Network systems in technology

![Smart grid diagram](image1)

**Smart grid**

![Amazon robotic warehouse](image2)

**Amazon robotic warehouse**

![Portland water network](image3)

**Portland water network**

![Industrial chemical plant](image4)

**Industrial chemical plant**
Network systems in sciences

**Sociology**: opinion dynamics, propagation of information, performance of teams

**Ecology**: ecosystems and foodwebs

**Economics**: input-output models

**Medicine/Biology**: compartmental systems
<table>
<thead>
<tr>
<th>Acknowledgments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gregory Toussaint</td>
</tr>
<tr>
<td>Todd Cerven</td>
</tr>
<tr>
<td>Jorge Cortés*</td>
</tr>
<tr>
<td>Sonia Martínez*</td>
</tr>
<tr>
<td>G. Notarstefano</td>
</tr>
<tr>
<td>Anurag Ganguli</td>
</tr>
<tr>
<td>Ketan Savla</td>
</tr>
<tr>
<td>Kurt Plarre*</td>
</tr>
<tr>
<td>Ruggero Carli*</td>
</tr>
<tr>
<td>Nikolaj Nordkvist</td>
</tr>
<tr>
<td>Sara Susca</td>
</tr>
<tr>
<td>Stephen Smith</td>
</tr>
<tr>
<td>Gábor Orosz*</td>
</tr>
<tr>
<td>Shaunak Bopardikar</td>
</tr>
<tr>
<td>Karl Obermeyer</td>
</tr>
<tr>
<td>Sandra Dandach</td>
</tr>
<tr>
<td>Joey Durham</td>
</tr>
<tr>
<td>Vaibhav Srivastava</td>
</tr>
<tr>
<td>Fabio Pasqualetti</td>
</tr>
<tr>
<td>A. Mirtabatabaei</td>
</tr>
<tr>
<td>Rush Patel</td>
</tr>
<tr>
<td>Pushkarini Agharkar</td>
</tr>
</tbody>
</table>

- Florian Dörfler  
  ETH
- John Simpson-Porco  
  Waterloo
- Noah Friedkin  
  UCSB
- Peng Jia  
  UCSB

- NSF
- AFOSR
- ARO
- ONR
- DOE
1. Intro to Network Systems
   Models, behaviors, tools, and applications

2. Power Flow
   “Synchronization in oscillator networks” by Dörfler et al, PNAS ’13
   “Voltage collapse in grids” by Simpson-Porco et al, NatureComm ’16

3. Social Influence
   “Opinion dynamics and social power” by Jia et al, SIREV ’15
Linear network systems

\[ x(k + 1) = Ax(k) + b \quad \text{or} \quad \dot{x}(t) = Ax(t) + b \]

1. systems of interest
2. asymptotic behavior
3. tools

network structure ⇔ function = asymptotic behavior
### Perron-Frobenius Theory

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>non-negative</td>
<td>$(A \geq 0)$</td>
</tr>
<tr>
<td>irreducible</td>
<td>$(\sum_{k=0}^{n-1} A^k &gt; 0)$</td>
</tr>
<tr>
<td>primitive</td>
<td>(there exists $k$ such that $A^k &gt; 0$)</td>
</tr>
</tbody>
</table>

#### If $A$ non-negative

1. Eigenvalue $\lambda \geq |\mu|$ for all other eigenvalues $\mu$
2. Right and left eigenvectors $v_{\text{right}} \geq 0$ and $v_{\text{left}} \geq 0$

#### If $A$ irreducible

3. $\lambda > 0$ and $\lambda$ is simple
4. $v_{\text{right}} > 0$ and $v_{\text{left}} > 0$ are unique

#### If $A$ primitive

5. $\lambda > |\mu|$ for all other eigenvalues $\mu$
6. $\lim_{k \to \infty} A^k/\lambda^k = v_{\text{right}} v_{\text{left}}^T$, with normalization $v_{\text{right}}^T v_{\text{left}} = 1$
**Algebraic graph theory**

**Powers of $A \sim$ paths in $G$:**

$$(A^k)_{ij} > 0 \quad \iff \quad \text{there exists directed path of length } k \text{ from } i \text{ to } j \text{ in } G$$

**Primitivity of $A \sim$ paths in $G$:**

$A$ is primitive

$$(A \geq 0 \text{ and } A^k > 0) \quad \iff \quad G \text{ strongly connected and aperiodic (exists path between any two nodes) and (exists no } k \text{ dividing each cycle length)}$$

**Digraph**

**Strongly connected components**

**Condensation**
Averaging systems

Swarming via averaging

\[ x_i^+ := \text{average}(x_i, \{x_j, j \text{ is neighbor of } i\}) \]

\[ x(k+1) = Ax(k) \]

A influence matrix:
row-stochastic: non-negative and row-sums equal to 1

For general \( G \) with multiple condensed sinks (assuming each condensed sink is aperiodic)

consensus at sinks
convex combinations elsewhere

consensus: \( \lim_{k \to \infty} x(k) = (v_{\text{left}} \cdot x(0)) \mathbb{1}_n \)
where \( v_{\text{left}} = \text{convex combination} = \text{influence centrality} \)
Compartmental flow systems

Water flow model for a desert ecosystem

\[ \dot{q}_i = \sum_j (F_{j \rightarrow i} - F_{i \rightarrow j}) - F_{i \rightarrow 0} + u_i \]

\[ F_{i \rightarrow j} = f_{ij} q_i, \quad F = [f_{ij}] \]

\[ \dot{q} = (F^T - \text{diag}(F \mathbb{1}_n + f_0)) q + u =: C q + u \]

**C compartmental matrix:**
- quasi-positive (off-diag \( \geq 0 \)), \( f_0 \geq 0 \) \( \implies \) weakly diag dominant
- Analysis tools: *PF for quasi-positive, inverse positivity, algebraic graph*

system (= each condensed sink) is outflow-connected

\[ \lim_{t \to \infty} q(t) = -C^{-1} u \geq 0 \]

\[ (-C^{-1} u)_i > 0 \iff \text{ith compartment is inflow-connected} \]
Nonlinear network systems

Rich variety of emerging behaviors

1. equilibria / limit cycles / extinction in populations dynamics
2. epidemic outbreaks in spreading processes
3. synchrony / anti-synchrony in coupled oscillators
Population systems in ecology

Mutualism between clownfish and anemones

Lotka-Volterra: $x_i = \text{quantity/density}$

$$\frac{\dot{x}_i}{x_i} = b_i + \sum_j a_{ij} x_j$$

$$\dot{x} = \text{diag}(x)(Ax + b)$$

A interaction matrix:

(+, +) mutualism, (+, −) predation, (−, −) competition

rich behavior: persistence, extinction, equilibria, periodic orbits, …

1. logistic growth: $b_i > 0$ and $a_{ii} < 0$
2. bounded resources: A Hurwitz (e.g., irreducible and neg diag dom)
3. mutualism: $a_{ij} \geq 0$

exists unique steady state $-A^{-1}b > 0$

$$\lim_{t \to \infty} x(t) = -A^{-1}b \text{ from all } x(0) > 0$$
Network propagation in epidemiology

Network SIS: \((x_i = \text{infected fraction})\)

\[
\dot{x}_i = \beta \sum_j a_{ij}(1 - x_i)x_j - \gamma x_i
\]

(rescaling)

\[
\dot{x} = (I_n - \text{diag}(x))Ax - x
\]

A contact matrix: irreducible with dominant pair \((\lambda, v_{\text{right}})\)

below the threshold: \(\lambda < 1\)

0 is unique stable equilibrium

\(v_{\text{right}}^T x(t) \rightarrow 0\) monotonically & exponentially

above the threshold: \(\lambda > 1\)

0 is unstable equilibrium

unique other equilibrium \(x^* > 0\)

\(\lim_{t \to \infty} x(t) = x^*\) from all \(x(0) \neq 0\)
**Analysis methods**

1. **nonlinear stability theory**
2. **passivity**
3. **cooperative/competitive system and monotone generalizations**

**Mutualistic Lotka-Volterra:**
\[ \dot{x} = \text{diag}(x)(Ax + b) \]
A quasi-positive and Hurwitz $\implies$ inverse positivity
cooperative systems theory: (if Jacobian is quasi-positive,
then almost all bounded trajectories converge to an equilibrium)

**Network SIS:**
\[ \dot{x} = (I_n - \text{diag}(x))Ax - x \]
A irreducible, above the threshold $\lambda > 1$
monotonic iterations and LaSalle invariance
Incomplete references on linear network systems

Averaging: multi-sink, concise proofs, etc


Compartmental and positive systems

Lotka-Volterra models


Network SI/SIS/SIR models

Lectures on Network Systems

Francesco Bullo

With contributions by
Jorge Cortés
Florian Dörfler
Sonia Martínez

Lectures on Network Systems, v. .85
For students: free PDF for download
For instructors: slides and answer keys

Linear Systems:
1. motivating examples from social, sensor and compartmental networks,
2. matrix and graph theory, with an emphasis on Perron–Frobenius theory and algebraic graph theory,
3. averaging algorithms in discrete and continuous time, described by static and time-varying matrices, and
4. positive and compartmental systems, described by Metzler matrices.

Nonlinear Systems:
5. formation control problems for robotic networks,
6. coupled oscillators, with an emphasis on the Kuramoto model and models of power networks, and
7. virus propagation models, including lumped and network models as well as stochastic and deterministic models, and
8. population dynamic models in multi-species systems.
1 Intro to Network Systems
Models, behaviors, tools, and applications

2 Power Flow
“Synchronization in oscillator networks” by Dörfler et al, PNAS ’13
“Voltage collapse in grids” by Simpson-Porco et al, NatureComm ’16

3 Social Influence
“Opinion dynamics and social power” by Jia et al, SIREV ’15
Power flow equations

- Voltage magnitude and phase
- Active and reactive power

1. Secure operating conditions
2. Feedback control
3. Economic optimization

While accurate numerical solvers in current use, much ongoing research on optimization,

\[ \text{network structure} \leftrightarrow \text{function} = \text{power transmission} \]
Power networks as quasi-synchronous AC circuits

generators and loads

physics: Kirchoff and Ohm laws

today's simplifying assumptions:

- quasi-sync: voltage and phase $V_i$, $\theta_i$
  active and reactive power $P_i$, $Q_i$
- lossless lines
- approximated decoupled equations

Decoupled power flow equations

active: $P_i = \sum_j a_{ij} \sin(\theta_i - \theta_j)$
reactive: $Q_i = -\sum_j b_{ij} V_i V_j$
Power Flow Equilibria

\[ P_i = \sum_j a_{ij} \sin(\theta_i - \theta_j) \quad Q_i = -\sum_j b_{ij} V_i V_j \]

As function of network structure/parameters

1. do equations admit solutions / operating points?
2. how much active / reactive power can network transmit?
3. how to quantify stability margins?

Active power dynamics and mechanical analogy

Coupled swing equations

\[ M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]

Kuramoto coupled oscillators

\[ \dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]
Lessons from linear spring networks

Force $\propto$ displacement:

$$F_i = \sum a_{ij}(x_j - x_i) = -(Lx)_i$$

Laplacian / stiffness matrix and connectivity strength:

$$L = \text{diag}(A1_n) - A$$

$$\lambda_2 = \text{second smallest eigenvalue of } L$$

$$x = L^\dagger F_{\text{load}}$$

$$x - x_{\text{rest}} = L^{-1}_{\text{grounded}} F_{\text{load}}$$
Given balanced $P$, do angles exist?

$$P_i = \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

**connectivity strength vs. power transmission:**

#1: “torques” $\sim$ active powers $P_i$
   “displacements” $\sim$ power angles $(\theta_i - \theta_j)$

#2: with **increasing power transmission**, $(\theta_i - \theta_j)$ approach $\pi/2 = \text{sync loss}$

**Equilibrium angles** (neighbors within $\pi/2$ arc) exist if

\[
\|\text{pairwise differences of } P\|_2 < \lambda_2(L) \quad \text{for all graphs}
\]
\[
\|\text{pairwise differences of } L^\dagger P\|_\infty < 1 \quad \text{for trees, 3/4-cycles, complete}
\]
Given reactive $Q_{\text{loads}}$, do voltages $V_{\text{loads}}$ exist?

$$Q_i = - V_i \sum_j b_{ij} (V_j - V_{\text{rest},j})$$

where $V_{\text{rest}} = \text{open-circuit voltages}$

**connectivity strength vs. power transmission:**

#1: “force” $\sim$ reactive load $Q_{\text{loads}}$

“displacement” $\sim$ relative voltage variation

#2: with **increasing inductive** $Q_{\text{loads}},$

$V_{\text{loads}}$ falls until **voltage collapse**

Equilibrium voltage (high-voltage solution) exist if

$$\left\| L_{\text{grounded,scaled}}^{-1} Q_{\text{loads}} \right\|_{\infty} < 1$$
Summary (Power Flow)

New physical insight

1. sharp sufficient conditions for equilibria
2. upper bounds on transmission capacity
3. stability margins as notions of distance from bifurcations

Applications

1. secure operating conditions:
   realistic IEEE testbeds (Dörfler et al, PNAS ’13)
2. feedback control:
   microgrid design (Simpson-Porco et al, TIE ’15)
3. economic optimization:
   convex voltage support (Todescato et al, CDC ’15)
Incomplete references on power flow equations


Our recent work


Outline

1 Intro to Network Systems
   Models, behaviors, tools, and applications

2 Power Flow
   “Synchronization in oscillator networks” by Dörfler et al, PNAS ’13
   “Voltage collapse in grids” by Simpson-Porco et al, NatureComm ’16

3 Social Influence
   “Opinion dynamics and social power” by Jia et al, SIREV ’15
Social power along issue sequences

- **Deliberative groups in social organization**
  - government: juries, panels, committees
  - corporations: board of directors
  - universities: faculty meetings

- **Natural social processes along sequences:**
  - levels of openness and closure?
  - influence accorded to others? emergence of leaders?
  - rational/irrational decision making?

**Groupthink** = “deterioration of mental efficiency . . . from in-group pressures,” by I. Janis, 1972

**Wisdom of crowds** = “group aggregation of information results in better decisions than individual’s” by J. Surowiecki, 2005
DeGroot opinion formation

\[ y(k + 1) = Ay(k) \]

Dominant eigenvector \( v_{\text{left}} \) is social power:

\[ \lim_{k \to \infty} y(k) = (v_{\text{left}} \cdot y(0))\mathbb{1}_n \]

- \( A_{ii} =: x_i \) are self-weights / self-appraisal
- \( A_{ij} \) for \( i \neq j \) are interpersonal accorded weights
- assume \( A_{ij} =: (1 - x_i)W_{ij} \) for constant \( W_{ij} \)

\[ A(x) = \text{diag}(x) + \text{diag}(\mathbb{1}_n - x)W \]

- \( w_{\text{left}} = (w_1, \ldots, w_n) = \) dominant eigenvector for \( W \)
Reflected appraisal phenomenon (Cooley 1902 and Friedkin 2012)

along issues $s = 1, 2, \ldots$, individual dampens/elevates self-weight according to prior influence centrality

self-weights $\rightarrow$ relative control on prior issues $=$ social power

self-appraisal $\xrightarrow{\text{reflected appraisal mechanism}}$ self-weights

$x(s + 1) = v_{\text{left}}(A(x(s)))$

$v_{\text{left}}(A(x)) = \left( \frac{w_1}{1 - x_1}, \ldots, \frac{w_n}{1 - x_n} \right) / \sum_{i=1}^{n} \frac{w_i}{1 - x_i}$
For strongly connected $W$ and non-trivial initial conditions

1. **convergence to unique fixed point** (implies forgets initial condition)

$$\lim_{s \to \infty} x(s) = \lim_{s \to \infty} v_{\text{left}}(x(s)) = x^*$$

2. **accumulation of social power and self-appraisal**
   - fixed point $x^* = x^*(w_{\text{left}}) > 0$ has same ordering of $w_{\text{left}}$
   - social power threshold $p$: $x^*_i \geq w_i \geq p$ and $x^*_i \leq w_i \leq p$
Emergence of democracy

If $W$ is doubly-stochastic:

1. The non-trivial fixed point is $\frac{1}{n}$
2. $\lim_{s \to \infty} x(s) = \lim_{s \to \infty} \nu_{\text{left}}(x(s)) = \frac{1}{n}$

- Uniform social power
- No power accumulation = evolution to democracy
Emergence of autocracy

If $\mathcal{W}$ has star topology with center $j$:

1. there are no non-trivial fixed points
2. $\lim_{s \to \infty} x(s) = \lim_{s \to \infty} v_{\text{left}}(x(s)) = e_j$

- Autocrat appears in center node of star topology
- Extreme power accumulation = evolution to autocracy
1. Existence of $x^*$ via Brower fixed point theorem

2. Monotonicity:
   $i_{\text{max}}$ and $i_{\text{min}}$ are forward-invariant

   \[ i_{\text{max}} = \arg\max_j \frac{x_j(0)}{x_j^*} \]

   \[ \implies i_{\text{max}} = \arg\max_j \frac{x_j(s)}{x_j^*}, \text{ for all subsequent } s \]

3. Convergence via variation on classic "max-min" Lyapunov function:

   \[ V(x) = \max_j \left( \ln \frac{x_j}{x_j^*} \right) - \min_j \left( \ln \frac{x_j}{x_j^*} \right) \]

   strictly decreasing for $x \neq x^*$
New perspective on influence networks and social power

- dynamics and feedback in influence networks
- novel mechanism for power accumulation / emergence of autocracy
- measurement models and empirical validation

Open directions

- robustness for distinct models of opinion dynamics and appraisal
- cognitive models for time-varying interpersonal appraisals
- appraisals and power accumulation mechanisms
Incomplete references on social power

**Social Influence**


**Our recent work**


Network systems in science and technology

- **Models, behaviors, tools, and applications**
  - PF and algebraic graphs for linear behaviors
  - Variety of nonlinearities — elegant methods and broad impact

- **Power Networks** and **Social Influence**
  - Fundamental prototypical problems
  - Nonlinear variations from linear framework
  - Key outstanding questions remain

- **Outreach and collaboration opportunity for control community**
  - Biologists, ecologists, economists, physicists ...