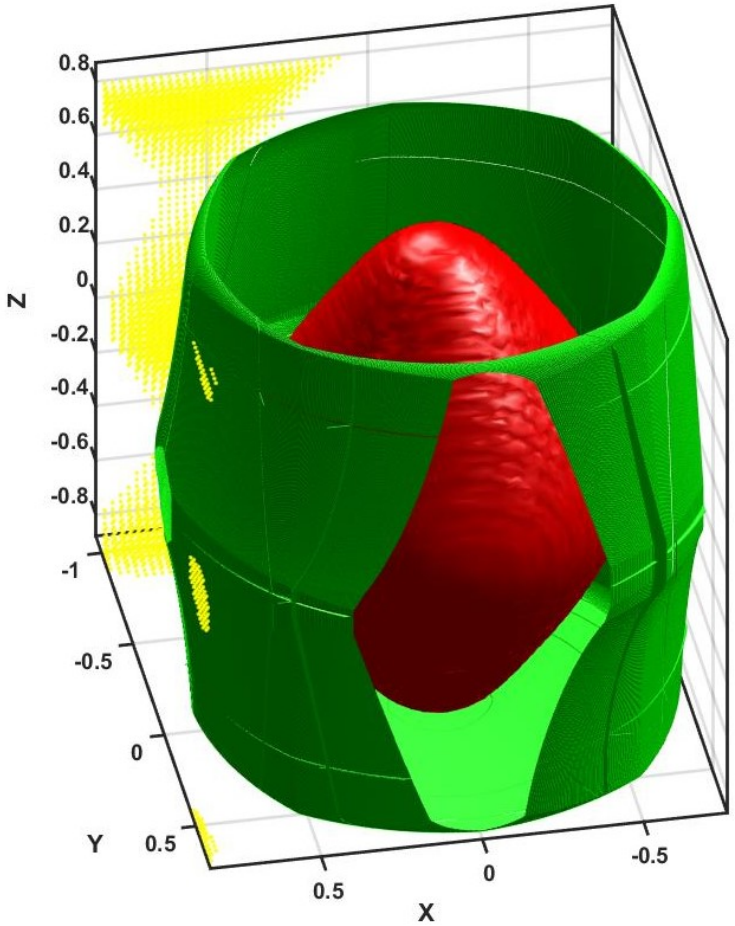

Figure: The red closed shape is a level set of it. The green surface is the boundary of the area where verification conditions (VP1) and (VP2) are satisfied. From the results we can conclude that there is exactly one equilibrium in the set bounded by the red surface, that it is exponentially stable, and that the red set is a subset of its basin of attraction.

Summary & Conclusions

- Described a method to compute contraction metrics for differential equations
  - generalized interpolation to solve a PDE numerically **or**
  - numerical integration and quadrature of a solution formula
  - CPA interpolation and verification
- Asserts the existence of an exponentially stable equilibrium **or**
- Asserts the existence of an exponentially stable periodic orbit
- Is robust to perturbations of the differential equation

Thank you for listening