Contraction of Continuous-Time Proximal Gradient Dynamics

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Convexity and contractivity

Kachurovskii's Theorem: For differentiable $f : \mathbb{R}^n \to \mathbb{R}$, equivalent statements:

1. f is **strongly convex** with parameter m
2. $-\nabla f$ is **(strongly) infinitesimally contracting** with respect to $\|\cdot\|_2$ with rate m

Also: global minimum of f = globally-exponentially stable equilibrium of $-\nabla f$

For strongly convex f, provides natural way to solve minimization with dynamical system

1. The minimization problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

2. strongly infinitesimally contracting dynamics

$$\dot{x} = -\nabla f(x)$$

How about more general minimization problems like $\min_{x \in \mathbb{R}^n} f(x) + g(x)$?

R. I. Kachurovskii. Monotone operators and convex functionals. *Uspekhi Matematicheskikh Nauk*, 15(4):213–215, 1960

Composite minimization and proximal gradient

For strongly convex + strongly smooth f, convex, closed, proper $g : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$x^\star = \operatorname*{argmin}_{x\in\mathbb{R}^n} f(x) + g(x) \qquad \Longleftrightarrow \qquad x^\star = \mathrm{prox}_{\gamma g}(x^\star - \gamma\nabla f(x))$$

$$\mathrm{prox}_{\gamma g}(z) = \operatorname*{argmin}_{x\in\mathbb{R}^n} g(x) + \frac{1}{2\gamma}\|x - z\|_2^2.$$

Transcription from

1. The minimization problem

$$\min_{x\in\mathbb{R}^n} f(x) + g(x)$$

2. to strongly infinitesimally contracting *proximal gradient* dynamics

$$\dot{x} = -x + \mathrm{prox}_{\gamma g}(x - \gamma\nabla f(x))$$

Contractivity of proximal gradient

$$\dot{x} = -x + \mathrm{prox}_{\gamma g}(x - \gamma \nabla f(x))$$

Contractivity properties

1. For $\gamma \in]0, 2/\ell[$, prox. gradient is contracting w.r.t. $\|\cdot\|_2$ with rate

$$c = 1 - \max\{|1 - \gamma m|, |1 - \gamma \ell|\}$$

   Optimal $\gamma^\star = \dfrac{2}{m + \ell}$

2. For $f(x) = \dfrac{1}{2} x^\top A x + b^\top x$, $A \succ 0$ prox. gradient is contracting w.r.t. $\|\cdot\|_{(\gamma A - I_n)}$ with rate

$$c = 1$$

   for every $\gamma \in]1/\lambda_{\min}(A), +\infty[$

Thank you