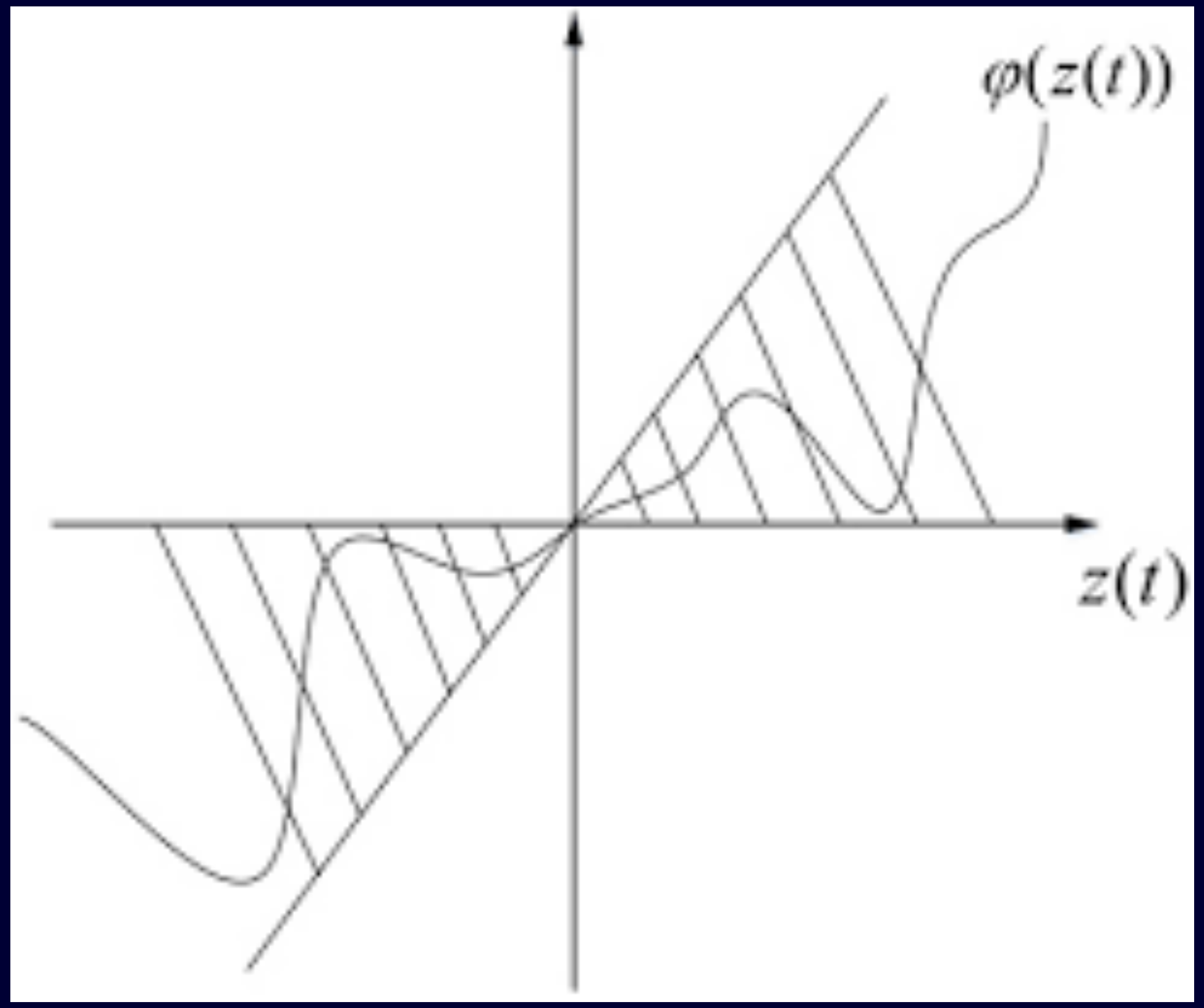


Non-quadratic S-Lemma and Contractivity of Lur’e-type Systems

Rethinking classical absolute stability theory

ANTON PROSKURNIKOV, POLITECNICO DI TORINO,

JOINT WORK WITH **ALEXANDER DAVYDOV AND FRANCESCO BULLO, UCSB**

ACC2023 Workshop W02 Contraction Theory for Systems, Control, and Learning.

May 30, 2023

Plan of the talk:

- **Absolute stability**

✓ Some terms. Two directions of absolute stability theory.

✓ Standard way of deriving stability conditions via S-Lemma

✓ Motivation for this work.

- **Main Results**

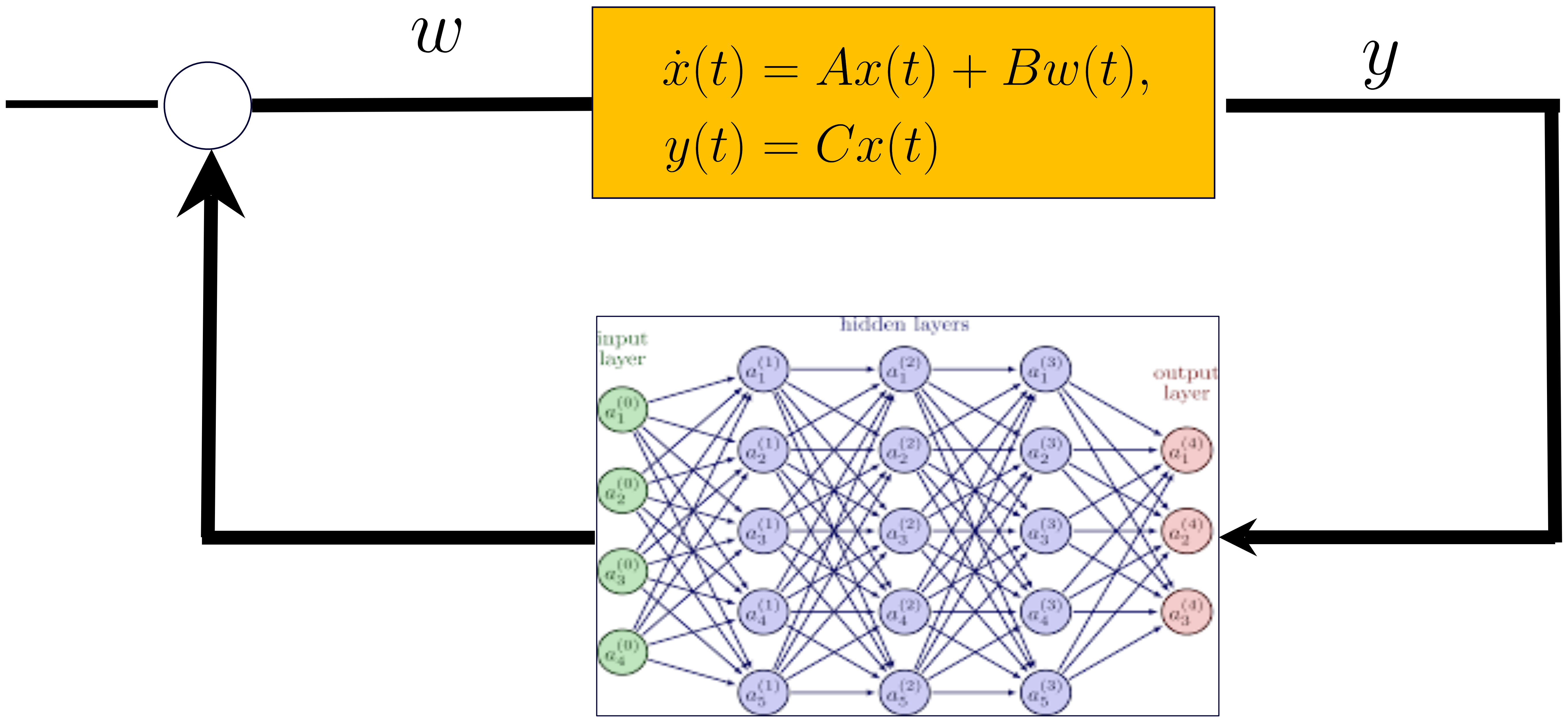
✓ Mathematical tools: log-norm and weak pairing

✓ Non-quadratic S-Lemma.

✓ Absolute contractivity criteria

- **Conclusions, Future works**

Absolute stability theory studies feedback interconnections of “simple and known” and “uncertain and (potentially) complex”



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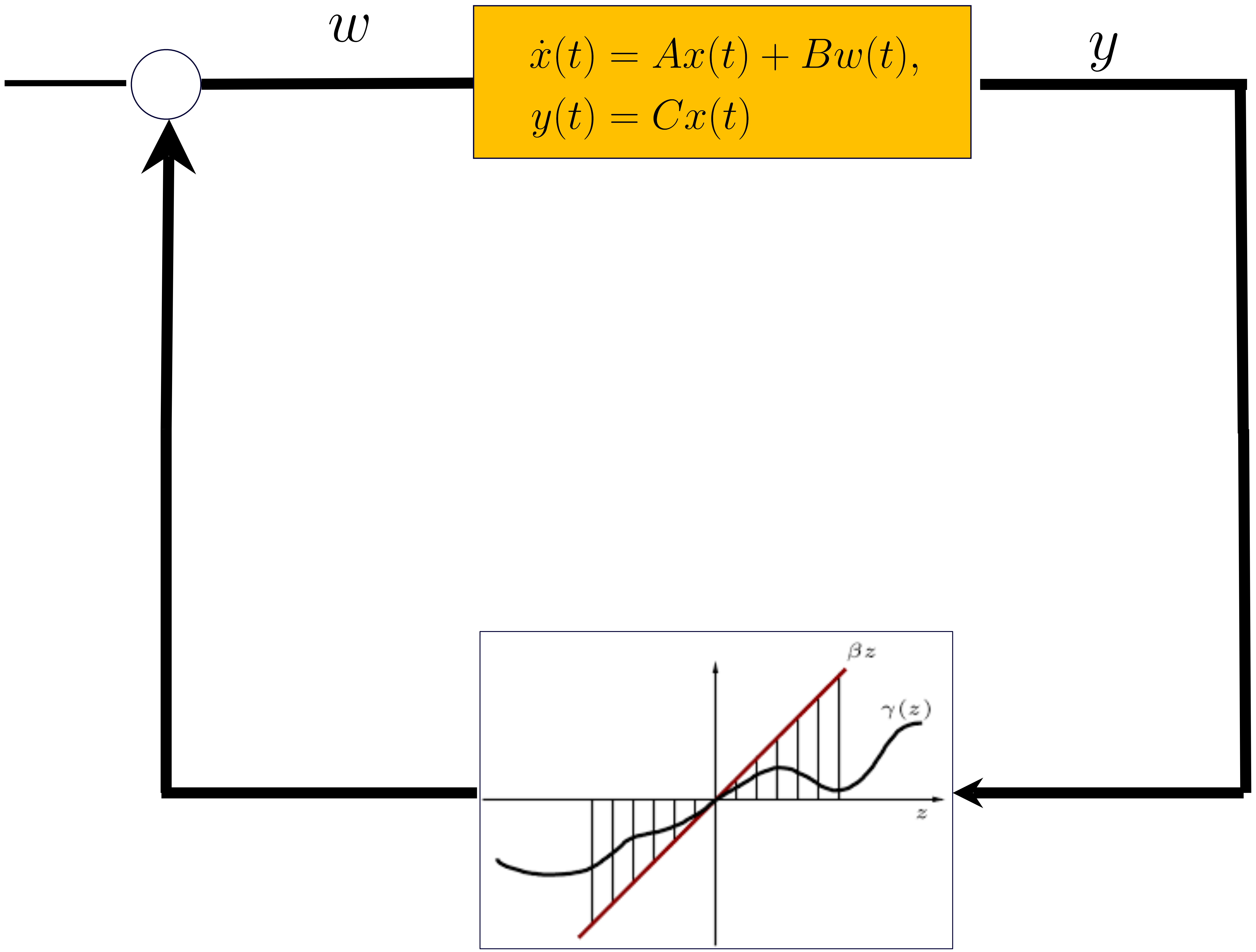

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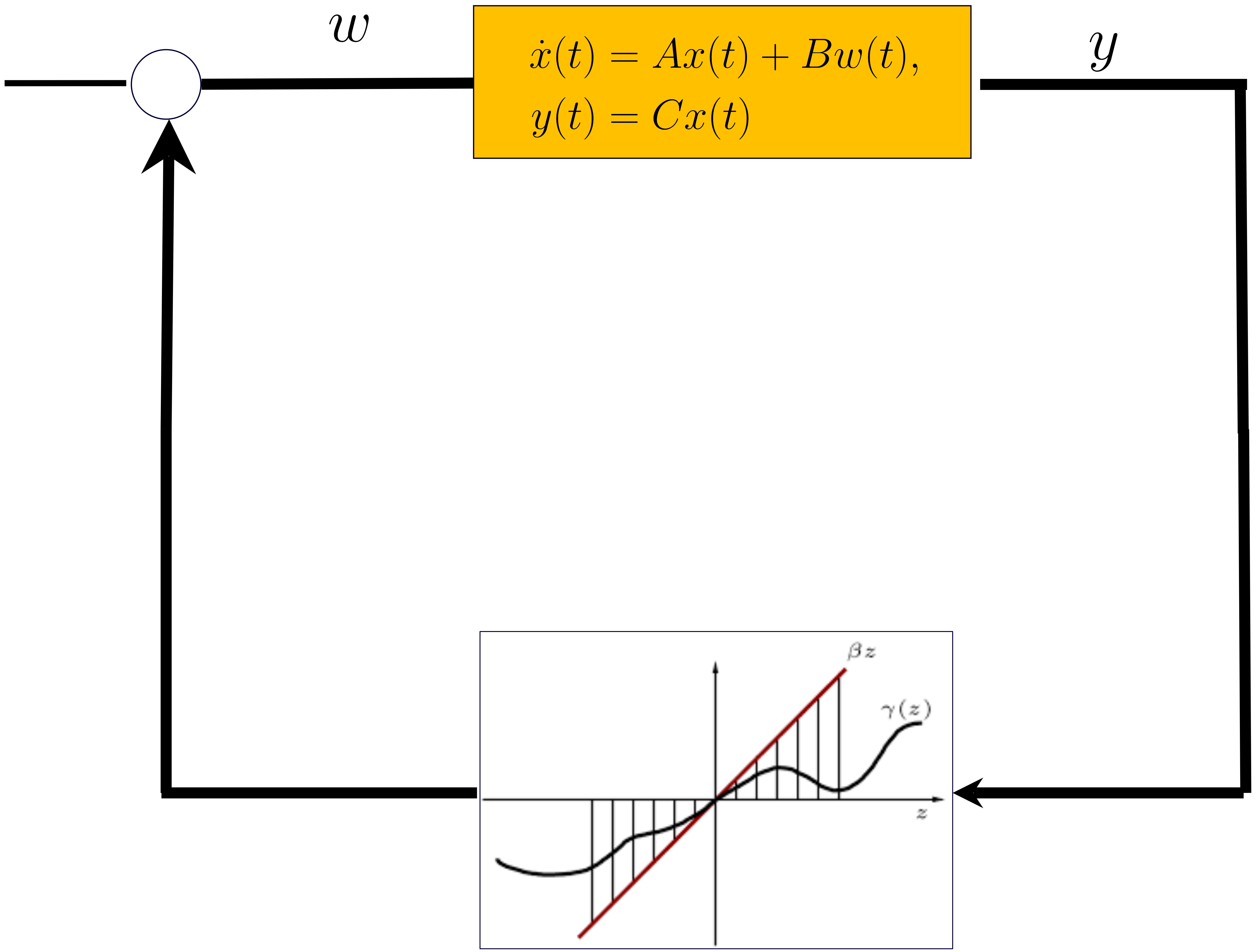

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1901-1980

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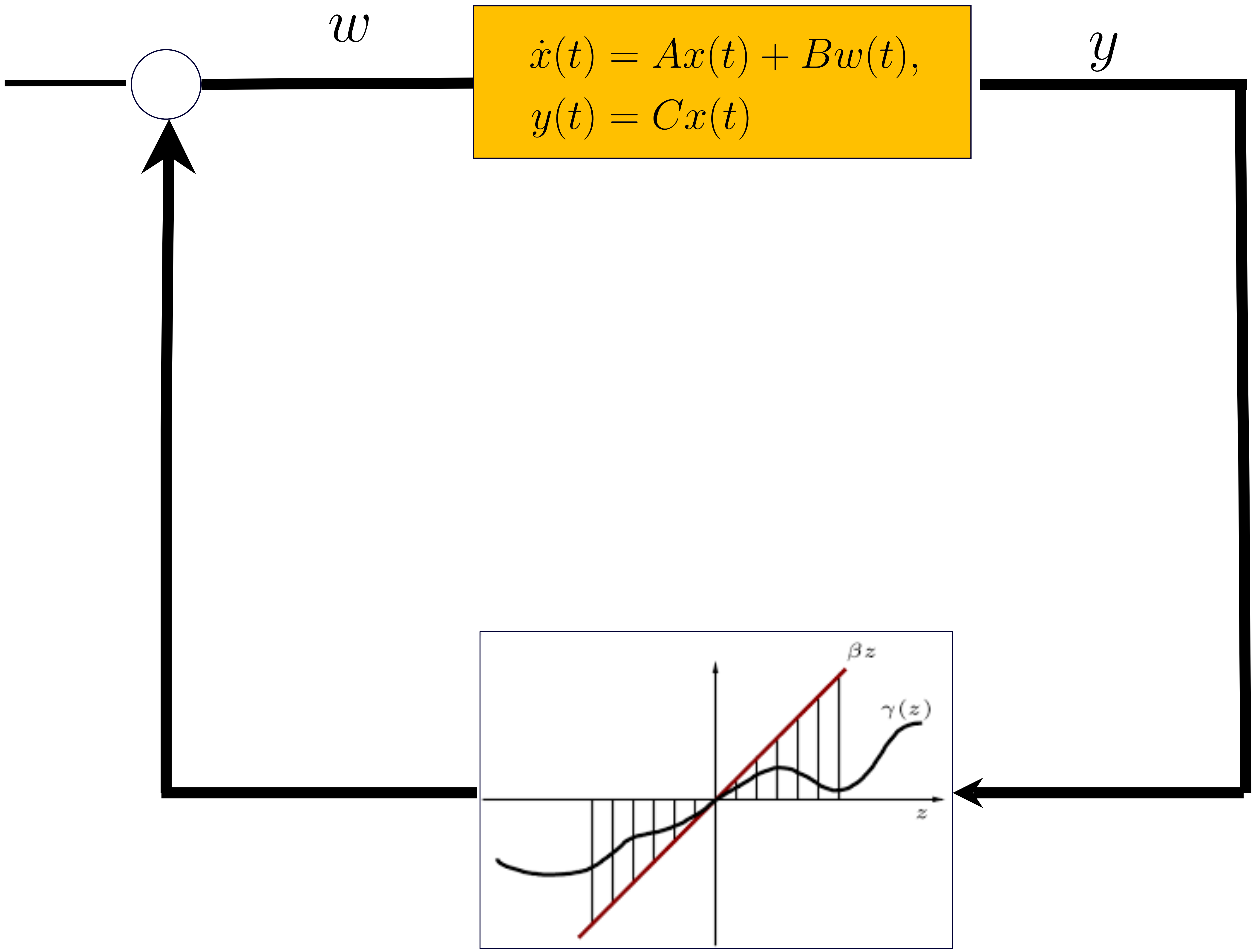

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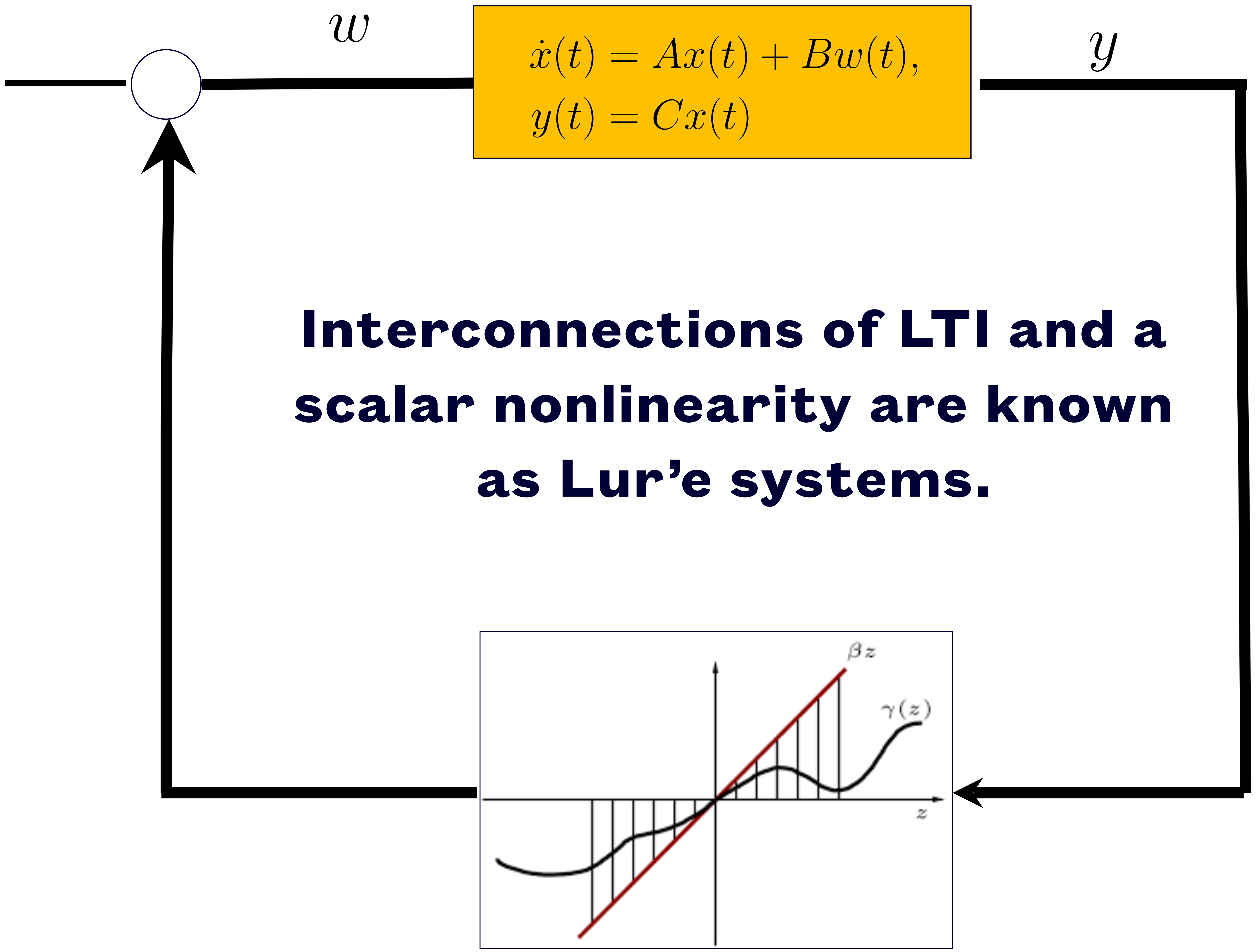

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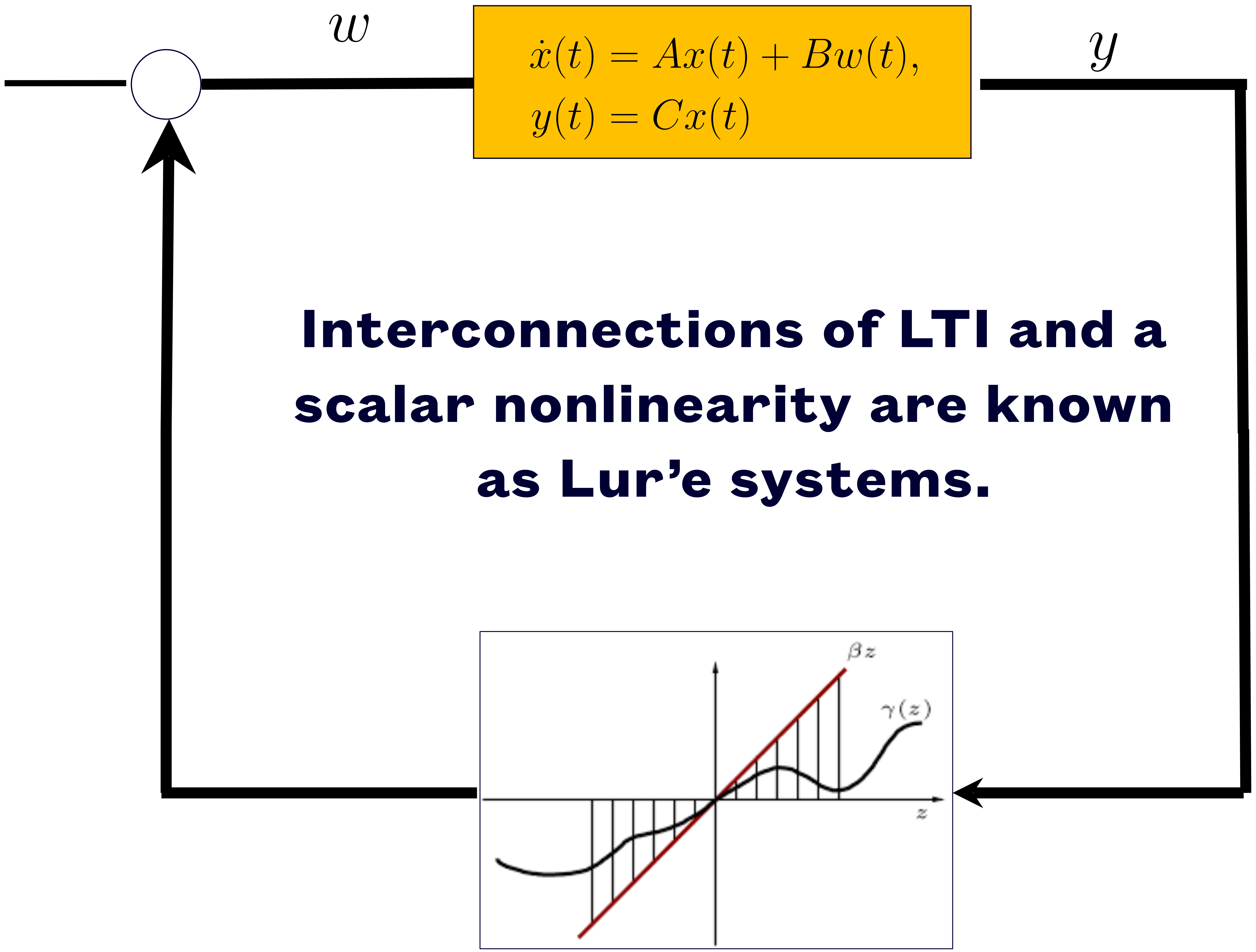

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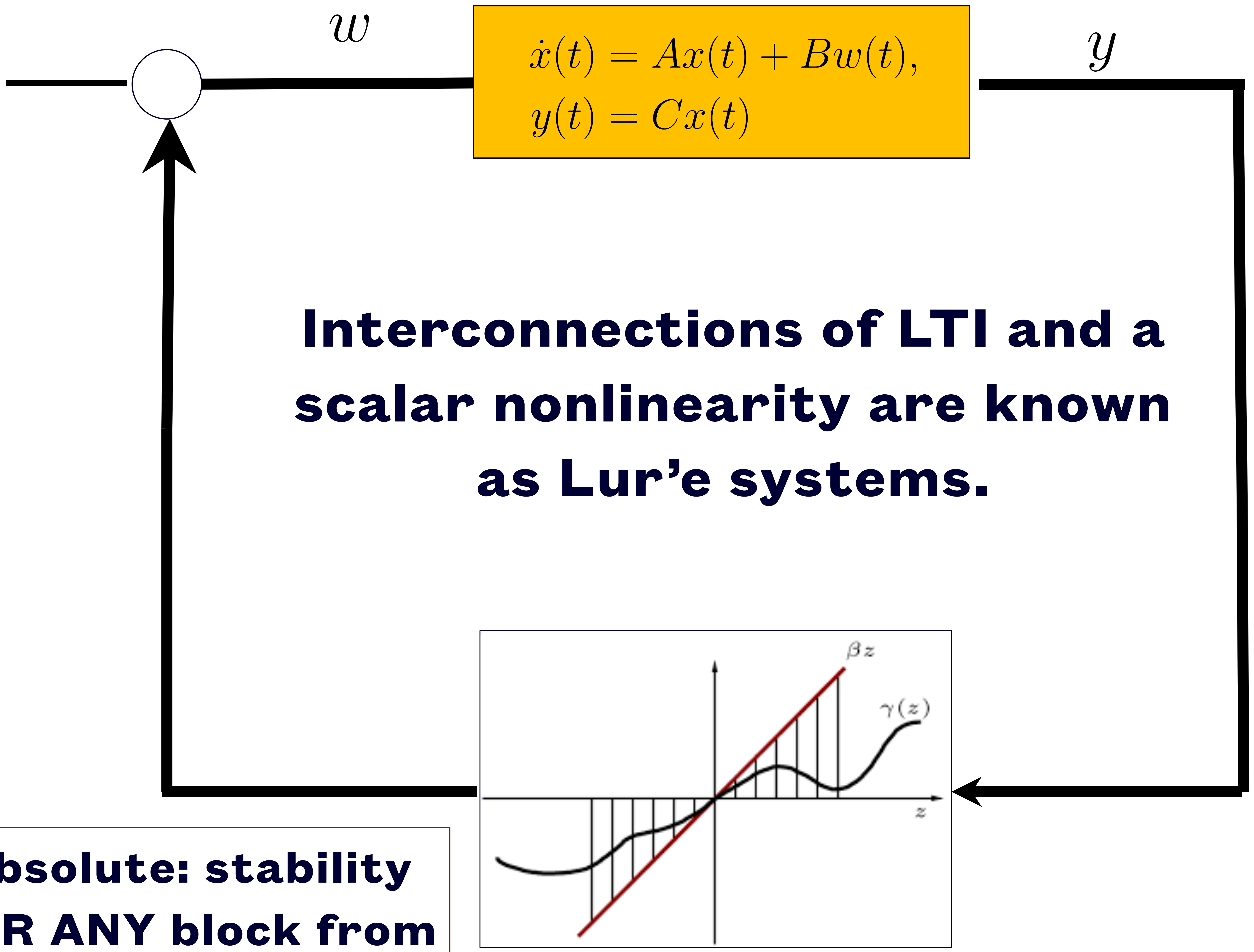

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Absolute: stability FOR ANY block from a given class: robustness

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Two tracks in absolute stability theory

- **Lyapunov stability certificates for the state space models**

  Lur'e, Barbashin, Persidskii, Kalman, Rosenwasser, Yakubovich, Popov, Narendra,…

$$V(x) = x^\top H x \qquad \left[+\int_0^y \varphi(s)ds \quad \text{(optional)}\right]$$

✓ **Lyapunov condition** $\dot{V} < 0$**: quadratic constraints + S-Lemma**

✓ **Existence of the LF (stability): frequency-domain via KYP lemma or (now) SDP.**

✓ **Also: contractivity, output stability, instability, oscillations, ….**

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- **Operator methods, functional spaces, integral quadratic constraints**

  Popov, Desoer, Vidyasagar, Yakubovich, Zames, Falb, Rasvan, Megretskii, Rantzer,…

✓ **General Volterra equations (also, infinite-dimensional)**

✓ **Stability in integral L2 norm, little information about transient process**

✓ **Conditions in frequency domain: hard to verify in MIMO case**

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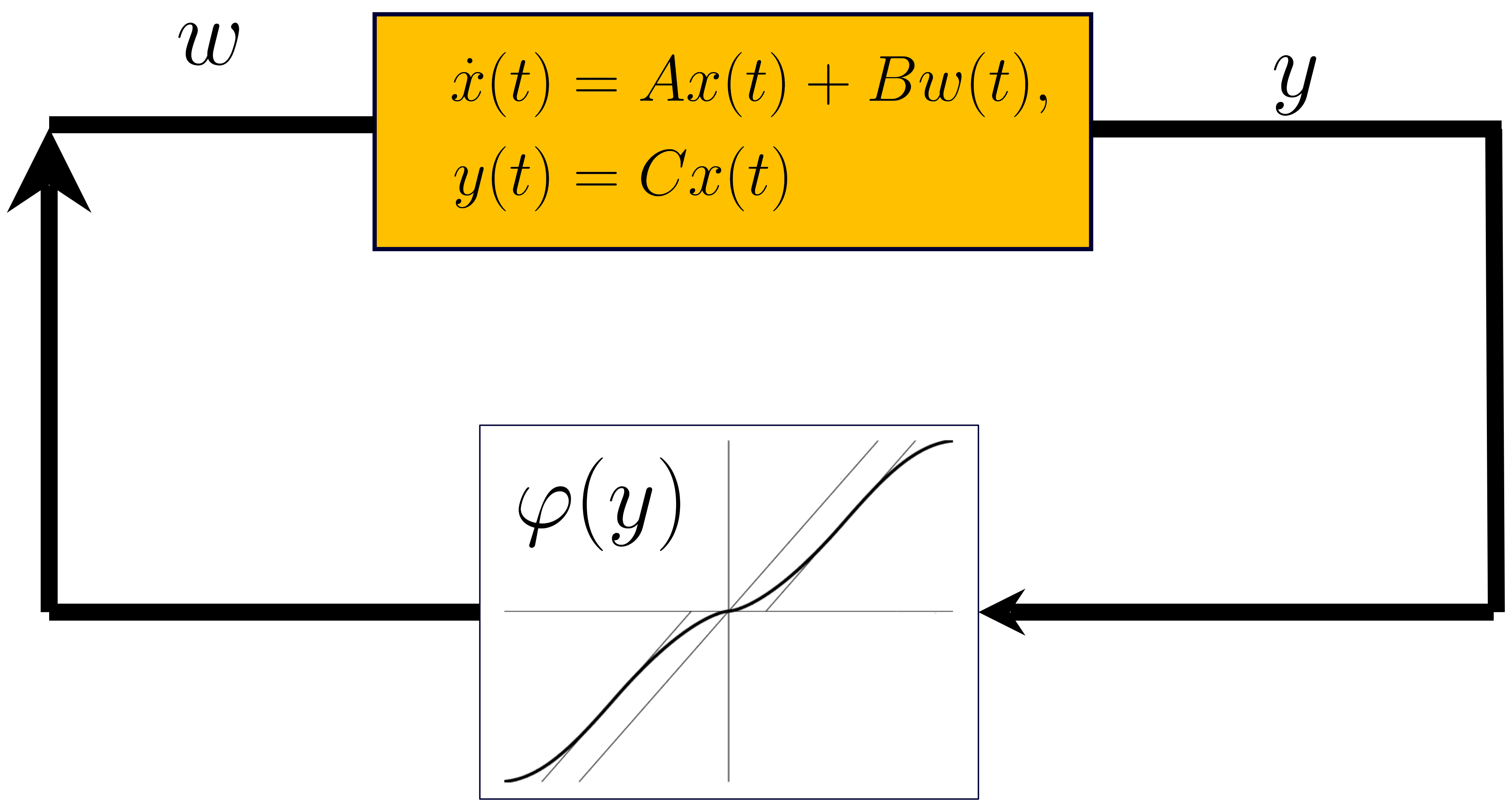

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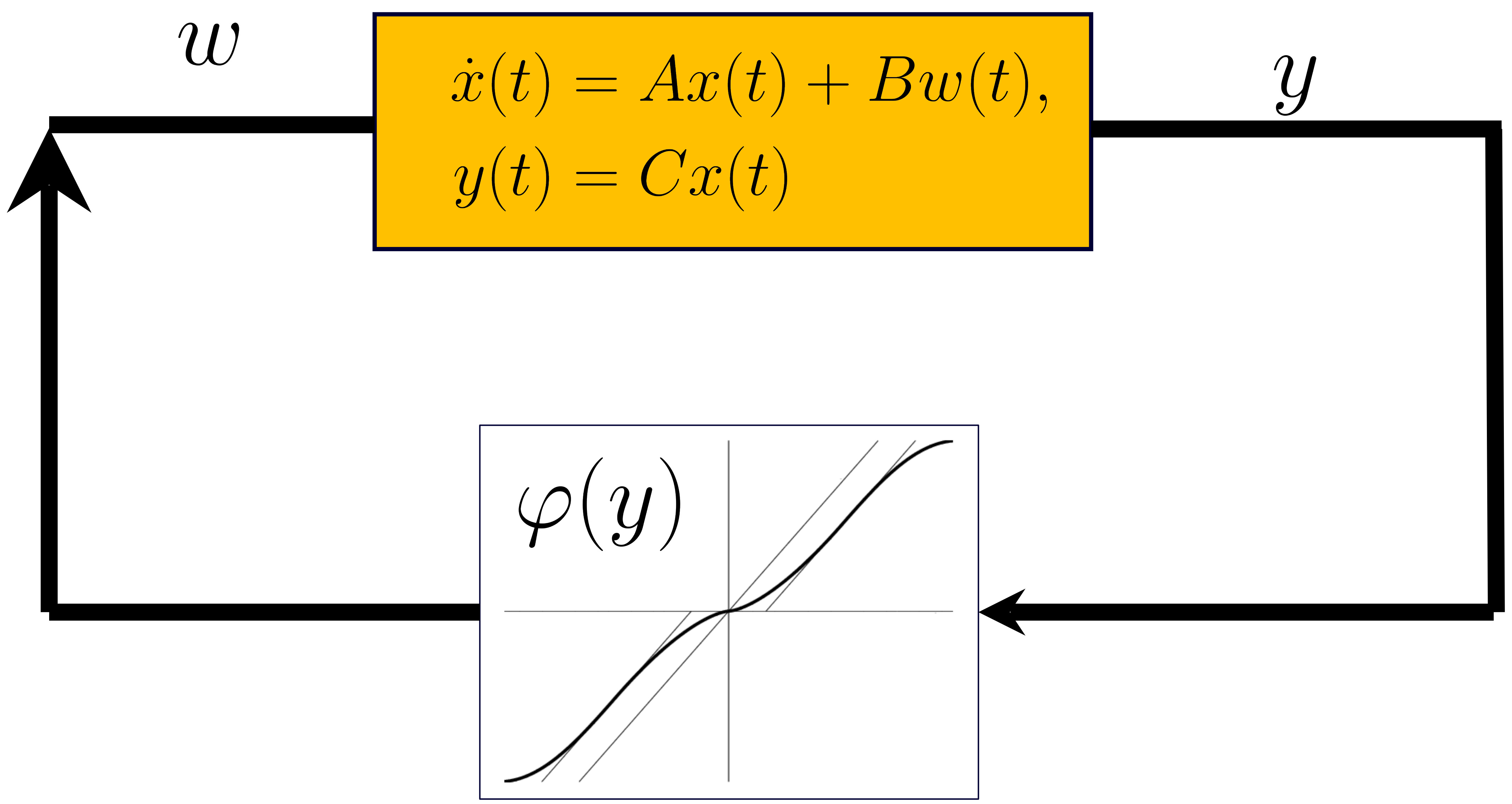

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$$V(\Delta x) = (\Delta x)^\top H(\Delta x) \Longrightarrow \dot{V} = 2(\Delta x)^\top H(A\Delta x + B\Delta w)$$

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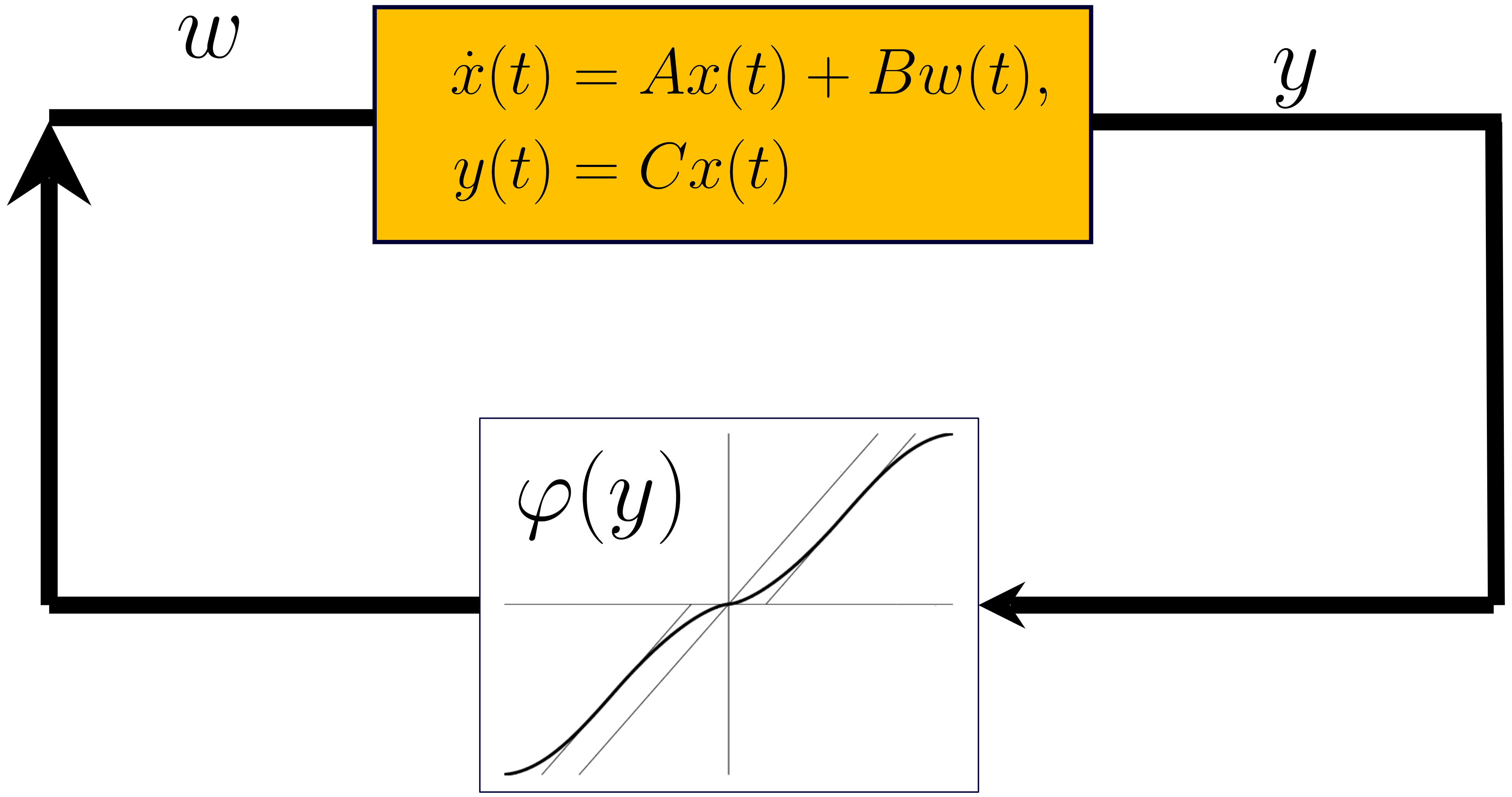

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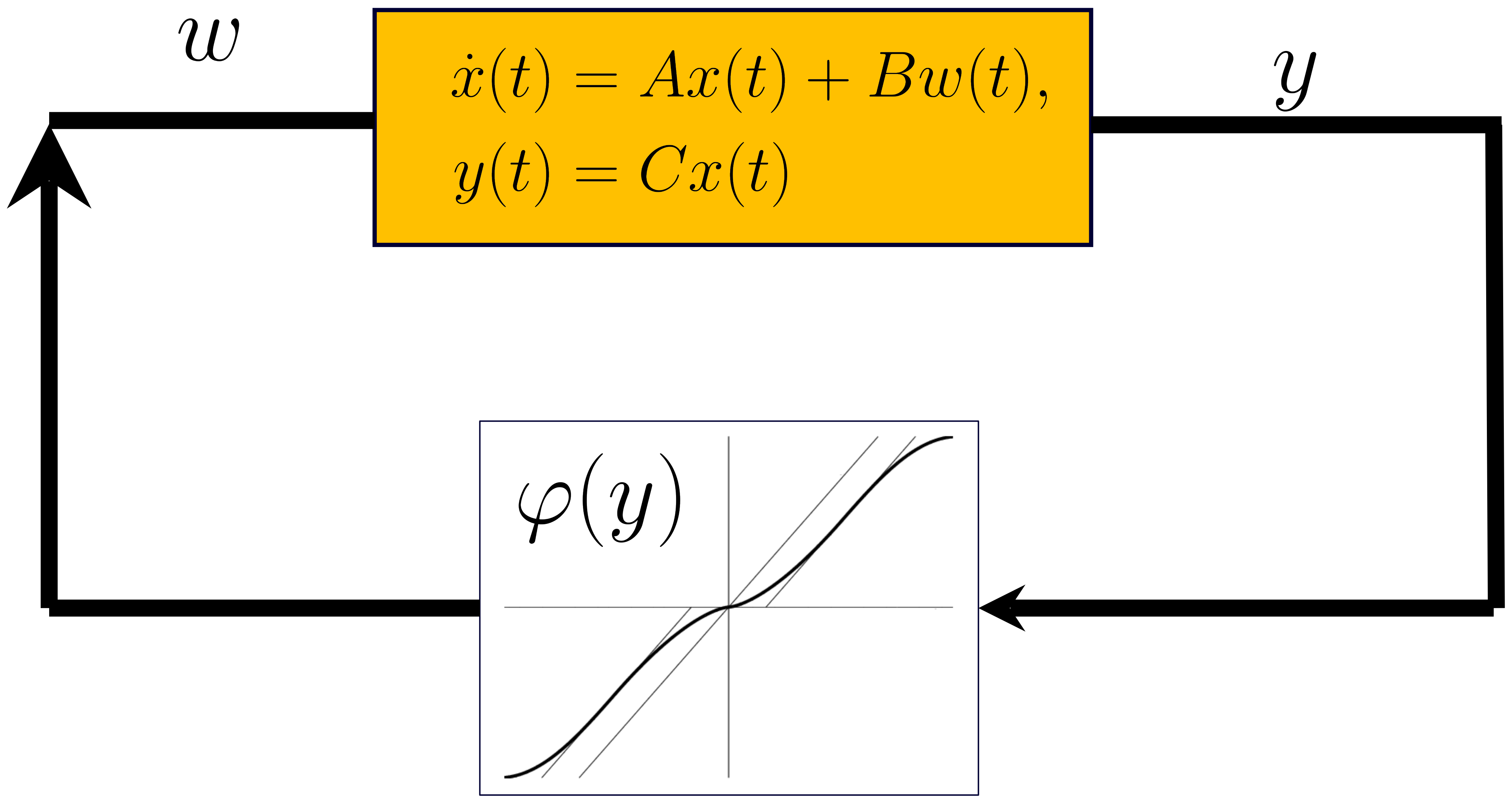

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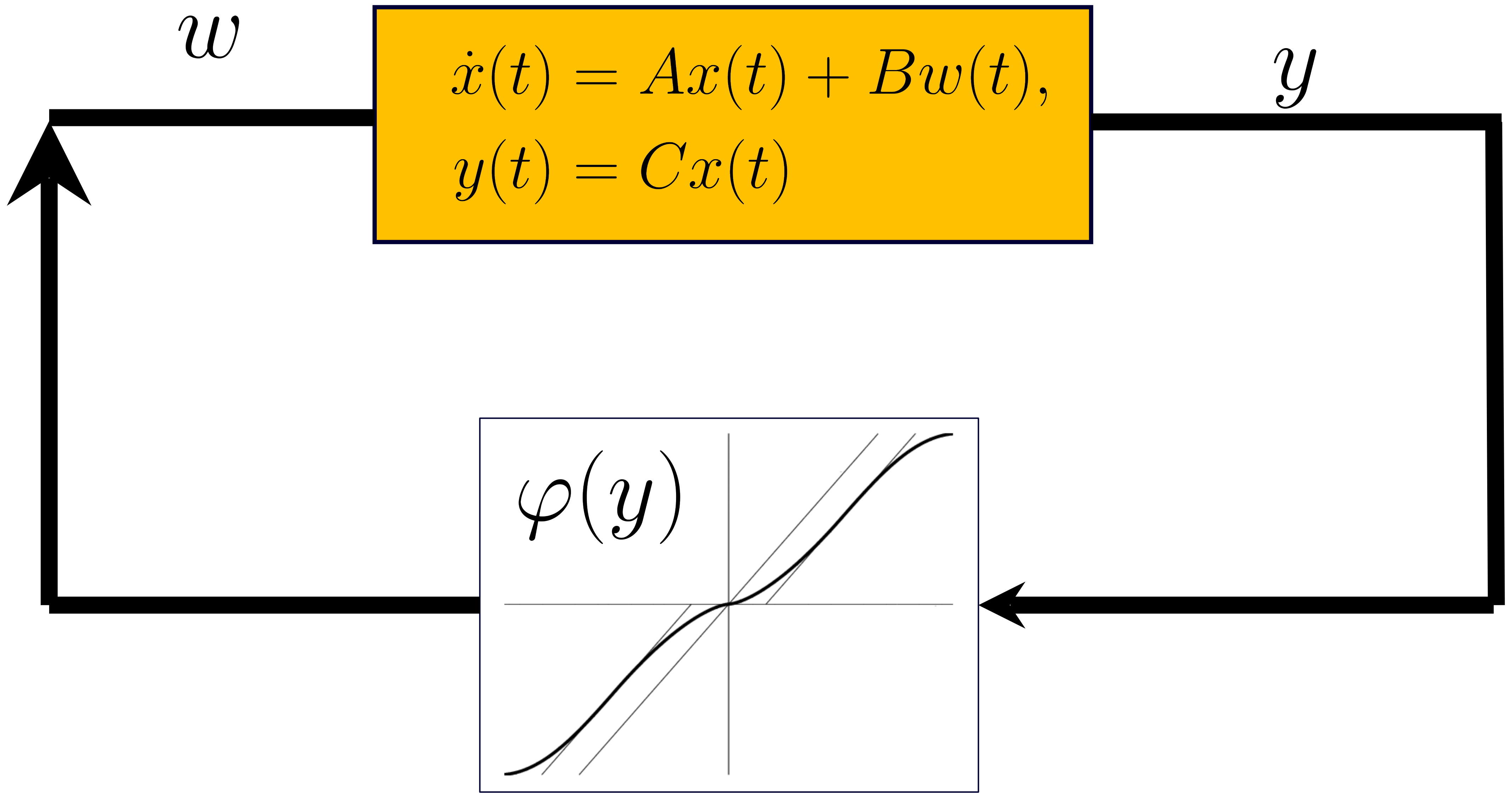

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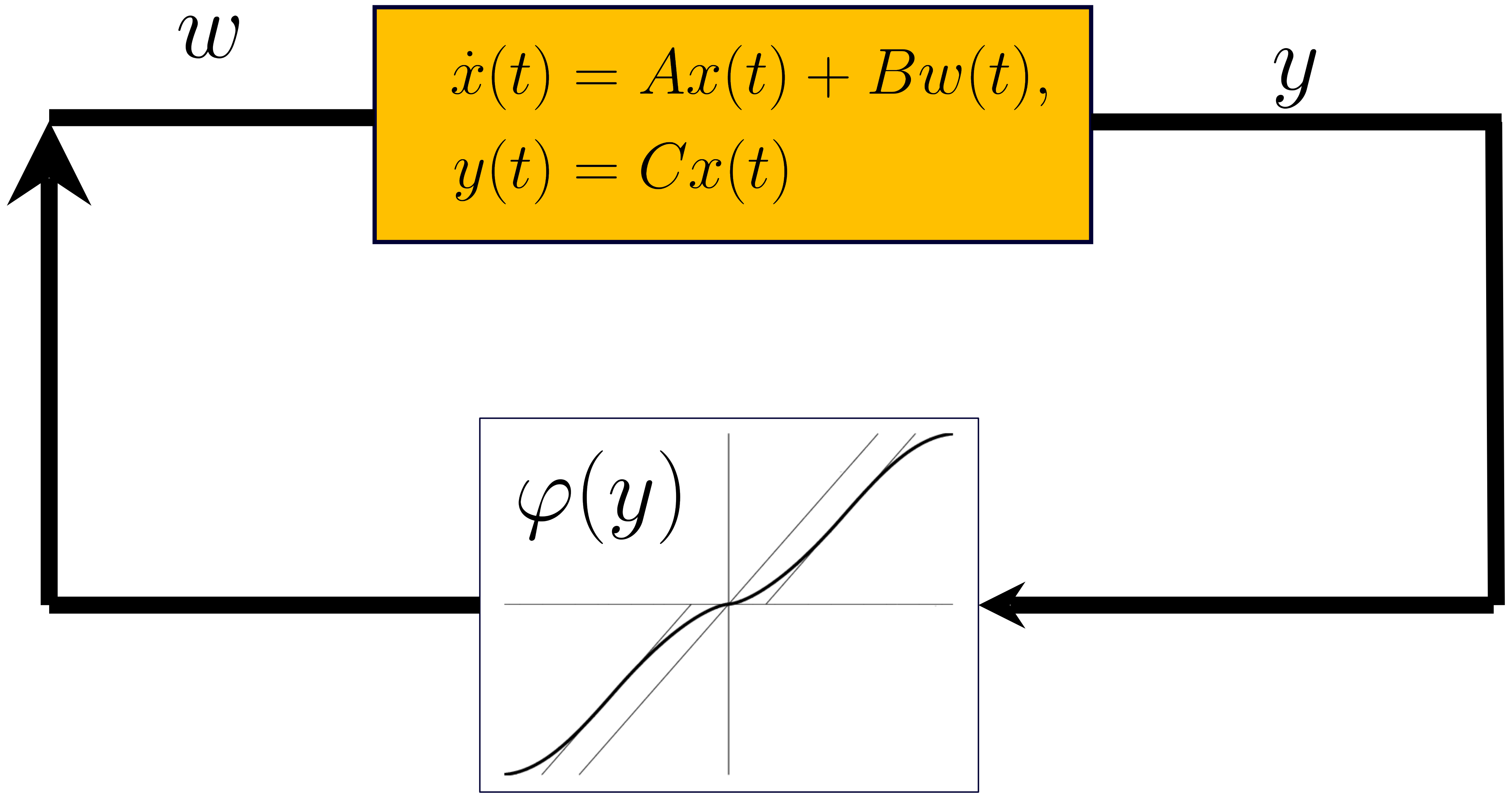

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for some $\tau \geq 0$ and all variables

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1926-2012

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Simple (if) part can be extended to "pseudo-quadratic mappings", LMI is replaced by some log-norm inequality.

Hard (only if) part can be proved in some situations, see journal version

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Norm on a finite-dimensional space $\|\cdot\|$

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Operator norm on matrices

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(i) (Subadditivity and continuity of first argument) $[\![x_1 + x_2, y]\!] \leq [\![x_1, y]\!] + [\![x_2, y]\!]$, for all $x_1, x_2, y \in \mathbb{R}^n$ and $[\![\cdot,\cdot]\!]$ is continuous in its first argument,

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Compatible WP is non-unique

Can always (and henceforth will be) chosen in such a way that

$$\mu(A) = \sup_{|x|=1} [Ax, x]$$

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(iii) (Positive definiteness) $[\![x, x]\!] > 0$, for all $x \neq \mathbb{0}_n$,

(iv) (Cauchy-Schwarz inequality) $|[\![x, y]\!]| \le [\![x, x]\!]^{1/2} [\![y, y]\!]^{1/2}$, for all $x, y \in \mathbb{R}^n$.

Compatible WP is non-unique

Can always (and henceforth will be) chosen in such a way that

$$\mu(A) = \sup_{|x|=1} [Ax, x]$$

$$\frac{d}{dt}\|x(t)\|^2 = [\dot{x}(t), x(t)] \quad \text{a.e.}$$

Norms, log-norms, "weak pairing": table for l_p norms

Norm	Weak pairing	Log norms and Lumer's equality
$\|x\|_2 = \sqrt{x^\top x}$	$[\![x,y]\!]_2 = x^\top y$	$\mu_2(A) = \frac{1}{2}\lambda_{\max}(A + A^\top) = \max_{\|x\|_2=1} x^\top A x$
$\|x\|_p = \left(\sum_i \|x_i\|^p\right)^{1/p}$, $1 < p < \infty$	$[\![x,y]\!]_p = \|y\|_p^{2-p}(y \circ \|y\|^{p-2})^\top x$	$\mu_p(A) = \max_{\|x\|_p=1} (x \circ \|x\|^{p-2})^\top A x$
$\|x\|_1 = \sum_i \|x_i\|$	$[\![x,y]\!]_1 = \|y\|_1 \operatorname{sign}(y)^\top x$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i\neq j} \|a_{ij}\|\right) = \sup_{\|x\|_1=1} \operatorname{sign}(x)^\top A x$
$\|x\|_\infty = \max_i \|x_i\|$	$[\![x,y]\!]_\infty = \max_{i\in I_\infty(y)} x_i y_i$	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j\neq i} \|a_{ij}\|\right) = \sup_{\|x\|_\infty=1} \max_{i\in I_\infty(x)} (Ax)_i x_i$

TABLE 1

Table of norms, weak pairings, and log norms for ℓ_2, ℓ_p for $p \in (1,\infty)$, ℓ_1, and ℓ_∞ norms. We adopt the shorthand $I_\infty(x) = \{i \in \{1,\dots,n\} \mid |x_i| = \|x\|_\infty\}$.

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Easy to find: p=1,2, ∞

Quite difficult to find for a general p, non-convex mathematical programming.

Operator p-norm computation is known to be NP hard.

Most probably, also log-norm.

Plan of the talk:

- **Absolute stability**

✓ Some terms. Two directions of absolute stability theory.

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✓ Motivation for this work.

- **Main Results**

✓ Mathematical tools: log-norm and weak pairing

✓ Non-quadratic S-Lemma.

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“Pseudo-quadratic” S-Lemma.

Suppose that one wishes to prove the following implication with “pseudo-quadratic” forms:

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The implication (?) is valid for each vector *x* if a parameter $\tau \geq 0$ exists such that $\mu(P - \tau Q) \leq 0.$

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Proof: Without loss of generality (since both inequalities are homogeneous), $\|x\| = 1.$

$$[Px, x] \leq [Px - \tau Qx, x] + [\tau Qx, x] \leq \mu(P - \tau Q) + \tau[Qx, x] \leq \tau[Qx, x].$$

Quadratic case S-Lemma: when the norm is usual l_2

Suppose that one wishes to prove the following implication for the standard inner-product:

$$x^\top P x = [Px, x] \leq 0 \quad \overset{?}{\Longleftarrow} \quad x^\top Q x = [Qx, x] \leq 0$$

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How does this condition look like in a simpler form?

$$\mu_2(P - \tau Q) = \frac{1}{2}\left[(P + P^\top) - \tau(Q + Q^\top)\right] \le 0.$$

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Yakubovich S-lemma: **the only if statement holds provided that for at least one vector** $x_0^\top Q x_0 < 0$.
This does not hold for the general norm (counter-examples available in the journal version).

Final remarks about non-quadratic S-lemma

$$x^\top P x = [Px, x] \leq 0 \quad \Longleftarrow \quad x^\top Q x = [Qx, x] \leq 0$$

- **S-lemma in the general case reduces to optimization problem (the implication holds if inf <= 0)**
- **The problem is convex**
- **The oracle computing the function (and gradient, if needed) is of high complexity, except for special norms, e.g. l_1 , l_2 , l_∞ .**
- **Leads to new absolute stability and absolute contraction criteria in non-Euclidean norms.**

Final remarks about non-quadratic S-lemma

$$x^\top P x = [Px, x] \leq 0 \quad \Longleftarrow \quad x^\top Q x = [Qx, x] \leq 0$$

$$\mu(P - \tau Q) \to \inf \quad \text{s.t. } \tau \geq 0$$

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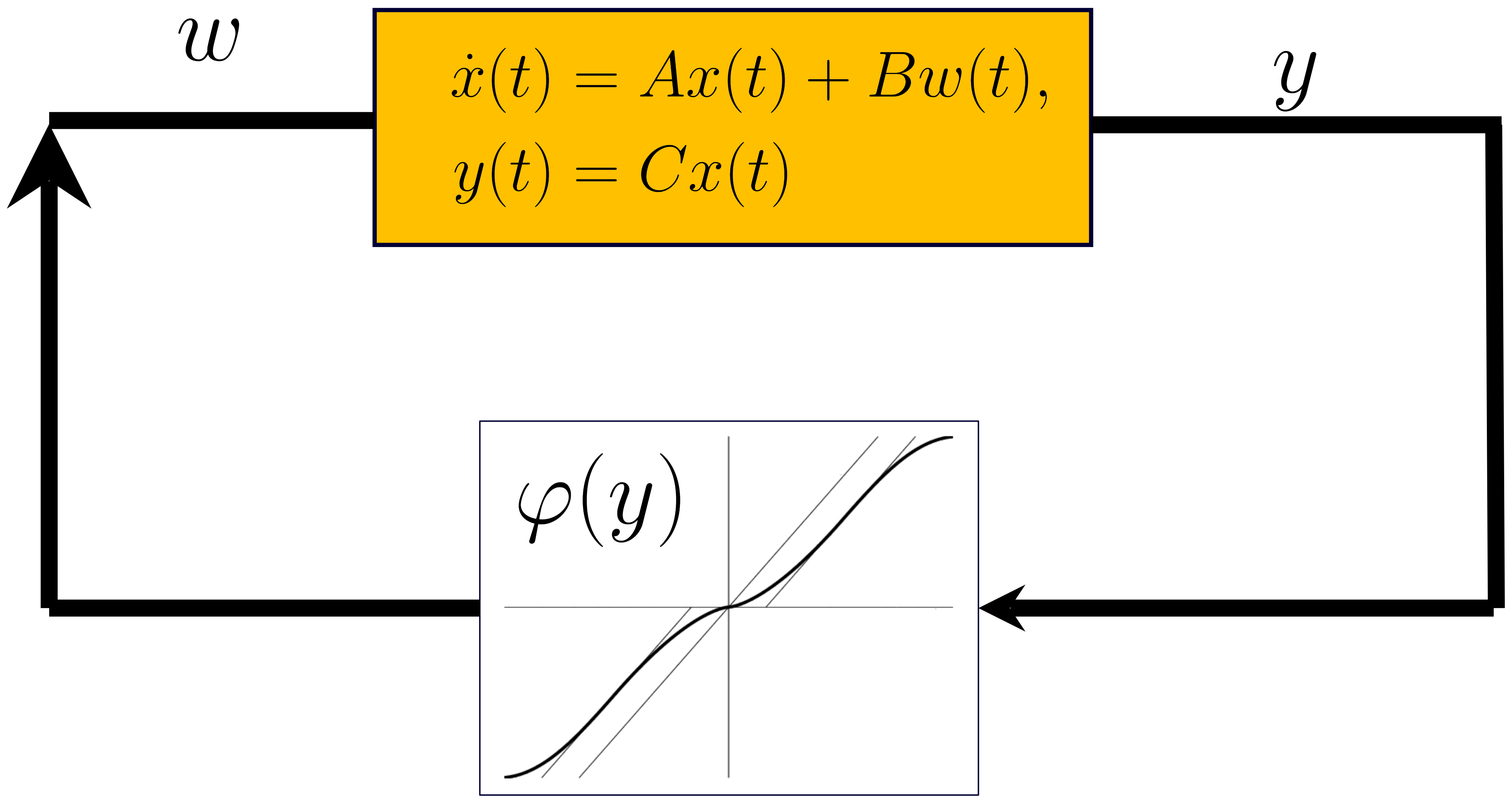
S-procedure for absolute contractivity: non-Euclidean norm case



$$0 \le \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \le \varkappa$$

It remains to apply S-lemma!

Program: repeat the trick with S-lemma.

- **Write the slope restriction as a "pseudo-quadratic" constraint**
- **Write the Lyapunov condition as a "pseudo-quadratic" constraint**
- **Apply S-lemma**
- **Principal difference: extended state-control vector needs to be used!**
- **We first assume that *R=I*.**

$$\dot V \le -c\|\Delta x\|_p^2 \iff [A\Delta x + c\Delta X + B\delta w, \Delta x]_p \le 0$$

$$\iff \left[\begin{pmatrix} A+cI & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta w\end{pmatrix}, \begin{pmatrix}\Delta x\\ \Delta w\end{pmatrix}\right]_p \le 0 \quad \xleftarrow{\;?\;} \quad \left[\begin{pmatrix} 0 & 0 \\ -C & \varkappa^{-1} \end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta w\end{pmatrix}, \begin{pmatrix}\Delta x\\ \Delta w\end{pmatrix}\right]_p \le 0$$

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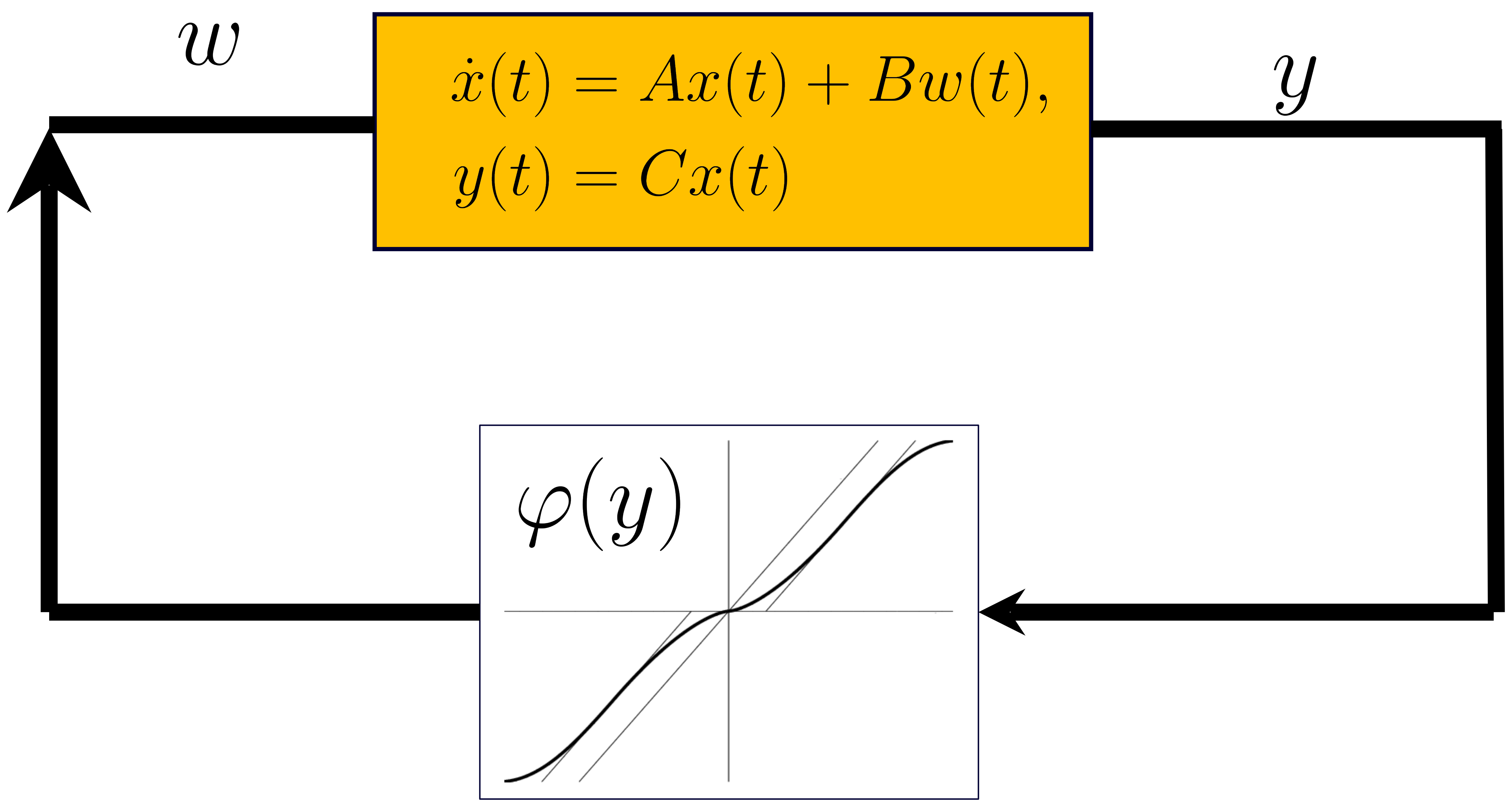

$$0 \le \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \le \varkappa$$

It remains to apply S-lemma!

$$V(\Delta x) = \|\Delta x\|_{p,R}^2 = \|R\Delta x\|_p^2, \ \det R \neq 0.$$

Program: repeat the trick with S-lemma.

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?

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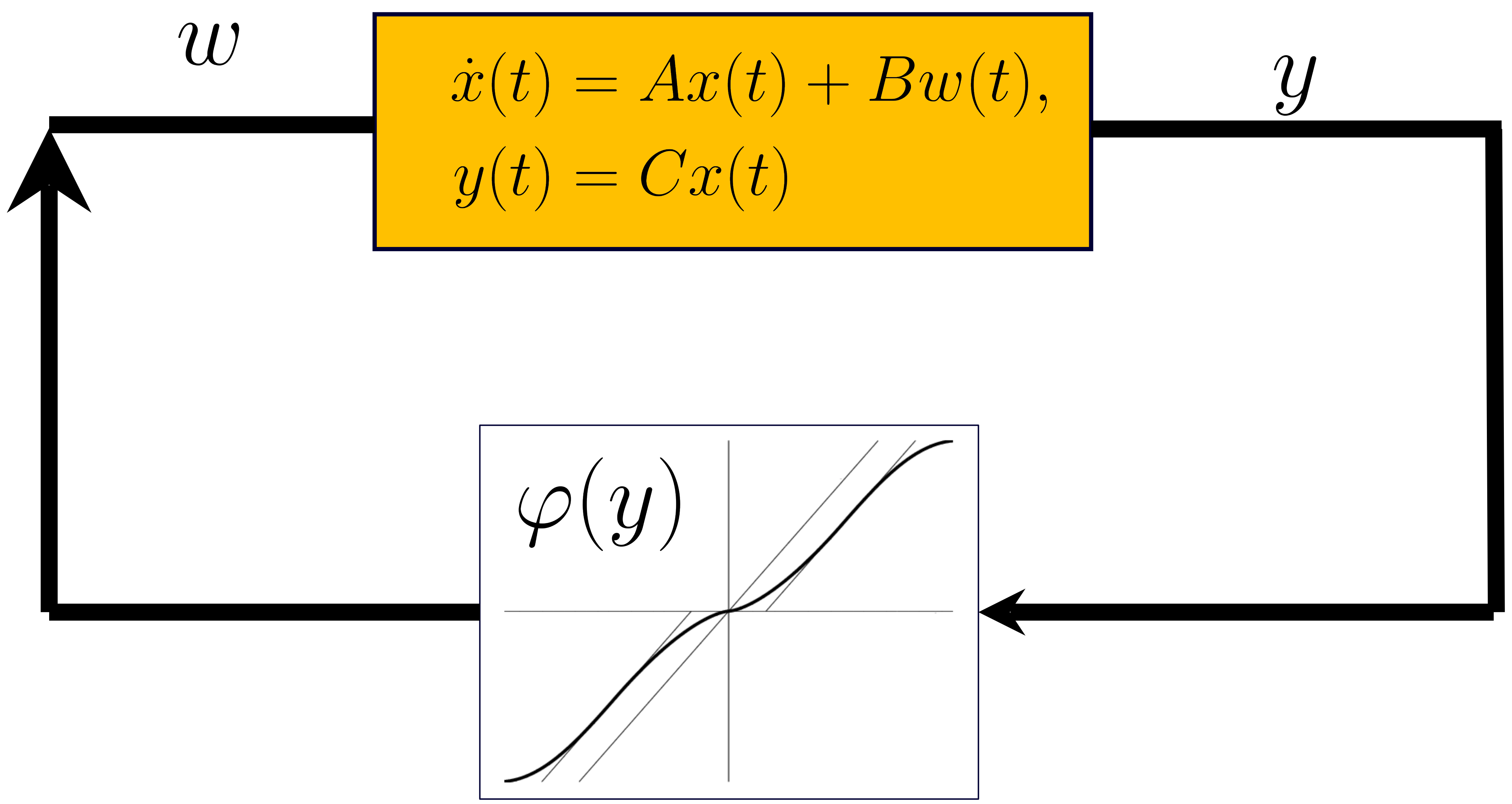
S-procedure for absolute contractivity: main result



$$0 \le \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \le \varkappa$$

Theorem. **The system is contractive with respect to the (non-Euclidean) norm**

$$\|\Delta x\|_{p,R} = \|R\Delta x\|_p$$

and rate *c* if τ ≥ *0* exists such that

$$\mu_p \begin{pmatrix} RAR^{-1} + cI & RB \\ \tau C R^{-1} & -\tau\varkappa^{-1} \end{pmatrix} \le 0$$

$$A + \varkappa BC$$

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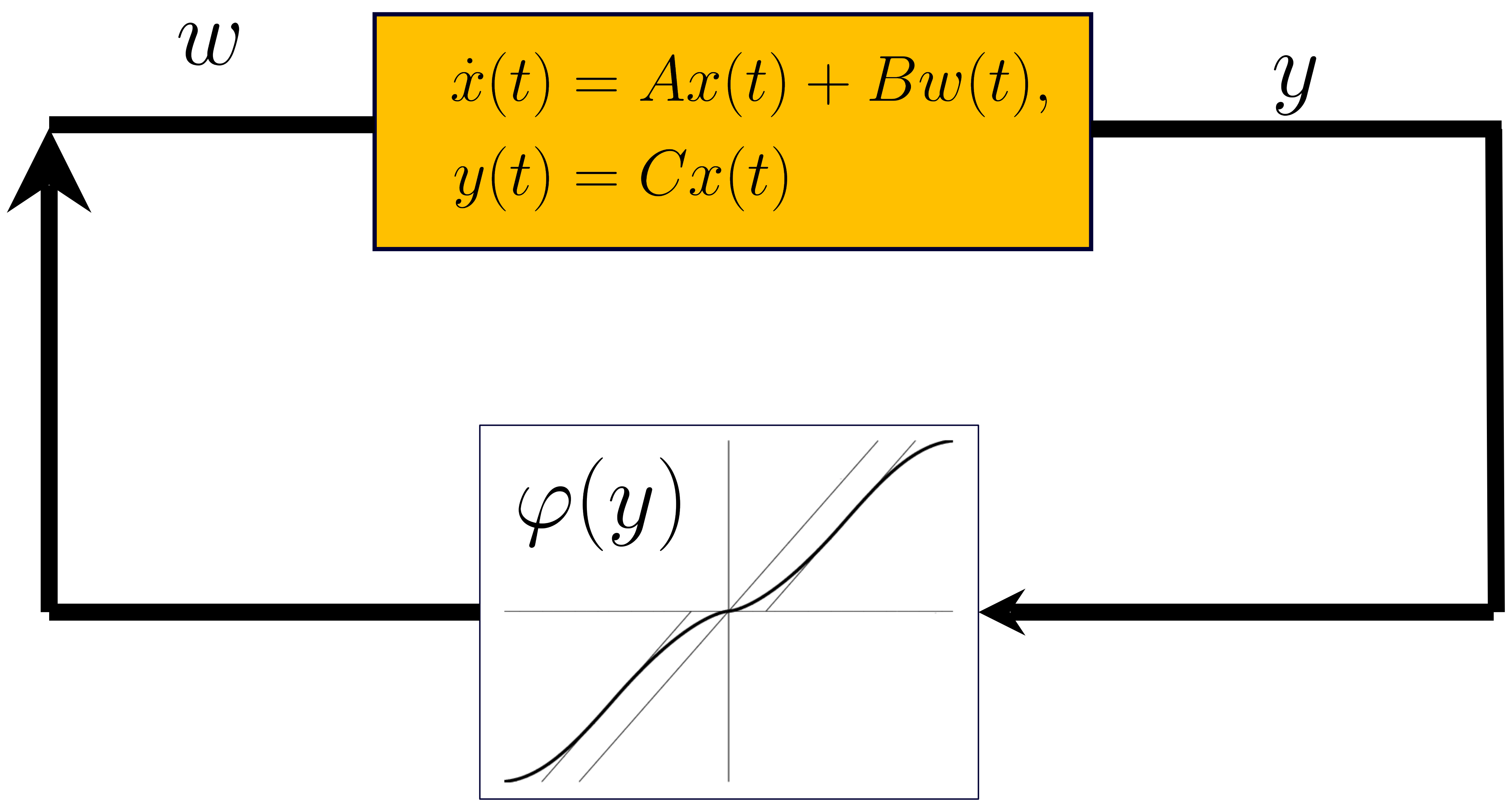

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Remarks:

- For a fixed weight matrix, the condition is convex (as usual in S-lemma).
- Non-convex with respect to the weights!
- But, there are situation where the problem reduces to **quasi-convex optimization (Davydov et al., arXiv:2110.08298)** **p=1 or ∞ + diagonal weight matrix R**
- **p=2: classical LMIs from absolute stability theory (see the journal text).**
- **Example in the paper: for positive linear part systems, contraction holds if and only if the matrix is Hurwitz** $A + \varkappa BC$

Strong version of Kalman conjecture for positive systems: stability under all linear feedbacks implies contractivity of all nonlinear systems with slope restriction.

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- **Main Results**

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- **Conclusions, Future works**

Journal Paper Upcoming

(accepted by SICON, out in 2-3 months) :

Arxiv: 2207.14579

More general form of the S-Lemma, conic log-norms, criteria for global stability (sector constraint instead of slope one) some examples (including the Kalman conjecture for positive system)

THE YAKUBOVICH S-LEMMA REVISITED: STABILITY AND CONTRACTIVITY IN NON-EUCLIDEAN NORMS*

ANTON V. PROSKURNIKOV†, ALEXANDER DAVYDOV‡, AND FRANCESCO BULLO‡

Abstract. The celebrated S-Lemma was originally proposed to ensure the existence of a quadratic Lyapunov function in the Lur'e problem of absolute stability. A quadratic Lyapunov function is, however, nothing else than a squared Euclidean norm on the state space (that is, a norm induced by an inner product). A natural question arises as to whether squared *non-Euclidean* norms $V(x) = \|x\|^2$ may serve as Lyapunov functions in stability problems. This paper presents a novel non-polynomial S-Lemma that leads to constructive criteria for the existence of such functions defined by weighted ℓ_p norms. Our generalized S-Lemma leads to new absolute stability and absolute contractivity criteria for Lur'e-type systems, including, for example, a new simple proof of the Aizerman and Kalman conjectures for positive Lur'e systems.

Key words. S-Lemma, Contraction, Absolute stability, Positive Systems.

AMS subject classifications. 34H05, 93C15

1. Introduction. The history of the S-Lemma dates back to early works on stability of nonlinear control systems with partially uncertain dynamics [2, 41]. In many situations, such a system may be represented in the Lur'e form, that is, as a *feedback superposition* of two blocks as shown in Fig. 1. One block is a known linear time-invariant system, whereas the other block may be nonlinear (and is traditionally referred to as the "nonlinearity") and uncertain or have no simple analytic representation as exemplified by neural network architectures [28] and lookup-table functions. A prototypical assumption on the uncertain block is that its input/output behavior satisfies some rough estimates. In the case of a static nonlinearity, such an estimate often takes the form of the *sector condition*

$$\alpha_1 \le \frac{w(t)}{y(t)} \le \alpha_2 \iff (y(t) - \alpha_1^{-1} w(t))(y(t) - \alpha_2^{-1} w(t)) \le 0, \tag{1.1}$$

where $-\infty \le \alpha_1 < \alpha_2 \le +\infty$. The classical Lur'e problem [38, 41] was to find conditions on the coefficients of the known LTI block and the sector slopes $\{\alpha_1, \alpha_2\}$ that ensure global asymptotic stability of the closed-loop system for *all* nonlinearities in the sector. Later the term *absolute stability* has been coined for such problems; the term "absolute" emphasizes the applicability of the stability criteria to all unknown systems whose "nonlinear" parts belong to a certain class.

Historically, the first approach to absolute stability theory [7, 41] was based on quadratic Lyapunov functions and their extensions (e.g., a quadratic form plus a definite integral of the nonlinearity). The validation of the Lyapunov property (the Lyapunov function's derivative along each trajectory is non-positive) leads to the following problem: *When is one quadratic inequality (the Lyapunov condition) implied by another quadratic inequality (the sector condition)?* More generally, when is a

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More general form of the S-Lemma, conic log-norms, criteria for global stability (sector constraint instead of slope one) some examples (including the Kalman conjecture for positive system)

I consider this paper as work-in-progress THINGS TO DO:

- **Efficient computational algorithms:**
  how to validate log-norm criteria?
- **Applications to robust certificates in neural networks, resilience to adversarial attacks.**

- **Complicated nonlinearities:**
  NNs, dynamical blocks, optimization solvers (control allocation logic, MPC etc.)
- **Does non-quadratic norms work better?**
  Can we outperform classical stability criteria?


thank you