

Perspectives on Coupled Oscillators: Geometry, Analysis and Computation

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Open Problems in Modeling, Analysis, and Control**
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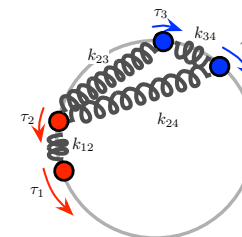
Outline

#1: Elastic and flow networks on the torus

$$\omega_i = \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j)$$

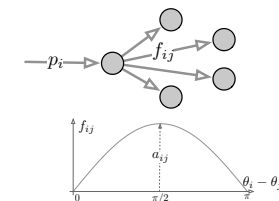
Spring network

- $\omega_i = \tau_i$: torque at i
- $a_{ij} = k_{ij}$: spring stiffness i, j
- $\sin(\theta_i - \theta_j)$: modulation
- elastic energy
 $\mathcal{E} = \sum_{ij} (1 - \cos(\theta_i - \theta_j))$



Power network

- $\omega_i = p_i$: injected power
- a_{ij} : max power flow i, j
- $\sin(\theta_i - \theta_j)$: modulation
- KCL flow conservation and Ohm's flow law

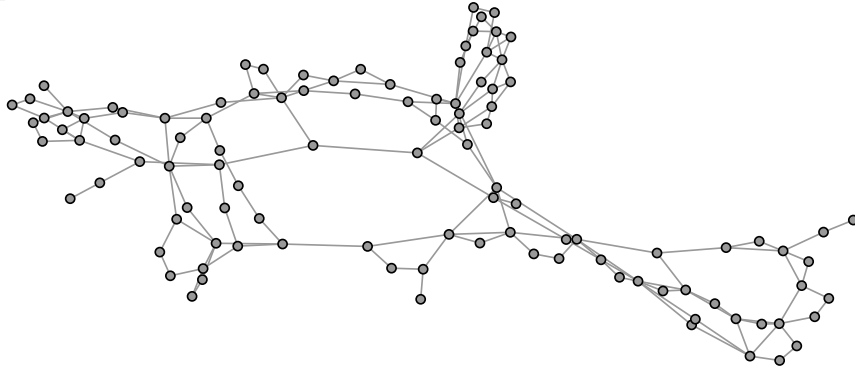


1 Recent progress

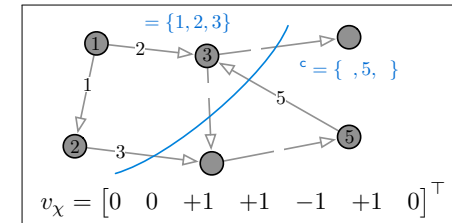
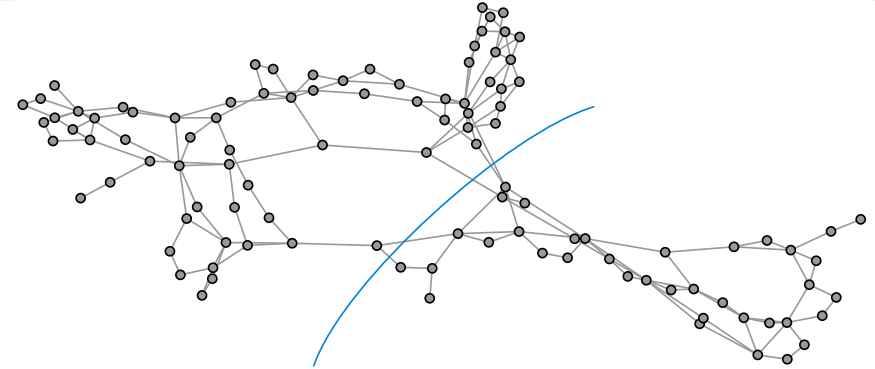
- Elastic and flow networks on the torus
- Cutset spaces
- Geometric graph theory on the n -torus
- Convexity, monotonicity, and contraction theory
- Multistability in phase-coupled oscillators
- Sync threshold: Approximate inverse via series methods
- Sync threshold: gap between necessary and sufficient conditions
- State-space oscillators

2 Open Problems

#2: Cutset spaces



#2: Cutset spaces



$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\text{Im}(B^\top)}_{\text{cutset space flow vectors}} \oplus \underbrace{\text{Ker}(BA)}_{\text{weighted cycle space cycle vectors}}$$

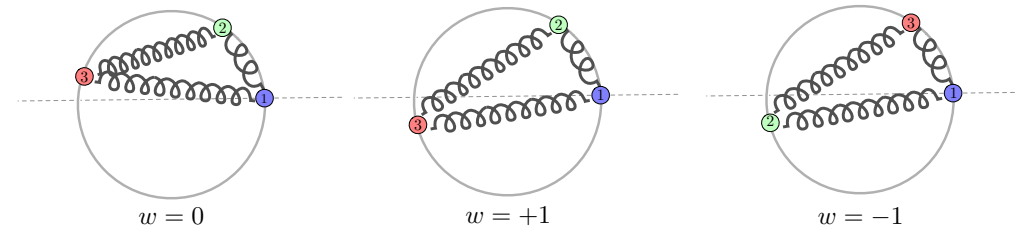
$$\mathcal{P} = B^\top L^\dagger BA = \text{cutset projection operator} \text{ — onto } \text{Im}(B^\top) \text{ parallel to } \text{Ker}(BA)$$

- 1 if G unweighted, then \mathcal{P} is orthogonal and $\|\mathcal{P}\|_2 = 1$
- 2 if G acyclic, then $\mathcal{P} = I_m$ and $\|\mathcal{P}\|_p = 1$
- 3 if G uniform complete or ring, then $\|\mathcal{P}\|_\infty = 2(n-1)/n \leq 2$
- 4 if θ is the minimal angle between the cutset space and the cycle space of G , then $\sin(\theta) = \|\mathcal{P}\|_2^{-1}$
- 5 if $R_{\text{eff}} \in \mathbb{R}^{n \times n}$ are effective resistances, then $\mathcal{P} = -\frac{1}{2} B^\top R_{\text{eff}} BA$
- 6 ...

#3: Winding numbers and partitions

Given a cycle $\sigma = (1, \dots, n_\sigma)$ and orientation

- 1 **winding number of $\theta \in \mathbb{T}^n$ along σ**
= number of times the **shortest-arc path wraps around torus**



- 2 given basis $\sigma_1, \dots, \sigma_r$ for cycles, **winding vector of θ** is

$$w(\theta) = (w_{\sigma_1}(\theta), \dots, w_{\sigma_r}(\theta))$$

Theorem: Kirchhoff angle law on \mathbb{T}^n

winding number is at most $\pm \lfloor n_\sigma/2 \rfloor - 1$



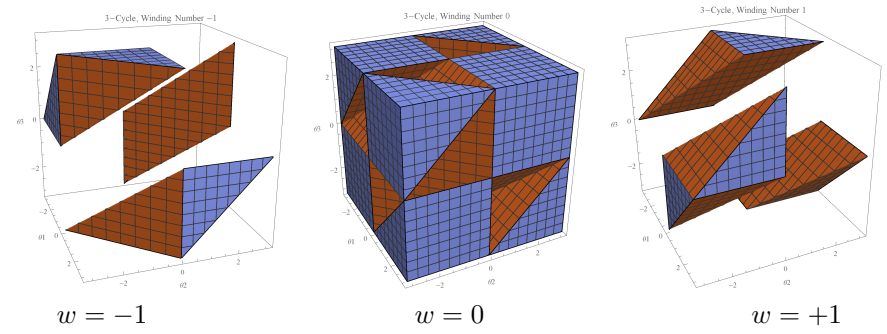
Theorem: Winding partition

For each possible winding vector u , define

$$\text{WindingCell}(u) := \{\theta \in \mathbb{T}^n \mid w(\theta) = u\}$$

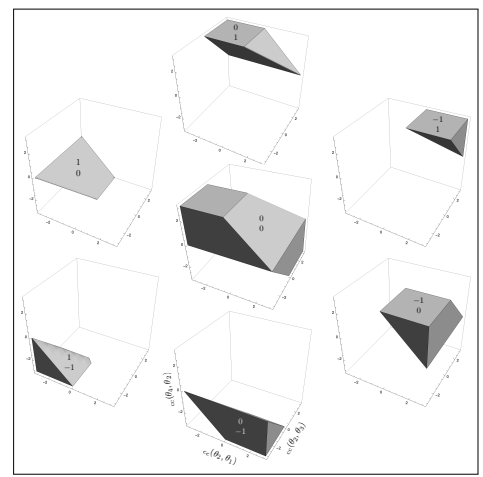
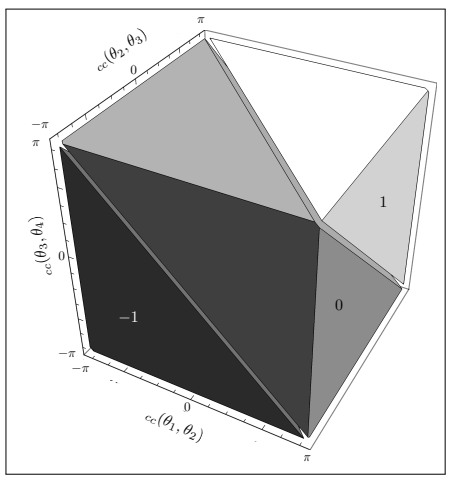
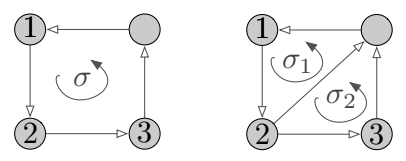
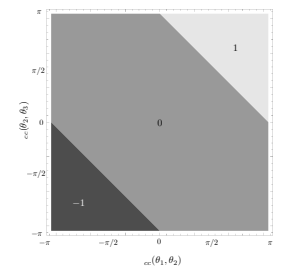
Then

$$\mathbb{T}^n = \cup_u \text{WindingCell}(u)$$

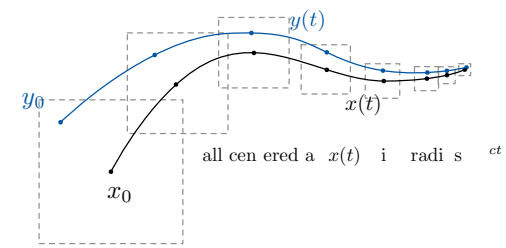


Theorem: Reduced cell is convex polytope

- each winding cell is connected and invariant under rotation
- **bijection:**
reduced winding cell \longleftrightarrow open convex polytope



#4: Analysis: Convexity, monotonicity, and contraction theory

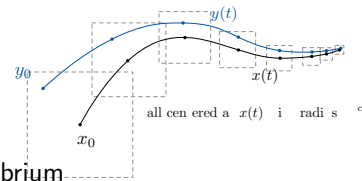


- 1 V is **strongly convex** with parameter m
- 2 $-\text{grad } V$ is **m -strongly contracting**, that is

$$(-\text{grad } V(x) + \text{grad } V(y))^T (x - y) \leq -m \|x - y\|_2^2$$

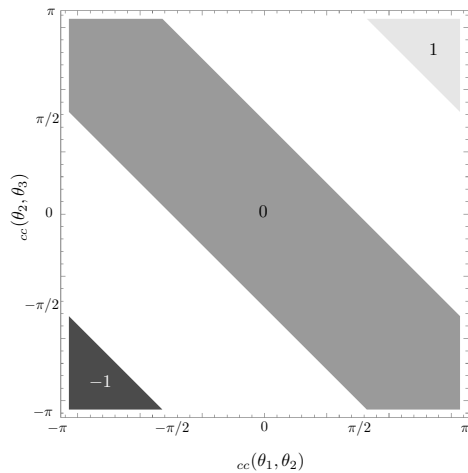
- 1 F is a **monotone operator** (or a **coercive operator**) with parameter m ,
- 2 $-F$ is **m -strongly contracting**

search for contraction properties
design engineering systems to be contracting

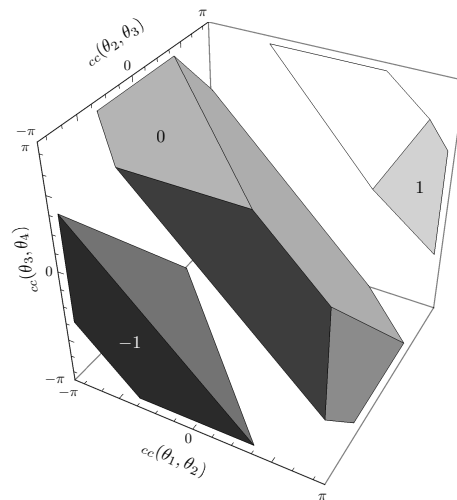


Highly ordered **transient** and **asymptotic** behavior:

- 1 time-invariant F: unique globally exponential stable equilibrium
two natural Lyapunov functions
- 2 periodic F: contracting system entrain to periodic inputs
- 3 accurate numerical integration and equilibrium computation
- 4 contractivity rate is natural measure/indicator of robust stability
input-to-state stability
finite input-state gain
contraction margin wrt unmodeled dynamics
input-to-state stability under delayed dynamics



(a)



()

#5: Multistable Sync = global partition + local contraction

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j)$$

in each winding cell

- 1 $\dot{\theta} = -\text{grad } \mathcal{E}(\theta)$, where

$$\mathcal{E}(\theta) = \sum_{i,j} (1 - \cos(\theta_i - \theta_j)) + \omega^\top \theta$$

- 2 Hessian $\mathcal{E}(\theta) = -\text{Cosine-Laplacian}(\theta) \preceq 0$
- 3 Hessian $\mathcal{E}(\theta) \preceq 0$ on the **cohesive subset** $|\theta_i - \theta_j| \leq \pi/2$
- 4 modulo the symmetry, the dynamics is strongly contracting

Theorem:

- 1 each winding cell has at most one cohesive equilibrium
- 2 contraction algorithm to decide/compute in each winding cell

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j + \phi_{ij})$$

same properties, by robustness of contracting dynamics

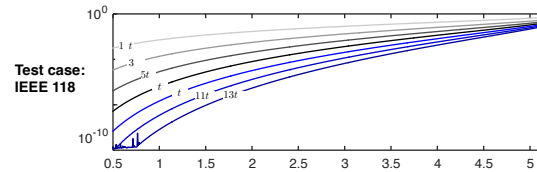
#6: Sync threshold: Approximate inverse via series methods

Projection onto to cutset space: $z = B^\top L^\dagger \omega$ and $x = B^\top \theta$
synchrony equilibrium equation is

$$z = \mathcal{P} \sin(x)$$

Given input z , unique solution is

$$x = \sum_{i=0}^{\infty} A_{2i+1}(z), \quad \begin{aligned} A_1(z) &= z &= B^\top L^\dagger \omega \\ A_3(z) &= \mathcal{P} \left(\frac{1}{3!} z^{\circ 3} \right) \\ A_5(z) &= \mathcal{P} \left(\frac{3}{3!} A_3(z) \circ z^{\circ 2} - \frac{1}{5!} z^{\circ 5} \right) \quad \dots \end{aligned}$$



Test Case	State of the Art Empirical Results on IEEE Test Cases			
	ratio of test prediction to numerical computation			
	$\ \cdot\ _2$	$\ \cdot\ _\infty$	$\alpha_\infty(\mathcal{P}) \approx 1$ approximate	numerical α_∞ (fmincon)
IEEE 9	16.5 %	73.7 %	92.1 %	85.1 % [†]
IEEE 14	8.3 %	59.4 %	83.1 %	81.3 % [†]
IEEE RTS 24	3.9 %	53.4 %	89.5 %	89.5 % [†]
IEEE 30	2.7 %	55.7 %	85.5 %	85.5 % [†]
IEEE 118	0.3 %	43.7 %	85.9 %	—*
IEEE 300	0.2 %	40.3 %	99.8 %	—*
Polish 2383	0.1 %	29.1 %	82.8 %	—*

[†] fmincon with 100 randomized initial conditions

* fmincon does not converge

#7: Sync threshold: gap between necessary and sufficient conditions

$$z = \mathcal{P} \sin(x)$$

given a norm, define

$$\alpha(\mathcal{P}) := \min \text{ amplification factor of } (\mathcal{P} \text{ diag}[\text{sinc}(x)]) < \|\mathcal{P}\|$$

Theorem: Sufficient Cohesive equilibrium angles exist if, in some norm,

$$\|B^\top L^\dagger \omega\| \leq \alpha(\mathcal{P})$$

Necessary Equilibrium angles do not exist if, in some norm

$$\|\mathcal{P}\| \leq \|B^\top L^\dagger \omega\|$$

Considering only first order term in expansion $\iff \alpha_\infty(\mathcal{P}) \approx 1$ (PNAS '13)

#8: State-space oscillators

Coupled networks of:

- 1 Stuart-Landau oscillator
- 2 FitzHugh–Nagumo neurons
- 3 Rössler chaotic oscillators
- 4 Lienard oscillators (Van Der Pol)
- 5 Biological Goodwin models
- 6 ...

semi-contraction theory

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^n a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

synchronization as function of

- 1 growth rate of the internal dynamics
- 2 strength of the diffusive coupling
- 3 heterogeneity of oscillators

Theorem: semi-contraction sufficient condition

If in some norm

$$\text{osLip}(f) < \lambda_2(L)$$

then

- 1 semi-contraction rate $\lambda_2(L) - \text{osLip}(f)$,
- 2 synchronization $\lim_{t \rightarrow \infty} \|x_i - x_j\| = 0$ for every i, j

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- 2 Open Problems

Our recent work

- 1 S. Jafarpour, E. Y. Huang, and F. Bullo. [Synchronization of Kuramoto oscillators: Inverse Taylor expansions](#). *SIAM Journal on Control and Optimization*, 57(5):3388–3412, 2019. doi:10.1137/18M1216262
- 2 S. Jafarpour and F. Bullo. [Synchronization of Kuramoto oscillators via cutset projections](#). *IEEE Transactions on Automatic Control*, 64(7):2830–2844, 2019. doi:10.1109/TAC.2018.2876786
- 3 S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo. [Flow and elastic networks on the \$n\$ -torus: Geometry, analysis and computation](#). *SIAM Review*, 64(1):59–104, 2022. doi:10.1137/18M1242056
- 4 S. Jafarpour, P. Cisneros-Velarde, and F. Bullo. [Weak and semi-contraction for network systems and diffusively-coupled oscillators](#). *IEEE Transactions on Automatic Control*, 67(3):1285–1300, 2022. doi:10.1109/TAC.2021.3073096
- 5 R. Delabays, S. Jafarpour, and F. Bullo. [Multistability and paradoxes in lossy oscillator networks](#). Submitted, February 2022. URL: <https://arxiv.org/pdf/2202.02439.pdf>

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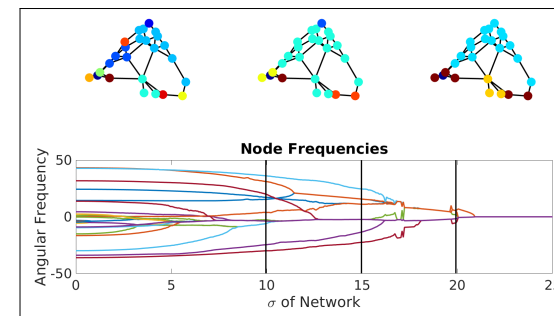
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Future research

- 1 Fundamental theory of phased-coupled oscillators
- 2 Fundamental theory of state-space-coupled oscillators
- 3 Applications in energy systems
- 4 Applications in machine learning and scientific computing

Fundamental theory of phased-coupled oscillators

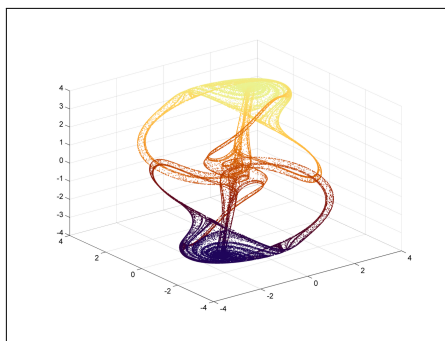
- 1 outside cohesive set: signed graphs, symbolic dynamics, ...
- 2 non-monotone phase couplings and higher-order dynamics
- 3 analysis and computation of cluster sync and bifurcation diagram



B. Gilg. *Critical Coupling and Synchronized Clusters in Arbitrary Networks of Kuramoto Oscillators*.
PhD thesis, Arizona State University, 2018

Fundamental theory of state-space-coupled oscillators

- 1 sharpest sync conditions for benchmarks
- 2 transverse contraction
- 3 fractal attractors via α -contraction theory



C. Wu, R. Pines, M. Margaliot, and J.-J. E. Slotine.
Generalization of the multiplicative and additive compounds of square matrices and contraction in the Hausdorff dimension.
IEEE Transactions on Automatic Control, 2022.
[doi:10.1109/TAC.2022.3162547](https://doi.org/10.1109/TAC.2022.3162547)

Applications in energy systems

- 1 understanding multi-stability in power flows
- 2 thick torus conjecture for active/reactive power flow and for OPF
- 3 paradoxes in lossy networks

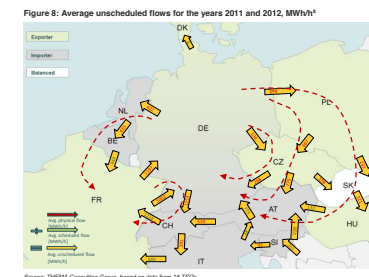
Practical observations:

sometimes undesirable power flows around loops

sometimes sizable difference between predicted and actual power flows



New York Independent System Operator, *Lake Erie Loop Flow Mitigation*, Technical Report, 2008



THEMA Consulting Group, *Loop-flows - Final advice*, Technical Report prepared for the European Commission, 2013

Applications in machine learning and scientific computing

- ❶ oscillator-based computing
- ❷ nanotech allows construction of massively-parallel analog fast low-power devices
CMOS, spin torque nano-oscillators (spintronics), MEMS resonators, optomechanical crystal cavities, ...
- ❸ Example applications:
 - ❶ NP-complete computing
 - ❷ associative memory
 - ❸ reservoir computing
- J. Von Neumann. [Non-linear capacitance or inductance switching, amplifying, and memory organs](#), December 1957. US Patent 2,815,488
- M. H. Matheny et al. [Exotic states in a simple network of nanoelectromechanical oscillators](#). *Science*, 363(6431), 2019.
[doi:10.1126/science.aav7932](#)
- G. Csaba and W. Porod. [Coupled oscillators for computing: A review and perspective](#). *Applied Physics Reviews*, 7(1):011302, 2020.
[doi:10.1063/1.5120412](#)

Conclusions

- ❶ Recent progress
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- ❷ Open Problems
 - ❶ Fundamental theory of phased-coupled oscillators
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 - ❸ Applications in energy systems
 - ❹ Applications in machine learning and scientific computing