Theory and Applications of Contracting Dynamical Systems

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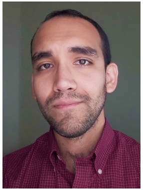

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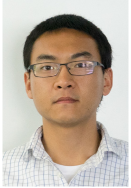

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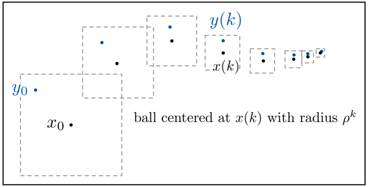
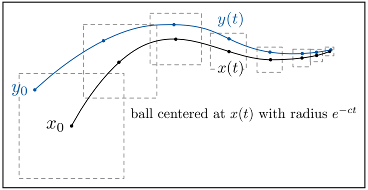
Outline and Aims

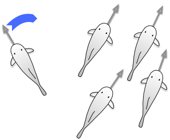
Dynamical Network Systems via Contraction Theory

1. structure and function of dynamical network systems
2. contractivity of dynamical systems
3. perspectives into artificial & biological neural networks

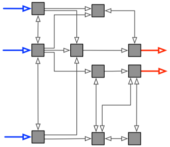




Structure and function for dynamical network systems

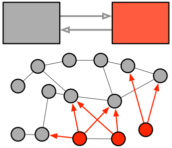


averaging

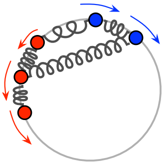


compartmental flows

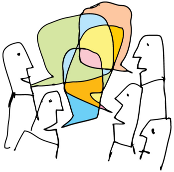
mutualism



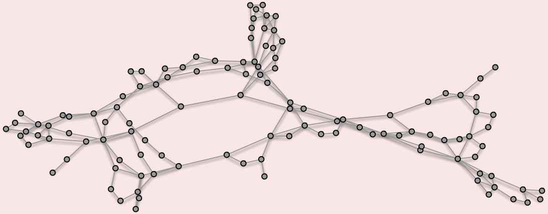
virus spread



coupled oscillators

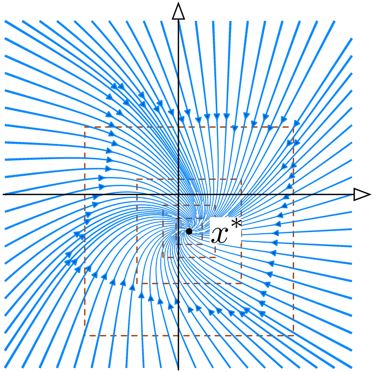


social power



network structure $\Longleftrightarrow$ **function = dynamic behavior**

Structure and function for dynamical network systems



function = dynamic behavior
highly-ordered transient and asymptotic behavior:

1. unique globally exponential stable equilibrium
   & two natural Lyapunov functions
2. robustness properties
   - bounded input, bounded output (iss)
   - robustness margin wrt unmodeled dynamics
   - robustness margin wrt delayed dynamics
3. periodic input, periodic output
4. modularity and interconnection properties
5. accurate numerical integration and equilibrium point computation

contracting dynamical systems

Contraction theory: historical notes

- **Origins**

  S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922. doi

  S. M. Lozinskii. Error estimate for numerical integration of ordinary differential equations. I. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 5:52–90, 1958. URL **http://mi.mathnet.ru/eng/ivm2980**. (in Russian)

  C. A. Desoer and H. Haneda. The measure of a matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Transactions on Circuit Theory*, 19(5):480–486, 1972. doi

- **Application in dynamics and control**: W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998. doi

- **Reviews**:

  M. Di Bernardo, D. Fiore, G. Russo, and F. Scafuti. Convergence, consensus and synchronization of complex networks via contraction theory. In J. Lü, X. Yu, G. Chen, and W. Yu, editors, *Complex Systems and Networks*, pages 313–339. Springer, 2016. ISBN 978-3-662-47824-0. doi

  P. Giesl, S. Hafstein, and C. Kawan. Review on contraction analysis and computation of contraction metrics, 2022. URL **https://arxiv.org/abs/2203.01367**. To appear in Journal of Computational Dynamics

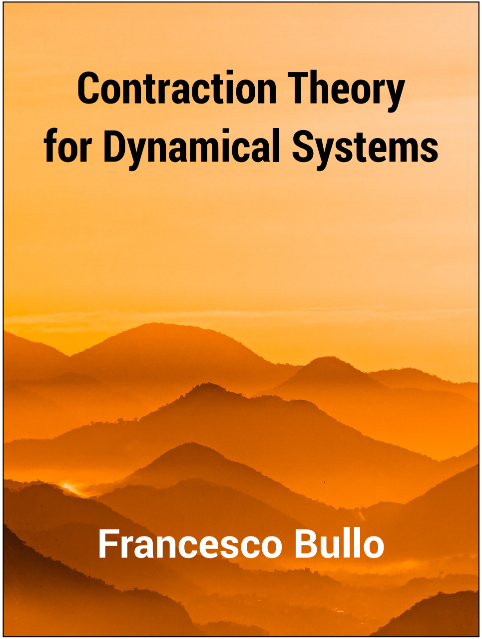
The Banach Contraction Theorem is also referred to as the *Picard-Banach-Caccioppoli*, because of the earlier work by Picard (1890) on the "method of successive approximations" and the later independent work by Renato Caccioppoli (1930).

Figure: Renato Caccioppoli (Napoli, 20 gennaio 1904 – Napoli, 8 maggio 1959) was an Italian mathematician

1921-1932 student and researcher @ Napoli
1931-1934 professor @ Padova
1934-1959 professor @ Napoli

R. Caccioppoli. Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale. *Rendiconti dell'Accademia Nazionale dei Lincei*, 11:794–799, 1930

1. **Lotka-Volterra population dynamics** (Lotka, 1920; Volterra, 1928):
   ℓ_1-weakly contracting (after a rescaling change of coordinates)
2. **Matrosov-Bellman interconnected stable systems** (Bellman, 1962; Matrosov, 1962):
   strongly contracting wrt composite norm
3. **Kuramoto coupled oscillators** (Kuramoto, 1975):
   strongly semicontracting wrt (ℓ_2, Π_n) norm, in neighb'd of each phase-cohesive equilibrium
4. **Yorke multigroup SIS epidemic model** (Lajmanovich and Yorke, 1976):
   equilibrium contracting wrt weighted ℓ_1/ℓ_∞ norms (at disease-free and endemic eq.)
5. **Hopfield and cellular neural networks** (Hopfield, 1982):
   ℓ_1/ℓ_∞-strongly contracting
6. **Daganzo cell transmission model for traffic networks** (Daganzo, 1994):
   ℓ_1-weakly contracting, when the dynamics is monotone
7. **Chua's diffusively-coupled dynamical systems** (Wu and Chua, 1995):
   strongly semi-contracting wrt $(2, p)$ tensor norm on $\mathbb{R}^n \otimes \mathbb{R}^k$
8. ...



Contraction Theory for Dynamical Systems, Francesco Bullo, KDP, 1.0 edition, 2022, ISBN 979-8836646806

1. Content:
(i) Banach contraction theorem and fixed point theory,
(ii) induced norms and induced log norms of matrices
(iii) strongly contracting dynamics over normed spaces,
(iv) weakly-contracting dynamics and monotone dynamics,
(v) semicontracting and partially contracting systems,

(vi) examples: Hopfield neural networks, systems in Lure' form, interconnected systems, gradient and primal dual flows of convex functions, Lotka-Volterra population dynamics, Daganzo traffic models, averaging flows, and diffusively-coupled synchronizing systems.

2. "Continuous improvement is better than delayed perfection"
Mark Twain

– Self-Published and Print-on-Demand at:
https://www.amazon.com/dp/B0B4K1BTF4
– PDF Freely available at
http://motion.me.ucsb.edu/book-ctds

Outline

1. On structure and function of dynamical network systems
2. Contractivity of dynamical systems
   - From discrete-time to continuous-time dynamics
   - From closed to open systems
   - From single systems to networks of systems
3. Perspectives into artificial & biological neural networks
4. Conclusions and future research
5. Advanced topics
   - Advanced Topic: Optimization and Fixed Point Theory
   - Advanced Topic: Contractivity with respect to Euclidean vs. nonEuclidean norms
   - Advanced Topic: Semicontraction Theory and Dual Seminorms
   - Advanced Topic: Indirect Optimal Control
   - Advanced Topic: Network contraction theory with delays
   - Advanced Topic: Riemannian manifolds

Linear algebra: induced norms

Vector norm	Induced matrix norm	Induced matrix log norm
$\lVert x\rVert_1 = \sum_{i=1}^n \lvert x_i\rvert$	$\lVert A\rVert_1 = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^n \lvert a_{ij}\rvert$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i=1,i\neq j}^n \lvert a_{ij}\rvert\right)$ = max column "absolute sum" of A
$\lVert x\rVert_2 = \sqrt{\sum_{i=1}^n x_i^2}$	$\lVert A\rVert_2 = \sqrt{\lambda_{\max}(A^\top A)}$	$\mu_2(A) = \lambda_{\max}\left(\frac{A+A^\top}{2}\right)$
$\lVert x\rVert_\infty = \max_{i\in\{1,\dots,n\}} \lvert x_i\rvert$	$\lVert A\rVert_\infty = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^n \lvert a_{ij}\rvert$	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j=1,j\neq i}^n \lvert a_{ij}\rvert\right)$ = max row "absolute sum" of A

Discrete-time dynamics and Lipschitz constants

$$x_{k+1} = \mathsf{F}(x_k) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced norm } \|\cdot\|$$

Lipschitz constant

$$\begin{aligned}\mathsf{Lip}(\mathsf{F}) &= \inf\{\ell > 0 \text{ such that } \|\mathsf{F}(x) - \mathsf{F}(y)\| \leq \ell \|x - y\| \quad \text{for all } x, y\} \\ &= \sup_x \|D\mathsf{F}(x)\|\end{aligned}$$

For **scalar map** f, $\quad \mathsf{Lip}(f) = \sup_x |f'(x)|$

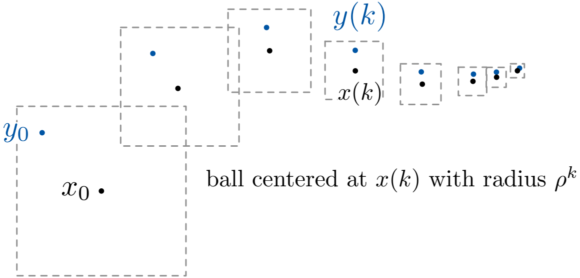
For **affine map** $\mathsf{F}_A(x) = Ax + a$

$$\begin{aligned}\|x\|_{2,P} &= (x^\top P x)^{1/2} & \mathsf{Lip}_{2,P}(\mathsf{F}_A) &= \|A\|_{2,P} \leq \ell & \iff \quad & A^\top P A \preceq \ell^2 P \\ \|x\|_{\infty,\eta} &= \max_i |x_i|/\eta_i & \mathsf{Lip}_{\infty,\eta}(\mathsf{F}_A) &= \|A\|_{\infty,\eta} \leq \ell & \iff \quad & \eta^\top |A| \leq \ell \eta^\top\end{aligned}$$

Banach contraction theorem for discrete-time dynamics:

If $\rho := \mathrm{Lip}(\mathsf{F}) < 1$, then

1. F is **contracting** $=$ distance between trajectories decreases exp fast (ρ^k)
2. F has a unique, glob exp stable equilibrium x^*



From induced norms to induced log norms

The **induced log norm** of $A \in \mathbb{R}^{n\times n}$ wrt to $\|\cdot\|$:

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

Basic properties:

$$\begin{array}{lll} \text{subadditivity:} & \mu(A+B) \leq \mu(A) + \mu(B) & \\ \text{scaling:} & \mu(bA) = b\mu(A), & \forall b \geq 0 \\ \text{convexity:} & \mu(\theta A + (1-\theta)B) \leq \theta\mu(A) + (1-\theta)\mu(B), & \forall \theta \in [0,1] \end{array}$$

spectral radius $\leq$ induced norm

spectral abscissa $\leq$ induced log norm

Example induced log norms

Vector norm	Induced matrix norm	Induced matrix log norm
$\|x\|_1 = \sum_{i=1}^{n} \lvert x_i \rvert$	$\|A\|_1 = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^{n} \lvert a_{ij} \rvert$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \Big(a_{jj} + \sum_{i=1, i\neq j}^{n} \lvert a_{ij} \rvert \Big)$ $=$ max column "absolute sum" of A
$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$	$\|A\|_2 = \sqrt{\lambda_{\mathsf{max}}(A^\top A)}$	$\mu_2(A) = \lambda_{\mathsf{max}}\Big(\frac{A + A^\top}{2}\Big)$
$\|x\|_\infty = \max_{i\in\{1,\dots,n\}} \lvert x_i \rvert$	$\|A\|_\infty = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^{n} \lvert a_{ij} \rvert$	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \Big(a_{ii} + \sum_{j=1, j\neq i}^{n} \lvert a_{ij} \rvert \Big)$ $=$ max row "absolute sum" of A

Continuous-time dynamics and one-sided Lipschitz constants

$$\dot{x} = \mathsf{F}(x) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced log norm } \mu(\cdot)$$

One-sided Lipschitz constant

$$\begin{aligned}\mathsf{osLip}(\mathsf{F}) &= \inf\{b \in \mathbb{R} \ \text{ such that } \ [\![\mathsf{F}(x) - \mathsf{F}(y), x - y]\!] \leq b\|x - y\|^2 \quad \text{for all } x, y\} \\ &= \sup_x \mu(D\mathsf{F}(x))\end{aligned}$$

For **scalar map** f, $\quad \mathsf{osLip}(f) = \sup_x f'(x)$

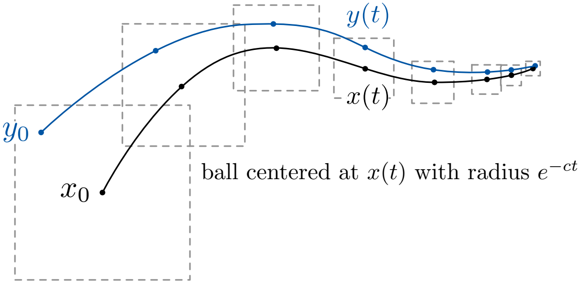
For **affine map** $\mathsf{F}_A(x) = Ax + a$

$$\begin{aligned}\mathsf{osLip}_{2,P}(\mathsf{F}_A) = \mu_{2,P}(A) \leq \ell \qquad &\Longleftrightarrow \qquad A^\top P + AP \preceq 2\ell P \\ \mathsf{osLip}_{\infty,\eta}(\mathsf{F}_A) = \mu_{\infty,\eta}(A) \leq \ell \qquad &\Longleftrightarrow \qquad a_{ii} + \sum_{j \neq i} |a_{ij}|\eta_i/\eta_j \leq \ell\end{aligned}$$

Banach contraction theorem for continuous-time dynamics:

If $-c := \mathrm{osLip}(\mathsf{F}) < 0$, then

1. F is **infinitesimally contracting** $=$ distance between trajectories decreases exp fast (e^{-ct})
2. F has a unique, glob exp stable equilibrium x^*



From inner products to weak pairings

A **weak pairing** is $[\![\cdot,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying

1. $[\![x_1 + x_2, y]\!] \leq [\![x_1, y]\!] + [\![x_2, y]\!]$ and $x \mapsto [\![x, y]\!]$ is continuous,
2. $[\![bx, y]\!] = [\![x, by]\!] = b\,[\![x, y]\!]$ for $b \geq 0$ and $[\![-x, -y]\!] = [\![x, y]\!]$,
3. $[\![x, x]\!] > 0$, for all $x \neq \mathbb{0}_n$,
4. $|\,[\![x, y]\!]\,| \leq [\![x, x]\!]^{1/2}\,[\![y, y]\!]^{1/2}$,

Given norm $\|\cdot\|$, compatibility: $[\![x, x]\!] = \|x\|^2$ for all x

Key properties

Curve norm derivative formula:	$\frac{1}{2}D^+\|x(t)\|^2 = [\![\dot{x}(t), x(t)]\!]$
Sup of non-Euclidean numerical range:	$\mu(A) = \sup_{\|x\|=1} [\![Ax, x]\!]$

A. Davydov, S. Jafarpour, and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. *IEEE Transactions on Automatic Control*, 2022a.

Example weak pairings

Norms	From inner products to sign and max pairings	From LMIs to log norms
$\|x\|_{2,P^{1/2}}^2 = x^\top P x$	$[\![x,y]\!]_{2,P^{1/2}} = x^\top P y$	$\mu_{2,P^{1/2}}(A) = \min\{b \mid A^\top P + PA \preceq 2bP\}$
$\|x\|_1 = \sum_i \|x_i\|$	$[\![x,y]\!]_1 = \|y\|_1 \operatorname{sign}(y)^\top x$	$\mu_1(A) = \max_j \Big(a_{jj} + \sum_{i\neq j} \|a_{ij}\|\Big)$
$\|x\|_\infty = \max_i \|x_i\|$	$[\![x,y]\!]_\infty = \max_{i\in I_\infty(y)} y_i x_i$	$\mu_\infty(A) = \max_i \Big(a_{ii} + \sum_{j\neq i} \|a_{ij}\|\Big)$

where $I_\infty(x) = \{i \in \{1,\dots,n\}$ such that $|x_i| = \|x\|_\infty\}$

Log Norm bound	Demidovich condition	One-sided Lipschitz condition
$\mu_{2,P}(D\mathsf{F}(x)) \leq b$	$PD\mathsf{F}(x) + D\mathsf{F}(x)^\top P \preceq 2bP$	$(x-y)^\top P\big(\mathsf{F}(x) - \mathsf{F}(y)\big) \leq b\\|x-y\\|_{P^{1/2}}^2$
$\mu_1(D\mathsf{F}(x)) \leq b$	$\operatorname{sign}(v)^\top D\mathsf{F}(x)v \leq b\\|v\\|_1$	$\operatorname{sign}(x-y)^\top(\mathsf{F}(x) - \mathsf{F}(y)) \leq b\\|x-y\\|_1$
$\mu_\infty(D\mathsf{F}(x)) \leq b$	$\max_{i \in I_\infty(v)} v_i\,(D\mathsf{F}(x)v)_i \leq b\\|v\\|_\infty^2$	$\max_{i \in I_\infty(x-y)} (x_i - y_i)(\mathsf{F}_i(x) - \mathsf{F}_i(y)) \leq b\\|x-y\\|_\infty^2$

Equivalent contractivity conditions

J. A. Jacquez and C. P. Simon. Qualitative theory of compartmental systems. *SIAM Review*, 35(1):43–79, 1993.

H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001.

G. Como, E. Lovisari, and K. Savla. Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing. *IEEE Transactions on Control of Network Systems*, 2(1):57–67, 2015.

Background on one-sided Lipschitz continuity

contraction conditions without Jacobians have been studied under many different names:

1. **uniformly decreasing maps** in: L. Chua and D. Green. A qualitative analysis of the behavior of dynamic nonlinear networks: Stability of autonomous networks. *IEEE Transactions on Circuits and Systems*, 23(6):355–379, 1976. doi
2. no-name in: A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer, 1988. ISBN 902772699X (Chapter 1, page 5)
3. **one-sided Lipschitz maps** in: E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I. Nonstiff Problems*. Springer, 1993. doi (Section 1.10, Exercise 6)
4. **maps with negative nonlinear measure** in: H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001. doi
5. **dissipative Lipschitz maps** in: T. Caraballo and P. E. Kloeden. The persistence of synchronization under environmental noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461(2059):2257–2267, 2005. doi
6. **maps with negative lub log Lipschitz constant** in: G. Söderlind. The logarithmic norm. History and modern theory. *BIT Numerical Mathematics*, 46(3):631–652, 2006. doi
7. **QUAD maps** in: W. Lu and T. Chen. New approach to synchronization analysis of linearly coupled ordinary differential systems. *Physica D: Nonlinear Phenomena*, 213(2):214–230, 2006. doi
8. **incremental quadratically stable maps** in: L. D'Alto and M. Corless. Incremental quadratic stability. *Numerical Algebra, Control and Optimization*, 3:175–201, 2013. doi

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    - Advanced Topic: Network contraction theory with delays
    - Advanced Topic: Riemannian manifolds

Fot time and input-dependent vector F,

$$\dot{x} = \mathsf{F}(t, x, u(t)), \qquad x(0) = x_0 \in \mathcal{X}, \qquad u(t) \in \mathcal{U} \tag{1}$$

Given norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{U}}$, assume constants $c, \ell > 0$ s.t.

- **osLip wrt x**: $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0$, uniformly in t, u
- **Lip wrt u**: $\mathrm{Lip}_u(\mathsf{F}) \leq \ell$, uniformly in t, x

Then

1. any soltns: $x(t)$ with input u_x and $y(t)$ with input u_y

$$D^+\|x(t)-y(t)\|_{\mathcal{X}} \;\leq\; -c\|x(t)-y(t)\|_{\mathcal{X}} \;+\; \ell\|u_x(t)-u_y(t)\|_{\mathcal{U}}$$

2. F is **incrementally ISS**, that is, for all x_0, y_0

$$\|x(t)-y(t)\|_{\mathcal{X}} \;\leq\; e^{-ct}\|x_0-y_0\|_{\mathcal{X}} \;+\; \frac{\ell(1-e^{-ct})}{c}\sup_{\tau\in[0,t]}\|u_x(\tau)-u_y(\tau)\|_{\mathcal{U}}$$

3. F has **incremental $\mathcal{L}^q_{\mathcal{X},\mathcal{U}}$ gain equal to ℓ/c, for $q\in[1,\infty]$**,

$$\|x(\cdot)-y(\cdot)\|_{\mathcal{X},q} \;\leq\; \frac{\ell}{c}\,\|u_x(\cdot)-u_y(\cdot)\|_{\mathcal{U},q} \qquad \text{(for } x_0=y_0\text{)}$$

Given norm $\|\cdot\|_{\mathcal{X}}$ on $\mathbb{R}^n$ (or $\|\cdot\|_{\mathcal{U}}$ on $\mathbb{R}^k$),

- $\mathcal{L}^q_{\mathcal{X}}$, $q \in [1, \infty]$, is vector space of continuous signals, $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, with well-defined bounded norm

$$\|x(\cdot)\|_{\mathcal{X},q} = \begin{cases} \left(\int_0^\infty \|x(t)\|^q_{\mathcal{X}} dt \right)^{1/q} & \text{if } q \in [1, \infty[\\ \sup_{t \geq 0} \|x(t)\|_{\mathcal{X}} & \text{if } q = \infty \end{cases} \tag{2}$$

- Input-state system has $\mathcal{L}^q_{\mathcal{X},\mathcal{U}}$**-induced gain** upper bounded by $\gamma > 0$ if, for all $u \in \mathcal{L}^q_{\mathcal{U}}$, the state x from zero initial state satisfies

$$\|x(\cdot)\|_{\mathcal{X},q} \leq \gamma \, \|u(\cdot)\|_{\mathcal{U},q} \tag{3}$$

From nominal to uncertain systems

Given a norm $\|\cdot\|$, consider

$$\dot{x} = \mathsf{F}(t,x) + \mathsf{G}(t,x) \tag{4}$$

Assume:

- $\text{osLip}_x(\mathsf{F}) \leq -c < 0$
- $\text{osLip}_x(\mathsf{G}) \leq d$

Then

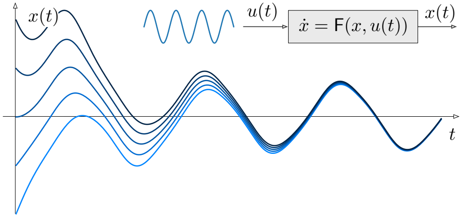
1. **(contractivity under perturbations)** if $d < c$,
   then $\mathsf{F}+\mathsf{G}$ is strongly contracting with rate $c-d$,
2. **(equilibria under perturbations)** if additionally F and G are time-invariant, then the unique equilibrium points x^* of F and x^{**} of $\mathsf{F}+\mathsf{G}$ satisfy

$$\|x^* - x^{**}\| \leq \frac{\|\mathsf{G}(x^*)\|}{c-d} \tag{5}$$

From time-invariant to periodic systems

For time-varying vector field F and norm $\|\cdot\|$

1. $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0$
2. F is T-periodic



Then

1. there exists a unique periodic solution $x^* : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ with period T
2. for every initial condition x_0,

$$\|x(t, x_0) - x^*(t)\| \leq \mathrm{e}^{-ct}\|x_0 - x^*(0)\| \tag{6}$$

G. Russo, M. Di Bernardo, and E. D. Sontag. Global entrainment of transcriptional systems to periodic inputs. *PLoS Computational Biology*, 6(4):e1000739, 2010.

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4. Conclusions and future research
5. Advanced topics
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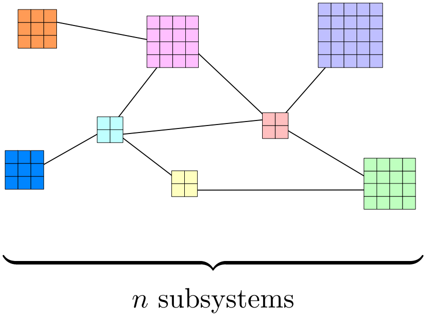
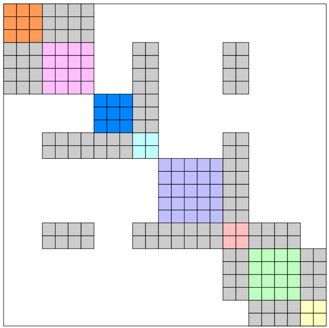
Composite norms





1. n *local norms* $\|\cdot\|_i$ on $\mathbb{R}^{N_i}$
2. an *aggregating norm* $\|\cdot\|_{\text{agg}}$ on $\mathbb{R}^n$
3. *composite norm*

G. Russo, M. Di Bernardo, and E. D. Sontag. A contraction approach to the hierarchical analysis and design of networked systems. *IEEE Transactions on Automatic Control*, 58(5):1328–1331, 2013.

Networks of contracting systems

Interconnected subsystems: $x_i \in \mathbb{R}^{N_i}$ and $x_{-i} \in \mathbb{R}^{N-N_i}$:

$$\dot{x}_i = \mathsf{F}_i(x_i, x_{-i}), \qquad \text{for } i \in \{1, \dots, n\}$$

Network contraction theorem

- **osLip wrt x_i**: $\mathrm{osLip}_{x_i}(\mathsf{F}_i) \leq -c_i$, uniformly in x_{-i}
- **Lip wrt to x_j**: $\mathrm{Lip}_{x_j}(\mathsf{F}_i) \leq \ell_{ij}$, uniformly in x_{-j}
- the Lipschitz constants matrix $\begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix}$ is **Hurwitz**

$\Longrightarrow$ the **interconnected system** is infinitesimally contracting

The network science of Metzler Hurwitz matrices

$$\begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix}$$ is **Metzler** (so that Perron-Frobenius Theorem applies)

Hurwitzness depends upon both topology and edge weights

- directed acyclic interconnections of contracting systems are strongly contracting
- For $n = 2$, Hurwitz if and only if **small gain condition**

$$\text{cycle gain} := \frac{\ell_{12}}{c_1}\frac{\ell_{21}}{c_2} < 1$$

- For $n \geq 3$, Hurwitz if and only if **network small-gain theorem for Metzler matrices**

Hurwitz Metzler Theorem

1. M is Hurwitz,
2. there exists $\eta \in \mathbb{R}^n_{>0}$ such that $\eta^\top M < \mathbb{0}_n^\top$ or, equivalently, $\mu_{1,[\eta]}(M) < 0$,
3. there exists $\xi \in \mathbb{R}^n_{>0}$ such that $M\xi < \mathbb{0}_n$ or, equivalently, $\mu_{\infty,[\xi]^{-1}}(M) < 0$, and
4. there exists a diagonal $P = P^\top \succ 0$ satisfying $M^\top P + PM \prec 0$ or, equivalently, $\mu_{2,P^{1/2}}(M) < 0$.

Input: a Metzler matrix $M \in \mathbb{R}^{n\times n}$
Output: polynomials $\{\gamma_{\mathcal{C}_2}, \dots, \gamma_{\mathcal{C}_n}\}$ in entries of M

$\mathcal{C} :=$ set of simple cycles of digraph associated to M
$\gamma_\phi :=$ gain of cycle $\phi \in \mathcal{C}$
for i from 2 to n
    $\mathcal{C}_i :=$ cycles in $\mathcal{C}$ passing through only nodes $1, \dots, i$
    $$\gamma_{\mathcal{C}_i} := \sum_{\phi\in\mathcal{C}_i} \gamma_\phi - \sum_{\substack{\phi,\psi\in\mathcal{C}_i\\ \phi\perp\psi}} \gamma_\phi\gamma_\psi + \sum_{\substack{\phi,\psi,\rho\in\mathcal{C}_i\\ \phi\perp\psi,\ \phi\perp\rho,\ \psi\perp\rho}} \gamma_\phi\gamma_\psi\gamma_\rho - \cdots$$

Network small-gain theorem for Metzler matrices

$$\text{Metzler } M \text{ is Hurwitz} \quad\Longleftrightarrow\quad \gamma_{\mathcal{C}_2} < 1, \cdots, \gamma_{\mathcal{C}_n} < 1$$

- not unique: distinct/equivalent conditions after renumbering, redundancy
- computational efficiency: after precomputation of simple cycles

X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic stability conditions for Metzler matrices and monotone systems. *SIAM Journal on Control and Optimization*, 59(5):3447–3471, 2021.

$$M = \begin{bmatrix} -c_1 & 0 & 0 & \ell_{14} \\ 0 & -c_2 & \ell_{23} & \ell_{24} \\ 0 & \ell_{32} & -c_3 & \ell_{34} \\ \ell_{41} & \ell_{42} & \ell_{43} & -c_4 \end{bmatrix}$$

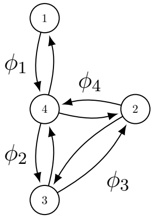


Figure: associated digraph and simple cycles

- $\gamma_{\phi_1} = \frac{\ell_{14}\ell_{41}}{c_1 c_4}$, $\gamma_{\phi_2} = \frac{\ell_{34}\ell_{43}}{c_3 c_4}$, $\gamma_{\phi_3} = \frac{\ell_{23}\ell_{32}}{c_2 c_3}$, and $\gamma_{\phi_4} = \frac{\ell_{24}\ell_{42}}{c_2 c_4}$
- $\mathcal{C}_2 = \emptyset$
- $\mathcal{C}_3 = \{\phi_3\}$: $\gamma_{\mathcal{C}_3} = \gamma_{\phi_3} < 1$ (redundant)
- $\mathcal{C}_4 = \{\phi_1, \ldots, \phi_4\}$: $\gamma_{\mathcal{C}_4} = \sum_{i=1}^4 \gamma_{\phi_i} - \gamma_{\phi_1}\gamma_{\phi_3} < 1$

$$\begin{bmatrix} -c_1 & 0 & 0 & 0 & \ell_{15} & \ell_{16} \\ 0 & -c_2 & 0 & \ell_{24} & \ell_{25} & 0 \\ 0 & 0 & -c_3 & \ell_{34} & 0 & \ell_{36} \\ 0 & \ell_{42} & \ell_{43} & -c_4 & 0 & 0 \\ \ell_{51} & \ell_{52} & 0 & 0 & -c_5 & 0 \\ \ell_{61} & 0 & \ell_{63} & 0 & 0 & -c_6 \end{bmatrix}$$

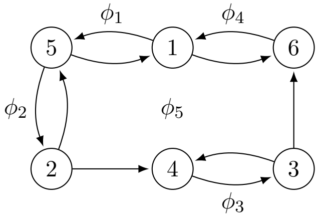


Figure: associated digraph and simple cycles

- $\mathcal{C}_2$, $\mathcal{C}_3$ empty
- $\mathcal{C}_4 = \{\phi_3\}$: $\gamma_3 < 1$ (redundant)
- $\mathcal{C}_5 = \{\phi_1, \phi_2, \phi_3\}$: $\gamma_{\mathcal{C}_5} = \gamma_1 + \gamma_2 + \gamma_3 - \gamma_1\gamma_3 - \gamma_2\gamma_3 < 1$
- $\mathcal{C}_6 = \{\phi_1, \ldots, \phi_5\}$: $\gamma_{\mathcal{C}_6} = \sum_{i=1}^{5} \gamma_i - \gamma_1\gamma_3 - \gamma_2\gamma_3 - \gamma_3\gamma_4 - \gamma_2\gamma_4 + \gamma_2\gamma_3\gamma_4 < 1$

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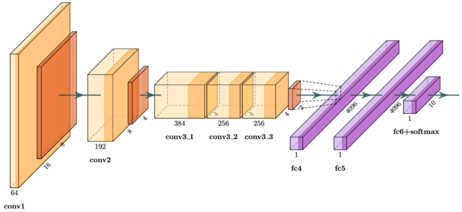
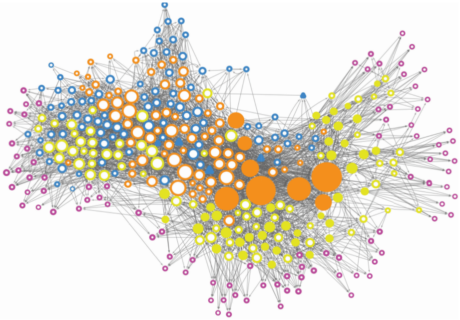
Artificial and biological neural networks



artificial neural network AlexNet '12



C. elegans connectome '17

Aim: understand the dynamics and functionality of neural networks, so that

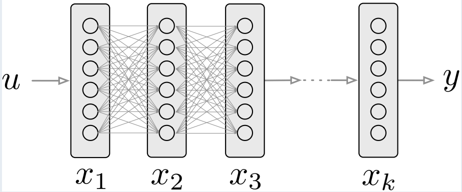
- **reproducible behavior, i.e., equilibrium response as function of stimuli**
- robust behavior in face of uncertain stimuli and dynamics
- functional/learning models, efficient computational tools, periodic behaviors ...

A. Krizhevsky, I. Sutskever, and G. E. Hinton. Imagenet classification with deep convolutional neural networks. *Advances in Neural Information Processing Systems*, 25, 2012

G. Yan, P. E. Vértes, E. K. Towlson, Y. L. Chew, D. S. Walker, W. R. Schafer, and A.-L. Barabási. Network control principles predict neuron function in the Caenorhabditis elegans connectome. *Nature*, 550(7677):519–523, 2017.

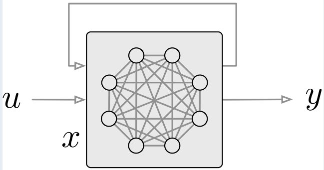
Artificial and biological neural networks – mathematization

Feedforward NN



$$x_{i+1} = \Phi(W_i x_i + b_i), \quad x_0 = u,$$
$$y = C x_k + d$$

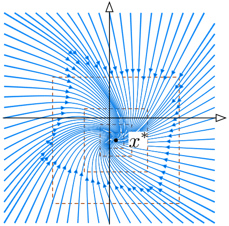
Implicit/Recurrent NN



$$x = \Phi(Wx + Bu + b),$$
$$y = Cx + d$$

Aim:

- well-posedness of the static model
- dynamic input/output models
- highly-ordered transient+asymptotic dynamic behavior
- biologically-plausible optimization



From continuous-time contracting dynamics to discrete-time computation

$$x = \mathsf{G}(x)$$

Banach contraction theorem

If $\mathsf{Lip}(\mathsf{G}) < 1$ that is $\|\mathsf{G}(u) - \mathsf{G}(v)\| \leq \mathsf{Lip}(\mathsf{G})\|u - v\|$,

then *Picard iteration* $x_{k+1} = \mathsf{G}(x_k)$ is a Banach contraction

For $\mathsf{Lip}(\mathsf{G}) \geq 1$, define the *average iteration*

$$x_{k+1} = (1 - \alpha)x_k + \alpha\mathsf{G}(x_k)$$

Infinitesimal Contraction Theorem

1. there exists $0 < \alpha < 1$ such that the average iteration is a Banach contraction
2. the map G satisfies $\mathsf{osLip}(\mathsf{G}) < 1$
3. the dynamics $\dot{x} = -x + \mathsf{G}(x)$ is infinitesimally contracting

Average iteration on the inner product space $(\mathbb{R}^n, \ell_2)$

Given $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$

$$x^* \in \mathrm{zero}(\mathsf{F}) \qquad \iff \qquad x^* \in \mathrm{fixed}(G), \text{ where } \mathsf{G} = \mathsf{Id} + \mathsf{F}$$

consider **forward step = Euler integration** for F $\quad$ = average iteration for G:

$$x_{k+1} = (\mathsf{Id} + \alpha \mathsf{F}) x_k = x_k + \alpha \mathsf{F}(x_k) \qquad = (1-\alpha)\,\mathsf{Id} + \alpha \mathsf{G}$$

Given *contraction rate* c and *Lipschitz constant* ℓ, define *condition number* $\kappa = \ell/c \geq 1$

1. the map $\mathsf{Id} + \alpha \mathsf{F}$ is a contraction map with respect to $\|\cdot\|_{2,P^{1/2}}$ for

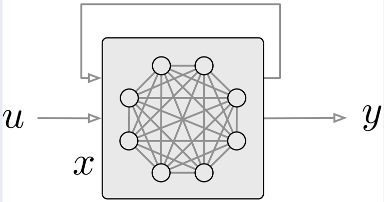
$$0 < \alpha < \frac{2}{c\kappa^2}$$

2. the optimal step size minimizing and minimum contraction factor:

$$\alpha_{\mathsf{E}}^* = \frac{1}{c\kappa^2}$$

$$\ell_{\mathsf{E}}^* = 1 - \frac{1}{2\kappa^2} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)$$

ℓ_∞-contracting recurrent neural networks



$$x = \Phi(Wx + Bu) \qquad \text{(\textit{fixed point})}$$

$$\dot{x} = -x + \Phi(Wx + Bu) \qquad \text{(\textit{recurrent NN})}$$

$$x_{k+1} = (1-\alpha)x_k + \alpha\Phi(Wx_k + Bu) \qquad \text{(\textit{average iter.n})}$$

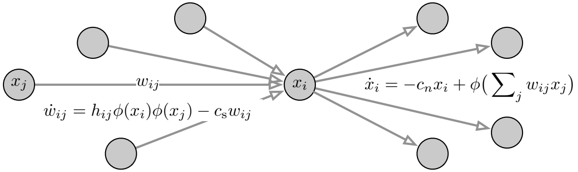
Maximizing a convex function over polytope:

$$\mathsf{osLip}_\infty\big(-x + \Phi(Wx + Bu)\big) = \sup_{x,u} \mu_\infty(-I_n + D\Phi \cdot W) = -1 + \mu_\infty(W)_+$$

If $\mu_\infty(W) < 1$ $\left(\text{i.e., } a_{ii} + \sum_j |a_{ij}| < 1 \text{ for all } i\right)$

- dynamics is contracting with rate $1 - \mu_\infty(W)_+$
- average iteration is Banach with factor $1 - \dfrac{1-\mu_\infty(W)_+}{1-\min_i(a_{ii})_-}$ at $\alpha = \dfrac{1}{1-\min_i(a_{ii})_-}$

Coupled neural-synaptic networks with Hebbian learning



coupled neural-synaptic dynamics

$$x_i = -c_{\mathsf{n}} x_i + \Phi\Big(\sum_{j=1}^{n} w_{ij}x_j + u_i\Big),$$

$$\dot{w}_{ij} = h_{ij}\Phi(x_i)\Phi(x_j) - c_{\mathsf{s}} w_{ij} + U_{ij}$$

network small gain condition:

$$c_{\mathsf{n}} c_{\mathsf{s}} > \text{interconnection strength}$$

V. Centorrino, F. Bullo, and G. Russo. Contraction analysis of Hopfield neural networks with Hebbian learning. In *IEEE Conf. on Decision and Control*, Dec. 2022. doi. To appear

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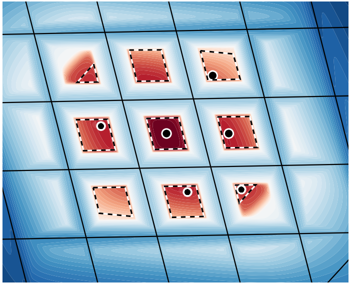
Conclusions and future research

Contraction theory for dynamical system

1. from discrete-time to continuous-time
2. from single system to networks of systems
3. Metzler Hurwitz, fixed point computation, ...
4. applications to neural networks



Future work

1. open problems
    1. local contractivity in multistable systems
    2. network theory of Metzler Hurwitz matrices
    3. contractivity of Lyapunov-diagonally-stable neural networks
2. applications to networks, control and optimization
3. learning strategies in neuroscience and ML

References

Contraction theory on normed spaces:

- A. Davydov, S. Jafarpour, and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. *IEEE Transactions on Automatic Control*, 2022a. doi
- S. Jafarpour, A. Davydov, and F. Bullo. Non-Euclidean contraction theory for monotone and positive systems. *IEEE Transactions on Automatic Control*, 2023. doi. To appear

Contracting neural networks:

- S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021. doi
- A. Davydov, A. V. Proskurnikov, and F. Bullo. Non-Euclidean contractivity of recurrent neural networks. In *American Control Conference*, pages 1527–1534, Atlanta, USA, May 2022c. doi
- A. Davydov, S. Jafarpour, A. V. Proskurnikov, and F. Bullo. Non-Euclidean monotone operator theory with applications to recurrent neural networks. In *IEEE Conf. on Decision and Control*, Dec. 2022b. doi. To appear
- V. Centorrino, F. Bullo, and G. Russo. Contraction analysis of Hopfield neural networks with Hebbian learning. In *IEEE Conf. on Decision and Control*, Dec. 2022. doi. To appear

Tutorial, text and extensions:

- K. D. Smith and F. Bullo. Contractivity of the method of successive approximations for optimal control. *IEEE Control Systems Letters*, Nov. 2022. doi. To appear
- F. Bullo, P. Cisneros-Velarde, A. Davydov, and S. Jafarpour. From contraction theory to fixed point algorithms on Riemannian and non-Euclidean spaces. In *IEEE Conf. on Decision and Control*, Dec. 2021. doi
- F. Bullo. *Contraction Theory for Dynamical Systems*. Kindle Direct Publishing, 1.0 edition, 2022. ISBN 979-8836646806. URL **http://motion.me.ucsb.edu/book-ctds**

Advanced Topics

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Optimization and Fixed Point Theory

S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021. doi

A. Davydov, S. Jafarpour, A. V. Proskurnikov, and F. Bullo. Non-Euclidean monotone operator theory with applications to recurrent neural networks. In *IEEE Conf. on Decision and Control*, Dec. 2022b. doi. To appear

For differentiable $V : \mathbb{R}^n \to \mathbb{R}$, equivalent statements:

1. V is **strongly convex** with parameter m
2. $-\operatorname{grad}V$ is m-**strongly infinitesimally contracting**, that is

$$\big(-\operatorname{grad}V(x) + \operatorname{grad}V(y)\big)^\top (x-y) \leq -m\|x-y\|_2^2$$

For map $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$, equivalent statements:

1. F is a **monotone operator**[a] (or a **coercive operator**) with parameter m,
2. $-\mathsf{F}$ is m-**strongly contracting**

[a] $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a *monotone operator* if $\langle\langle \mathsf{F}(x) - \mathsf{F}(y), x-y \rangle\rangle \geq 0$

On fixed point algorithms and Banach contractions

$$x = \mathsf{G}(x)$$

Banach Contraction Theorem

If $\mathsf{Lip}(\mathsf{G}) < 1$ that is $\|\mathsf{G}(u) - \mathsf{G}(v)\| \leq \mathsf{Lip}(\mathsf{G})\|u - v\|$,
then *Picard iteration* $x_{k+1} = \mathsf{G}(x_k)$ is a Banach contraction

For $\mathsf{Lip}(\mathsf{G}) \geq 1$, define the *average iteration*

$$x_{k+1} = (1 - \alpha)x_k + \alpha\mathsf{G}(x_k)$$

Infinitesimal Contraction Theorem

1. there exists $0 < \alpha < 1$ such that the average iteration is a Banach contraction
2. the map G satisfies $\mathsf{osLip}(\mathsf{G}) < 1$
3. the dynamics $\dot{x} = -x + \mathsf{G}(x)$ is infinitesimally strongly contracting

Robustness of fixed point algorithms

Robustness based upon Contraction

x_u^* is a fixed point of $x = \mathsf{G}(x,u)$ and $\mathsf{Lip}_x \mathsf{G} < 1$, then

$$\|x_u^* - x_v^*\| \leq \frac{\mathsf{Lip}_u \mathsf{G}}{1 - \mathsf{Lip}_x \mathsf{G}} \|u - v\|$$

Robustness based upon Infinitesimal Contraction

x_u^* is a fixed point of $x = \mathsf{G}(x,u)$
x_v^* is a fixed point of $x = \mathsf{G}(x,v) + \mathsf{D}(x,v)$, and
$\mathsf{osLip}_x(\mathsf{G}+\mathsf{D}) < 1$, then

$$\|x_u^* - x_v^*\| \leq \frac{1}{1 - \mathsf{osLip}_x(\mathsf{G}+\mathsf{D})} \Big(\mathsf{Lip}_u(\mathsf{G}+\mathsf{D})\|u - v\| + \|\mathsf{D}(x_u^*, u)\| \Big)$$

Average iteration on the inner product space $(\mathbb{R}^n, \ell_2)$

Given $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$

$$x^* \in \mathrm{zero}(\mathsf{F}) \qquad \iff \qquad x^* \in \mathrm{fixed}(G), \text{ where } \mathsf{G} = \mathsf{Id} + \mathsf{F}$$

consider **forward step = Euler integration** for F = average iteration for G:

$$x_{k+1} = (\mathsf{Id} + \alpha \mathsf{F}) x_k = x_k + \alpha \mathsf{F}(x_k) \qquad = (1-\alpha)\,\mathsf{Id} + \alpha \mathsf{G}$$

Given *contraction rate* c and *Lipschitz constant* ℓ, define *condition number* $\kappa = \ell / c \geq 1$

1. the map $\mathsf{Id} + \alpha \mathsf{F}$ is a contraction map with respect to $\| \cdot \|_{2,P^{1/2}}$ for

$$0 < \alpha < \frac{2}{c\kappa^2}$$

2. the optimal step size minimizing and minimum contraction factor:

$$\alpha^*_{\mathsf{E}} = \frac{1}{c\kappa^2}$$

$$\ell^*_{\mathsf{E}} = 1 - \frac{1}{2\kappa^2} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)$$

Average iteration on the normed vector spaces $(\mathbb{R}^n, \ell_1/\ell_\infty)$

Consider a norm $\|\cdot\|$ with compatible weak pairing $[\![\cdot,\cdot]\!]$
Recall **forward step method** $\quad x_{k+1} = (\mathsf{Id} + \alpha\mathsf{F})x_k = x_k + \alpha\mathsf{F}(x_k)$

Given *contraction rate* c and *Lipschitz constant* ℓ, define *condition number* $\kappa = \ell/c \geq 1$

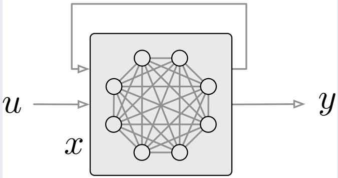
1. the map $\mathsf{Id} + \alpha\mathsf{F}$ is a contraction map with respect to $\|\cdot\|$ for

$$0 < \alpha < \frac{1}{c\kappa(1+\kappa)}$$

2. the optimal step size minimizing and minimum contraction factor:

$$\alpha^*_{\mathsf{nE}} = \frac{1}{c}\Big(\frac{1}{2\kappa^2} - \frac{3}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)\Big)$$
$$\ell^*_{\mathsf{nE}} = 1 - \frac{1}{4\kappa^2} + \frac{1}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)$$

Application: ℓ_∞-contracting neural networks



$$x = \Phi(Ax + Bu + b) \qquad \text{(\textit{INN fixed point})}$$
$$\dot{x} = -x + \Phi(Ax + Bu + b) \qquad \text{(\textit{Recurrent NN})}$$
$$x_{k+1} = (1-\alpha)x_k + \alpha\Phi(Ax_k + Bu + b) \qquad \text{(\textit{Average iter.n})}$$

If

$$\mu_\infty(A) < 1 \qquad \left(\text{i.e., } a_{ii} + \sum_j |a_{ij}| < 1 \text{ for all } i\right)$$

- dynamics is contracting with rate $1 - \mu_\infty(A)_+$
- average iteration is Banach with factor $1 - \dfrac{1-\mu_\infty(A)_+}{1-\min_i(a_{ii})_-}$ at $\alpha = \dfrac{1}{1-\min_i(a_{ii})_-}$
- input-output Lipschitz constant $\mathrm{Lip}_{u\to y} = \dfrac{\|B\|_\infty \|C\|_\infty}{1-\mu_\infty(A)_+}$

Background on Infinitesimal Contraction Theorem

1. there exists $0 < \alpha < 1$ such that the average iteration is a Banach contraction
2. the map G satisfies $\mathrm{osLip}(\mathsf{G}) < 1$
3. the dynamics $\dot{x} = \mathsf{F}(x) := -x + \mathsf{G}(x)$ is infinitesimally contracting

- the equivalence (2) $\iff$ (3) is just a transcription:
  - $\mathsf{F} = -\mathrm{Id} + \mathsf{G}$ contracting with rate $c \iff \mathrm{osLip}(\mathsf{F}) < -c \iff \mathrm{osLip}(\mathsf{G}) < 1 - c$, for $c > 0$
  - in (ℓ_2, P), $\mathrm{osLip}(\mathsf{F}) < -c$ is usual Krasovskii: $PJ(x) + J(x)^\top P \preceq -2cP$ for all x and $J = D\mathsf{F}$
- (2) $\implies$ (1): known in monotone operator theory (page 15 "forward step method" in[1])
  - vector field F is contracting with rate $c \iff -\mathsf{F}$ is strongly monotone with parameter c
- Theorem 1 in[2] proves the equivalence (1) $\iff$ (2) for any norm, i.e., the implication (2) $\implies$ (1) for any norm (with proper osLip definitions) and the converse direction (1) $\implies$ (2) for ℓ_2, P. Theorem 3 in[2] proves partly the "Robustness based upon infinitesimal contraction".

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[1] E. K. Ryu and S. Boyd. Primer on monotone operator methods. *Applied Computational Mathematics*, 15(1):3–43, 2016

[2] S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021.

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Euclidean vs. non-Euclidean contractions

Most foundational results in systems theory are based on ℓ_2 linear-quadratic theory;
their ℓ_1/ℓ_∞ analogs are yet to be worked out.

NonEuclidean contractions: biological transcriptional systems (Russo et al., 2010), Hopfield neural networks (Fang and Kincaid, 1996; Qiao et al., 2001), chemical reaction networks (Al-Radhawi and Angeli, 2016), traffic networks (Coogan and Arcak, 2015; Como et al., 2015; Coogan, 2019), multi-vehicle systems (Monteil et al., 2019), and coupled oscillators (Russo et al., 2013; Aminzare and Sontag, 2014)

Advantages of non-Euclidean approaches

1. *especially well suited for certain class of systems*
2. *computational advantages*: non-Euclidean log-norm constraints lead to LPs, whereas ℓ_2 constraints leads to LMIs. Parametrization of log-norm constrained matrices is polytopic.

   A. Rantzer. Scalable control of positive systems. *European Journal of Control*, 24:72–80, 2015.
3. *guaranteed robustness to structural perturbations*: ℓ_∞ contractivity ensures:
   1. absolute contractivity $=$ with respect to a class of activation functions
   2. total contractivity $=$ remove any node and all its incident connections
   3. connective contractivity $=$ remove any set of edges
4. *adversarial input-output analysis*

   ℓ_∞ better suited for the analysis of adversarial examples than ℓ_2: in high dimensions, large inner product between two vectors is possible even when one vector has small ℓ_∞ norm

   I. J. Goodfellow, J. Shlens, and C. Szegedy. Explaining and harnessing adversarial examples. In *International Conference on Learn Representations (ICLR)*, 2015. URL **https://arxiv.org/abs/1412.6572**
5. *fully asynchronous distributed model*: ℓ_∞ contractions

   D. P. Bertsekas. Distributed asynchronous computation of fixed points. *Mathematical Programming*, 27(1):107–120, 1983.

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Semicontraction Theory, Dual Seminorms and Ergodicity

A. A. Markov. Extensions of the law of large numbers to dependent quantities. *Izvestiya Fiziko-matematicheskogo obschestva pri Kazanskom universitete*, 15, 1906. (in Russian)

S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922.

W. Wang and J. J. Slotine. On partial contraction analysis for coupled nonlinear oscillators. *Biological Cybernetics*, 92(1):38–53, 2005.

S. Jafarpour, P. Cisneros-Velarde, and F. Bullo. Weak and semi-contraction for network systems and diffusively-coupled oscillators. *IEEE Transactions on Automatic Control*, 67(3):1285–1300, 2022.

G. De Pasquale, K. D. Smith, F. Bullo, and M. E. Valcher. Dual seminorms, ergodic coefficients, and semicontraction theory. *Technical Report*, 2022.

Semicontraction: history and setup

For *row-stochastic* A, consider averaging and dynamical flow systems:

$$x(k+1) = Ax(k) \qquad \text{(averaging flow, consensus algorithm)}$$

$$\pi(k+1) = A^\top \pi(k) \qquad \text{(dynamical flow system, Markov chain)}$$

Similarly, let L be a Laplacian matrix and consider the continuous-time counterparts:

$$\dot{x}(t) = -Lx(t) \qquad \text{(Laplacian flow)}$$

$$\dot{\pi}(t) = -L^\top \pi(t) \qquad \text{(continuous-time Markov chain, routing dynamics)}$$

For row-stochastic A, define the *Markov-Dobrushin ergodic coefficient*:

$$\tau_1(A) := \max_{\|z\|_1 = 1,\, z^\top \mathbb{1}_n = 0} \|A^\top z\|_1$$

Simple calculations and remarkable similarity

For $\pi(k+1) = A^\top \pi(k)$, Markov showed any two solutions $\pi(k), \sigma(k)$ satisfy

$$d_{\mathrm{TV}}\big(\pi(k) - \sigma(k)\big) \leq \ \tau_1(A)^k \ d_{\mathrm{TV}}\big(\pi(0) - \sigma(0)\big) \tag{9}$$

$$d_{\mathrm{TV}}(\pi, \sigma) = \tfrac{1}{2} \sum\nolimits_i |\pi_i - \sigma_i|$$

(**total variation distance**)

For $x(k+1) = Ax(k)$, it is known in the consensus literature that

$$|||x(k)|||_{\mathrm{dist},\infty} \leq \ \tau_1(A)^k \ |||x(0)|||_{\mathrm{dist},\infty} \tag{10}$$

$$|||x|||_{\mathrm{dist},\infty} = \tfrac{1}{2}\Big(\max_i\{x_i\} - \min_j\{x_j\}\Big)$$

(**disagreement seminorm**)

Open questions

1. Why is the same ergodic coefficient τ_1 relevant for the contraction properties of both dynamical flows and averaging? Is it the tightest such bound?
2. What is the relationship between d_{TV} and $|||\cdot|||_{\mathrm{dist},\infty}$? How does one generalize bounds (9) and (10) to τ_p ergodic coefficients defined wrt ℓ_p norms (instead of ℓ_1 in (62))?
3. What are canonical Lyapunov functions for both systems, whose discrete-time variation along the flow is described by $\tau_p(A)$?
4. How does one define ergodic coefficients for continuous-time systems?
5. Is there a *contraction theoretic framework* that applies to time-varying and nonlinear systems with generalized invariance or conservation properties?

Seminorms

A function $|||\cdot||| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *seminorm* on $\mathbb{R}^n$ if, for all $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$:

$$\begin{aligned} \text{(homogeneity):}\ & |||ax||| = |a|\,|||x|||, \text{ and} \\ \text{(subadditivity):}\ & |||x+y||| \leq |||x||| + |||y|||. \end{aligned}$$

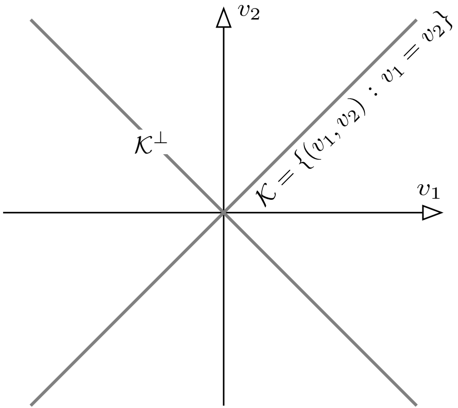
The *kernel* of $|||\cdot|||$ is the vector subspace $\mathcal{K} = \ker(|||\cdot|||) = \{x \in \mathbb{R}^n : |||x||| = 0\}$

1. *dual seminorm* is the function $|||\cdot|||_\star : V^\star \to \mathbb{R}$ defined by

$$|||y|||_\star \triangleq \max_{\substack{|||x|||\leq 1 \\ x \perp \mathcal{K}}} \langle y, x \rangle$$

2. *induced matrix seminorm on* $\mathbb{R}^{n\times n}$ $|||\cdot||| : n \times n \to \mathbb{R}_{\geq 0}$ is

$$|||A||| \triangleq \max_{\substack{|||x|||\leq 1 \\ x \perp \mathcal{K}}} |||Ax|||$$



- On $\mathbb{R}^2$, the function $(v_1, v_2) \mapsto \sqrt{(v_1 - v_2)^2} = |v_1 - v_2|$ is seminorm
- $\mathcal{K} = \{(v_1, v_2)$ such that $v_1 = v_2\} = \mathrm{span}\{(1,1)^\top\}$ and $\mathcal{K}^\perp = \mathrm{span}\{(1,-1)^\top\}$.
- The orthogonal projection matrices onto $\mathcal{K}$ and $\mathcal{K}^\perp$ are

$$\Pi_\parallel = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \Pi_\perp = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Projection and distance seminorms

Given any subspace $\mathcal{K}$, let

$$\Pi_{\perp} \in \mathbb{R}^{n \times n} \;\; := \;\; \text{orthogonal projection onto } \mathcal{K}^{\perp}$$

For each $p \in [1, \infty]$, define the *projection seminorm*

$$|||x|||_{\mathrm{proj},p} \triangleq \|\Pi_{\perp} x\|_p$$

and the *distance seminorm*

$$|||x|||_{\mathrm{dist},p} \triangleq \mathrm{dist}_p(x, \mathcal{K}) = \min_{u \in \mathcal{K}} \|x - u\|_p.$$

Consensus seminorm = a seminorm with kernel $\mathcal{K} = \mathrm{span}\{\mathbb{1}_n\}$

1. define $x_{\text{avg}} = \frac{1}{n}\mathbb{1}_n^\top x$:

$$|||x|||_{\text{proj},1} = \sum_{i=1}^{n} |x_i - x_{\text{avg}}|\,, \qquad |||x|||_{\text{proj},\infty} = \max_i |x_i - x_{\text{avg}}|$$

$$|||x|||_{\text{proj},2} = \Big(\frac{1}{n}\sum\nolimits_{i,j}(x_i - x_j)^2\Big)^{1/2}$$

2. sort the entries of x according to $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$:

$$|||x|||_{\text{dist},1} = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} x_{(i)} - \sum_{i=\lceil \frac{n}{2} \rceil + 1}^{n} x_{(i)}, \qquad |||x|||_{\text{dist},\infty} = \frac{1}{2}\big(x_{(1)} - x_{(n)}\big)\,.$$

$$|||x|||_{\text{dist},2} = \Big(\frac{1}{n}\sum\nolimits_{i,j}(x_i - x_j)^2\Big)^{1/2}$$

Total variation d_{TV} and ℓ_1 projection seminorm: $d_{\text{TV}}(x,y) = \frac{1}{2}|||x-y|||_{\text{proj},1}$ for $x, y \in \Delta_n$

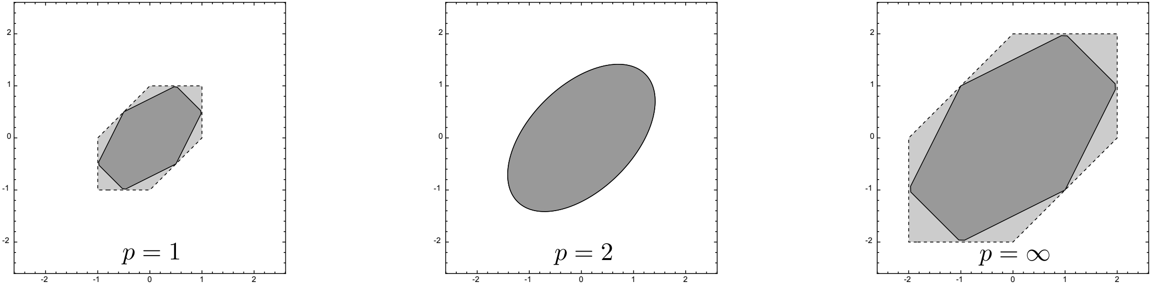


Figure: Two-dimensional sections of three-dimensional unit disks of projection (solid contours) and distance (dashed contours) consensus seminorms. We plot the sections corresponding to $(x_1, x_2, x_3 = 0)$.

1. ℓ_p and ℓ_q norms are dual, for $1/p + 1/q = 1$

$$|||\cdot|||_p = (|||\cdot|||_q)_\star \qquad\qquad |||\cdot|||_q = (|||\cdot|||_p)_\star$$

2. dual norm satisfies (sharp) *Hölder inequality*: $x^\top y \leq \|x\|_p \|y\|_q$
3. dual norm induces duality: $\|A\|_p = \|A^\top\|_q$
4. induced norm is submultiplicative: $\|AB\| \leq \|A\| \|B\|$

Key theorems about dual and induced seminorms

Projection and distance seminorms are dual

$$|||\cdot|||_{\mathrm{dist},p} = (|||\cdot|||_{\mathrm{proj},q})_\star \qquad\qquad |||\cdot|||_{\mathrm{proj},q} = (|||\cdot|||_{\mathrm{dist},p})_\star$$

Properties of dual and induced seminorms

1. dual seminorm satisfies (sharp) *Markov inequality*: $x^\top \Pi_\perp y \leq |||x|||_{\mathrm{dist},p} |||y|||_{\mathrm{proj},q}$
2. dual seminorm induces duality: $|||A|||_{\mathrm{dist},p} = |||A^\top|||_{\mathrm{proj},q}$
3. induced seminorm is submultiplicative: $|||AB||| \leq |||A|||\,|||B|||$ if $A\mathcal{K} \subseteq \mathcal{K}$ or $B\mathcal{K}^\top \subseteq \mathcal{K}^\top$

Ergodic coefficients are induced seminorms

If $A\mathcal{K} \subseteq \mathcal{K}$, then $$|||A|||_{\mathrm{dist},p} = |||A^\top|||_{\mathrm{proj},q} = \tau_q(\mathcal{K}, A) := \max_{\|z\|_q=1,\ z\perp\mathcal{K}} \|A^\top z\|_q$$

How Markov and Banach's results meet

Given $\mathcal{K} \subset \mathbb{R}^n$ and $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$, consider $\{A(k)\}_{k \in \mathbb{Z}_{\geq 0}} \subset \mathbb{R}^{n \times n}$ satisfying:

$$A(k)\mathcal{K} \subseteq \mathcal{K} \quad \text{for all } k \in \mathbb{Z}_{\geq 0}, \qquad \text{(invariance)}$$

$$\rho \triangleq \sup_{k \in \mathbb{Z}_{\geq 0}} \tau_p(\mathcal{K}, A(k)) < 1. \qquad \text{(semicontraction)}$$

1. the *averaging system*

$$x(k+1) = A(k)x(k) + b, \quad b \in \mathbb{R}^n,$$

is *strongly semicontracting with rate ρ wrt $|||\cdot|||_{\mathrm{dist},q}$*

$$|||x(k) - y(k)|||_{\mathrm{dist},q} \leq \rho^k |||x(0) - y(0)|||_{\mathrm{dist},q}$$

2. the *dynamical flow system*

$$x(k+1) = A^\top(k)x(k) + b, \quad b \in \mathbb{R}^n,$$

is *strongly semicontracting with rate ρ wrt $|||\cdot|||_{\mathrm{proj},p}$* and, for any $x(0) - y(0) \in \mathcal{K}^\perp$,

$$|||x(k) - y(k)|||_{\mathrm{proj},p} \leq \rho^k |||x(0) - y(0)|||_{\mathrm{proj},p}$$

Continuous-time semicontraction theory

The *induced log seminorm* of $A \in \mathbb{R}^{n\times n}$ is

$$\mu_{|||\cdot|||}(A) \triangleq \lim_{h\to 0^+} \frac{|||I_n + hA||| - 1}{h}$$

Theorem (Dual logarithmic seminorms)

Let $p, q \in [1, \infty]$ such that $p^{-1} + q^{-1} = 1$. For any matrix $M \in \mathbb{R}^{n\times n}$, and any kernel $\mathcal{K}$,

$$\mu_{\mathrm{dist},p}(M) = \mu_{\mathrm{proj},q}(M^\top)$$

Formulas for induced log seminorm of Laplacian matrices

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Indirect optimal control via contraction theory and iISS

K. D. Smith and F. Bullo. Contractivity of the method of successive approximations for optimal control. *IEEE Control Systems Letters*, Nov. 2022. . To appear

Optimal control problem:

$$\begin{cases} \dot{x} = \mathsf{F}(x,u) \\ \mathcal{J}[u] = \int_0^T \text{running cost} + \text{final cost} \end{cases}$$

Pontryagin minimum principle

$$\dot{x} = \mathsf{F}(x,u)$$

$$\dot{\lambda} = \mathsf{Adjoint}(x,u,\lambda)$$

$$u = \operatorname{argmin}_{\tilde{u}} \mathcal{H}(x,\tilde{u},\lambda)$$

Method of successive approximations

Input: initial guess $u^{(0)}$, init value x_0

1: **for** $i \in \{1,\ldots,N\}$ **do**
2: $\quad x^{(i)} \leftarrow$ forward with $u^{(i-1)}$
3: $\quad \lambda^{(i)} \leftarrow$ backward with $x^{(i)}$ and $u^{(i-1)}$
4: $\quad u^{(i)} \leftarrow \operatorname{argmin}_{\tilde{u}} H(x^{(i)}, \lambda^{(i)}, \tilde{u})$
5: **end for**
6: **return** $u^{(N)}$

Contractivity of adjoint dynamics and MSA

1. $\mathsf{osLip}_{\|\cdot\|^*}(\mathsf{Adjoint}^{\leftarrow}) = \mathsf{osLip}_{\|\cdot\|}(\mathsf{F})$
2. $\mathsf{Lip}(\mathsf{MSA}) \to 0^+$ as $T \to 0^+$ or $\mathsf{osLip}(\mathsf{F}) \to -\infty$
3. MSA is a contraction for (short T or highly contracting F)

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Incremental ISS for strongly contracting delay ODEs

$$\dot{x}(t) = f\big(x(t), x(t-s), u(t)\big), 0 \leq s \leq S, \qquad \|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{U}} \tag{11}$$

assume there exist positive constants c, $\ell_{\mathcal{U}}$, $\ell_{\mathcal{X}}$ such that, for all variables,

$$\begin{aligned}
&\textbf{osL } x: && [\![f(x,d,u) - f(y,d,u), x-y]\!]_{\mathcal{X}} \leq -c\|x-y\|_{\mathcal{X}}^2 && (12)\\
&\textbf{Lip } x(t-s): && \|f(x,x_1,u) - f(x,x_2,u)\|_{\mathcal{X}} \leq \ell_{\mathcal{X}}\|x_1-x_2\|_{\mathcal{X}} && (13)\\
&\textbf{Lip } u: && \|f(x,d,u) - f(x,d,v)\|_{\mathcal{X}} \leq \ell_{\mathcal{U}}\|u-v\|_{\mathcal{U}} && (14)
\end{aligned}$$

By the curve norm derivative formula, subadditivity, and Cauchy-Schwarz inequality,

$$\begin{aligned}
\|x(t)-y(t)\|_{\mathcal{X}} D^+\|x(t)-y(t)\|_{\mathcal{X}} &= [\![f(x(t),x(t-s),u_x(t)) - f(y(t),y(t-s),u_y(t)), x(t)-y(t)]\!]_{\mathcal{X}}\\
&\leq [\![f(x(t),x(t-s),u_x(t)) - f(y(t),x(t-s),u_x(t)), x(t)-y(t)]\!]_{\mathcal{X}}\\
&\quad + [\![f(y(t),x(t-s),u_x(t)) - f(y(t),y(t-s),u_x(t)), x(t)-y(t)]\!]_{\mathcal{X}}\\
&\quad + [\![f(y(t),y(t-s),u_x(t)) - f(y(t),y(t-s),u_y(t)), x(t)-y(t)]\!]_{\mathcal{X}}\\
&\leq -c\|x(t)-y(t)\|_{\mathcal{X}}^2 + \ell_{\mathcal{X}}\|x(t-s)-y(t-s)\|_{\mathcal{U}}\|x(t)-y(t)\|_{\mathcal{X}},\\
&\quad + \ell_{\mathcal{U}}\|u_x(t)-u_y(t)\|_{\mathcal{U}}\|x(t)-y(t)\|_{\mathcal{X}}.
\end{aligned}$$

Thus, with $m(t) = \|x(t)-y(t)\|_{\mathcal{X}}$, delay differential inequality:

$$D^+ m(t) \leq -cm(t) + \ell_{\mathcal{X}} \sup_{0\leq s\leq S} m(t-s) + \ell_{\mathcal{U}}\|u_x(t)-u_y(t)\|_{\mathcal{U}}, \tag{15}$$

Halanay inequality is applicable. If $c > \ell_{\mathcal{X}}$, then

$$m(t) \leq m_0 \mathrm{e}^{-\rho(t-t_0)} + \ell_{\mathcal{U}} \int_{t_0}^{t} \mathrm{e}^{-\rho(t-\tau)} \|u_x(\tau)-u_y(\tau)\|_{\mathcal{U}} d\tau, \tag{16}$$

where $\rho > 0$ is the unique positive root of $\rho = c - \ell_{\mathcal{X}}\mathrm{e}^{\rho S}$ and $m_0 = \sup_{0\leq s\leq S} m(t_0-s)$.

Networks of contracting systems with time delays

Interconnected subsystems $i \in \{1, \dots, n\}$

$$\dot{x}_i = f_i(x_i, x_{-i}, x_{-i}(t-s), u_i), \qquad 0 \le s \le S, \qquad \|\cdot\|_i, \|\cdot\|_{i,\mathcal{U}} \tag{17}$$

Assume there exist positive constants st

osL x_i : $\quad [\![f_i(x_i, \dots) - f_i(y_i, \dots), x_i - y_i]\!]_i \le -c_i \|x_i - y_i\|_i^2$

Lip x_{-i} : $\quad \|f_i(\dots, x_{-i}, \dots) - f_i(\dots, y_{-i}, \dots)\|_i \le \sum_{j=1, j\neq i}^{n} \gamma_{ij} \|x_j - y_j\|_j$

Lip x_{-1}^{-s} : $\quad \|f_i(\dots, x_{-i}^{-s}, \dots) - f_i(\dots, y_{-i}^{-s}, \dots)\|_i \le \sum_{j=1, j\neq i}^{n} \widehat{\gamma}_{ij} \|x_j^{-s} - y_j^{-s}\|_j$

Lip u_i : $\quad \|f_i(\dots, u_i) - f_i(\dots, v_i)\|_i \le \ell_{i,\mathcal{U}} \|u_i - v_i\|_{i,\mathcal{U}}$

With $m_i(t) = \|x_i(t) - y_i(t)\|_i$, delay differential inequality:

$$D^+ m(t) \le -Cm(t) + \Gamma m(t) + \widehat{\Gamma} \sup_{0 \le s \le S} m(t-s) + \ell_{\mathcal{U}} \|u_x(t) - u_y(t)\|_{\mathcal{U}}$$

and, if the Metzler matrix $-C + \Gamma + \widehat{\Gamma}$ is Hurwitz, then (17) is incremental ISS

F. Mazenc, M. Malisoff, and M. Krstic. Vector extensions of Halanay's inequality. *IEEE Transactions on Automatic Control*, 67(3):1453–1459, 2022.

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Contraction theory on Riemannian manifolds

Contraction theory on Riemannian manifolds originates in

W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998.

A formal coordinate-free analysis (with connection to monotone operators) is given in

J. W. Simpson-Porco and F. Bullo. Contraction theory on Riemannian manifolds. *Systems & Control Letters*, 65:74–80, 2014.

In the differential geometry literature, geodesically monotonic vector fields are studied by

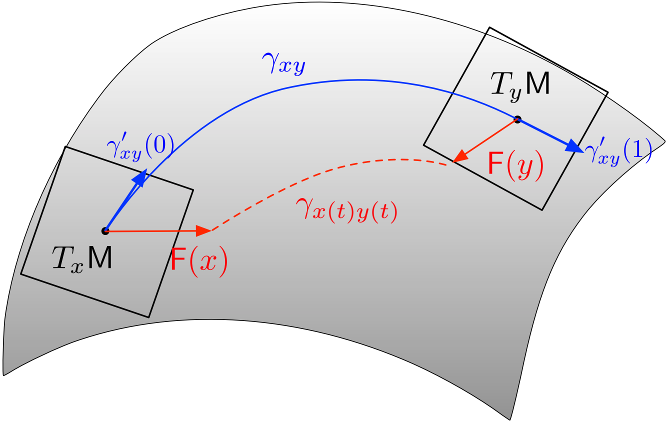
S. Z. Németh. Geodesic monotone vector fields. *Lobachevskii Journal of Mathematics*, 5:13–28, 1999. URL **http://mi.mathnet.ru/eng/ljm145**

J. X. Da Cruz Neto, O. P. Ferreira, and L. R. Lucambio Pérez. Contributions to the study of monotone vector fields. *Acta Mathematica Hungarica*, 94(4):307–320, 2002.

J. H. Wang, G. López, V. Martín-Márquez, and C. Li. Monotone and accretive vector fields on Riemannian manifolds. *Journal of Optimization Theory and Applications*, 146(3):691–708, 2010.

Contraction theory on Riemannian manifold $(\mathsf{M}, \mathbb{G})$

F **contracting** if geodesic distances from x to y diminishes along the flow of F



integral test: the inner product between F and the geodesic velocity vector γ'_{xy} at x and y

differential test: condition on covariant differential of F

$$\mathbb{G}(x)D\mathsf{F}x(x) + D\mathsf{F}x(x)^\top\mathbb{G}(x) + \dot{\mathbb{G}}(x) \preceq -2c\mathbb{G}(x)$$

Strong infinitesimal contraction on a Riemannian manifold

Given a time-independent vector field X on a Riemannian manifold $(\mathsf{M}, \mathbb{G})$ and $c > 0$, the following statements are equivalent:

1. for any $x, y \in \mathsf{M}$ and geodesic curve $\gamma_{xy} : [0,1] \to \mathsf{M}$ with $\gamma_{xy}(0) = x$, $\gamma_{xy}(1) = y$,
$$\langle\!\langle X(y), \gamma'_{xy}(1) \rangle\!\rangle_{\mathbb{G}} - \langle\!\langle X(x), \gamma'_{xy}(0) \rangle\!\rangle_{\mathbb{G}} \leq -c\, \mathrm{d}_{\mathbb{G}}(x,y)^2$$
2. for all $v_x \in T_x\mathsf{M}$
$$\langle\!\langle A_X(x) v_x, v_x \rangle\!\rangle_{\mathbb{G}} \leq -c \|v_x\|^2_{\mathbb{G}},$$
where the *covariant differential* $A_X(x) : T_x\mathsf{M} \to T_x\mathsf{M}$ is defined by $A_X(x) v_x = \nabla_{v_x} X(x)$
3. $D^+ \mathrm{d}_{\mathbb{G}}(x(t), y(t)) \leq -c\, \mathrm{d}_{\mathbb{G}}(x(t), y(t))$, for all solutions $x(\cdot), y(\cdot)$