Parsimonious Models for Opinion Dynamics and Structural Balance

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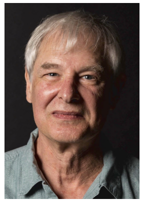

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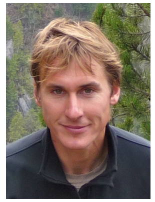

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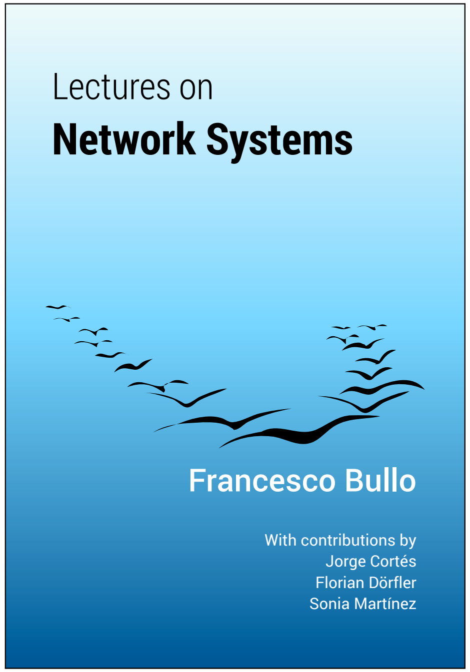

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William of Ockham, 1287–1347, stained glass window in Surrey, UK

Today's topic: opinions and appraisals

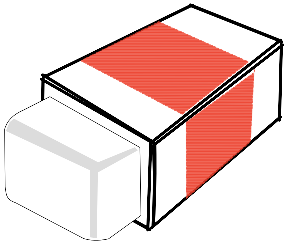
"Entities should not be multiplied without necessity" William of Occam

$\rightsquigarrow$ "The simplest solution is most likely the right one"

"Everything should be made as simple as possible, but not simpler"

Albert Einstein

1. fewer assumptions, same behavior, or same assumptions, richer behavior



Occam's famous eraser

2. semantics (meaning of a sentence) vs syntax (arrangement of words to create well-formed sentences)

Outline

1. **Parsimony in opinion dynamics**
   W. Mei, F. Bullo, G. Chen, and F. Dörfler. Occam's razor in opinion dynamics: The weighted-median influence process.
   September 2019.
   URL: `https://arxiv.org/abs/1909.06474`

2. **Parsimony in structural balance dynamics**
   P. Cisneros-Velarde, N. E. Friedkin, A. V. Proskurnikov, and F. Bullo. Structural balance via gradient flows over signed graphs.
   September 2019.
   URL: `https://arxiv.org/pdf/1909.11281.pdf`

3. **Parsimony in opinion dynamics over signed graphs**
   P. Cisneros-Velarde, K. S. Chan, and F. Bullo. Polarization and fluctuations in signed social networks.
   February 2019.
   URL: `https://arxiv.org/pdf/1902.00658.pdf`

French-DeGroot model

$$x_i(t+1) = \sum_{j=1}^{n} w_{ij} x_j(t), \qquad \text{or:} \qquad x(t+1) = Wx(t)$$

- individual opinions denoted by real numbers
- opinions updated by **weighted averaging**
- $W = (w_{ij})_{n\times n}$ is row-stochastic and defines **influence network** $\mathcal{G}(W)$
- If $\mathcal{G}(W)$ contains a globally reachable & aperiodic SCC,

$$\lim_{t\to\infty} x(t) = \text{consensus}$$

[1] J. R. P. French, *Psychological Review*, 63(3):181-194, 1956

Extensions of French-DeGroot model

1. French-DeGroot model with absolutely stubborn agents
   - $\exists$ individual i s.t. $w_{ii} = 1$
2. Friedkin-Johnsen model: persistent attachment to initial belief
   - $x(t+1) = (I_n - \Lambda)Wx(t) + \Lambda x(0)$
3. Bounded-confidence model: influence truncated at confidence radius

$$x_i(t+1) = \frac{\sum_{j:\,|x_j(t)-x_i(t)|<r_i} x_j(t)}{\#\{j \,|\, |x_j - x_i| < r_i\}}$$

4. Altafini model: French-DeGroot model over signed graphs

Countless extensions

- opinion dynamics with time-varying graph / switching topology
- gossip dynamics
- negative weights
- quantized dynamics
- multiple issues with logical constraints
- evolution of social power along issue sequence
- state-dependent interpersonal influence
- unilateral bounded-confidence model
- opinion dynamics with time-delay and noise

and random combinations of them:

- gossip-like opinion dynamics with negative weights
- convergence rate of opinion dynamics with negative weights
- evolution of social power with time-varying communication graphs
- multiple issues with heterogeneous logical constraints

Proposed topic for Wenjun's research:

Convergence Rate of Gossip-like Quantized Opinion Dynamics with non-Euclidean Spaces and Unilateral Confidence Bounds on Switching Topology with Delays and Antagonistic Interactions

Parsimony in opinion dynamics

"Simplicity is the ultimate sophistication," Leonardo Da Vinci

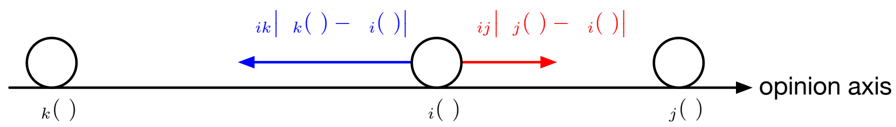
Proposed topic for Wenjun's research:
develop new model

- as simple as classic French-DeGroot model (no additional params)
- based on equally (or more) reasonable microscopic mechanisms

- rich in macroscopic behavior
- wider domain of applicability
- able to capture various real phenomena that other models fail to
  1. multi-modal opinion distributions
  2. vulnerability of peripheral nodes to extremism
  3. lower consensus likelihood in large groups

weighted averaging = taken-for-granted, but perhaps unrealistic micro-foundation

- **opinion "attractiveness" $\sim$ opinion distance**
- the core behind the consensus prediction of French-DeGroot model
- inherited by all its extensions



$$_i(\ +1) = \ _i(\ ) + \ _{ik}(\ _k(\ ) - \ _i(\ )) + \ _{ij}(\ _j(\ ) - \ _i(\ ))$$

Cognitive dissonance as microfoundation

- **cognitive dissonance** caused by disagreement [2]

$$u_i(x_i, x_{-i}) = \sum_{j=1}^{n} w_{ij} |x_i - x_j|^{\alpha}$$

- Best response to minimize the dissonance: $x_i^+ = \mathrm{argmin}_z\, u_i(z, x_{-i})$
- $\alpha = 2$ for French-DeGroot model [3]
  why should cognitive dissonance grow quadratically?
- $\alpha > 1$: encouragement to move towards distant opinions
- $\alpha = 1 \implies$ **weighted-median opinion dynamics**

[2] L. Festinger, "A Theory of Cognitive Dissonance." Stanford University Press, 1962.

[3] D. Bindel, J. Kleinberg, and S. Oren, *Games and Economic Behavior*, 92:248-265, 2015

[4] P. Groeber, J. Lorenz and F. Schweitzer, *J of Mathematical Sociology*, 38:147-174, 2014

Weighted-median opinion dynamics: model set-up

$$x_i(t+1) = \mathrm{argmin}_{z\in\mathbb{R}} \sum_{j=1}^{n} w_{ij} \big|z - x_j(t)\big|$$

$$\implies \quad x_i(t+1) = \mathrm{Med}_i\big(x(t); W\big)$$

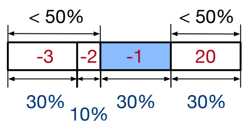
What is weighted median?
nonlinear average, independent under monotone scaling, opinion ordering

Given $x = (x_1, \dots, x_m)$ and weights $w = (w_1, \dots, w_m)$,
weighted median of x is $x^* \in \{x_1, \dots, x_m\}$ such that

$$\sum_{i\,:\,x_i < x^*} w_i \leq 50\% \quad \text{and} \quad \sum_{i\,:\,x_i > x^*} w_i \leq 50$$

(uniqueness: $\sum_{j\in\theta} w_{ij} \neq 1/2$, for any $\theta \subset \{1, \dots, n\}$)

$= (-1, -3, 20, -2)$
$= (30\%, 30\%, 30\%, 10\%)$ $\implies$

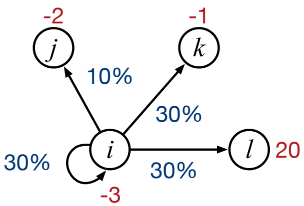


$\implies$ Weighted average: 4.6
Weighted median: -1

Weighted-median opinion dynamics: model set-up

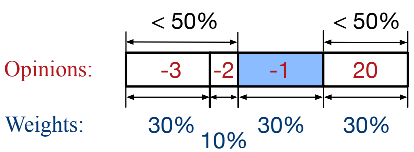
Weighted-median opinion dynamics

$$x_i(t+1) = \mathrm{Med}_i\big(x(t); W\big) \qquad \text{for all } i$$
$$= \text{weighted median of } x(t) \text{ by } i\text{-th row of } W$$



$\Rightarrow$

For node i: Weighted average: 4.6
Weighted median: -1



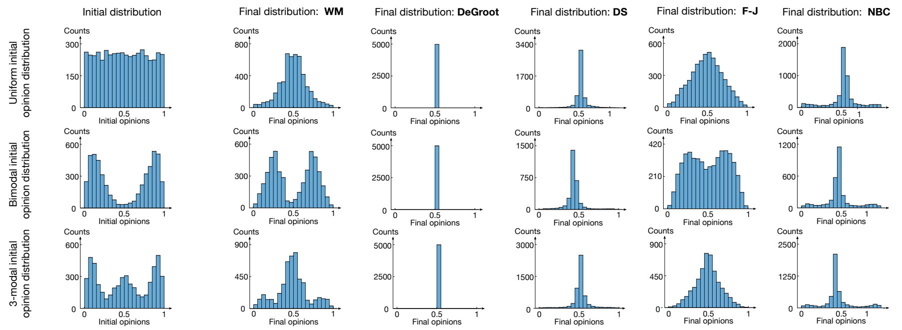
inconspicuous microscopic change:
from weighted average to weighted median
$\implies$ dramatic macroscopic consequences

- Broader applicability than French-DeGroot:
  ordered multiple-choice issues, eg, common in political debate
  no requirement to map opinions onto real numbers
- More realistic predictions (numerical comparisons)
- More sophisticated dynamical behavior (theoretical analysis)
- Higher robustness to the perturbation of influence networks

Numerical comparisons 1/3

Various types of steady public opinion distributions [5]

- Empirically observed: uni-modal, bimodal, multi-modal
- Premise of multi-party political system

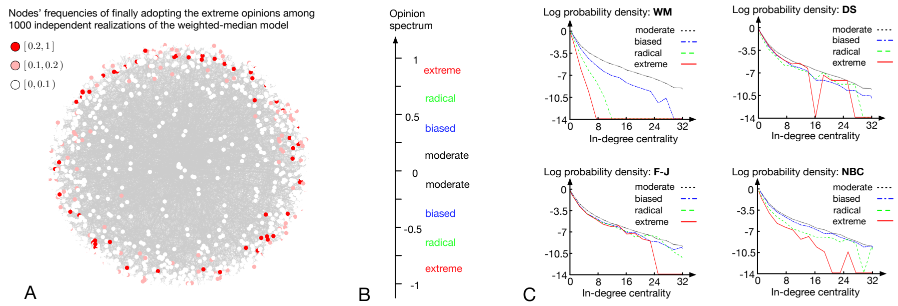


Acronyms: **WM** = the weighted-median model; **DS** = the DeGroot model with absolutely stubborn agents; **F-J** = the Friedkin-Johnsen model; **NBC** = the networked bounded-confidence model.

[5] A. Downs, *Journal of Political Economy*, 65(2):135-150, 1957

Numerical comparisons 2/3

Peripheral nodes are more vulnerable to extreme opinions. [5]

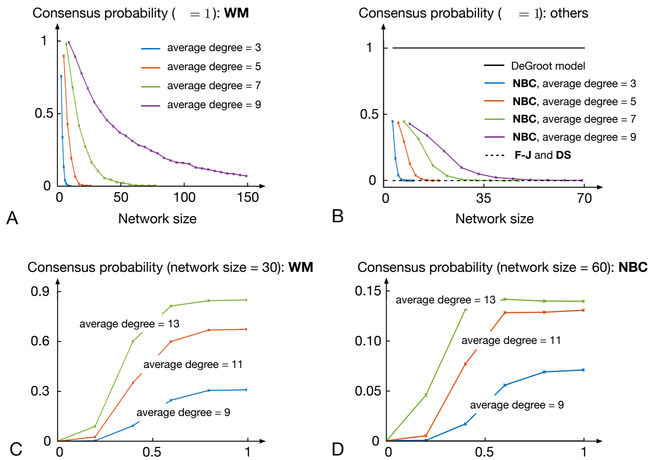


Acronyms: **WM** = the weighted-median model; **DS** = the DeGroot model with absolutely stubborn agents; **F-J** = the Friedkin-Johnsen model; **NBC** = the networked bounded-confidence model.

[6] C. McCauley and S. Moskalenko, *Terrorism and Political Violence*, 20(3):415-433, 2008

Numerical comparisons 3/3

Lower consensus likelihoods in larger or more clustered groups [6]



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F-J = the Friedkin-Johnsen model; **NBC** = the networked bounded-confidence model.

[7] A. P. Hare, *American Sociological Review*, 17(3):261-267, 1952

Theoretical analysis

Important concepts

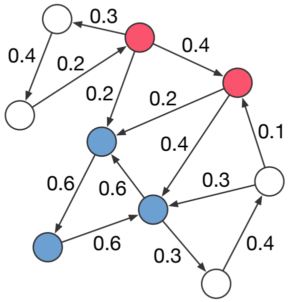
- cohesive set [7]: a subset of nodes ("echo chamber")
  - M is cohesive if $\sum_{j\in M} w_{ij} \geq 1/2$ for any $i \in M$
  - more generalized definition used in *linear threshold diffusion model* [8]
- maximal cohesive set
  - M is cohesive & $M \cup \{i\}$ is not cohesive for any $i \notin M$
- cohesive expansion: $E(M)$
  - $M \to M \cup \{i\}$ $(i \notin M)$ as long as $M \cup \{i\}$ remains cohesive

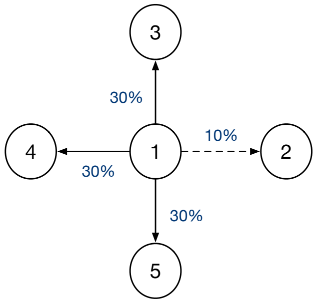
[7] S. Morris. "Contagion." *The Review of Economic Studies*, 2000

[8] D. Acemoglu et al. "Diffusion of innovations in social networks." *CDC*, 2011

- Cohesive expansion is unique, independent of addition order
- M is cohesive $\implies$ $E(M)$ = smallest maximal cohesive containing M
- M is cohesive $\implies$
  - $E(M) = \{1, \dots, n\}$, or
  - $E(M)$ and $\{1, \dots, n\} \setminus E(M)$ are both maximally cohesive

- decisive link: (i, j) is decisive if
  - $\exists \theta \subset \mathcal{N}_i$ s.t. $j \in \theta$, $\sum_{k\in\theta} w_{ik} \geq 1/2$, and $\sum_{k\in\theta\setminus\{j\}} w_{ik} < 1/2$
- (i, j) is indecisive $\implies$ $x_i(t+1)$ is independent of $x_j(t)$





Theoretical analysis

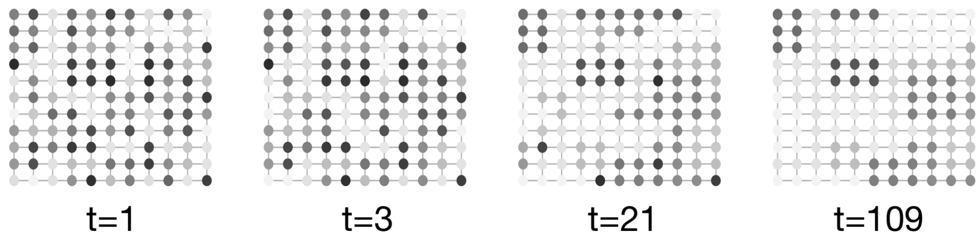
Weighted-median opinion dynamics

At each time, randomly pick i

$$x_i(t+1) = \mathrm{Med}_i\big(x(t); W\big)$$

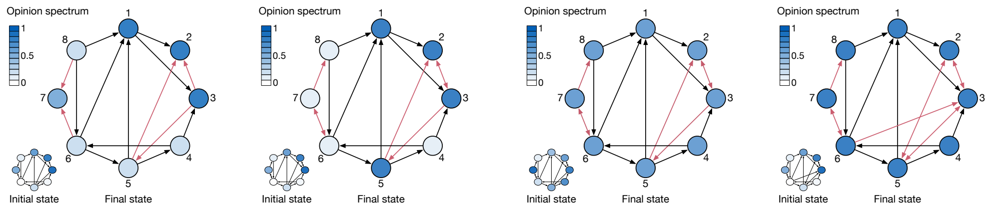
or synchronous

$$x(t+1) = \mathrm{Med}\big(x(t); W\big)$$



Dynamical behavior

- almost-sure convergence to an equilibrium in finite time
- $\{1, \dots, n\}$ is the only max cohesive $\implies$ almost-sure consensus
- $\exists\, M \subsetneq \{1, \dots, n\}$ that is maximal cohesive
  $\implies$ almost sure disagreement from initial conditions in set of positive measure
- $\mathcal{G}_{\text{decisive}}(W)$ no glob reach node $\implies$ almost-sure disagreement
- Conjecture: $\mathcal{G}_{\text{decisive}}(W)$ has a globally reachable node
  $\implies$ $\exists$ non-zero-measure set of initial conditions and update sequence leading to consensus in finite time

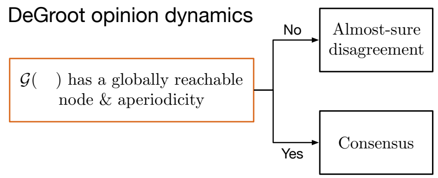


A. Network 1 B. Network 2, Case 1 C. Network 2, Case 2 D. Network 3

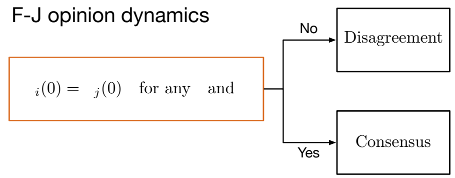
Comparison with weighted-averaging models

- more robust
- dependence on more delicate network structure
- richer dynamical behavior

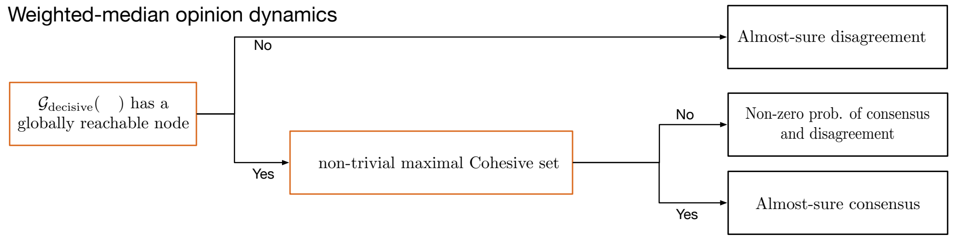
DeGroot opinion dynamics



F-J opinion dynamics



Weighted-median opinion dynamics



Summary and future research

Summary

Novel model of microscopic influence and macroscopic implications

Future research

- weighted-median model with compromise behavior (working paper)
  - Discrete-time with inertia: $x_i(t+1) = (1-\epsilon_i)\mathrm{Med}_i\big(x(t); W\big) + \epsilon_i x_i(t)$
  - Continuous-time model: $\dot{x} = \mathrm{Med}(x; W) - x$
- selected extensions, as for French-DeGroot model
  - time-varying graph
  - antagonistic interactions
- consensus conditions for different cognitive dissonance functions

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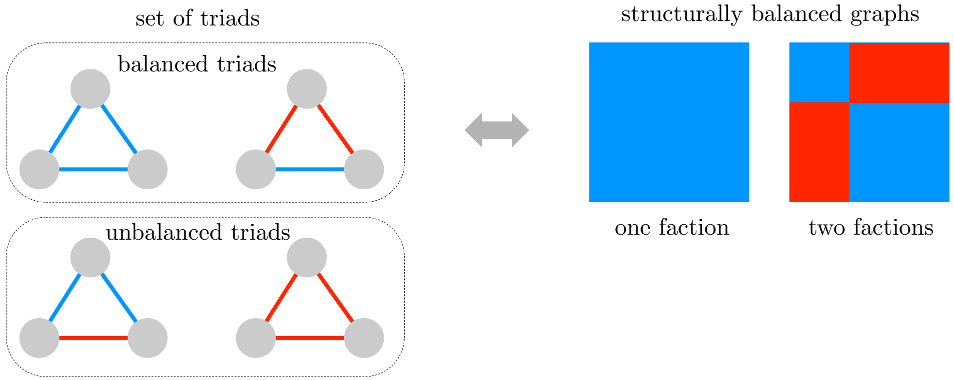
   February 2019.
   URL: https://arxiv.org/pdf/1902.00658.pdf

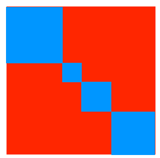
Heider's axioms and structural balance

- Signed graphs = friendly and antagonistic relations
  x_{ij} interpersonal appraisal of j held by i
- Heider's axioms:
  1. "the friend of a friend is a friend"
  2. "the enemy of a friend is an enemy"
  3. "the friend of an enemy is an enemy"
  4. "the enemy of an enemy is a friend"
- Heider's axioms $\iff$ $x_{ij} \sim x_{ik}x_{kj}$
- violation of axiom elicits **cognitive dissonance**
  and effort to resolve contradiction and reduce discomfort

$$x_{ij}^+ \approx x_{ij} + x_{ik}x_{kj}$$
$$\dot{x}_{ij} \approx x_{ik}x_{kj} \qquad \text{(plus saturation)}$$

From microscopic relations to macroscopic allowable structures:





Recent years: dynamic structural balance

The *Kułakowski et al. model*

$$x_{ij} \sim x_{ik}x_{kj} \tag{1}$$

$$\rightsquigarrow \quad \dot{x}_{ij} \sim x_{ik}x_{kj} \tag{2}$$

$$\rightsquigarrow \quad \dot{x}_{ij} = \sum_k x_{ik}x_{kj} \tag{3}$$

In matrix form:

$$\dot{X} = X^2 \tag{4}$$

K. Kułakowski, P. Gawroński, and P. Gronek. The Heider balance: A continuous approach. *International Journal of Modern Physics C*, 16(05):707–716, 2005

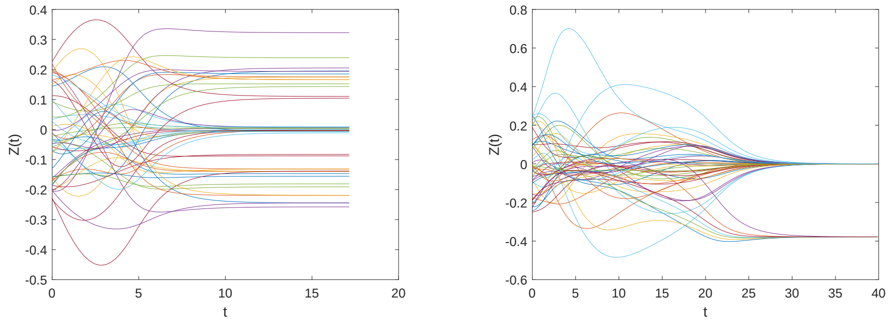
$X(t) = \text{scaling}(t) Z(t)$, where Z lives in Frobenious unit-sphere

$$\dot{X} = X^2 \quad \iff \quad \dot{Z} = Z^2 + \mathcal{D}(Z)Z$$

Phenomenology of Kułakowski et al model

$Z(0) = Z(0)^\top$: good behavior

$Z(0) \neq Z(0)^\top$: poor behavior



(a) $\lim_{t\to\infty} Z(t)$: not balanced

(b) $\lim_{t\to\infty} Z(t)$: diagonal

Three assumptions, ready for Occam's eraser

$$\dot{X} = X^2 \quad \Longrightarrow \quad \dot{x}_{ij} = \sum_{k=1}^{n} x_{ik}x_{kj}$$

that is

$$\dot{x}_{ij} = \sum_{\substack{k=1 \\ k\neq i,j}}^{n} x_{ik}x_{kj} + x_{ij}(x_{ii} + x_{jj})$$

$$\dot{x}_{ii} = x_{ii}^2 + \sum_{\substack{k=1 \\ k\neq i}}^{n} x_{ik}x_{ki}$$

"It is more parsimonious to assume that the sun goes around the Earth, that atoms at the smallest scale operate in accordance with the same rules that objects at larger scales follow, and that we perceive what is really out there." David Eagleman

Parsimony in structural balance dynamics

- new model by simply removing self-appraisals
  1. **sociologically better motivated**
  2. **gradient flow** of dissonance function
  3. much larger set of initial conditions leads to structural balance (*)

The *dissonance function* is (− appraisal product on directed cycle)

$$\mathcal{D}(X) = -\sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{n} x_{ij}x_{jk}x_{ki} = -\,\text{trace}(X^3) = -\langle\langle X^2, X^\top \rangle\rangle_F$$

The pure-influence model

pure-influence model is

$$\dot{x}_{ij} = \sum_{\substack{k=1 \\ k\neq i,j}}^{n} x_{ik}x_{kj}, \qquad i \neq j \tag{5}$$

In matrix form, with $X(0)$ with zero diagonal,

$$\dot{X} = X^2 - \mathrm{diag}(X^2)$$

and in matrix projected form (on Frobenious sphere)

$$\dot{Z} = Z^2 + \mathcal{D}(Z)Z - \mathrm{diag}(Z^2)$$

What is more parsimonious: $\dot{X} = X^2$ or $\dot{X} = X^2 - \mathrm{diag}(X^2)$?
syntax versus semantics

Symmetric pure-influence models are gradient flows

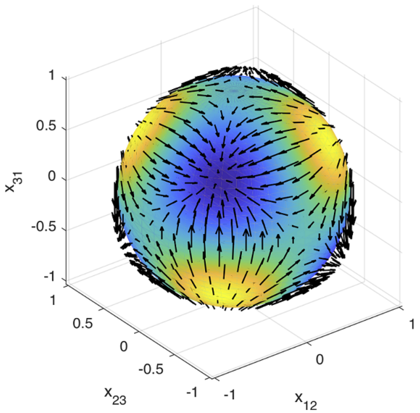
Theoretical analysis for symmetric matrices, numerical for asymmetric

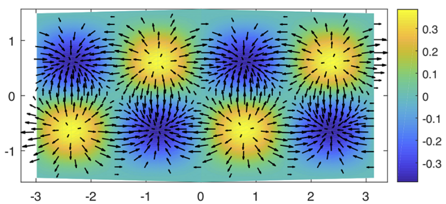
1. **pure influence model**
   leaves invariant set of symmetric zero-diagonal matrices

$$\dot{X} = X^2 - \mathrm{diag}(X^2) \quad = \quad -\frac{1}{3}\,\mathrm{grad}\,\mathcal{D}(X)$$

2. **projected pure-influence model**
   leaves invariant set of unit-norm symmetric zero-diagonal matrices

$$\dot{Z} = Z^2 + \mathcal{D}(Z)Z - \mathrm{diag}(Z^2) \quad = \quad -\tfrac{1}{3}\mathcal{P}_{Z^\perp}\big(\,\mathrm{grad}\,\mathcal{D}(Z)\big)$$





For $n = 3$, any symmetric unit-norm zero-diagonal Z is determined by upper-right triangle (z_{12}, z_{23}, z_{31}) with $z_{12}^2 + z_{23}^2 + z_{31}^2 = 1$

figures: sphere with heatmap of $\mathcal{D}(Z)$ and gradient vector field

note: four global minima = configurations of structural balance

Equilibria

Proposition (Balanced equilibria, I)

If Z^* is equilibrium point with **a single positive eigenvalue** *for projected pure-influence model, then*

1. $$Z^* = \left[\begin{array}{c|c} Z' & \mathbb{0}_{n_1\times n-n_1} \\ \hline \mathbb{0}_{n-n_1\times n_1} & \mathbb{0}_{n-n_1\times n-n_1} \end{array}\right]$$
   with $n_1 \leq n$ and $Z' = \frac{1}{\sqrt{n_1(n_1-1)}}(ss^\top - I_{n_1})$, for some $s \in \{-1,+1\}^{n_1}$;
2. *$G(Z')$ satisfies structural balance*
   s characterizes individual-faction assignment

Result not shown: we also characterized all symmetric equilibria

Equilibria

Proposition (Balanced equilibria, II)

If Z^* is equilibrium point with **a single positive eigenvalue** *and* **irreducible ($G(Z^*)$ is connected graph)**, *then*

1. *$G(Z^*)$ satisfies structural balance*
2. *Z^* is global minimizer of*

$$\begin{aligned} \underset{Z \in \mathbb{R}^{n\times n}}{\text{minimize}} \quad & \mathcal{D}(Z) \\ \text{subject to} \quad & Z \text{ is unit-norm, zero-diagonal and symmetric} \end{aligned}$$

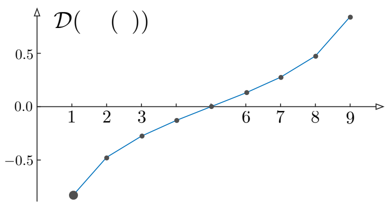


Figure: $\mathcal{D}$ at irreducible equilibria with k positive eigenvalues, $n = 10$.

Proposition (Convergence results and dynamical properties)

For pure-influence model with zero-diagonal symmetric $X(0)$ and projected pure-influence model with $Z(0) = \frac{X(0)}{\|X(0)\|_F}$,

1. *$Z(t)$ converges to a critical point of $\mathcal{D}$*
2. *the number of negative eigenvalues of $Z(t)$ is non-decreasing*

Moreover, if $X(0)$ has one positive and no zero eigenvalue, then

3. *$\lim_{t\to+\infty} Z(t) = Z^*$, with Z^* as in last proposition*
4. *$\mathrm{sign}(X(t)) = \mathrm{sign}(Z^*)$ in finite time*

Numerical experiments

Probability estimation: 27K numerical experiments at each generic

- symmetric $Z(0)$ and $n \in \{3, 5, 6, 15\}$
- asymmetric $Z(0)$ and $n \in \{5, 6\}$

Result: 99% confidence level: there is at least 0.99 probability that $Z(t)$ converges to structural balance in finite time.

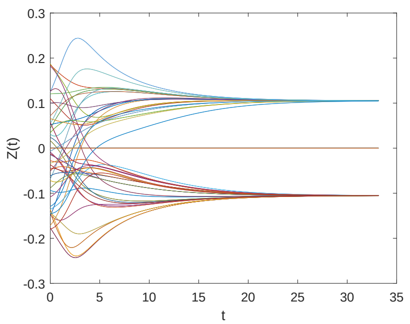


Figure: Projected model, structural balance from generic $Z(0) = Z(0)^\top$, $n = 10$

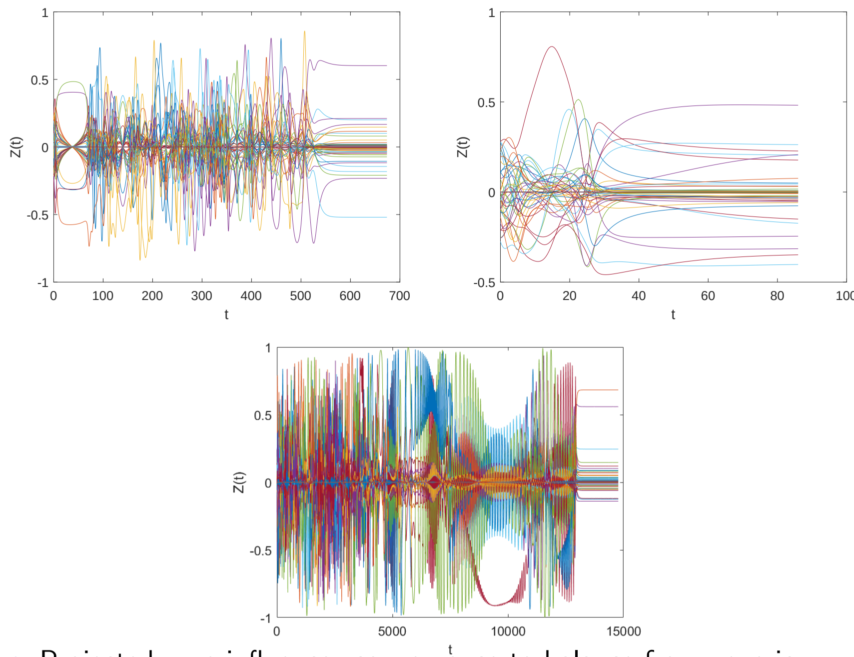


Figure: Projected pure-influence: convergence to balance from generic asymmetric, $n = 7, 7, 10$. Same initial conditions as for Kułakowski et al model

Summary and future research

Summary

inconspicuous microscopic change: remove selfweights

$\Longrightarrow$ dramatic macroscopic consequences

gradient flow and converges to balance from much larger initials

Future research

1. pure-influence model: less conservative sufficient conditions
2. dynamic models with sociologically-justified transient behavior

Outline

1. **Parsimony in opinion dynamics**
   W. Mei, F. Bullo, G. Chen, and F. Dörfler. Occam's razor in opinion dynamics: The weighted-median influence process.
   September 2019.
   URL: https://arxiv.org/abs/1909.06474
2. **Parsimony in structural balance dynamics**
   P. Cisneros-Velarde, N. E. Friedkin, A. V. Proskurnikov, and F. Bullo. Structural balance via gradient flows over signed graphs.
   September 2019.
   URL: https://arxiv.org/pdf/1909.11281.pdf
3. **Parsimony in opinion dynamics over signed graphs**
   P. Cisneros-Velarde, K. S. Chan, and F. Bullo. Polarization and fluctuations in signed social networks.
   February 2019.
   URL: https://arxiv.org/pdf/1902.00658.pdf

Opinions over signed graphs

- How do opinions evolve as a function of interpersonal relationships?
- What are the implications of friendly and antagonistic relationships?
- We propose: new, simple and intuitive model that incorporates the **boomerang effect** on opinion dynamics.

The boomerang effect

- The *boomerang effect* has been studied in social psychology [1].
  - Why do two individuals who engage in communication end up with their attitudes more diverse instead of more agreeable?
  - Possible explanation [2]: because of *"the relative distance between subjects' attitudes and position of communication"*.
- Based on studies on interpersonal attraction [3], our model assumes two friendly agents will be closer in their attitudes and perspectives than two unfriendly agents.

[1] [R. Abelson and J. C. Miller, 1967; S. Byrne and P. Solomon Hart, 2009; A. Cohen, 1962]
[2] [C. I. Hovland, et al., 1957]
[3] [In N. J. Smelser and P. B. Baltes, 2001]

Polarization in the Altafini model

Definition (Gossip Altafini model)

1. G is signed graph with edges $\mathcal{E}_+ \cup \mathcal{E}_-$
2. each agent $x_i(0) \in [-1, +1]$ and self-weight $w_i \in (0, 1)$
3. at each discrete time, positive probability to select $\{i, j\}$
4. update the opinions of i (and j) according to:

$$x_i^+ = \begin{cases} w_i x_i + (1 - w_i) x_j & \text{if } \{i, j\} \in \mathcal{E}_+ \\ w_i x_i + (1 - w_i)(-x_j) & \text{if } \{i, j\} \in \mathcal{E}_- \end{cases} \tag{6}$$

C. Altafini. Consensus problems on networks with antagonistic interactions.
IEEE Transactions on Automatic Control, 58(4):935–946, 2013

G. Shi, M. Johansson, and K. H. Johansson. How agreement and disagreement evolve over random dynamic networks.
IEEE Journal of Selected Areas in Communication, 31(6):1061–1071, 2013

W. Xia, M. Cao, and K. H. Johansson. Structural balance and opinion separation in trust–mistrust social networks.
IEEE Transactions on Control of Network Systems, 3(1):46–56, 2015

The affine boomerang model

Definition (Affine boomerang/repelling model)

1. G is signed graph with edges $\mathcal{E}_+ \cup \mathcal{E}_-$
2. each agent $x_i(0) \in [-1, +1]$ and self-weight $w_i \in (0, 1)$
3. at each discrete time, positive probability to select $\{i, j\}$
4. update the opinions of i (and j) according to:

$$x_i^+ = \begin{cases} w_i x_i + (1 - w_i) x_j & \text{if } \{i, j\} \in \mathcal{E}_+ \\ w_i x_i + (1 - w_i)\operatorname{sign}^*(x_i - x_j) & \text{if } \{i, j\} \in \mathcal{E}_- \end{cases} \tag{7}$$

where $\operatorname{sign}^*(0) = +1$.

Note: convex averaging between x_i and $\operatorname{sign}^*(x_i - x_j)$, instead of $-x_j$
Note: repelling effect

Consensus and polarization in signed graphs

For balanced graph with k factions, affine boomerang model:

1. Consensus: if $k = 1$, then a.s. $\lim_{t\to\infty} x(t) = c\mathbb{1}_n$
2. Polarization: if $k = 2$, then a.s.

$$\begin{aligned} &\lim_{t\to\infty} x_i(t) = -1 && \text{for each } i \text{ in one faction, and} \\ &\lim_{t\to\infty} x_j(t) = +1 && \text{for each } j \text{ in other faction} \end{aligned}$$

Gossip Altafini predicts same consensus, polarization properties, and

$$\lim_{t\to\infty} x(t) = 0, \qquad \text{if signed graph is not structurally balanced}$$

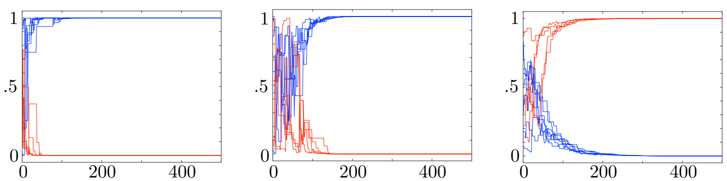


Figure: Polarization in structurally balanced graph, self-weights: 0.25, 0.50, 0.5.

Affine boomerang model leads to polarization in "almost structurally balanced" graphs

Numerical evidence for polarization also for "almost structurally balanced"

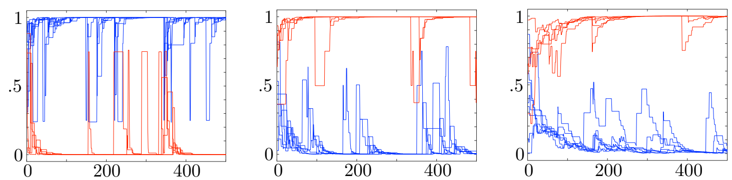


Figure: 8 agents, organized in 2 factions $(4 + 4)$, but with 3 edges violating structural balance. Self-weights: 0.25, 0.50, 0.75.

Affine boomerang model exhibits polarization+fluctuations in signed graphs with clustering balance

signed graph with clustering balance with factions $\{F_1, \dots, F_k\}$, $k \geq 3$ (assume at least one negative edge between any pair of factions)

assume polarization of two factions:

$x_i(0) = -1$ for $i \in F_1$ and $x_i(0) = +1$ for $i \in F_2$

let j be in F_k, $k \geq 3$. If $x_j(0) \in (-1, +1)$, then, for $0 < \epsilon < 1$,

$$\mathbb{P}[x_j(t) \in (-1, -1+\epsilon) \cup (+1-\epsilon, +1) \text{ i.o.}] = 1$$

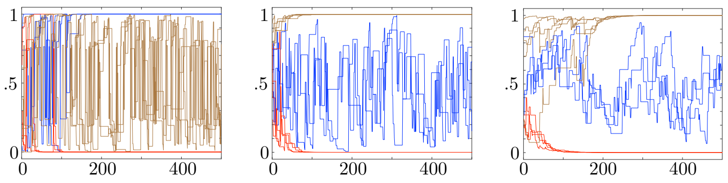


Figure: Polarized inits, clustering balance, self-weights: 0.25, 0.50, 0.75.

Outline

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   URL: https://arxiv.org/pdf/1902.00658.pdf

Conclusions

1. 3 new models for opinion dynamics and balance dynamics
2. fewer assumptions, same behavior, or
   same assumptions, richer behavior

"The easiest reading is damned hard writing," Thomas Hood

Future research

- publish three papers
- further model development and lots of conjectures
- human subject protocols

"Prediction is very hard, especially about the future" Yogi Berra