# Contraction-guided reachability analysis of neural network controlled systems

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# Acknowledgements

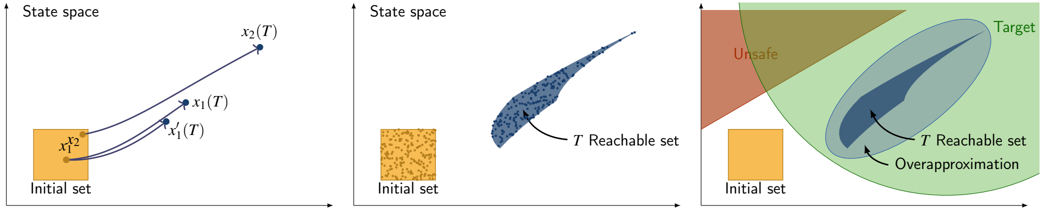
- S. Jafarpour, A. Harapanahalli, S. Coogan, "Interval Reachability of Nonlinear Dynamical Systems with Neural Network Controllers", *L4DC*, 2023, arXiv:2301.07912
- A. Harapanahalli, S. Jafarpour, S. Coogan, "Contraction-Guided Adaptive Partitioning for Reachability Analysis of Neural Network Controlled Systems", in submission to *IEEE CDC*, arXiv:2304.03671
- S. Jafarpour and S. Coogan, "Monotonicity and Contraction on Polyhedral Cones", in submission to *IEEE TAC*, arXiv:2210.11576

Saber Jafarpour

Akash Harapanahalli

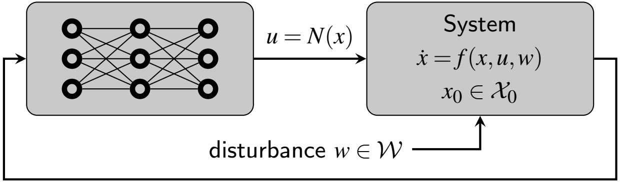
# Reachable sets of dynamical systems

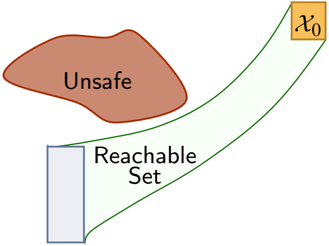
**System:** $\dot{x} = f(x, w)$ **State:** $x \in \mathbb{R}^n$ **Disturbance:** $w \in \mathcal{W} \subset \mathbb{R}^m$




- Reachable sets characterize possible system evolution
- Overapproximations of reachable sets are appropriate for verification and safety

# Neural network feedback controllers: The reachability problem




- **Goal:** compute reachable sets of closed-loop system

**Challenges:** soundness, efficiency-vs-conservatism tradeoff

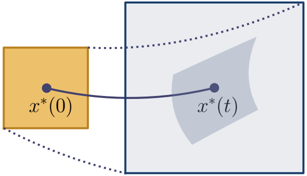
In this talk:

- Interval-based reachability using monotone systems theory
- Contraction-based adaptive partitioning
- Contraction with alternate partial orders for improved fidelity

# A starting point: Reachability estimates from contraction theory

For $\dot{x} = f(x,w)$, $x \in \mathbb{R}^n$ and $w \in \mathcal{W} \subseteq \mathbb{R}^m$, define **reachable set** as

$$\mathcal{R}^f(t;X_0) = \{x(t) : x(\cdot) \text{ is sol'n for some } w(\cdot) \text{ with } x_0 \in X_0\}.$$




Suppose

$$\mu\left(\frac{\partial f}{\partial x}(x,w)\right) \le c \quad \text{and} \quad \left\|\frac{\partial f}{\partial w}(x,w)\right\|_{w\to x} \le \ell.$$

If $X_0 = B(r_1, x_0^*)$ and $\mathcal{W} = B(r_2, w^*)$ where $B(\cdot,\cdot)$ is ball with radius and center, then

$$\mathcal{R}^f(t;X_0) \subseteq B\left(e^{ct}r_1 + \frac{\ell}{c}(e^{ct}-1)r_2, x^*(t)\right)$$

where $\dot{x}^*(\tau) = f(x^*(\tau), w^*)$ for all $\tau$.

(Pf. Rewriting of Grönwall Comparison Lemma, [Bullo, Corollary 3.17, p. 81].)

# Monotone dynamical systems

The system $\dot{x} = f(x,w)$, $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ is *monotone*[1] if

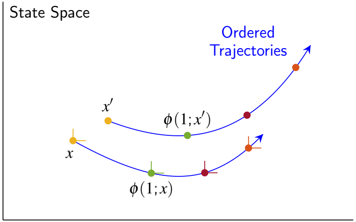
$$x_0 \preceq_{K_x} x_0' \text{ implies that } x(t) \preceq_{K_x} x'(t) \text{ for all time,}$$

for any $w(\cdot)$ and $w'(\cdot)$ such that $w(t) \preceq_{K_w} w'(t)$ for all $t$, where $\preceq_K$ is some partial order induced by cone $K$ ($K_x \subset \mathbb{R}^n$ or $K_w \subset \mathbb{R}^m$).

**Test for monotonicity** (standard order $\leq$):

$$\frac{\partial f}{\partial x}(x,w) \text{ is Metzler } (\geq 0 \text{ off-diag. entries})$$

$$\frac{\partial f}{\partial w}(x,w) \geq 0$$




[1] D. Angeli and E. Sontag, "Monotone Control Systems", *IEEE TAC*, 2003

# Reachability estimates for monotone systems

**Reachability Analysis for Monotone Systems.** For a monotone system,

$$\text{Reachable set} \subseteq [\text{lower trajectory}, \text{upper trajectory}].$$

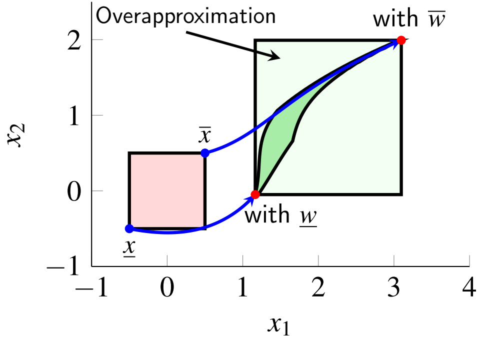
**Monotone System:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 - x_1 + w \\ x_1 \end{bmatrix}$$

$$[\underline{x}, \overline{x}] = [(-0.5, -0.5), (0.5, 0.5)]$$

$$w \in [\underline{w}, \overline{w}] = [2.2, 2.3]$$

$$T = 1$$




Reachability estimates from monotonicity are tight.

# When do monotone systems contract rectangles? [2,3]

**Theorem.** A monotone system $\dot{x} = f(x)$ (wrt. $\leq$) contracts at rate $c$ any box with sidelengths a multiple of $\eta \in \mathbb{R}^n_{>0}$ if (equivalently, contracts w.r.t. $\|\cdot\|_{\infty,[\eta]^{-1}}$):

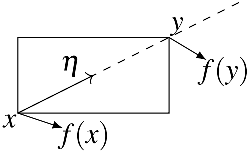
i) $\left(\dfrac{\partial f}{\partial x}(x)\right)\eta \leq c\eta$ for all $x$ or, equivalently,

ii) $f(y) - f(x) \leq c(y - x)$ for all $x$ and $y = x + \beta\eta$ for some $\beta > 0$

Proof for i): Follows from the fact that $\dfrac{\partial f}{\partial x}(x)$ is Metzler and that

$$\mu_\infty(A) = \max_i A_{ii} + \sum_{j \neq i} |A_{ij}|.$$

Proof sketch for ii):




[2][Bullo, Ch. 4.5, Lemma 4.17, p. 117]

[3]S. Coogan, "A contractive approach to separable Lyapunov functions for monotone systems", *Automatica*, 2019.

# Example

$$\bullet \longleftrightarrow x_1 \longleftrightarrow x_2 \longleftrightarrow x_3 \longleftrightarrow x_4 \longleftrightarrow x_5$$

- Consensus on line graph with anchored node 0:

$$\dot{x} = \underbrace{\begin{bmatrix} -2 & 1 & 0 & & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & 0 & 0 & 1 & -1 \end{bmatrix}}_{A} x$$

- $A\mathbb{1} \leq 0$, so hypersquares are nonexpansive

- Distribute“excess negativity” from first column by choosing

$$\eta = \begin{bmatrix} 1 \\ 1+\varepsilon_1 \\ 1+\varepsilon_1+\varepsilon_2 \\ \vdots \\ 1+\varepsilon_1+\ldots+\varepsilon_{n-1} \end{bmatrix}$$

with $1 > \varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_{n-1} > 0$, then

$$A\eta < 0.$$

# Reachability for nonmonotone systems

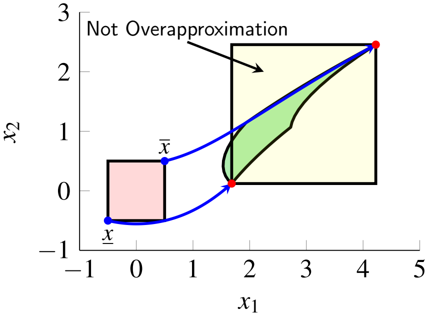
- Generally, cannot bound the reachable set between two extreme trajectories for nonmonotone systems

**Nonmonotone System:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^2 + 2 \\ x_1 \end{bmatrix}$$

$$[\underline{x}, \overline{x}] = [(-0.5, -0.5), (0.5, 0.5)]$$

$$T = 1$$



# Mixed monotonicity embeds nonmonotone systems in a monotone system

- Given $\dot{x} = f(x,w)$, disturbance input $w \in \mathcal{W} = [\underline{w}, \overline{w}] = \{w : \underline{w} \le w \le \overline{w}\}$

Mixed monotone approach: find decomposition functions $\underline{d}$, $\overline{d}$ such that:

1. $\underline{d}(x,x,w,w) = \overline{d}(x,x,w,w)$ for all $x, w$ and
2. the $2n$ dimensional *embedding system*[4,5]

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\overline{x}} \end{bmatrix} = \begin{bmatrix} \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}) \\ \overline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}) \end{bmatrix}$$

is monotone w.r.t the *southeast order* $\le_{\mathsf{SE}}$ on $\mathbb{R}^{2n}$ defined as:

$$(x, \widehat{x}) \le_{\mathsf{SE}} (y, \widehat{y}) \text{ if and only if } x \le y \text{ and } \widehat{y} \le \widehat{x}.$$

[4] G. Enciso, H. Smith, E. Sontag, "Non-monotone systems decomposable into monotone systems with negative feedback", *Journal of Diff. Eq.*, 2006

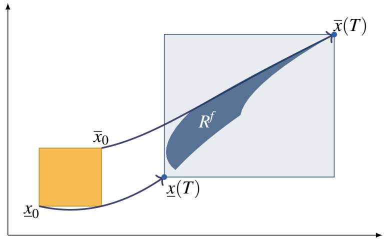
[5] H. Smith, "Global stability for mixed monotone systems", *Journal of Difference Equations and Applications*, 2008

# Reachability from embedding system

Reachable set from embedding trajectory:

$$\mathcal{R}^f(T;[\underline{x}_0,\overline{x}_0]) \subseteq [\underline{x}(T),\overline{x}(T)]$$

where $\underline{x}(t),\overline{x}(t)$ is embedding system solution with initial condition $(\underline{x}_0,\overline{x}_0)$




- **MM is fast**: A single trajectory of the deterministic embedding bounds reachable sets of the original system
- **MM is scalable**: If $(\underline{x}_{eq},\overline{x}_{eq})$ is an equilibrium for embedding system, then hyperrectangle $[\underline{x}_{eq},\overline{x}_{eq}]$ is robustly forward invariant[6]

[6] S. Coogan, "Mixed monotonicity for reachability and safety in dynamical systems", IEEE CDC, 2020 (tutorial paper)

# Example

**Mixed Monotone System:**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2^2+2\\ x_1\end{bmatrix}$$

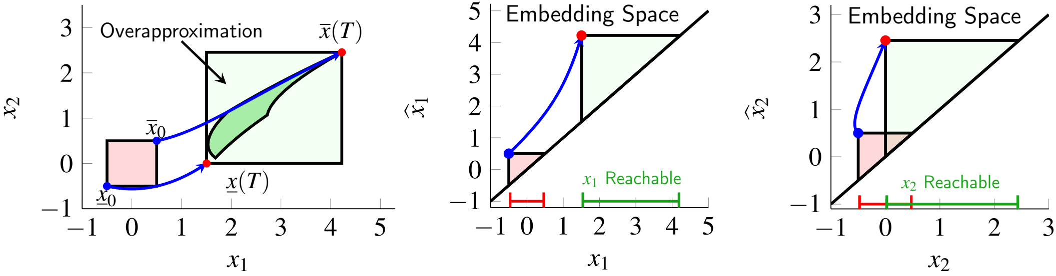
$$[\underline{x}_0, \overline{x}_0] = [(-0.5,-0.5),(0.5,0.5)]$$

$$T = 1$$

**Decomposition Function:**

$$\underline{d}_1(x,\widehat{x}) = \begin{cases} x_2^2+2 & \text{if } x_2 \geq 0 \text{ and } x_2 \geq -\widehat{x}_2, \\ \widehat{x}_2^2+2 & \text{if } \widehat{x}_2 \leq 0 \text{ and } x_2 < -\widehat{x}_2, \\ 2 & \text{if } x_2 < 0 \text{ and } \widehat{x}_2 > 0. \end{cases}$$

$$\underline{d}_2(x,\widehat{x}) = x_1, \qquad \overline{d}(x,\widehat{x}) = \underline{d}(\widehat{x},x)$$



# Stability in the embedding space implies robust invariance

**Mixed Monotone System:**

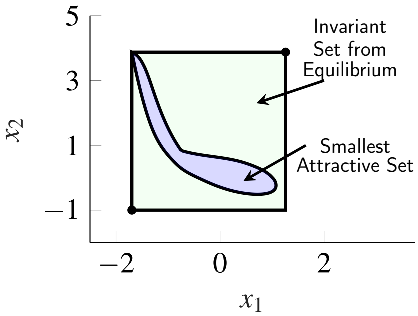
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_1^3 - x_2 - w\\ x_1^2 - x_2 + w^3\end{bmatrix}$$

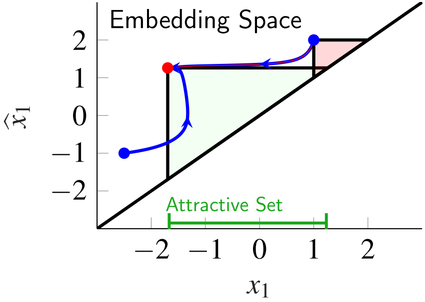
$$\mathcal{W} = [-1, 1]$$

**Decomposition Function:**

$$d_1(x, w, \widehat{x}, \widehat{w}) = -x_1^3 - \widehat{x}_2 - \widehat{w}$$

$$d_2(x, w, \widehat{x}, \widehat{w}) = \begin{cases} x_1^2 - x_2 + w^3 & \text{if } x_1 \geq 0 \text{ and } x_1 \geq -\widehat{x}_1, \\ \widehat{x}_1^2 - x_2 + w^3 & \text{if } \widehat{x}_1 \leq 0 \text{ and } x_1 < -\widehat{x}_1, \\ -x_2 + w^3 & \text{if } x_1 < 0 \text{ and } \widehat{x}_1 > 0. \end{cases}$$



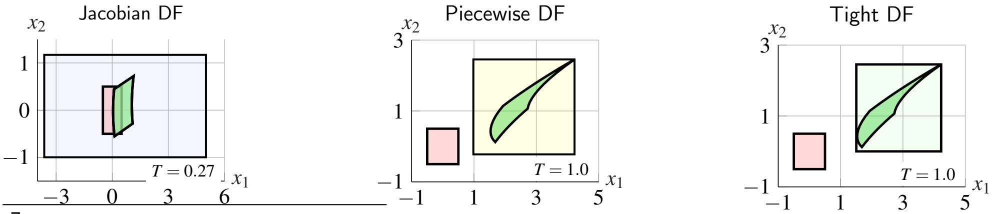
# Decomposition functions

**Theorem**[7]**.** Any system has best (i.e, tightest) decomposition functions satisfying

$$\underline{d}_i(x,\widehat{x},w,\widehat{w}) = \min_{y\in[x,\widehat{x}],y_i=x_i,z\in[w,\widehat{w}]} f_i(y,z)$$

$$\overline{d}_i(x,\widehat{x},w,\widehat{w}) = \max_{y\in[x,\widehat{x}],y_i=\widehat{x}_i,z\in[w,\widehat{w}]} f_i(y,z)$$

- Closed form sometimes possible, otherwise, several automated approaches for finding decomposition functions[8] (using, e.g., interval arithmetic, Jacobian bounds)




[7] M. Abate, M. Dutreix, S. Coogan, *IEEE L-CSS*, 2020.

[8] A. Harapanahalli, S. Jafarpour, S. Coogan, `Python-based toolbox`, forthcoming.

# Contraction of the tight embedding is same as original system

**Theorem.** Let $\begin{bmatrix} \dot{\underline{x}} \\ \dot{\overline{x}} \end{bmatrix} = \begin{bmatrix} \underline{d}(\underline{x},\overline{x},\underline{w},\overline{w}) \\ \overline{d}(\underline{x},\overline{x},\underline{w},\overline{w}) \end{bmatrix} =: e(\underline{x},\overline{x})$ be the embedding system construction from tight decomposition functions. For any $\eta \in \mathbb{R}^n$,

$$\mu_{\infty,[\eta]^{-1}}\left(\frac{\partial f}{\partial x}(x,w)\right) \leq c \quad \text{for all } x$$

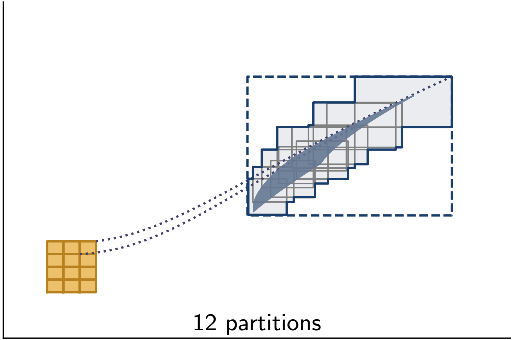
if and only if

$$\mu_{\infty,[(\eta,\eta)]^{-1}}\left(\frac{\partial e}{\partial(\underline{x},\overline{x})}(\underline{x},\overline{x})\right) \leq c \quad \text{for all } \underline{x},\overline{x}.$$

- Reachable sets computed from a trajectory of the tight embedding system are at least as good as using contraction alone on the original system

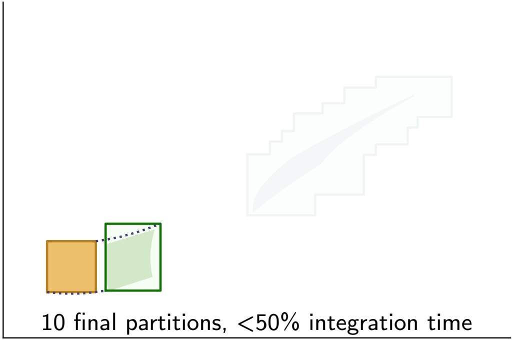
# Idea of adaptive partitioning

Nonadaptive partition



12 partitions

Adaptive partitioning

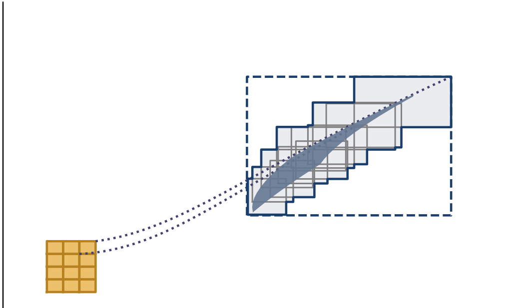


10 final partitions, <50% integration time

- Fixing # of partitions, adaptive is faster and can be more accurate
- Fixing computation time, adaptive allows for more partitions

# Idea of adaptive partitioning

Nonadaptive partition



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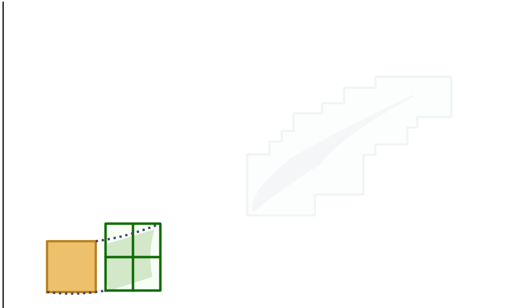

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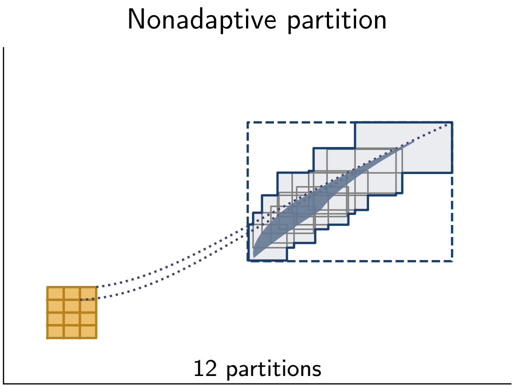


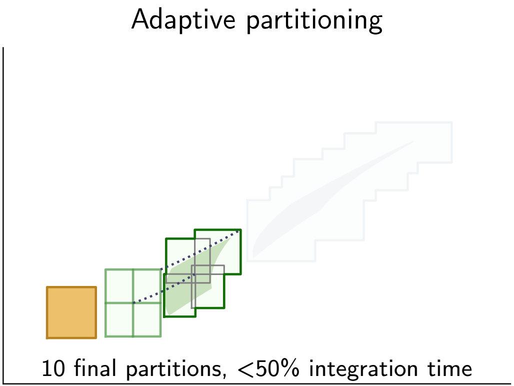


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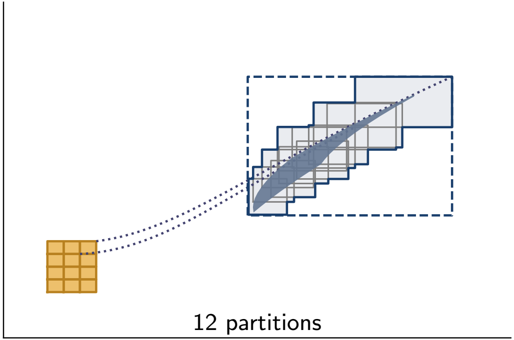
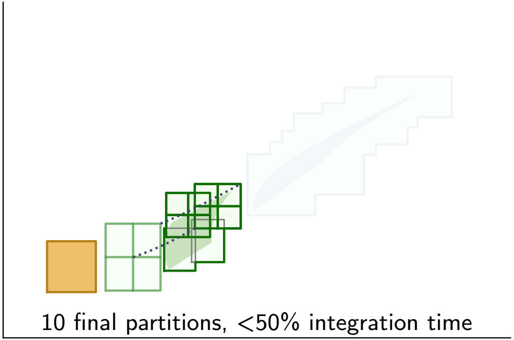



Adaptive partitioning

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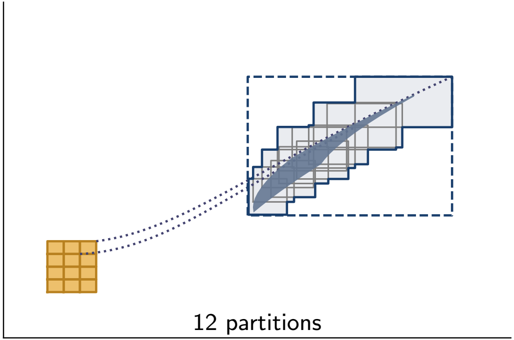

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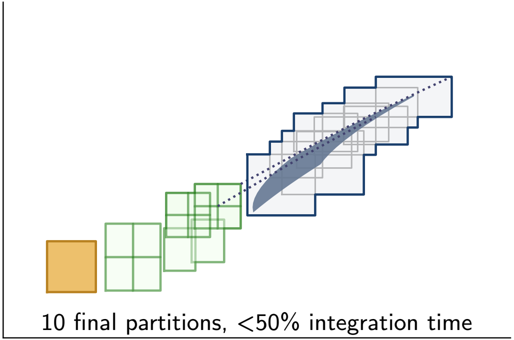

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# Algorithm overview

1. Given

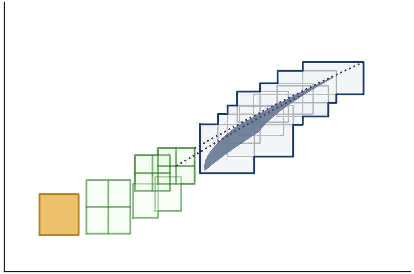
$$\dot{x} = f^{ol}(x,u,w) \quad \text{(open-loop system)}$$
$$u = N(x) \quad \text{(NN controller)},$$

obtain decomposition function for closed-loop system

$$\dot{x} = f^{cl}(x,w) = f^{ol}(x,N(x),w)$$

from

- Open-loop decomposition function for $f^{ol}$ and
- Bounding functions for open-loop controller $N(x)$ (next slides)



2. Efficiently evolve embedding system to obtain overapproximation set
3. Adaptively (in space and time) partition to avoid compounding conservatism

# Neural network bounding functions

Given NN $N(x)$ and bounds $[\underline{z},\overline{z}]$, $\underline{N}_{(\underline{z},\overline{z})}(\underline{x},\overline{x})$ and $\overline{N}_{(\underline{z},\overline{z})}(\underline{x},\overline{x})$ are *bounding functions* for $N(x)$ valid on $[\underline{z},\overline{z}]$ if

$$\underline{N}_{(\underline{z},\overline{z})}(\underline{x},\overline{x}) \leq N(x) \leq \overline{N}_{(\underline{z},\overline{z})}(\underline{x},\overline{x}) \quad \text{for all } x \in [\underline{x},\overline{x}] \subseteq [\underline{z},\overline{z}].$$

Example: CROWN[9] is a NN verifier that produces affine bounds:

$$\underline{A}_{(\underline{z},\overline{z})}x + \underline{b}_{(\underline{z},\overline{z})} \leq N(x) \leq \overline{A}_{(\underline{z},\overline{z})}x + \overline{b}_{(\underline{z},\overline{z})}$$

**Lemma.** Given CROWN bounds $\underline{A}_{(\underline{z},\overline{z})}x + \underline{b}_{(\underline{z},\overline{z})} \leq N(x) \leq \overline{A}_{(\underline{z},\overline{z})}x + \overline{b}_{(\underline{z},\overline{z})}$, then

$$\underline{N}_{(\underline{z},\overline{z})}(\underline{x},\overline{x}) = [\underline{A}_{(\underline{z},\overline{z})}]^{+}\underline{x} + [\underline{A}_{(\underline{z},\overline{z})}]^{-}\overline{x} + \underline{b}_{(\underline{z},\overline{z})},$$
$$\overline{N}_{(\underline{z},\overline{z})}(\underline{x},\overline{x}) = [\overline{A}_{(\underline{z},\overline{z})}]^{+}\overline{x} + [\overline{A}_{(\underline{z},\overline{z})}]^{-}\underline{x} + \overline{b}_{(\underline{z},\overline{z})}$$

are bounding functions for $N(x)$ valid on $[\underline{z},\overline{z}]$.

[9] Zhang, Weng, Chen, Hsieh,Daniel, "Efficient neural network robustness certification with general activation functions." *Advances in neural information processing systems*, 2018.

# Mixed monotone reachability of neural network controlled systems

Assume we have:

1. Open-loop decomposition function $d^{ol}$ for $f^{ol}(x,u,w)$
2. Ability to construct NN bounding functions $\underline{N}_{(\underline{z},\overline{z})}$, $\overline{N}_{(\underline{z},\overline{z})}$
   - Construction is expensive while queries are cheap (e.g., CROWN vs matrix multiply)

For reachability of $f^{cl}(x,w) = f^{ol}(x,N(x),w)$, we propose an algorithm that features:

1. Separation of bounding function construction from queries
2. Spatially and temporally adaptive partitioning for accuracy

# Closed-loop decomposition function

**Theorem.**[10] Let $\underline{N}_{(\underline{z},\overline{z})}$, $\overline{N}_{(\underline{z},\overline{z})}$ be bounding functions for $N(x)$ valid on some $[\underline{z},\overline{z}]$, and let $d^{ol}$ be a decomposition function for $f^{ol}$. Then $d_i$ defined by

$$\underline{d}_i(\underline{x},\overline{x},\underline{w},\overline{w}) = d_i^{ol}(\underline{x},\overline{x},\underline{\eta},\overline{\eta},\underline{w},\overline{w}),$$
$$\overline{d}_i(\underline{x},\overline{x},\underline{w},\overline{w}) = d_i^{ol}(\overline{x},\underline{x},\overline{\nu},\underline{\nu},\overline{w},\underline{w}),$$

where

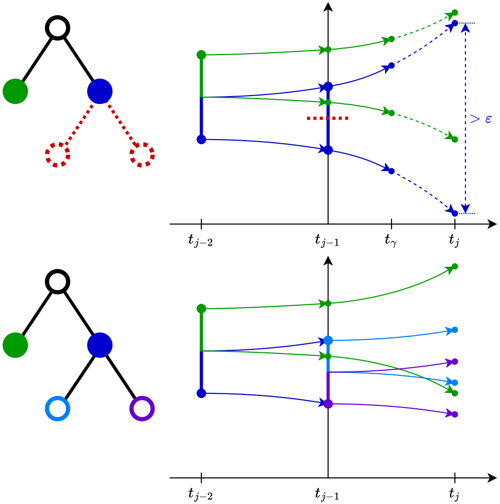
$$\underline{\eta} = \underline{N}_{(\underline{z},\overline{z})}(\underline{x},\overline{x}_{[i:\underline{x}]}) \quad \overline{\eta} = \overline{N}_{(\underline{z},\overline{z})}(\underline{x},\overline{x}_{[i:\underline{x}]}),$$
$$\underline{\nu} = \underline{N}_{(\underline{z},\overline{z})}(\overline{x},\underline{x}_{[i:\overline{x}]}), \quad \overline{\nu} = \overline{N}_{(\underline{z},\overline{z})}(\overline{x},\underline{x}_{[i:\overline{x}]}),$$

is a decomposition function for $f^{cl}$ for any $(\underline{x},\overline{x})$ such that $[\underline{x},\overline{x}] \subseteq [\underline{z},\overline{z}]$, where $v_{[i:v']}$ is the vector $v$ with its $i$-th component replaced with that of $v'$.

[10] S. Jafarpour, A. Harapanahalli, S. Coogan, "Interval Reachability of Nonlinear Dynamical Systems with Neural Network Controllers", *L4DC*, 2023

# Spatially and temporally adaptive partitioning for accuracy

1. Evolving tree structure tracks partitionings, leaves are current partitions
2. Along an evolution step (e.g., control update step) if partition expansion exceeds threshold, add a partition and reevolve
3. NN bounding construction performed at some nodes, used for all descendent leaves



# Case study: kinematic bicycle

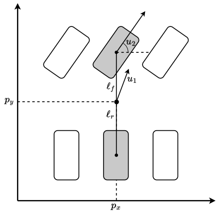
$$x = (p_x, p_y, \phi, v) \in \mathbb{R}^4$$

$$\dot{p_x} = v\cos(\phi + \beta(u_2))$$

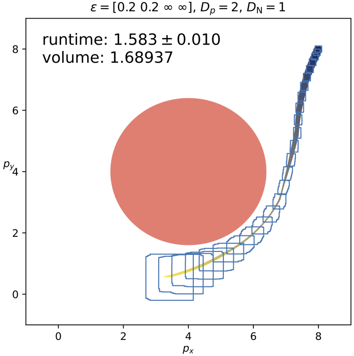
$$\dot{p_y} = v\sin(\phi + \beta(u_2))$$

$$\dot{\phi} = \frac{v}{\ell_r}\sin(\beta(u_2))$$

$$\dot{v} = u_1 + w$$




- Obstacle-avoiding go-to-origin NN controller ($4 \times 100 \times 100 \times 2$ ReLU) trained from nonlinear MPC data

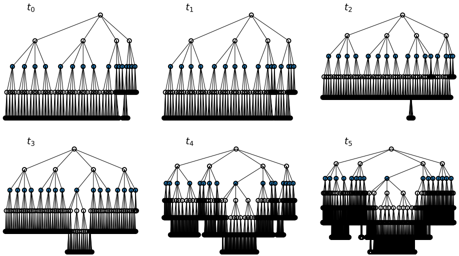


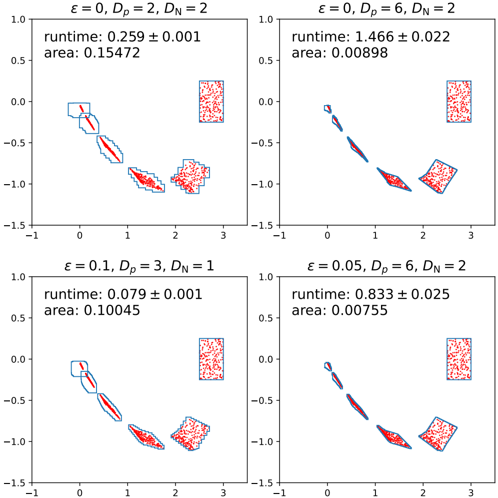

| | $D_p$ | $D_N$ | Runtime (s) | Volume |
|---|---|---|---|---|
| non-adaptive | 2 | 1 | $1.851 \pm 0.010$ | 1.988 |
| adaptive | 2 | 1 | $1.583 \pm 0.010$ | 1.689 |
| non-adaptive | 2 | 2 | $4.274 \pm 0.023$ | 0.803 |
| adaptive | 2 | 2 | $3.332 \pm 0.012$ | 0.787 |

# Benchmark: Double integrator

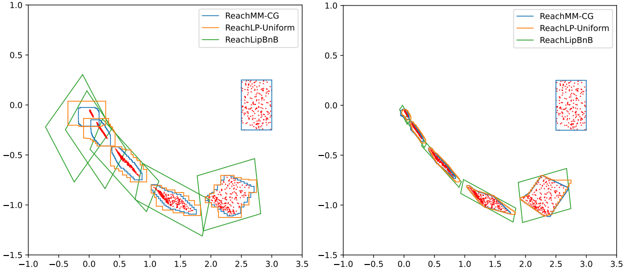
$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_t$$

- $2 \times 10 \times 5 \times 1$ ReLU NN controller from literature



# Benchmark: Double integrator



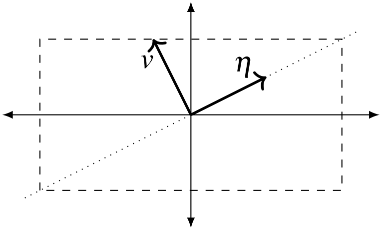

| Method | Setup | Runtime (s) | Area |
|---|---|---|---|
| ReachMM-CG | $(0.1,\ 3,\ 1)$ | $\mathbf{0.079 \pm 0.001}$ | $\mathbf{1.0 \cdot 10^{-1}}$ |
| (our method) | $(0.05,\ 6,\ 2)$ | $\mathbf{0.833 \pm 0.025}$ | $\mathbf{7.5 \cdot 10^{-3}}$ |
| ReachLP-Unif | 4 | $0.212 \pm 0.002$ | $1.5 \cdot 10^{-1}$ |
| | 16 | $3.149 \pm 0.004$ | $1.0 \cdot 10^{-2}$ |
| ReachLP-GSG | 55 | $0.913 \pm 0.031$ | $5.3 \cdot 10^{-1}$ |
| | 205 | $2.164 \pm 0.042$ | $8.8 \cdot 10^{-2}$ |
| ReachLipBnB | 0.1 | $0.956 \pm 0.067$ | $5.4 \cdot 10^{-1}$ |
| | 0.001 | $3.681 \pm 0.100$ | $1.2 \cdot 10^{-2}$ |

# Generalizing rectangular compatibility of monotonicity and contraction

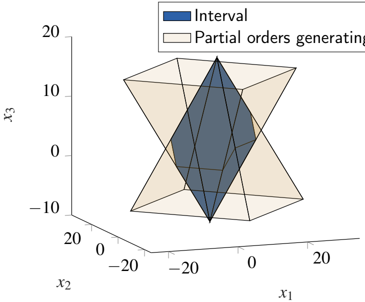
Given a partial order $\preceq_K$ induced by a cone $K$, for any $\eta \in \mathrm{int}(K)$, the corresponding *gauge norm* (or seminorm) is defined as

$$\|v\|_{\eta,K} = \inf\{\lambda \in \mathbb{R}_{\geq 0} \mid -\eta \preceq_K v \preceq_K \eta\}.$$

- For the standard order $\leq$, gauge norms are weighted $\ell_\infty$ norms and intervals




- More generally, any interval is a gauge norm ball



# Monotonicity w.r.t. polyhedral cones

A *polyhedral cone* has the form $K = \underbrace{\{x \in \mathbb{R}^n \mid Hx \geq \mathbb{0}_m\}}_{\text{Halfspace rep}} = \underbrace{\{Vx \in \mathbb{R}^n \mid x \geq \mathbb{0}_m\}}_{\text{Vertex rep}}$

**Theorem.**[11] The system $\dot{x} = f(x)$ is monotone w.r.t. $\preceq_K$, $K$ a polyhedral cone, if any of the following equivalent conditions hold:

1. There exists $\alpha(x,u)$ such that
$$H\left(\frac{\partial f}{\partial x}(x) + \alpha(x,u) I_n\right) V \geq \mathbb{0}_{m\times m} \quad \text{for all } x, u,$$
2. There exists Metzler $P(x,u)$ such that
$$H \frac{\partial f}{\partial x}(x) = P(x,u) H \quad \text{for all } x, u,$$
3. There exists Metzler $Q(x,u)$ such that
$$\frac{\partial f}{\partial x}(x) V = V Q(x,u) \quad \text{for all } x, u.$$

[11] S. Jafarpour and S. Coogan, "Monotonicity and Contraction on Polyhedral Cones", in submission to *IEEE TAC*.

# Contraction with respect to gauge norms in monotone systems

**Theorem.**[12] Let $\dot{x} = f(x)$ be monotone w.r.t. $\preceq_K$, $K$ polyhedral. Let $\eta \in \mathrm{int}(K)$, $c \in \mathbb{R}$. The following are equivalent:

1. $\frac{\partial f}{\partial x}(x)\eta \preceq_K c\eta$,
2. $H\frac{\partial f}{\partial x}(x) \leq cH\eta$,
3. $\|y(t) - x(t)\|_{\eta,K} \leq e^{ct}\|y(0) - x(0)\|_{\eta,K}$ for any two trajectories $x(t), y(t)$

[12] S. Jafarpour and S. Coogan, "Monotonicity and Contraction on Polyhedral Cones", in submission to *IEEE TAC*.

# Example: Inverted pendulum

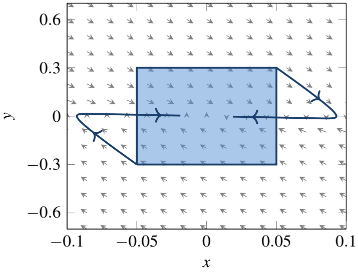
- Inverted pendulum:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \sin(x_1) + u$$

- Choose $u = -k_1 x_1 - k_2 x_2$, to stabilize origin, then

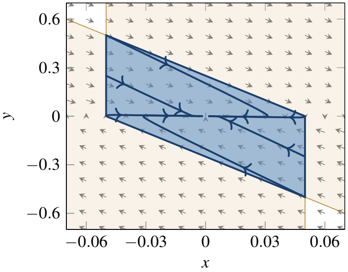
$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} 0 & 1 \\ \cos(x_1) - k_1 & -k_2 \end{bmatrix}$$

- No controller contracts a rectangle around the origin




- Alternative cone

$$K = \left\{ x \mid \begin{bmatrix} 1 & 0 \\ 1 & \gamma \end{bmatrix} x \geq \mathbb{0}_m \right\}$$



# Example: Network flow

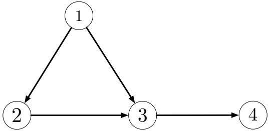
- Nodal flows:

$$\dot{x} = -\underbrace{B_v B_v^T}_{L} x, \quad B_v = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Edge flows: $z = B_v^T x$.
- Non-pointed $K = \{x \in \mathbb{R}^n : B_v^T x \geq \mathbb{0}_4\}$:

$$-B_v L = \underbrace{\begin{bmatrix} -3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Metzler}} B_v^T,$$

so edge flow dynamic are monotone:
$B_v^T x(0) \leq B_v^T y(0) \implies B_v^T x(t) \leq B_v^T y(t) \forall t$




- For $\eta = [1.5 \ 1.4 \ 1 \ 0.1]^T$,

$$-B_v^T L \eta \leq \frac{-3}{4} B_v^T \eta,$$

so edge flows contract with rate $3/4$.

- However, with coordinates $z = B_v^T x$,

$$\dot{z} = B_v^T \dot{x} = -\underbrace{B_v^T B_v}_{L_\mathcal{E}} \underbrace{B_v x}_{z} = -L_\mathcal{E} z,$$

but $L_\mathcal{E}$ is not Metzler nor Hurwitz.

# Conclusions

In this talk:

1. Contraction (and [mixed] monotonicity) for computing reachable sets, applications to learning-based controllers
2. Gauge norms induced by partial orders are natural for computing with intervals
3. Opportunities for "computational contraction theory"

`coogan.ece.gatech.edu` for papers and code

Thank you

# Bounding accuracy

**Theorem** (informal).

$$\begin{aligned}\|\text{partition}(t)\|_\infty \leq & e^{ct}\|\text{partition}(0)\|_\infty + \frac{L_w^{\mathrm{ol}}(e^{ct}-1)}{c}\|\overline{w}-\underline{w}\|_\infty \\ & + \frac{L_u^{\mathrm{ol}}(e^{ct}-1)}{c}\sup_z \max\{\|\underline{N}(z,z)-N(z)\|_\infty, \|\overline{N}(z,z)-N(z)\|_\infty\}\end{aligned}$$

where

$$\begin{aligned} c &= \text{local contraction rate of closed-loop embedding system} \\ L_u^{\mathrm{ol}} &= \text{local Lipschitz constant of open-loop embedding system w.r.t. } u \\ L_w^{\mathrm{ol}} &= \text{local Lipschitz constant of open-loop embedding system w.r.t. } w \end{aligned}$$

- Separation $\Longrightarrow$ improve term 1 independent of term 3
- Spatial awareness $\Longrightarrow$ improved local Lipschitz constant $L_w^{\mathrm{ol}}$, $L_u^{\mathrm{ol}}$
- Temporal awareness $\Longrightarrow$ controlled $e^{ct}\|\text{partition}(0)\|_\infty$

# Bounding closed-loop contraction rate

**Theorem** (informal). Contraction rate $c$ of closed loop system satisfies

$$c \leq c^{\mathrm{ol}} + L_u^{\mathrm{ol}} \sup_{(\underline{x},\overline{x})} \max \left\{ \left\| \frac{\partial \underline{N}}{\partial(\underline{x},\overline{x})}(\underline{x},\overline{x}) \right\|_\infty, \left\| \frac{\partial \overline{N}}{\partial(\underline{x},\overline{x})}(\underline{x},\overline{x}) \right\|_\infty \right\}$$

where $c^{\mathrm{ol}}$ is contraction rate of the open-loop embedding system.

Moreover, when the open-loop decomposition function is tight, $c^{\mathrm{ol}}$ matches the contraction rate of the open-loop system.