

# Non-Euclidean Contraction Theory for Network Systems

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S. Jafarpour, A. Davydov, and F. Bullo. Non-Euclidean contraction theory for monotone and positive systems. *IEEE Transactions on Automatic Control*, Sept. 2021b. URL <https://arxiv.org/abs/2104.01321>

## Lectures on **Network Systems**



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# Prototypical nonlinear network systems

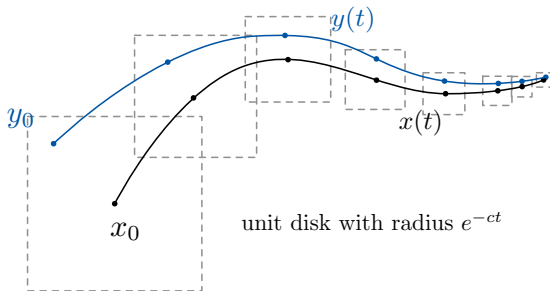
In chronological order

- 1 the Lotka-Volterra population dynamics (Lotka, 1920; Volterra, 1928),
- 2 Matrosov-Bellman interconnection of stable systems (and the method of vector Lyapunov functions) (Bellman, 1962; Matrosov, 1962),
- 3 Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981),
- 4 Yorke network epidemic model (Lajmanovich and Yorke, 1976),
- 5 Hopfield and cellular neural networks (Hopfield, 1982),
- 6 Daganzo cell transmission model for traffic networks (Daganzo, 1994),
- 7 Chua's diffusively-coupled dynamical systems (Wu and Chua, 1995), and
- 8 compartmental systems in biology, medicine, and ecology (Sandberg, 1978; Maeda et al., 1978), earlier models fading back into the past.

# Contraction theory: a brief overview

## Definition

$f$  is contractive if its flow is a contraction map



# Contraction theory: a brief overview

## Historical notes


- **Origins**


G. Dahlquist. *Stability and error bounds in the numerical integration of ordinary differential equations*. PhD thesis, (Reprinted in Trans. Royal Inst. of Technology, No. 130, Stockholm, Sweden, 1959), 1958

B. P. Demidovič. Dissipativity of a nonlinear system of differential equations. *Uspekhi Matematicheskikh Nauk*, 16(3(99)):216, 1961


C. A. Desoer and H. Haneda. The measure of a matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Transactions on Circuit Theory*, 19(5):480–486, 1972.




- **Application in control theory:** W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998. 

- **Non-Euclidean contraction:** S. Coogan. A contractive approach to separable Lyapunov functions for monotone systems. *Automatica*, 106:349–357, 2019. 

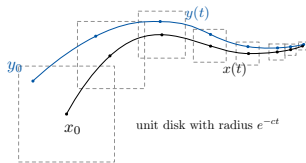
- **Reviews:**

M. Di Bernardo, D. Fiore, G. Russo, and F. Scafuti. Convergence, consensus and synchronization of complex networks via contraction theory. In J. Lü, X. Yu, G. Chen, and W. Yu, editors, *Complex Systems and Networks*, pages 313–339. Springer, 2016. ISBN 978-3-662-47824-0. 

Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *IEEE Conf. on Decision and Control*, pages 3835–3847, Dec. 2014. 

# Contraction theory: a brief overview

## Properties of contracting systems



Highly ordered **transient** and **asymptotic** behavior:

- 1 initial conditions are forgotten, and monotonic decrease (no overshoot) in distance between trajectories
- 2 time-invariant  $f$ : unique globally exponentially stable equilibrium  
two natural Lyapunov functions
- 3 periodic  $f$ : unique globally exponentially stable periodic, and solution contracting system entrain to periodic inputs
- 4 robustness properties:
  - input-to-state stability
  - finite input-state gain
  - contraction margin wrt unmodeled dynamics
  - input-to-state stability under delayed dynamics
  - $\implies$  contractivity rate is natural measure/indicator of robust stability
- 5 accurate numerical integration and fixed point computation

# Contraction theory: a brief overview

## Contraction theory vs Lyapunov stability theory

Contraction theory combines in unified coherent framework results from:

- 1 stability notions and Lyapunov functions for incremental stability or stability of equilibrium points and trajectories,
- 2 Banach contraction and Brower fixed point theorems,
- 3 monotone systems theory, and
- 4 geometry of Banach, Riemannian and Finsler spaces

**nonlinear robust stability theory**



- 1 Main equivalence theorem
- 2 Analysis of dynamic behavior and robustness
- 3 Network contraction theory: Small gain theorems
- 4 Network contraction theory: Weakly contracting systems
- 5 Network contraction theory: Semi-contracting systems

# Contraction analysis for scalar vector field

$f : \mathbb{R} \rightarrow \mathbb{R}$  is **one-sided Lipschitz (osL) continuous** if there exists  $b \in \mathbb{R}$  s.t.

$$f'(x) \leq b, \quad \forall x \quad (\mathbf{d-osL})$$

$$\iff f(x) - f(y) \leq b(x - y), \quad \forall x > y$$

$$\iff (x - y)(f(x) - f(y)) \leq b(x - y)^2, \quad \forall x, y \quad (\mathbf{osL})$$

- $f$  is osL with  $b = 0$  iff  $f$  weakly decreasing
- if  $f$  is Lipschitz with bound  $\ell$ , then  $f$  is osL with  $b = \ell$
- For

$$\dot{x} = f(x) \quad (1)$$

the Grönwall lemma implies  $|x(t) - y(t)| \leq e^{bt}|x(0) - y(0)|$

# Contraction theorem on $\mathbb{R}^n$ and $\ell_2$ norm

For  $x \in \mathbb{R}^n$  and differentiable time-dep

$$\dot{x} = f(t, x) \quad (2)$$

For  $P = P^\top \succ 0$ , define  $\|x\|_P^2 = x^\top P x$

## Main equivalences:

- 1 **osL** :  $(f(t, x) - f(t, y))^\top P(x - y) \leq b\|x - y\|_P^2$ , for all  $x, y, t$
- 2 **d-osL** :  $P Df(t, x) + Df(t, x)^\top P \preceq 2bP$  for all  $x, t$
- 3 **d-IS** :  $D^+ \|x(t) - y(t)\|_P \leq b\|x(t) - y(t)\|_P$ , for all soltns  $x(\cdot), y(\cdot)$
- 4 **IS** :  $\|x(t) - y(t)\|_P \leq e^{b(t-t_0)} \|x(t_0) - y(t_0)\|_P$ , for all soltns  $x(\cdot), y(\cdot)$

If  $f$  not differentiable, then **osL**  $\iff$  **d-IS**  $\iff$  **IS**

# Equivalent rewriting with inner products

For  $x \in \mathbb{R}^n$  and differentiable time-dep

$$\dot{x} = f(t, x) \quad (3)$$

For  $P = P^\top \succ 0$ , define  $\langle\langle x, y \rangle\rangle_P = x^\top P y$  and  $\|x\|_P^2 = x^\top P x$ ,

## Main equivalences:

- osL** :  $\langle\langle f(t, x) - f(t, y), x - y \rangle\rangle_P \leq b \|x - y\|_P^2$ , for all  $x, y, t \geq 0$
- d-osL** :  $\langle\langle Df(t, x)v, v \rangle\rangle_P \leq b \|v\|_P^2$  for all  $x, v, t \geq 0$ ,
- d-IS** :  $D^+ \|x(t) - y(t)\|_P \leq b \|x(t) - y(t)\|_P$ , for all soltns  $x(\cdot), y(\cdot)$
- IS** :  $\|x(t) - y(t)\|_P \leq e^{b(t-t_0)} \|x(t_0) - y(t_0)\|_P$ , for all soltns  $x(\cdot), y(\cdot)$

# Linear algebra detour: Matrix measures

The **matrix measure** of  $A \in \mathbb{R}^{n \times n}$  wrt to  $\|\cdot\|$ :

$$\mu(A) := \lim_{h \rightarrow 0^+} \frac{\|I_n + hA\| - 1}{h}$$

$$\mu_2(A) = \frac{1}{2} \lambda_{\max}(A + A^T)$$

$$\mu_1(A) = \max_j (a_{jj} + \sum_{i \neq j} |a_{ij}|) \qquad \mu_\infty(A) = \max_i (a_{ii} + \sum_{j \neq i} |a_{ij}|)$$

## Basic properties:

subadditivity:  $\mu(A + B) \leq \mu(A) + \mu(B)$

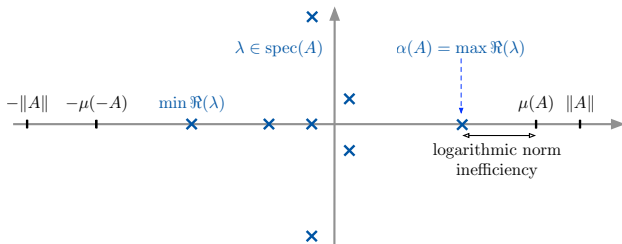
scaling:  $\mu(bA) = b\mu(A), \qquad \forall b \geq 0$

convexity:  $\mu(\theta A + (1 - \theta)B) \leq \theta\mu(A) + (1 - \theta)\mu(B), \qquad \forall \theta \in [0, 1]$

norm/spectrum:

$$\operatorname{Re}(\lambda) \leq \mu(A) \leq \|A\|, \quad \forall \lambda \in \operatorname{spec}(A)$$

$$\forall \lambda \in \operatorname{spec}(A)$$



A norm  $\|\cdot\|$  is

- 1 **logarithmically optimal** for  $A$  if  $\mu(A) = \alpha(A)$ , and
- 2 **logarithmically  $\epsilon$ -efficient** for  $A$  if  $\alpha(A) \leq \mu(A) \leq \alpha(A) + \epsilon$ .

$$\ell_2 : \quad \forall A, \quad \exists P \succ 0, \text{ s.t. } \|\cdot\|_P \text{ is efficient}$$

$$\ell_p, p \in [1, \infty] : \quad \forall M \text{ Metzler}, \quad \exists \eta > 0_n, \text{ s.t. } \|\cdot\|_{p, [\eta]} \text{ is efficient}$$

# Connection between osL and matrix measure

## The best osL constant:

$$\begin{aligned}\text{osL}_2(f) &:= \sup_{x \neq y} \frac{\langle\langle f(x) - f(y), x - y \rangle\rangle}{\|x - y\|_2^2} \\ &= \sup_x \mu_2(Df(x))\end{aligned}\quad \text{if } f \text{ differentiable}$$

## Proof for affine vector fields:

$$\begin{aligned}\text{osL}_2(Ax + b) &= \sup_{x \neq y} \frac{\langle\langle Ax - Ay, x - y \rangle\rangle}{\|x - y\|_2^2} \\ &= \sup_{x \neq 0_n} \frac{x^\top Ax}{x^\top x} && \text{(Rayleigh quotient)} \\ &= \frac{1}{2} \lambda_{\max}(A + A^\top) && \text{(sup of numerical range)} \\ &= \mu_2(A)\end{aligned}$$

# Weak pairing

A **weak pairing** (WP) is  $[\cdot, \cdot] : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying

- 1  $[x_1 + x_2, y] \leq [x_1, y] + [x_2, y]$  and  $x \mapsto [x, y]$  is continuous,
- 2  $[bx, y] = [x, by] = b[x, y]$  for  $b \geq 0$  and  $[-x, -y] = [x, y]$ ,
- 3  $[x, x] > 0$ , for all  $x \neq 0_n$ ,
- 4  $|[x, y]| \leq [x, x]^{1/2} [y, y]^{1/2}$ ,

Given norm  $\|\cdot\|$ , compatibility:  $[x, x] = \|x\|^2$  for all  $x$

Sup of non-Euclidean numerical range:

$$\mu(A) = \sup_{x \neq 0} \frac{[Ax, x]}{\|x\|^2}$$

Norm derivative formula:

$$\frac{1}{2} D^+ \|x(t)\|^2 = [\dot{x}(t), x(t)]$$

G. Lumer. Semi-inner-product spaces. *Transactions of the American Mathematical Society*, 100: 29–43, 1961. 



## Norm

## WP

## Matrix measure

$$\|x\|_{2,p^{1/2}} = \sqrt{x^T P x}$$

$$[[x, y]]_{2,p^{1/2}} = x^T P y$$

$$\begin{aligned} \mu_{2,p^{1/2}}(A) &= \min\{b \mid A^T P + P A \preceq 2bP\} \\ &= \frac{1}{2} \lambda_{\max}(P A P^{-1} + A^T) \\ &= \max_{\|x\|_P=1} x^T P A x \end{aligned}$$

$$\|x\|_p = \left( \sum_i |x_i|^p \right)^{1/p}$$

$$[[x, y]]_p = \|y\|_p^{2-p} (y \circ |y|^{p-2})^T x$$

$$\mu_p(A) = \max_{\|x\|_p=1} (x \circ |x|^{p-2})^T A x$$

$$p \in ]1, \infty[$$

$$\|x\|_1 = \sum_i |x_i|$$

$$[[x, y]]_1 = \|y\|_1 \operatorname{sign}(y)^T x$$

$$\begin{aligned} \mu_1(A) &= \max_{j \in \{1, \dots, n\}} \left( a_{jj} + \sum_{i \neq j} |a_{ij}| \right) \\ &= \sup_{\|x\|_1=1} \operatorname{sign}(x)^T A x \end{aligned}$$

$$\|x\|_\infty = \max_i |x_i|$$

$$[[x, y]]_\infty = \max_{i \in I_\infty(y)} y_i x_i$$

$$\begin{aligned} \mu_\infty(A) &= \max_{i \in \{1, \dots, n\}} \left( a_{ii} + \sum_{j \neq i} |a_{ij}| \right) \\ &= \max_{\|x\|_\infty=1} \max_{i \in I_\infty(x)} x_i (A x)_i \end{aligned}$$

Table of norms, WPs, and matrix measures for weighted  $\ell_2$ ,  $\ell_p$  for  $p \in ]1, \infty[$ ,  $\ell_1$ , and  $\ell_\infty$  norms. Note:  $I_\infty(x) = \{i \in \{1, \dots, n\} \mid |x_i| = \|x\|_\infty\}$ .

# Contraction theorem on $\mathbb{R}^n$ with arbitrary norm

For  $x \in \mathbb{R}^n$  and differentiable time-dep

$$\dot{x} = f(t, x) \quad (4)$$


For norm  $\|\cdot\|$  with matrix measure  $\mu(\cdot)$  and compatible WP  $\llbracket \cdot, \cdot \rrbracket$ ,


## Main equivalences:


- 1 **osL** :  $\llbracket f(t, x) - f(t, y), x - y \rrbracket \leq b \|x - y\|^2$  for all  $x, y, t \geq 0$ ,
- 2 **d-osL** :  $\llbracket Df(t, x)v, v \rrbracket \leq b \|v\|^2$ , for all  $v, x, t \geq 0$ , or  
 $\mu(Df(t, x)) \leq b$ , for all  $x, t \geq 0$ ,
- 3 **d-IS** :  $D^+ \|x(t) - y(t)\| \leq b \|x(t) - y(t)\|$ , for soltns  $x(\cdot), y(\cdot)$ ,
- 4 **IS** :  $\|x(t) - y(t)\| \leq e^{b(t-t_0)} \|x(t_0) - y(t_0)\|$ , for all soltns  $x(\cdot), y(\cdot)$

Measure bound	Demidovich condition	One-sided Lipschitz condition
$\mu_{2,P}(Df(x)) \leq b$	$PDf(x) + Df(x)^\top P \preceq 2bP$	$(x - y)^\top P(f(x) - f(y)) \leq b\ x - y\ _{P^{1/2}}^2$
$\mu_p(Df(x)) \leq b$	$(v \circ  v ^{p-2})^\top Df(x)v \leq b\ v\ _p^p$	$((x - y) \circ  x - y ^{p-2})^\top (f(x) - f(y)) \leq b\ x - y\ _p^p$
$\mu_1(Df(x)) \leq b$	$\text{sign}(v)^\top Df(x)v \leq b\ v\ _1$	$\text{sign}(x - y)^\top (f(x) - f(y)) \leq b\ x - y\ _1$
$\mu_\infty(Df(x)) \leq b$	$\max_{i \in I_\infty(v)} v_i (Df(x)v)_i \leq b\ v\ _\infty^2$	$\max_{i \in I_\infty(x-y)} (x_i - y_i)(f_i(x) - f_i(y)) \leq b\ x - y\ _\infty^2$

Table of equivalences between measure bounded Jacobians, differential Demidovich and one-sided Lipschitz conditions. Note:  $I_\infty(v) = \{i \in \{1, \dots, n\} \mid |v_i| = \|v\|_\infty\}$ .

J. A. Jacquez and C. P. Simon. Qualitative theory of compartmental systems. *SIAM Review*, 35(1):43–79, 1993. 

H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001. 

G. Como, E. Lovisari, and K. Savla. Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing. *IEEE Transactions on Control of Network Systems*, 2(1):57–67, 2015. 

# Metzler matrices and monotone systems

- For Metzler  $M$  and monotonic  $\|\cdot\|$ ,  $\mu(M) = \sup_{x \geq 0_n} \frac{\llbracket Ax, x \rrbracket}{\|x\|}$ .
- For  $\eta, \xi \in \mathbb{R}_{>0}^n$ ,

$$\mu_{1, [\eta]}(M) = \max(\eta^\top M [\eta]^{-1}) = \min\{b \in \mathbb{R} \mid \eta^\top M \leq b \eta^\top\}$$
$$\mu_{\infty, [\xi]^{-1}}(M) = \max([\xi]^{-1} M \xi) = \min\{b \in \mathbb{R} \mid M \xi \leq b \xi\}$$

$f$  **monotone** if  $Df(x)$  Metzler for all  $x$

- ① **osL** :  $\llbracket f(x) - f(y), x - y \rrbracket \leq b \|x - y\|^2$  for all  $x \geq y$
- ② **d-osL** :  $\llbracket Df(x)v, v \rrbracket \leq b \|v\|^2$ , for all  $v \geq 0$  and  $x$

$$\mu_{1, [\eta]}(Df(x)) \leq b \quad \eta^\top Df(x) \leq b \eta^\top \quad \eta^\top (f(x) - f(y)) \leq b \eta^\top (x - y) \text{ for all } x \geq y$$

$$\mu_{\infty, [\xi]^{-1}}(Df(x)) \leq b \quad Df(x)\xi \leq b\xi \quad f(x) - f(y) \leq b(x - y) \text{ for all } x = y + \beta\xi, \beta > 0$$

# Infinitesimally contracting systems

Given vector field  $f$  over normed space with WP:

$$\llbracket f(t, x) - f(t, y), x - y \rrbracket \leq b \|x - y\|^2, \quad \text{for all } x, y, t \geq 0$$

- $b = -c, c > 0$  : strongly contracting with rate  $c$
- $b = 0$  and strict inequality : strictly contracting
- $b = 0$  : weakly contracting (or non-expansive)

For differentiable  $V$  over convex set  $C$ , equivalent statements:

- 1  $V$  is **strongly convex** with parameter  $m$
- 2  $-\text{grad}V$  is **strongly contracting** with rate  $m$  wrt  $\ell_2$ , that is

$$\left( -\text{grad}V(x) + \text{grad}V(y) \right)^\top (x - y) \leq -m \|x - y\|_2^2$$

- 1 Main equivalence theorem
- 2 Analysis of dynamic behavior and robustness
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- 4 Network contraction theory: Weakly contracting systems
- 5 Network contraction theory: Semi-contracting systems

# Globally exponentially stable equilibrium

For time-invariant vector field  $f$  and norm  $\|\cdot\|$

- 1 there exists a convex and  $f$ -invariant set  $C$ ,
- 2  $f$  is strongly contracting with rate  $c$  on  $C$

Then

- 1 flow of  $f$  is a contraction, i.e., distance between solutions exponentially decreases with rate  $c$
- 2 there exists an equilibrium  $x^*$ , unique, globally exponentially stable with global Lyapunov functions

$$x \mapsto \|x - x^*\|^2 \quad \text{and} \quad x \mapsto \|f(x)\|^2$$

# Globally exponentially stable periodic orbits

a.k.a., Entrainment to periodic inputs


For time-varying vector field  $f$  and norm  $\| \cdot \|$

- 1 there exists a convex, closed  $f$ -invariant set  $C$ ,
- 2  $f$  is strongly contracting with rate  $c$  on  $C$ ,
- 3  $f$  is  $T$ -periodic.

Then

- 1 there exists a unique periodic solution  $x^* : \mathbb{R}_{\geq 0} \rightarrow C$  with period  $T$
- 2 for every initial condition  $x_0 \in C$ ,

$$\|x(t, x_0) - x^*(t)\| \leq e^{-ct} \|x_0 - x^*(0)\| \quad (5)$$

G. Russo, M. Di Bernardo, and E. D. Sontag. Global entrainment of transcriptional systems to periodic inputs. *PLoS Computational Biology*, 6(4):e1000739, 2010. 



For a time and input-dependent vector  $f$ ,

$$\dot{x} = f(t, x, u(t)), \quad x(0) = x_0 \in \mathbb{R}^n, u(t) \in \mathbb{R}^k \quad (6)$$

Assume  $\|\cdot\|_{\mathcal{X}}$  with compatible  $\llbracket \cdot, \cdot \rrbracket_{\mathcal{X}}$ , a norm  $\|\cdot\|_{\mathcal{U}}$ , and  $c, \ell > 0$  such that

- **osL**:  $\llbracket f(t, x, u) - f(t, y, u), x - y \rrbracket_{\mathcal{X}} \leq -c\|x - y\|_{\mathcal{X}}^2$ , for all  $t, x, y, u$ ,
- **Lip**:  $\|f(t, x, u) - f(t, x, v)\|_{\mathcal{X}} \leq \ell\|u - v\|_{\mathcal{U}}$ , for all  $t, x, u, v$ .

Then

- 1 any two soltns  $x(t)$  and  $y(t)$  to (6) with inputs  $u_x, u_y$

$$D^+ \|x(t) - y(t)\|_{\mathcal{X}} \leq -c \|x(t) - y(t)\|_{\mathcal{X}} + \ell \|u_x(t) - u_y(t)\|_{\mathcal{U}}$$

- 2  $f$  is **incrementally input-to-state stable**, i.e., for all  $x_0, y_0$

$$\|x(t) - y(t)\|_{\mathcal{X}} \leq e^{-ct} \|x_0 - y_0\|_{\mathcal{X}} + \frac{\ell(1 - e^{-ct})}{c} \sup_{\tau \in [0, t]} \|u_x(\tau) - u_y(\tau)\|_{\mathcal{U}}$$

- 3  $f$  has **incremental  $\mathcal{L}_{\mathcal{X}, \mathcal{U}}^q$  gain equal to  $\ell/c$ , for  $q \in [1, \infty]$ ,**

$$\|x(\cdot) - y(\cdot)\|_{\mathcal{X}, q} \leq \frac{\ell}{c} \|u_x(\cdot) - u_y(\cdot)\|_{\mathcal{U}, q} \quad (\text{for } x_0 = y_0)$$

Given norm  $\|\cdot\|_{\mathcal{X}}$  on  $\mathbb{R}^n$  (or  $\|\cdot\|_{\mathcal{U}}$  on  $\mathbb{R}^k$ ),

- $\mathcal{L}_{\mathcal{X}}^q$ ,  $q \in [1, \infty]$ , is vector space of continuous signals,  $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ , with well-defined bounded norm

$$\|x(\cdot)\|_{\mathcal{X},q} = \begin{cases} \left( \int_0^{\infty} \|x(t)\|_{\mathcal{X}}^q dt \right)^{1/q} & \text{if } q \in [1, \infty[ \\ \sup_{t \geq 0} \|x(t)\|_{\mathcal{X}} & \text{if } q = \infty \end{cases} \quad (7)$$

- Input-state system has  $\mathcal{L}_{\mathcal{X},\mathcal{U}}^q$ -induced gain upper bounded by  $\gamma > 0$  if, for all  $u \in \mathcal{L}_{\mathcal{U}}^q$ , the state  $x$  from zero initial state satisfies

$$\|x(\cdot)\|_{\mathcal{X},q} \leq \gamma \|u(\cdot)\|_{\mathcal{U},q} \quad (8)$$

# Robustness to unmodeled dynamics

Given a norm  $\|\cdot\|$ , consider

$$\dot{x} = f(t, x) + g(t, x) \quad (9)$$

If  $f$  has one-sided Lipschitz constant  $-c < 0$  and  $g$  has one-sided Lipschitz constant  $d > 0$ , then

- 1 **(contractivity under perturbations)** if  $d < c$ , then  $f + g$  is strongly contracting with rate  $c - d$ ,
- 2 **(equilibrium point under perturbations)** if additionally  $f$  and  $g$  are time-invariant, then the unique equilibrium point  $x^*$  of  $f$  and  $x^{**}$  of  $f + g$  satisfy

$$\|x^* - x^{**}\| \leq \frac{\|g(x^*)\|}{c - d} \quad (10)$$

- 1 Main equivalence theorem
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# Networks of contracting systems

Interconnected subsystems

$$\dot{x}_i = f_i(t, x_i, x_{-i}), \quad \text{for } i \in \{1, \dots, n\}, \quad (11)$$

where  $x_i \in \mathbb{R}^{N_i}$ ,  $N = \sum_{i=1}^n N_i$ , and  $x_{-i} \in \mathbb{R}^{N-N_i}$ .

If

- **osL**:  $\|f_i(t, x_i, x_{-i}) - f_i(t, y_i, x_{-i}), x_i - y_i\|_i \leq -c_i \|x_i - y_i\|_i^2$
- **Lip**:  $\|f_i(t, x_i, x_{-i}) - f_i(t, x_i, y_{-i})\|_i \leq \sum_{j=1, j \neq i}^n \gamma_{ij} \|x_j - y_j\|_j$
- the gain matrix  $\begin{bmatrix} -c_1 & \dots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \dots & -c_n \end{bmatrix}$  is Hurwitz


then the **interconnected system** is strongly contracting

wrt appropriate composite norm and with rate = (-) abscissa gain matrix

## Hurwitz Metzler Theorem

- 1  $M$  is Hurwitz,
- 2 there exists  $\eta \in \mathbb{R}_{>0}^n$  such that  $\eta^\top M < \mathbb{0}_n^\top$  or, equivalently,  $\mu_{1, [\eta]}(M) < 0$ ,
- 3 there exists  $\xi \in \mathbb{R}_{>0}^n$  such that  $M\xi < \mathbb{0}_n$  or, equivalently,  $\mu_{\infty, [\xi]^{-1}}(M) < 0$ , and
- 4 there exists a diagonal  $P = P^\top \succ 0$  satisfying  $M^\top P + PM \prec 0$  or, equivalently,  $\mu_{2, P^{1/2}}(M) < 0$ .

- 1 DAG interconnections of contracting systems are strongly contracting
- 2 2-dimensional matrix: small gain condition  $c_1 c_2 > l_1 l_2$

X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic stability conditions for Metzler matrices and monotone systems. *SIAM Journal on Control and Optimization*, 59(5):3447–3471, 2021. 

# Networks of ISS systems

Interconnections scalar ISS subsystems

$$\dot{x}_i = -a_i(x_i) + \sum_{j \neq i} \gamma_{ij}(x_j) + u_i, \quad \text{for } i \in \{1, \dots, n\}. \quad (12)$$

where  $a_i$  are of class  $\mathcal{K}_\infty$  and  $\gamma_{ij}$  are of class  $\mathcal{K}$ . Define

$$A_i(x) = a_i(x_i), \quad \text{and } \Gamma_i(x) = \sum_{j \neq i} \gamma_{ij}(x_j)$$

If there exist  $\eta \in \mathbb{R}_{>0}^n$  and  $c > 0$  satisfying

$$\eta^\top (A(v) - A(w)) \geq \eta^\top (\Gamma(v) - \Gamma(w) + c(v - w)), \quad \text{for all } v \geq w \geq 0_n$$

then the **interconnected system** is strongly contracting

with respect to  $\|\cdot\|_{1, [\eta]}$  and with rate  $c$

Proof:  $\text{osL}_{1, [\eta]}(f) \leq b$  if and only if  $\eta^\top (f(x) - f(y)) \leq b\eta^\top (x - y)$



- 1 Main equivalence theorem
- 2 Analysis of dynamic behavior and robustness
- 3 Network contraction theory: Small gain theorems
- 4 Network contraction theory: Weakly contracting systems
- 5 Network contraction theory: Semi-contracting systems

# Contraction theory for networks

**Challenge:** many real-world networks are not contracting.



**conservation law:**  $\mathbb{1}_n^\top x(t) = \text{const}$     **invariance, symmetry:**  $f(x + \alpha \mathbb{1}_n) = f(x)$

For a vector field  $f$  and positive vectors  $\eta, \xi \in \mathbb{R}_{>0}^n$ ,

conservation law                       $\eta^\top f(x) = \eta^\top f(y) \quad \forall x, y$                        $\iff$                        $\eta^\top Df(x) = 0 \quad \forall x$

translation invariance     $f(x + \alpha \xi) = f(x) \quad \forall x, \alpha$                        $\iff$                        $Df(x)\xi = 0 \quad \forall x$

If  $f$  satisfies a conservation or resp. invariance, then

- 1  $\text{osL}(f) \geq 0$ ,
- 2 if, additionally,  $f$  is monotone, then  $\text{osL}_{1, [\eta]}(f) = 0$  or resp.  $\text{osL}_{\infty, [\xi]^{-1}}(f) = 0$

# Weakly-contracting systems

## Definition and examples

$\dot{x} = f(t, x)$  is **weakly-contracting** wrt  $\|\cdot\|$ :

$$\text{osL}(f) \leq 0$$

- 1 Lotka-Volterra population dynamics (Lotka, 1920; Volterra, 1928) ( $\ell_1$ -norm for mutualistic)
- 2 Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981) ( $\ell_1$ -norm and  $\ell_\infty$ -norm)
- 3 Daganzo's cell transmission model for traffic networks (Daganzo, 1994), ( $\ell_1$ -norm for non-FIFO intersection)
- 4 compartmental systems in biology, medicine, and ecology (Sandberg, 1978; Maeda et al., 1978). ( $\ell_1$ -norm)
- 5 saddle-point dynamics for optimization of weakly-convex functions (Arrow et al., 1958). ( $\ell_2$ -norm)

# Weakly-contracting systems

## Part I: Dichotomy in asymptotic behavior

**Theorem: Dichotomy for weakly-contracting systems** For a weakly-contracting system  $\dot{x} = f(x)$ , either

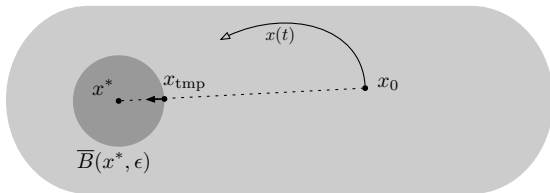
- 1  $f$  has no equilibrium and every trajectory is unbounded, or
- 2  $f$  has at least one equilibrium  $x^*$  and every trajectory is bounded.

# Weakly-contracting systems

## Part II: bounded trajectory case

**Theorem** If  $\dot{x} = f(x)$  is weakly-contracting and  $f$  has at least one equilibrium  $x^*$  then:

- (i) each equilibrium  $x^{**}$  is stable with weak Lyapunov function  $x \mapsto \|x - x^{**}\|$ ,
- (ii) if the norm  $\|\cdot\|$  is a  $(p, R)$ -norm,  $p \in \{1, \infty\}$  and  $f$  is piecewise real analytic, then every trajectory converges to the set of equilibria,
- (iii)  $x^*$  is locally asy stable  $\implies x^*$  is globally asy stable.



# Example: Distributed primal-dual algorithm

Optimization problem

$$\min_{x \in \mathbb{R}^k} f(x) = \min_{x \in \mathbb{R}^k} \sum_{i=1}^n f_i(x)$$

Distributed implementation

- $n$  agents communicate over a undirected weighted graph  $G$ ,
- agent  $i$  have access to function  $f_i$  and can exchange  $x_i$  with its neighbors.

$$\begin{aligned} \min_{x \in \mathbb{R}^k} \quad & \sum_{i=1}^n f_i(x_i) \\ & x_1 = x_2 = \dots = x_n \end{aligned}$$

In matrix form by assuming  $x = (x_1^\top, \dots, x_n^\top)^\top \in \mathbb{R}^{nk}$ :

$$\begin{aligned} \min_{x \in \mathbb{R}^k} \quad & \sum_{i=1}^n f_i(x_i) \\ & (L \otimes I_k)x = \mathbb{0}_{nN} \end{aligned}$$

# Example: Distributed primal-dual algorithm

If each  $f_i$  is continuously differentiable in  $x_i$ :

## Lagrangian

$$\mathcal{L}(x, \nu) = \sum_{i=1}^n f_i(x_i) + \nu^\top (L \otimes I_k)x$$

Distributed primal-dual algorithm (component form):

$$\dot{x}_i = -\frac{\partial \mathcal{L}}{\partial x_i} = -\nabla f_i(x_i) - \sum_{j=1}^n a_{ij}(\nu_i - \nu_j),$$
$$\dot{\nu}_i = \frac{\partial \mathcal{L}}{\partial \nu_i} = \sum_{j=1}^n a_{ij}(x_i - x_j)$$

Distributed primal-dual algorithm (vector form):

$$\dot{x} = -\nabla f(x) - (L \otimes I_k)\nu,$$
$$\dot{\nu} = (L \otimes I_k)x$$

# Example: Distributed primal-dual algorithm

- 1  $f$  has a minimum  $x^* \in \mathbb{R}^k$ ,
- 2  $f_i$  is twice differentiable,  $\nabla^2 f_i(x) \succeq 0$  for all  $x$ , and  $\nabla^2 f_i(x^*) \succ 0$ , and
- 3 the undirected weighted graph  $G$  is connected with Laplacian  $L$ .

**Theorem: Distributed primal-dual dynamics** The distributed primal-dual algorithm

- 1 is weakly-contracting wrt  $\ell_2$ -norm,
- 2  $(x(t), \nu(t)) \rightarrow (\mathbf{1}_n \otimes x^*, \mathbf{1}_n \otimes \nu^*)$ , with  $\nu^* = \sum_{i=1}^n \nu_i(0)$ ,
- 3 exponential convergence rate is  $-\alpha_{\text{ess}} \left( \begin{bmatrix} -\nabla^2 f(x^*) & -L \otimes I_k \\ L \otimes I_k & 0 \end{bmatrix} \right)$  where

$$\alpha_{\text{ess}}(A) := \max\{\Re(\lambda) \mid \lambda \in \text{spec}(A) \setminus \{0\}\}.$$

Proof:  $\mu_2(A) = 0$  and there exists locally asy stable equilibrium point



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# Semi-contracting systems

## Semi-norms

### Definition: Semi-norms

$\|\cdot\|$  is a **semi-norm** if

- 1  $\|cv\| = |c|\|v\|$ , for every  $v \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ ;
- 2  $\|v + w\| \leq \|v\| + \|w\|$ , for every  $v, w \in \mathbb{R}^n$ .

- define the subspace  $\text{Ker } \|\cdot\| = \{v \in \mathbb{R}^n \mid \|v\| = 0\}$
- Example: for  $k < n$ ,  $R \in \mathbb{R}^{k \times n}$ , and norm  $\|\cdot\|$ , we get  $\|x\|_R = \|Rx\|$

# Semi-contracting systems

## Matrix semi-measures

The **matrix semi-measure** of  $A \in \mathbb{R}^{n \times n}$  wrt  $\|\cdot\|$ :

$$\mu_{\|\cdot\|}(A) = \lim_{h \rightarrow 0^+} \frac{\|I_n + hA\| - 1}{h}.$$

if  $\text{Ker } \|\cdot\|$  is invariant under  $A$ ,

then  $\Re(\lambda) \leq \mu_{\|\cdot\|}(A)$ , for every  $\lambda \in \text{spec}_{\text{Ker } \|\cdot\|^\perp}(A^\top)$ .

# Semi-norms for network systems

For undirected  $G$  with edge set  $\mathcal{E}$ , incidence matrix  $B$  and Laplacian  $L$

$$\|x\|_{\mathcal{E}} := \max_{(i,j) \in \mathcal{E}} |x_i - x_j| = \|B^T x\|_{\infty}$$

For connected graphs  $\text{Ker } \|\cdot\|_{\mathcal{E}} = \text{span}\{\mathbf{1}_n\}$

The orthogonal projection  $\Pi_n : \mathbb{R}^n \rightarrow \text{span}\{\mathbf{1}_n\}^{\perp}$

$$\Pi_n = I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T = \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{bmatrix} \quad \text{or } 0$$

Logarithmically-optimal semi-norm

$$\mu_{2, \Pi_n}(-L) = -\lambda_2(L)$$

# Semi-contracting systems

## Definition and examples

$\dot{x} = f(t, x)$  is **semi-contracting** wrt the semi-norm  $\|\cdot\|$  with rate  $c > 0$ :

$$\text{osL}_{\|\cdot\|}(f) \leq -c$$

or, for differentiable systems,  $\mu_{\|\cdot\|}(Df(t, x)) \leq -c$

- 1 Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981), ( **$\ell_1$ -norm**)
- 2 Chua's diffusively-coupled circuits (Wu and Chua, 1995), ( **$\ell_2$ -norm**)
- 3 morphogenesis in developmental biology (Turing, 1952), ( **$\ell_1$ -norm, over some param ranges**)
- 4 Goodwin model for oscillating auto-regulated gene (Goodwin, 1965). ( **$\ell_1$ -norm, over some param ranges**)

# Semi-contracting systems

## Semi-contraction and asymptotic behavior

Consider  $\dot{x} = f(t, x)$  with  $f$  continuously differentiable in  $x$  and assume

- $f$  is semi-contracting wrt the semi-norm  $\|\cdot\|$  with rate  $c > 0$ , and
- **(Affine invariance)**: there exists  $x^*$  such that  $f(t, x^* + \text{Ker } \|\cdot\|) \subseteq \text{Ker } \|\cdot\|$

Then,

- 1 for every trajectory  $x(t)$ ,

$$\|x(t) - x^*\| \leq e^{-ct} \|x(0) - x^*\|, \quad \text{for every } t \geq 0.$$

- 2 every trajectory converges to  $x^* + \text{Ker } \|\cdot\|$ .

## Example: Diffusively-coupled oscillators

- $n$  agents connected by a weighted undirected graph  $G$ ,
- identical internal dynamics  $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^k \rightarrow \mathbb{R}^k$

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^n a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

- **synchronization:**

$$\lim_{t \rightarrow \infty} \|x_i - x_j\| = 0 \quad \text{for every } i, j$$

- synchronization of diffusively-coupled oscillators:
  - 1 contractivity of the internal dynamics
  - 2 strength of the diffusive coupling

Introduce **local-global mixed norm**:  $(2, p)$ -tensor norm on  $\mathbb{R}^{nk} = \mathbb{R}^n \otimes \mathbb{R}^k$

$$\|u\|_{(2,p)} = \inf \left\{ \left( \sum_{i=1}^r \|v^i\|_2^2 \|w^i\|_p^2 \right)^{\frac{1}{2}} \mid u = \sum_{i=1}^r v^i \otimes w^i \right\}.$$

- **global norm**:  $\ell_2$ -norm for the interactions between agents
- **local norm**:  $\ell_p$ -norm for internal dynamics of each agent
- closely related to, but different from, the projective tensor product norm  
R. A. Ryan. *Introduction to Tensor Products of Banach Spaces*. Springer, 2002. ISBN 9781852334376

$(\prod_n \otimes I_k)x$  measures **dissimilarity** of the states  $x_j$ :

$$x = \mathbb{1}_n \otimes x^* \quad \implies \quad (\prod_n \otimes I_k)x = (\prod_n \otimes I_k)(\mathbb{1}_n \otimes x^*) = \prod_n \mathbb{1}_n \otimes x^* = \mathbb{0}_{(n-1) \times k}$$



## Example: Diffusively-coupled oscillators

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^n a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

$G$  is an undirected weighted graph with Laplacian  $L$

Suppose there exist  $p \in [1, \infty]$ ,  $Q \in \mathbb{R}^{k \times k}$

$$\text{osL}_{p,Q}(f) < \lambda_2(L)$$

then semi-contraction wrt  $\|\cdot\|_{(2,p),(\Pi_n \otimes Q)}$  and rate  $c = \lambda_2(L) - \text{osL}_{p,Q}(f)$ ,

1 for every trajectory  $x(t)$ ,

$$\|x(t) - \mathbb{1}_n \otimes x_{\text{ave}}(t)\|_{(2,p),(\Pi_n \otimes Q)} \leq e^{-ct} \|x(0) - \mathbb{1}_n \otimes x_{\text{ave}}(0)\|_{(2,p),(\Pi_n \otimes Q)}$$

2 synchronization:  $\lim_{t \rightarrow \infty} x(t) = \mathbb{1}_n \otimes x_{\text{ave}}(t)$  where  $x_{\text{ave}}(t) = \frac{1}{n} \sum_{i=1}^n x_i(t)$

## Example: Diffusively-coupled oscillators

$$\text{osL}_{p,Q}(f) \leq \lambda_2(L)$$

- trade off between **internal dynamics** and **coupling strength**
- $f$  time-invariant: every trajectory converges to the unique equilibrium point.
- $f$  periodic: every trajectory converges to the unique periodic orbit.
- for any Lipschitz unstable dynamics  $f$ ,  
there exists sufficiently strong coupling  $\lambda_2(L)$  s.t. the network synchronizes.

# Summary

- main equivalence theorem
  - notion of one-sided Lipschitz constant and weak pairing
- characterization of contraction wrt non-Eulidean  $\ell_1, \ell_\infty$  norms
- robustness and iss properties
- network contraction theory
  - small-gain theorems
  - weak contraction
  - semi-contraction

## Future work

- 1 forthcoming: fixed point algorithms and theorems
- 2 maybe: Halanay inequalities
- 3 maybe: optimal control problems for osL control systems

# Incremental ISS for strongly contracting delay ODEs

$$\dot{x}(t) = f(x(t), x(t-s), u(t)), 0 \leq s \leq S, \quad \|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{U}} \quad (13)$$

assume there exist positive constants  $c, \ell_{\mathcal{U}}, \ell_{\mathcal{X}}$  such that, for all variables,

$$\text{osL } x : \quad \llbracket f(x, d, u) - f(y, d, u), x - y \rrbracket_{\mathcal{X}} \leq -c \|x - y\|_{\mathcal{X}}^2 \quad (14)$$

$$\text{Lip } x(t-s) : \quad \llbracket f(x, x_1, u) - f(x, x_2, u) \rrbracket_{\mathcal{X}} \leq \ell_{\mathcal{X}} \|x_1 - x_2\|_{\mathcal{X}} \quad (15)$$

$$\text{Lip } u : \quad \llbracket f(x, d, u) - f(x, d, v) \rrbracket_{\mathcal{X}} \leq \ell_{\mathcal{U}} \|u - v\|_{\mathcal{U}} \quad (16)$$

By the curve norm derivative formula, subadditivity, and Cauchy-Schwarz inequality,

$$\begin{aligned} \|x(t) - y(t)\|_{\mathcal{X}} D^+ \|x(t) - y(t)\|_{\mathcal{X}} &= \llbracket f(x(t), x(t-s), u_x(t)) - f(y(t), y(t-s), u_y(t)), x(t) - y(t) \rrbracket_{\mathcal{X}} \\ &\leq \llbracket f(x(t), x(t-s), u_x(t)) - f(y(t), x(t-s), u_x(t)), x(t) - y(t) \rrbracket_{\mathcal{X}} \\ &\quad + \llbracket f(y(t), x(t-s), u_x(t)) - f(y(t), y(t-s), u_x(t)), x(t) - y(t) \rrbracket_{\mathcal{X}} \\ &\quad + \llbracket f(y(t), y(t-s), u_x(t)) - f(y(t), y(t-s), u_y(t)), x(t) - y(t) \rrbracket_{\mathcal{X}} \\ &\leq -c \|x(t) - y(t)\|_{\mathcal{X}}^2 + \ell_{\mathcal{X}} \|x(t-s) - y(t-s)\|_{\mathcal{U}} \|x(t) - y(t)\|_{\mathcal{X}}, \\ &\quad + \ell_{\mathcal{U}} \|u_x(t) - u_y(t)\|_{\mathcal{U}} \|x(t) - y(t)\|_{\mathcal{X}}. \end{aligned}$$

Thus, with  $m(t) = \|x(t) - y(t)\|_{\mathcal{X}}$ , delay differential inequality:

$$D^+ m(t) \leq -cm(t) + \ell_{\mathcal{X}} \sup_{0 \leq s \leq S} m(t-s) + \ell_{\mathcal{U}} \|u_x(t) - u_y(t)\|_{\mathcal{U}}, \quad (17)$$

Halanay inequality is applicable. If  $c > \ell_{\mathcal{X}}$ , then

$$m(t) \leq m_0 e^{-\rho(t-t_0)} + \ell_{\mathcal{U}} \int_{t_0}^t e^{-\rho(t-\tau)} \|u_x(\tau) - u_y(\tau)\|_{\mathcal{U}} d\tau, \quad (18)$$

where  $\rho > 0$  is the unique positive root of  $\rho = c - \ell_{\mathcal{X}} e^{\rho S}$  and  $m_0 = \sup_{0 \leq s \leq S} m(t_0 - s)$ .

# Networks of contracting systems with time delays

Interconnected subsystems  $i \in \{1, \dots, n\}$

$$\dot{x}_i = f_i(x_i, x_{-i}, x_{-i}(t-s), u_i), \quad 0 \leq s \leq S, \quad \|\cdot\|_i, \|\cdot\|_{i,\mathcal{U}} \quad (19)$$

Assume there exist positive constants st

**osL**  $x_i$  :  $\llbracket f_i(x_i, \dots) - f_i(y_i, \dots), x_i - y_i \rrbracket_i \leq -c_i \|x_i - y_i\|_i^2$

**Lip**  $x_{-i}$  :  $\|f_i(\dots, x_{-i}, \dots) - f_i(\dots, y_{-i}, \dots)\|_i \leq \sum_{j=1, j \neq i}^n \gamma_{ij} \|x_j - y_j\|_j$

**Lip**  $x_{-1}^{-s}$  :  $\|f_i(\dots, x_{-i}^{-s}, \dots) - f_i(\dots, y_{-i}^{-s}, \dots)\|_i \leq \sum_{j=1, j \neq i}^n \hat{\gamma}_{ij} \|x_j^{-s} - y_j^{-s}\|_j$

**Lip**  $u_i$  :  $\|f_i(\dots, u_i) - f_i(\dots, v_i)\|_i \leq \ell_{i,\mathcal{U}} \|u_i - v_i\|_{i,\mathcal{U}}$

With  $m_i(t) = \|x_i(t) - y_i(t)\|_i$ , delay differential inequality:

$$D^+ m(t) \leq -Cm(t) + \Gamma m(t) + \hat{\Gamma} \sup_{0 \leq s \leq S} m(t-s) + \ell_{\mathcal{U}} \|u_x(t) - u_y(t)\|_{\mathcal{U}}$$

and, if the Metzler matrix  $-C + \Gamma + \hat{\Gamma}$  is Hurwitz, then (19) is incremental ISS

# Background on one-sided Lipschitz continuity

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- 4 **nonlinear measure** in: H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001. [doi](#)
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- 9 equivalent differential conditions!