On the Dynamics of Opinions and Influence Systems

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Workshop on Distributed Control and Multi-Agent Systems
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New text “Lectures on Network Systems”

For students: free PDF for download
For instructors: slides and answer keys
http://motion.me.ucsb.edu/book-lns
https://www.amazon.com/dp/1986425649
300 pages (plus 200 pages solution manual)
3K downloads since Jun 2016
150 exercises with solutions

Linear Systems:
- social, sensor, robotic & compartmental examples,
- matrix and graph theory, with an emphasis on
  Perron–Frobenius theory and algebraic graph theory,
- averaging algorithms in discrete and continuous time,
  described by static and time-varying matrices, and
- positive & compartmental systems, dynamical flow
  systems, Metzler matrices.

Nonlinear Systems:
- nonlinear consensus models,
- population dynamic models in multi-species systems,
- coupled oscillators, with an emphasis on the
  Kuramoto model and models of power networks

Educational introduction to network systems

What are fundamental dynamic phenomena over networks?

Examples drawn from:
- social networks
- Markov chains
- epidemic propagation
- population dynamic models
- evolutionary game theory
- parallel computing
- dynamical flow systems: transmission and traffic networks
- coupled oscillators
- multi-agent coordination
- network science
Dynamics and learning in social systems

**Dynamic phenomena on dynamic social networks**

1. **dynamics**: opinion formation, but also information propagation, task execution, strategic network formation
2. interpersonal network structures: influence systems, but also appraisal systems, transactive memory systems and other group psychological constructs

**Questions on collective intelligence and rationality:**

- wisdom of crowds vs. group think
- influence centrality (democracy versus autocracy)

**Selected literature on math sociology and systems/control**

  ISBN 0691148201
  ISBN 0521195330

Exploding literature on social networks from sociology, physics, CS/engineering

**Selected literature on opinion dynamics**


Characterization of average consensus, 15 years before DeGroot

**Theorem 14.** A strong group attains unanimity at the arithmetic mean of the initial opinions if and only if its matrix $M$ is doubly stochastic.

Outline

1 Influence systems: basic models and statistical results on empirical data

doi:10.15195/v3.a20

doi:10.1073/pnas.1710603114

Influence systems: the mathematics of social power

Opinion dynamics and social power along sequences

Deliberative groups in social organization
- government: juries, panels, committees
- corporations: board of directors
- universities: faculty meetings

Natural social processes along sequences
- opinion dynamics for single issue?
- levels of openness and closure along sequence?
- influence accorded to others? emergence of leaders?

Postulated mechanisms for opinion dynamics 1/2

French-DeGroot averaging model

\[ y_{i}^{+} := \text{average}(y, \{y_j, j \text{ is neighbor of } i\}) \]

\[ y(k + 1) = Ay(k) \]

where \( A \) is nonnegative and row-stochastic

Consensus under mild connectivity assumptions:

\[ \lim_{k \to \infty} y(k) = (c^T y(0)) 1_n \]

self-weight = level of closure: \( a_{ii} \) diagonal entries of influence matrix

social power: \( c_i \) entries of dominant left eigenvector

Postulated mechanisms for opinion dynamics 2/2

Averaging (French-DeGroot model)

\[ y(k + 1) = Ay(k) \quad \lim_{k \to \infty} y(k) = (c^T y(0)) 1_n \]

Averaging + attachment to initial opinion (F-J model)

\[ y(k + 1) = (I_n - \Lambda)Ay(k) + \Lambda y(0), \quad \Lambda = \text{diag}(A) \]

Convergence under mild connectivity+stubborness assumptions:

\[ \lim_{k \to \infty} y(k) = V \cdot y(0), \quad \text{for } V = (I_n - (I_n - \Lambda)^{-1} \Lambda) \]

\[ c = V^T 1_n / n = \text{average contribution of each agent} \]

self-weight = level of closure: \( a_{ii} \) diagonal entries of influence matrix

social power: \( c_i \) entries of centrality vector
Today we skip these proofs

Analysis of French-DeGroot and F-J models well-understood:
- Jordan normal form
- Perron-Frobenius theory
- Algebraic graph theory (connectivity, periodicity, etc)

Experiments on opinion formation and influence networks
domains: risk/reward choice, analytical reliability, resource allocation

- 30 groups of 4 subjects in a face-to-face discussion
- Sequence of 15 issues
- Each issue is risk/reward choice:
  - What is your minimum level of confidence (scored 0-100) required to accept a risky option with a high payoff rather than a less risky option with a low payoff?
  - E.g.: medical, financial, professional, etc
- “please, reach consensus” pressure
- On each issue, each subject recorded (privately/chronologically):
  1. An initial opinion prior to the group discussion,
  2. A final opinion after the group-discussion (3-27 mins),
  3. An allocation of “100 influence units”
     ("these allocations represent your appraisal of the relative influence of each group member’s opinion on yours").

(1/3) Prediction of individual final opinions

Balanced random-intercept multilevel longitudinal regression

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<tr>
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<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>F-J prediction</td>
<td>0.897***</td>
<td>1.157***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.018)</td>
<td>(0.032)</td>
<td></td>
</tr>
<tr>
<td>Initial opinions</td>
<td>-0.282***</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.031)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>-8579.835</td>
<td>-7329.003</td>
<td>-7241.097</td>
</tr>
</tbody>
</table>

Standard errors are in parentheses; ** $p \leq 0.01$, *** $p \leq 0.001$; maximum likelihood estimation with robust standard errors; $n = 1,800$.

FJ averaging model is predictive for risk/reward choice issues

Extensions to: intellective and resource allocation issues

Risk/reward choice

Intellective issue = Problem solving

Two medical teams are working independently to achieve a cure for a disease. Team A succeeds if problems $A_1$ and $A_2$ with $P[A_1] = 0.60$ and $P[A_2] = 0.45$.

Team B succeeds if problems $B_1$, $B_2$, and $B_3$, with $P[B_1] = 0.80$, $P[B_2] = 0.85$, $P[B_3] = 0.95$.

What is your estimate of the probability that the disease will be cured?

Multidimensional resource allocation


What do you recommend as min and max percent of food consumption in terms of (1) Fruits or Vegetables, (2) Grains, and (3) Meats?

What are your ideal percentages in your preferred min/max ranges?
Opinion averaging models are predictive

From Wikipedia

1. Reflected appraisal = a person’s perception of how others see and evaluate him or her.
2. This process has been deemed important to the development of a person’s self-esteem, because it includes interaction with people outside oneself.
3. The reflected appraisal process concludes that people come to think of themselves in the way they believe others think of them.

Reflected appraisal process (Cooley 1902 and Friedkin 2011)

Along issues $s = 1, 2, \ldots$, individual dampens/elevates self-weight according to prior influence centrality

self-weights := relative control on prior issues = social power

Opinion dynamics along sequences
Postulated mechanism for network evolution

From Sociological Sciences 2016

doi:10.1073/pnas.1710603114

Empirical evidence that (1) FJ model substantially clarifies how truth wins in groups engaged in sequences of intellective issues (2) learning and reflected appraisal take place

Submitted, November 2017.
Submitted

Empirical evidence that (1) FJ model provides quantitative mechanistic explanation for uncertain multi-objective decision making problem and (2) FJ provides detailed explanation for group satisficing solutions

(3/3) Prediction of cumulative influence centrality

Balanced random-intercept multilevel longitudinal regression

individual’s “closure to influence” as predicted by:

- individual’s prior centrality $c_i(s)$
- individual’s time-averaged centrality $\bar{c}_i(s) = \frac{1}{s} \sum_{t=1}^{s} c_i(t)$

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<tbody>
<tr>
<td>$c_i(s)$</td>
<td></td>
<td>0.336***</td>
<td></td>
</tr>
<tr>
<td>$\bar{c}_i(s)$</td>
<td></td>
<td></td>
<td>0.404**</td>
</tr>
<tr>
<td>$s$</td>
<td></td>
<td>0.002</td>
<td>-0.018***</td>
</tr>
<tr>
<td>$s \times c_i(s)$</td>
<td></td>
<td>0.171</td>
<td></td>
</tr>
<tr>
<td>$s \times \bar{c}_i(s)$</td>
<td></td>
<td></td>
<td>0.095***</td>
</tr>
<tr>
<td>log likelihood</td>
<td>-367.331</td>
<td>-327.051</td>
<td>-293.656</td>
</tr>
</tbody>
</table>

prior and cumulative prior centrality predicts individual closure

individuals accumulate influence centralities at different rates, and their time-average centrality stabilizes to constant values
Influence systems: statistical results on empirical data


Opinion dynamics and social power along issue sequences

Reflected appraisal phenomenon (Cooley 1902 and Friedkin 2011) along issues $s = 1, 2, \ldots$, individual dampens/elevates self-weight according to prior influence centrality

self-weights $\rightarrow$ relative control on prior issues = social power

French-DeGroot averaging model

$$y(k + 1) = Ay(k)$$

Consensus under mild assumptions:

$$\lim_{k \to \infty} y(k) = (v_{\text{left}}(A) : y(0))1_n$$

where $v_{\text{left}}(A)$ is social power

- $A_{ij} =: x_i$ are self-weights / self-appraisal = level of closure
- let $W_{ij}$ be relative interpersonal accorded weights

\[
A(x) = \text{diag}(x) + \text{diag}(1_n - x)W
\]

- $v_{\text{left}}(W) = (w_1, \ldots, w_n)$ = dominant eigenvector for $W$

Dynamics of the influence network

Existence and stability of equilibria?
Role of network structure and parameters?
Emergence of autocracy and democracy?

Theorem: For strongly connected $W$ and non-trivial initial conditions

1. unique fixed point $x^* = x^*(w_1, \ldots, w_n)$
2. convergence $\Rightarrow$ forgets initial condition

$$\lim_{s \to \infty} x(s) = \lim_{s \to \infty} v_{\text{left}}(A(x(s))) = x^*$$

3. accumulation of social power and self-appraisal

- fixed point $x^*$ has same ordering of $(w_1, \ldots, w_n)$
- $x^*$ is an extreme version of $(w_1, \ldots, w_n)$
Emergence of democracy

If $W$ is doubly-stochastic:
1. The non-trivial fixed point is $\frac{1}{n}$
2. $\lim_{s \to \infty} x(s) = \lim_{s \to \infty} v_{left}(A(x(s))) = \frac{1}{n}$

- Uniform social power
- No power accumulation = evolution to democracy

Emergence of autocracy

If $W$ has star topology with center $j$:
1. There are no non-trivial fixed points
2. $\lim_{s \to \infty} x(s) = \lim_{s \to \infty} v_{left}(A(x(s))) = e_j$

- Autocrat appears in center node of star topology
- Extreme power accumulation = evolution to autocracy

Analysis methods

1. Existence of $x^*$ via Brower fixed point theorem
2. Monotonicity: $i_{\text{max}}$ and $i_{\text{min}}$ are forward-invariant
$$i_{\text{max}} = \arg \max_j \frac{x_j(0)}{x_j^*}$$
$$\Rightarrow i_{\text{max}} = \arg \max_j \frac{x_j(s)}{x_j^*}, \text{ for all subsequent } s$$

3. Convergence via variation on classic "max-min" Lyapunov function:
$$V(x) = \max_j \left( \ln \frac{x_j}{x_j^*} \right) - \min_j \left( \ln \frac{x_j}{x_j^*} \right) \text{ strictly decreasing for } x \neq x^*$$

Reducible interpersonal networks

- $W$ reducible
- Two cases: single sink and multiple sinks in condensation
- Generalized analysis with similar and related results
Stochastic models with cumulative memory

1. Assume noisy interpersonal weights $W(s) = W_0 + N(s)$
2. Assume noisy perception of social power $x(s+1) = v_{left}(A(x(s))) + n(s)$

**Thm:** practical stability of $x^*$

- Assume self-weight := cumulative average of prior social power
- $x(s+1) = (1 - \alpha(s))x(s) + \alpha(s)\left(v_{left}(A(x(s))) + n(s)\right)$

**Thm:** a.s. convergence to $x^*$ (under technical conditions)

Recent extensions on social power evolution


Summary

**New perspective on influence networks and social power**
- Designed/executed/analyzed experiments on group discussions
- Proposed/analyzed/validated dynamical models with feedback
- Novel mechanism for power accumulation / emergence of autocracy

**Open directions**
- Robustness to modelling assumptions
- Dynamics of interpersonal appraisals
- Larger-scale online experiments
- Intervention strategies for optimal group discussions

*No one speaks twice, until everyone speaks once*  
*Robert’s Rules of Order & parliamentary procedures*