From Contracting Dynamics to Contracting Algorithms on Euclidean, Riemannian and non-Euclidean spaces

Francesco Bullo

Center for Control,
Dynamical Systems & Computation
University of California at Santa Barbara

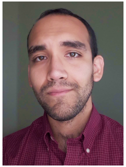
http://motion.me.ucsb.edu

IEEE CDC 21, Tutorial Session **Contraction Theory for Machine Learning**

Many thanks to: Soon-Jo Chung @Caltech Jean-Jacques Slotine @MIT
Chuchu Fan @MIT Brett Lopez @UCLA Winfried Lohmiller @Airbus

Acknowledgments

Saber Jafarpour



Pedro Cisneros-Velarde

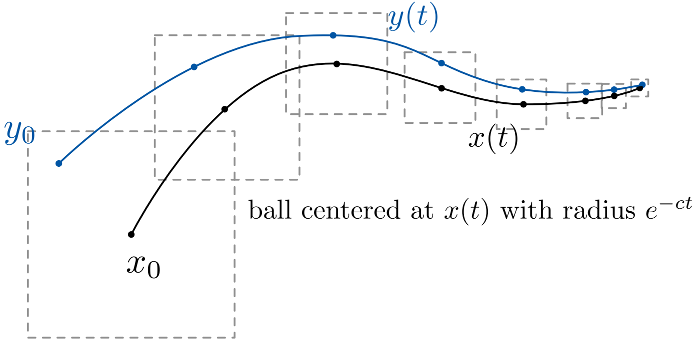
Alex Davydov

Anton Proskurnikov

- S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Nov. 2021. URL **http://arxiv.org/abs/2106.03194**
- A. Davydov, S. Jafarpour, and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. *IEEE Transactions on Automatic Control*, July 2021. URL **https://arxiv.org/abs/2103.12263**. Submitted
- A. Davydov, A. V. Proskurnikov, and F. Bullo. Non-Euclidean contractivity of recurrent neural networks. In *American Control Conference*, 2022. URL **https://arxiv.org/abs/2110.08298**. Submitted

Contraction theory: definition

Given $\dot{x} = \mathsf{F}(t, x)$, vector field F is contractive if its flow is a contraction map



Contraction theory: historical notes

- **Origins**

  S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922.

  S. M. Lozinskii. Error estimate for numerical integration of ordinary differential equations. I. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 5:52–90, 1958

  C. A. Desoer and H. Haneda. The measure of a matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Transactions on Circuit Theory*, 19(5):480–486, 1972.

- **Application in control theory**: W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998.

- **Reviews**:

  Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *IEEE Conf. on Decision and Control*, pages 3835–3847, Dec. 2014.

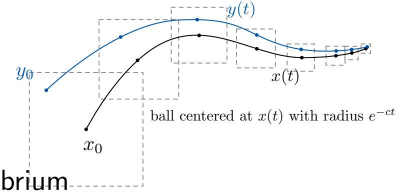
  M. Di Bernardo, D. Fiore, G. Russo, and F. Scafuti. Convergence, consensus and synchronization of complex networks via contraction theory. In J. Lü, X. Yu, G. Chen, and W. Yu, editors, *Complex Systems and Networks*, pages 313–339. Springer, 2016. ISBN 978-3-662-47824-0.

  H. Tsukamotoa, S.-J. Chung, and J.-J. E. Slotine. Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview, 2021. URL **https://arxiv.org/abs/2110.00675**

- contraction conditions on vector field do not necessarily involve Jacobians
- contraction conditions without Jacobians have been studied under many different names:

1. **uniformly decreasing maps** in: L. Chua and D. Green. A qualitative analysis of the behavior of dynamic nonlinear networks: Stability of autonomous networks. *IEEE Transactions on Circuits and Systems*, 23(6):355–379, 1976.
2. **one-sided Lipschitz maps** in: E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I. Nonstiff Problems*. Springer, 1993. (Section 1.10, Exercise 6)
3. **maps with negative nonlinear measure** in: H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001.
4. **dissipative Lipschitz maps** in: T. Caraballo and P. E. Kloeden. The persistence of synchronization under environmental noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461(2059):2257–2267, 2005.
5. **maps with negative lub log Lipschitz constant** in: G. Söderlind. The logarithmic norm. History and modern theory. *BIT Numerical Mathematics*, 46(3):631–652, 2006.
6. **QUAD maps** in: W. Lu and T. Chen. New approach to synchronization analysis of linearly coupled ordinary differential systems. *Physica D: Nonlinear Phenomena*, 213(2):214–230, 2006.
7. **incremental quadratically stable maps** in: L. D'Alto and M. Corless. Incremental quadratic stability. *Numerical Algebra, Control and Optimization*, 3:175–201, 2013.

Contraction theory: properties of contracting systems



Highly ordered **transient** and **asymptotic** behavior:

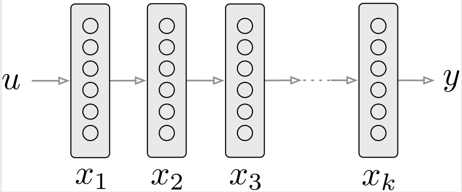
1. time-invariant F: unique globally exponential stable equilibrium
   two natural Lyapunov functions
2. periodic F: contracting system entrain to periodic inputs
3. contractivity rate is natural measure/indicator of robust stability
4. accurate numerical integration, and
5. there exist efficient methods for their **fixed point computation**

Why fixed point computations?

Fixed point strategies in data science = simplifying and unifying framework to model, analyze, and solve advanced convex optimization methods, Nash equilibria, monotone inclusions, etc.

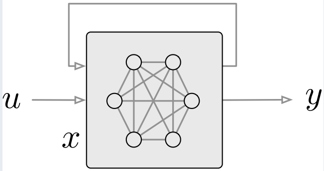
P. L. Combettes and J.-C. Pesquet. Fixed point strategies in data science. *IEEE Transactions on Signal Processing*, 2021.

Feedforward NN



$$x_{i+1} = \phi(A_i x_i + b_i), \quad x_0 = u,$$
$$y = C x_k + d$$

Implicit/Recurrent NN



$$x = \phi(Ax + Bu + b),$$
$$y = Cx + d$$

Advantages of implicit/equilibrium/fixed point formulation: simplicity, analogy with neural circuits, accuracy, memory efficiency, input-output robustness, etc

Recent literature on implicit NNs

1. S. Bai, J. Z. Kolter, and V. Koltun. Deep equilibrium models. In *Advances in Neural Information Processing Systems*, 2019. URL **https://arxiv.org/abs/1909.01377**
2. L. El Ghaoui, F. Gu, B. Travacca, A. Askari, and A. Y. Tsai. Implicit deep learning. 2019. URL **https://arxiv.org/abs/1908.06315**
3. E. Winston and J. Z. Kolter. Monotone operator equilibrium networks. In *Advances in Neural Information Processing Systems*, 2020. URL **https://arxiv.org/abs/2006.08591**
4. M. Revay, R. Wang, and I. R. Manchester. Lipschitz bounded equilibrium networks. 2020. URL **https://arxiv.org/abs/2010.01732**
5. A. Kag, Z. Zhang, and V. Saligrama. RNNs incrementally evolving on an equilibrium manifold: A panacea for vanishing and exploding gradients? In *International Conference on Learning Representations*, 2020. URL **https://openreview.net/forum?id=HylpqA4FwS**
6. K. Kawaguchi. On the theory of implicit deep learning: Global convergence with implicit layers. In *International Conference on Learning Representations*, 2021. URL **https://openreview.net/forum?id=p-NZIuwqhI4**
7. S. W. Fung, H. Heaton, Q. Li, D. McKenzie, S. Osher, and W. Yin. Fixed point networks: Implicit depth models with Jacobian-free backprop, 2021. URL **https://arxiv.org/abs/2103.12803**. ArXiv e-print

Literature on recurrent NN ODEs

1. J. J. Hopfield. Neurons with graded response have collective computational properties like those of two-state neurons. *Proceedings of the National Academy of Sciences*, 81(10):3088–3092, 1984.
2. E. Kaszkurewicz and A. Bhaya. On a class of globally stable neural circuits. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 41(2):171–174, 1994.
3. M. Forti, S. Manetti, and M. Marini. Necessary and sufficient condition for absolute stability of neural networks. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 41(7):491–494, 1994.
4. Y. Fang and T. G. Kincaid. Stability analysis of dynamical neural networks. *IEEE Transactions on Neural Networks*, 7(4):996–1006, 1996.
5. H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001.
6. W. He and J. Cao. Exponential synchronization of chaotic neural networks: a matrix measure approach. *Nonlinear Dynamics*, 55:55–65, 2009.
7. H. Zhang, Z. Wang, and D. Liu. A comprehensive review of stability analysis of continuous-time recurrent neural networks. *IEEE Transactions on Neural Networks and Learning Systems*, 25(7): 1229–1262, 2014.

Primer on monotone operator theory and contractions

$$x = \mathsf{G}(x)$$

Banach Contraction Theorem

If $\mathrm{Lipschitz}(\mathsf{G}) < 1$, then Picard iteration $x_{k+1} = \mathsf{G}(x_k)$ is a Banach contraction

For $\mathrm{Lipschitz}(\mathsf{G}) \geq 1$, define the *average/damped/Mann-Krasnosel'skii iteration*

$$x_{k+1} = (1-\alpha)x_k + \alpha\mathsf{G}(x_k)$$

Infinitesimal Contraction Theorem

1. there exists $0 < \alpha < 1$ such that the average iteration is a Banach contraction
2. the dynamics $\dot{x} = -x + \mathsf{G}(x)$ is contracting

Addendum: Perturbed fixed point theorems

Classical Lim's lemma

x_u^* is a fixed point of $x = \mathsf{G}(x, u)$ and $\mathrm{Lip}_x \mathsf{G} < 1$, then

$$\|x_u^* - x_v^*\| \leq \frac{\mathrm{Lip}_u \mathsf{G}}{1 - \mathrm{Lip}_x \mathsf{G}} \|u - v\|$$

T. C. Lim. On fixed point stability for set-valued contractive mappings with applications to generalized differential equations. *Journal of Mathematical Analysis and Applications*, 110(2):436–441, 1985.

Generalized Lim's lemma

x_u^* is a fixed point of $x = \mathsf{G}(x, u)$ and $\mathrm{osL}_x \mathsf{G} < 1$, then

$$\|x_u^* - x_v^*\| \leq \frac{\mathrm{Lip}_u \mathsf{G}}{1 - \mathrm{osL}_x \mathsf{G}} \|u - v\|$$

Outline

For $x \in \mathbb{R}^n$ and differentiable time-dep

$$\dot{x} = \mathsf{F}(t, x)$$

For $P = P^\top \succ 0$, define $\|x\|^2_{2,P^{1/2}} = x^\top P x$

Main equivalences: For $c > 0$, map **F is c-strongly contracting** if

1. **osL** : $(\mathsf{F}(t,x) - \mathsf{F}(t,y))^\top P(x-y) \leq -c\|x-y\|^2_{2,P^{1/2}}$, for all x, y, t
2. **d-osL** : $P D\mathsf{F}(t,x) + D\mathsf{F}(t,x)^\top P \preceq -2cP$ for all x, t
3. **d-IS** : $D^+\|x(t) - y(t)\|_{2,P^{1/2}} \leq -c\|x(t) - y(t)\|_{2,P^{1/2}}$, for all soltns $x(\cdot), y(\cdot)$

For differentiable $V : \mathbb{R}^n \to \mathbb{R}$, equivalent statements:

1. V is **strongly convex** with parameter m
2. $-\operatorname{grad} V$ is m-**strongly contracting**, that is

$$\big(-\operatorname{grad} V(x) + \operatorname{grad} V(y)\big)^\top (x-y) \leq -m\|x-y\|_2^2$$

For map $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$, equivalent statements:

1. F is a **monotone operator** (or a **coercive operator**) with parameter m,
2. $-\mathsf{F}$ is m-**strongly contracting**

E. K. Ryu and W. Yin. *Large-Scale Convex Optimization via Monotone Operators*. Cambridge, 2022

Equilibria of contracting vector fields:

For a time-invariant F, c-strongly contracting with respect to $\|\cdot\|_{2,P^{1/2}}$

1. flow of F is a contraction,
   i.e., distance between solutions exponentially decreases with rate c
2. there exists an equilibrium x^*, that is unique, globally exponentially stable with global Lyapunov functions

$$x \mapsto \|x - x^*\|_{2,P^{1/2}}^2 \quad \text{and} \quad x \mapsto \|\mathsf{F}(x)\|_{2,P^{1/2}}^2$$

Given $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$

$$x^* \in \mathrm{zero}(\mathsf{F}) \qquad \iff \qquad x^* \in \mathrm{fixed}(G), \text{ where } \mathsf{G} = \mathsf{Id} + \mathsf{F}$$

consider **forward step = Euler integration** for F = averaged iteration for G:

$$x_{k+1} = (\mathsf{Id} + \alpha \mathsf{F}) x_k = x_k + \alpha \mathsf{F}(x_k) \qquad = (1-\alpha)\,\mathsf{Id} + \alpha \mathsf{G}$$

Given *contraction rate* c and *Lipschitz constant* ℓ, define *condition number* $\kappa = \ell / c \geq 1$

1. the map $\mathsf{Id} + \alpha \mathsf{F}$ is a contraction map with respect to $\| \cdot \|_{2,P^{1/2}}$ for

$$0 < \alpha < \frac{2}{c\kappa^2}$$

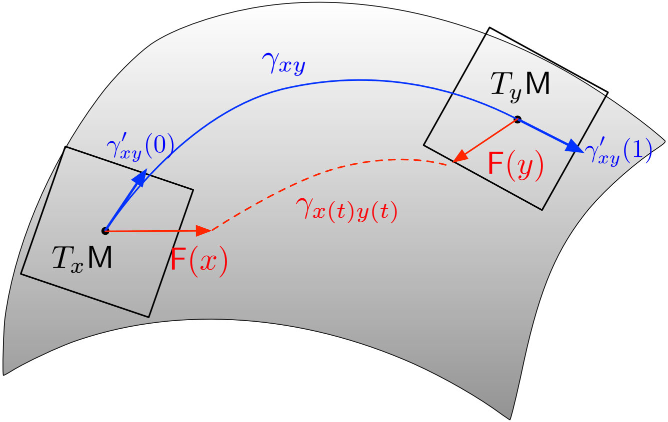
2. the optimal step size minimizing and minimum contraction factor:

$$\alpha^*_{\mathsf{E}} = \frac{1}{c\kappa^2}$$
$$\ell^*_{\mathsf{E}} = 1 - \frac{1}{2\kappa^2} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)$$

Outline

1. Overview and motivation
2. Contraction on Euclidean and inner product spaces
3. Contraction on Riemannian manifolds
4. Contraction on non-Euclidean normed vector spaces

F **contracting** if geodesic distances from x to y diminishes along the flow of F



integral test: the inner product between F and the geodesic velocity vector γ'_{xy} at x and y
differential test: condition on covariant differential of F

$$\mathbb{G}(x)\frac{\partial \mathsf{F}}{\partial x}(x) + \frac{\partial \mathsf{F}}{\partial x}(x)^\top \mathbb{G}(x) + \dot{\mathbb{G}}(x) \preceq -2c\mathbb{G}(x)$$

Outline

1. Overview and motivation
2. Contraction on Euclidean and inner product spaces
3. Contraction on Riemannian manifolds
4. Contraction on non-Euclidean normed vector spaces

Norms	From inner products to sign and max pairings	From LMIs to log norms
$\|x\|_{2,P^{1/2}}^2 = x^\top P x$	$[\![x,y]\!]_{2,P^{1/2}} = x^\top P y$	$\mu_{2,P^{1/2}}(A) = \min\{b \mid A^\top P + PA \preceq 2bP\}$
$\|x\|_1 = \sum_i \|x_i\|$	$[\![x,y]\!]_1 = \|y\|_1 \operatorname{sign}(y)^\top x$	$\mu_1(A) = \max_j \Big(a_{jj} + \sum_{i\neq j} \|a_{ij}\|\Big)$
$\|x\|_\infty = \max_i \|x_i\|$	$[\![x,y]\!]_\infty = \max_{i \in I_\infty(y)} y_i x_i$	$\mu_\infty(A) = \max_i \Big(a_{ii} + \sum_{j\neq i} \|a_{ij}\|\Big)$

where $I_\infty(x) = \{i \in \{1,\dots,n\} \mid |x_i| = \|x\|_\infty\}$

A **weak pairing** is $[\![\cdot,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying

1. $[\![x_1 + x_2, y]\!] \leq [\![x_1, y]\!] + [\![x_2, y]\!]$ and $x \mapsto [\![x, y]\!]$ is continuous,
2. $[\![bx, y]\!] = [\![x, by]\!] = b\,[\![x, y]\!]$ for $b \geq 0$ and $[\![-x, -y]\!] = [\![x, y]\!]$,
3. $[\![x, x]\!] > 0$, for all $x \neq \mathbb{0}_n$,
4. $|\,[\![x, y]\!]\,| \leq [\![x, x]\!]^{1/2}\,[\![y, y]\!]^{1/2}$,

Given norm $\|\cdot\|$, compatibility: $[\![x, x]\!] = \|x\|^2$ for all x

Sup of non-Euclidean numerical range: $$\mu(A) = \sup_{x \neq 0} \frac{[\![Ax, x]\!]}{\|x\|^2}$$

Norm derivative formula: $$\tfrac{1}{2} D^+ \|x(t)\|^2 = [\![\dot{x}(t), x(t)]\!]$$

A. Davydov, S. Jafarpour, and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. *IEEE Transactions on Automatic Control*, July 2021. URL **https://arxiv.org/abs/2103.12263**. Submitted

The **log norm** of $A \in \mathbb{R}^{n\times n}$ wrt to $\|\cdot\|$:

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

Basic properties:

subadditivity:	$\mu(A+B) \leq \mu(A) + \mu(B)$	
scaling:	$\mu(bA) = b\mu(A),$	$\forall b \geq 0$
convexity:	$\mu(\theta A + (1-\theta)B) \leq \theta\mu(A) + (1-\theta)\mu(B),$	$\forall \theta \in [0,1]$

T. Ström. On logarithmic norms. *SIAM Journal on Numerical Analysis*, 12(5):741–753, 1975.

For $x \in \mathbb{R}^n$ and differentiable time-dep

$$\dot{x} = \mathsf{F}(t,x) \tag{1}$$

For norm $\|\cdot\|$ with log norm $\mu(\cdot)$ and compatible weak pairing $[\![\cdot,\cdot]\!]$

Main equivalences: for $c > 0$

1. **osL** : $[\![\mathsf{F}(t,x) - \mathsf{F}(t,y), x-y]\!] \leq -c\|x-y\|^2$ for all x, y, $t \geq 0$,
2. **d-osL** : $\mu(D\mathsf{F}(t,x)) \leq -c$, for all $x, t \geq 0$
3. **d-IS** : $D^+\|x(t) - y(t)\| \leq -c\|x(t) - y(t)\|$, for soltns $x(\cdot), y(\cdot)$

Consider a norm $\|\cdot\|$ with compatible weak pairing $[\![\cdot,\cdot]\!]$
Recall **forward step method** $x_{k+1} = (\mathsf{Id} + \alpha \mathsf{F})x_k = x_k + \alpha\mathsf{F}(x_k)$

Given *contraction rate* c and *Lipschitz constant* ℓ, define *condition number* $\kappa = \ell/c \geq 1$

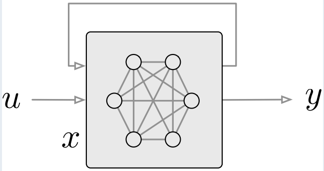
1. the map $\mathsf{Id} + \alpha\mathsf{F}$ is a contraction map with respect to $\|\cdot\|$ for

$$0 < \alpha < \frac{1}{c\kappa(1+\kappa)}$$

2. the optimal step size minimizing and minimum contraction factor:

$$\alpha^*_{\mathsf{nE}} = \frac{1}{c}\Big(\frac{1}{2\kappa^2} - \frac{3}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)\Big)$$
$$\ell^*_{\mathsf{nE}} = 1 - \frac{1}{4\kappa^2} + \frac{1}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)$$

Example: ℓ_∞-contracting neural networks



Recurrent neural network dynamics

$$\dot{x} = -x + \Phi(Ax + Bu)$$

Average iteration

$$x_{k+1} = (1-\alpha)x_k + \alpha\Phi(Ax_k + Bu)$$

If

$$\mu_\infty(A) < 1 \qquad \Big(\text{i.e., } a_{ii} + \sum_j |a_{ij}| < 1 \text{ for all } i\Big)$$

Then, with norm $\|\cdot\|_\infty$,

- dynamics is contracting with rate $1 - \mu_\infty(A)_+$
- average iteration is contracting with factor $1 - \dfrac{1-\mu_\infty(A)_+}{1-\min_i(a_{ii})_-}$ at $\alpha = \dfrac{1}{1-\min_i(a_{ii})_-}$

Conclusions

From Contracting Dynamics to Contracting Algorithms:

1. contraction theory and monotone operator theory are deeply connected
2. well established methodologies to tackle control, optimization and learning problems via fixed point strategies
3. same methods on Euclidean, Riemannian and non-Euclidean spaces
4. example application to recurrent neural networks