Dynamic Vehicle Routing for Robotic Networks
Lecture #4: The multi-vehicle DVR problem

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Workshop at the 2010 American Control Conference
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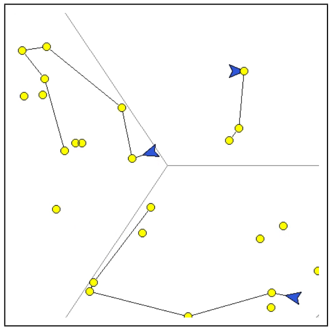
Load balancing in DVR via territory partitioning



1. Resource allocation in DVR is transcribed into partitioning!
2. Focus of this lecture is mutivehicle DVR via optimal partitioning

Lecture outline

Territory partitioning is ... art

Ocean Park Paintings, by Richard Diebenkorn (1922-1993)

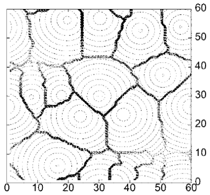
Territory partitioning: optimality and behaviors

DESIGN of performance metrics

1. how to cover a region with n minimum-radius overlapping disks?
2. how to design a minimum-distortion (fixed-rate) vector quantizer?
3. where to place mailboxes in a city / cache servers on the internet?

ANALYSIS of cooperative distributed behaviors

how do animals share territory?
how do they decide foraging ranges?
how do they decide nest locations?



4. what if each robot goes to "center" of own dominance region?
5. what if each robot moves away from closest vehicle?

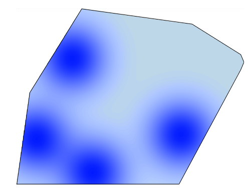
Multi-center functions

Expected wait time (in light load)

$$\mathcal{H}(p, v) = \int_{v_1} \|x - p_1\| dx + \cdots + \int_{v_n} \|x - p_n\| dx$$

- n robots at $p = \{p_1, \dots, p_n\}$
- environment is partitioned into $v = \{v_1, \dots, v_n\}$

$$\mathcal{H}(p, v) = \sum_{i=1}^{n} \int_{v_i} f(\|x - p_i\|) \varphi(x) dx$$



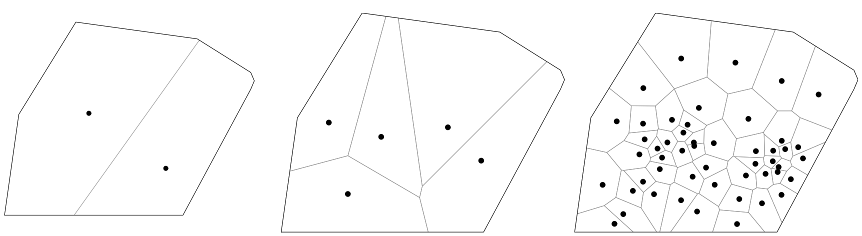
- $\varphi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ density
- $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ penalty function

F. Bullo, J. Cortés, and S. Martínez. *Distributed Control of Robotic Networks*. Applied Mathematics Series. Princeton University Press, 2009. Available at http://www.coordinationbook.info

Optimal partitioning

The **Voronoi partition** $\{V_1, \dots, V_n\}$ generated by points $(p_1, \dots, p_n)$

$$\begin{aligned} V_i(p) &= \{x \in \mathcal{Q} \mid \|x - p_i\| \leq \|x - p_j\|, \forall j \neq i\} \\ &= \mathcal{Q} \bigcap_j (\text{half plane between } i \text{ and } j, \text{ containing } i) \end{aligned}$$

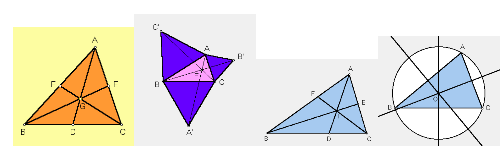


Descartes 1644, Dirichlet 1850, Voronoi 1908, Thiessen 1911, Fortune 1986 (sweepline algorithm $O(n \log(n))$)

Optimal centering (for region v with density φ)

function of p	minimizer = center
$p \mapsto \int_v \|x - p\| \varphi(x) dx$	**median** (or **Fermat–Weber point**)
$p \mapsto \int_v \|x - p\|^2 \varphi(x) dx$	**centroid** (or **center of mass**)
$p \mapsto \text{area}(v \cap \text{disk}(p, r))$	**r-area center**
$p \mapsto$ radius of largest disk centered at p enclosed inside v	**incenter**
$p \mapsto$ radius of smallest disk centered at p enclosing v	**circumcenter**

From online Encyclopedia of Triangle Centers



How to compute the median of a convex set

For convex planar set $\mathcal{Q}$ with strictly positive density φ,

$$\mathcal{H}_{\mathrm{FW}}(p) = \int_{\mathcal{Q}} \|p - x\| \varphi(x) dx$$

1. $\mathcal{H}_{\mathrm{FW}}$ is strictly convex
2. the global minimum point is in $\mathcal{Q}$ and is called *median* of $\mathcal{Q}$
3. compute median via gradient flow with

$$\frac{d}{dp}\mathcal{H}_{\mathrm{FW}}(p) = \int_{\mathcal{Q}} \frac{p - x}{\|p - x\|} \varphi(x) dx$$

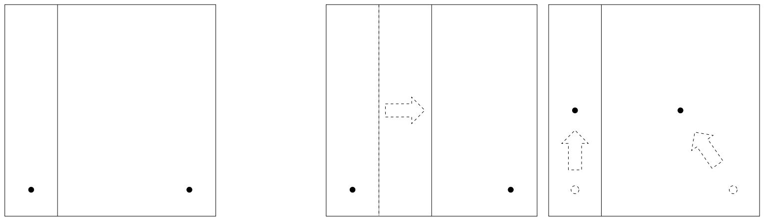
From optimality conditions to algorithms

$$\mathcal{H}(p, v) = \sum_{i=1}^{n} \int_{v_i} f(\|x - p_i\|)\varphi(x) dx$$

Theorem (Alternating Algorithm, Lloyd '57)

1. *at fixed positions, optimal partition is Voronoi*
2. *at fixed partition, optimal positions are "generalized centers"*
3. *alternate v-p optimization*

$\implies$ *local optimum = center Voronoi partition*



Gradient algorithm for multicenter function

After assuming v is Voronoi partition,

$$\mathcal{H}(p) = \sum_{j=1}^{n} \int_{V_j(p)} f(\|x - p_j\|)\varphi(x) dx$$

For f smooth, note simplifications for boundary terms

$$\frac{\partial \mathcal{H}}{\partial p_i}(p) = \int_{V_i(p)} \frac{\partial}{\partial p_i} f\left(\|x - p_i\|\right) \varphi(x) dx$$

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$$\frac{\partial \mathcal{H}}{\partial p_i}(p) = \int_{V_i(p)} \frac{\partial}{\partial p_i} f\left(\|x - p_i\|\right) \varphi(x) dx + \int_{\partial V_i(p)} f\left(\|x - p_i\|\right) \langle n_i(x), \frac{\partial x}{\partial p_i} \rangle \varphi(x) dx$$

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$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial p_i}(p) = &\int_{V_i(p)} \frac{\partial}{\partial p_i} f\left(\|x - p_i\|\right)\varphi(x)dx \\ &+ \int_{\partial V_i(p)} f\left(\|x - p_i\|\right)\langle n_i(x), \frac{\partial x}{\partial p_i}\rangle\varphi(x)dx \\ &+ \underbrace{\sum_{j \text{ neigh } i} \int_{\partial V_j(p)\cap\partial V_i(p)} f\left(\|x - p_j\|\right)\langle n_{ji}(x), \frac{\partial x}{\partial p_i}\rangle\varphi(x)dx}_{\text{contrib from neighbors}}\end{aligned}$$

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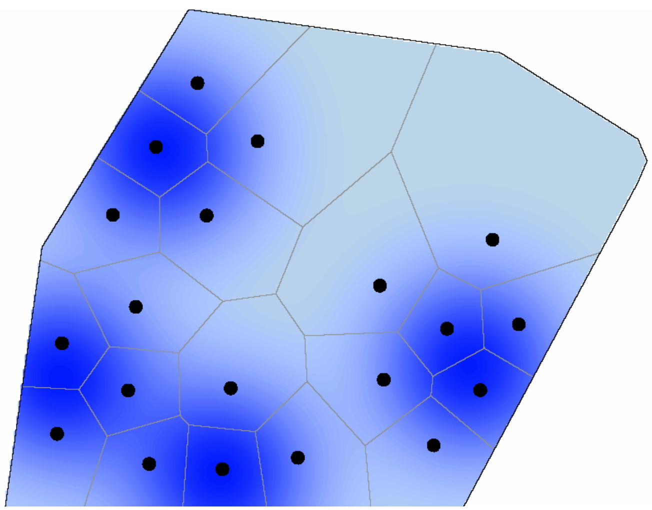
Example optimal partition



Lecture outline

1. Territory Partitioning
2. The multi-vehicle DVR problem
3. Multi-vehicle DVR policies based on partitioning

D. J. Bertsimas and G. J. van Ryzin. Stochastic and dynamic vehicle routing with general interarrival and service time distributions. *Advances in Applied Probability*, 25:947–978, 1993

Multi-vehicle DVR problem

- results on single-vehicle DVR generalize easily to the multi-vehicle case
- previous methodology (locational optimization, queueing and control theory, combinatorics) applicable to this case
- main new idea: **partitioning**

Light-load lower bound

Multi - Median

- minimizer $p^* = \{p_1^*, \dots, p_m^*\}$ of

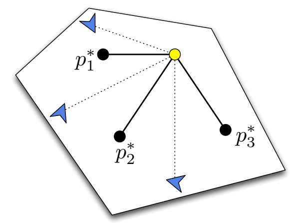
$$p \mapsto \mathbb{E}_\varphi[\min_i \|X - p_i\|] = \sum_{i=1}^{m} \int_{V_i} \|x - p_i\| \varphi(x) dx$$

Lower bound (most useful when $\lambda \to 0^+$)

For all policies π: $\overline{T}_\pi \geq \mathbb{E}_\varphi[\min_i \|X - p_i^*\|]/v + \overline{s}$

Proof sketch:

- multi-median: best a priori location to reach a newly arrived demand



Heavy-load lower bound

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spatially-unbiased policies: $\overline{T}_\pi \geq \frac{\beta_{\mathrm{TSP}}^2}{2} \frac{\lambda \left(\int_Q \varphi^{1/2}(x) dx\right)^2}{m^2 v^2 (1-\varrho)^2}$ as $\varrho \to 1^-$

spatially-biased policies: $\overline{T}_\pi \geq \frac{\beta_{\mathrm{TSP}}^2}{2} \frac{\lambda \left(\int_Q \varphi^{2/3}(x) dx\right)^3}{m^2 v^2 (1-\varrho)^2}$ as $\varrho \to 1^-$

Proof sketch (for unbiased policies):

- Recall inter-demand distance $\overline{D} \geq \beta_{\mathrm{TSP}} \frac{\int_Q \varphi^{1/2}(x) dx}{\sqrt{2\overline{N}}}$, as $\varrho \to 1^-$
- for stability with m vehicles:

$$\overline{s} + \frac{\overline{D}}{v} \leq \frac{m}{\lambda} \implies \overline{s} + \beta_{\mathrm{TSP}} \frac{\int_Q \varphi^{1/2}(x) dx}{v\sqrt{2\overline{N}}} \leq m/\lambda$$

- $\overline{N} = \lambda \overline{W}$ and $\overline{T} = \overline{W} + \overline{s} \implies \overline{T}^* \geq \frac{\beta_{\mathrm{TSP}}^2}{2} \frac{\lambda\left(\int_Q \varphi^{1/2}(x) dx\right)^2}{m^2 v^2 (1-\varrho)^2}$

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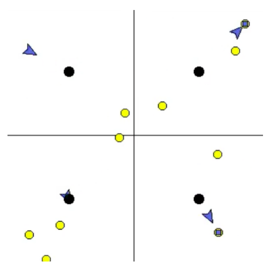
An optimal light-load policy

m Stochastic Queueing Median (mSQM)

Compute multi-median p^* and assign one vehicle at each median point. Then:

1: Assign demand that falls in V_i to vehicle i
2: each vehicles service demands in FCFS order
3: each vehicle returns to p^*_k after each service is completed



Proof sketch of optimality

- As $\lambda \to 0^+$, $\mathbb{P}\,[\text{demand generated when system is empty}] \to 1$
- $\Rightarrow$ all demands are generated with the vehicles at p^*

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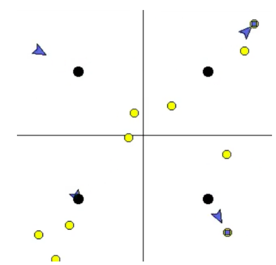

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An optimal spatially-unbiased heavy-load policy

Unbiased TSP (UTSP)

Partition $\mathcal{Q}$ into r subregions $\mathcal{Q}_k$ with $\int_{\mathcal{Q}_k} \varphi(x)dx = 1/r$. Then:

1: within each subregion form sets of size n/r
2: deposit sets in a queue
3: service sets FCFS with the first available vehicle by following a TSP tour

Optimize over n.

Optimality of UTSP policy

$$\lim_{\varrho \to 1^-} \overline{T}_{\mathrm{UTSP}}(r)/\overline{T}^*_{\mathrm{U}} \leq 1 + 1/r$$

Proof sketch of optimality (r=1)

- reduction to GI/G/m

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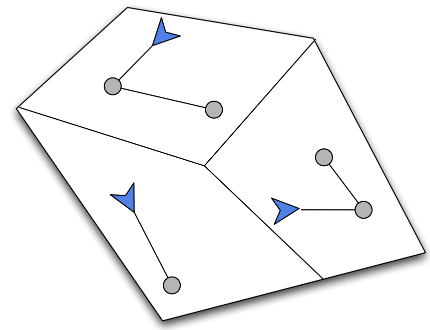
M. Pavone, E. Frazzoli, and F. Bullo. Distributed and adaptive algorithms for vehicle routing in a stochastic and dynamic environment. *IEEE Transactions on Automatic Control*, May 2010. (Submitted, Apr 2009) to appear

Partitioning policies

Definition (π-partitioning policy)

Given m vehicles and single-vehicle policy π:

1. Workspace divided into m subregions
2. One-to-one correspondence vehicles/subregions
3. Each agent executes the single-vehicle policy π within its own subregion



Motivation

Performance:

- light load: problem reduces to locational optimization
- heavy load:
  1. delay of optimal single vehicle policy scales as $\lambda\,|\mathcal{Q}|$
  2. by (equitably) partitioning, delay reduces to $\frac{\lambda}{m}\frac{|\mathcal{Q}|}{m} = \frac{\lambda|\mathcal{Q}|}{m^2}$
  3. $\Rightarrow$ delay scales as m^{-2}, as in the lower bound

Implementation:

- systematic approach to lift adaptive single-vehicle policies to multi-vehicle policies
- coupled with **distributed** partitioning algorithms, provides distributed multi-vehicle policies

distributed multi-vehicle policy = single-vehicle policy + optimal partitioning + distributed algorithm for partitioning

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Optimal partitioning in heavy load

Intuition

- per-vehicle workload is $\propto \lambda \int_{\mathcal{Q}_k} \varphi(x)dx$
- per-vehicle service capacity is $\propto \lambda \int_{\mathcal{Q}_k} \varphi^{1/2}(x)dx$
- optimal partitioning = **equalizing** per-vehicle workload **and** service capacity

Definition

A partition $\{\mathcal{Q}_k\}_{k=1}^m$ is:

- equitable if $\int_{\mathcal{Q}_k} \varphi(x)dx = \int_{\mathcal{Q}} \varphi(x)dx/m$
- simultaneously equitable if
  1. $\int_{\mathcal{Q}_k} \varphi(x)dx = \int_{\mathcal{Q}} \varphi(x)dx/m$, and
  2. $\int_{\mathcal{Q}_k} \varphi^{1/2}(x)dx = \int_{\mathcal{Q}} \varphi^{1/2}(x)dx/m$

Simultaneously equitable partitions exist for any $\mathcal{Q}$ and φ

(S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink, 2000)

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FB, EF, MP, KS, SLS (UCSB, MIT) Dynamic Vehicle Routing (Lecture 4/8) 29jun10 @ Baltimore, ACC 22 / 27

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Optimal partitioning in heavy load

Theorem

Given single-vehicle optimal policy π^*, a π^*-partitioning policy using a simultaneously equitable partition is an optimal unbiased policy

Proof sketch

- $\mathbb{P}[\text{demand arrives in } \mathcal{Q}_k] = \int_{\mathcal{Q}_k} \varphi(x)\,dx = 1/m$
- arrival rate in region k: $\lambda_k = \lambda/m$
- $\Rightarrow \varrho_k = \lambda_k \bar{s} = \lambda \bar{s}/m = \varrho < 1 \Rightarrow$ system is stable
- conditional density for region k: $\varphi(x)/\left(\int_{\mathcal{Q}_k} \varphi(x)\,dx\right) = m\,\varphi(x)$
- $\overline{T} = \sum_{k=1}^m \left(\int_{\mathcal{Q}_k} \varphi(x)\,dx \; \frac{\beta_{\mathrm{TSP}}^2}{2} \frac{\lambda_k}{v^2\,(1-\varrho_k)^2} \left[\int_{\mathcal{Q}_k} \sqrt{\frac{\varphi(x)}{\int_{\mathcal{Q}_k}\varphi(x)\,dx}}\,dx \right]^2 \right)$
  $= \sum_{k=1}^m \frac{1}{m} \overline{T}_{\pi^*} \frac{1}{m^2}$

Comments

If $\{\mathcal{Q}_k\}_{k=1}^m$ is only equitable wrt to $\varphi^{1/2}$...

- $\exists\,\bar{k}$ such that $\varrho_{\bar{k}} = \lambda\,(1/m + \varepsilon)\,\bar{s} = \varrho + \varepsilon\lambda\bar{s}$
- potentially, policy **unstable** for $\varrho < 1$!

If $\{\mathcal{Q}_k\}_{k=1}^m$ is only equitable wrt to φ...

- per-vehicle service capacity is unbalanced $\Rightarrow$ policy stable but **not optimal**
- guaranteed to be within m of optimal unbiased performance

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Special cases

Case $\bar{s} = 0$:

- stability not an issue:

$$\underbrace{\lambda}_{\text{generation rate}} \quad - \quad \underbrace{m \cdot \frac{n}{\mathrm{TSPlength}(n)}}_{\text{service rate}} \quad = \quad \text{demand growth rate}$$

- since $\mathrm{TSPlength}(n) \propto \sqrt{n} \Rightarrow$ stability for all λ, m
- **equitability only wrt to $\varphi^{1/2}$ provides optimal performance**

Case $\varphi = \text{uniform}$:

- equitable wrt to $\varphi \Rightarrow$ equitable wrt to $\varphi^{1/2}$
- no need to use algorithms for simultaneous equitability

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Lecture outline

Workshop Structure and Schedule

8:00-8:30am	*Coffee Break*	
8:30-9:00am	Lecture #1:	Intro to dynamic vehicle routing
9:05-9:50am	Lecture #2:	Prelims: graphs, TSPs and queues
9:55-10:40am	Lecture #3:	The single-vehicle DVR problem
10:40-11:00am	*Break*	
11:00-11:45pm	Lecture #4:	The multi-vehicle DVR problem
11:45-1:10pm	*Lunch Break*	
1:10-2:10pm	Lecture #5:	Extensions to vehicle networks
2:15-3:00pm	Lecture #6:	Extensions to different demand models
3:00-3:20pm	*Coffee Break*	
3:20-4:20pm	Lecture #7:	Extensions to different vehicle models
4:25-4:40pm	Lecture #8:	Extensions to different task models
4:45-5:00pm		Final open-floor discussion