Network Systems and Kuramoto Oscillators

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NSF AFOSR ARO ONR DOE

Outline

1. Intro to Network Systems and Power Flow

2. **Known tests and a conjecture**
   F. Dörfler, M. Chertkov, and F. Bullo. Synchronization in complex oscillator networks and smart grids.
   Proc National Academy of Sciences, 110(6):2005–2010, 2013.
   doi:10.1073/pnas.1212134110

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   S. Jafarpour, E. Huang, and F. Bullo. Synchronization of coupled oscillators: The Taylor expansion of the inverse Kuramoto map.
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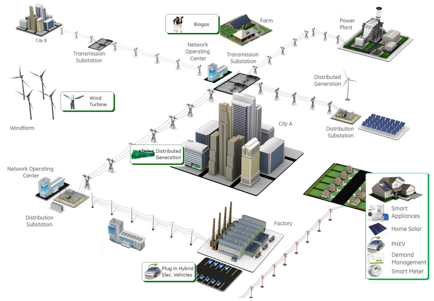
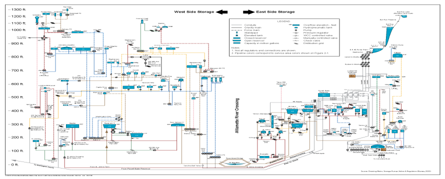
Example network systems



Smart grid

Amazon robotic warehouse

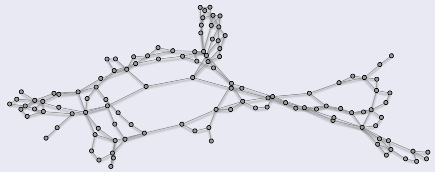


Portland water network

Industrial chemical plant

Linear network systems

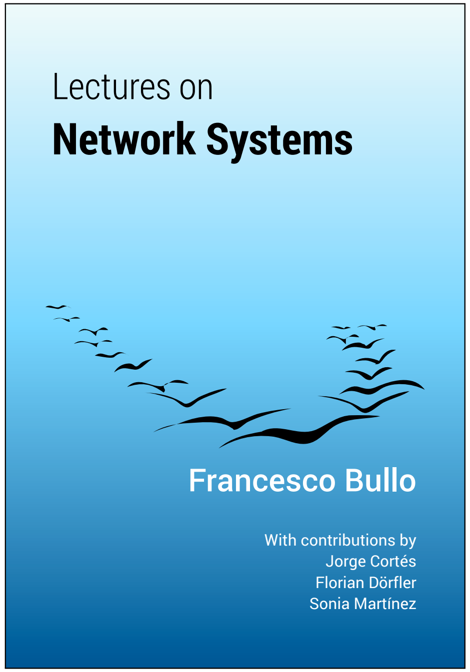
$$x(k+1) = Ax(k) + b \qquad \text{or} \qquad \dot{x}(t) = Ax(t) + b$$



1. systems of interest
2. asymptotic behavior
3. tools

network structure $\iff$ function = asymptotic behavior

New text "Lectures on Network Systems"



Lectures on Network Systems, F. Bullo, 1 edition, CreateSpace, ISBN 978-1-986425-64-3

For students: free PDF for download
For instructors: slides and answer keys
`http://motion.me.ucsb.edu/book-lns`
`https://www.amazon.com/dp/1986425649`
300 pages (plus 200 pages solution manual)
3K downloads since Jun 2016
150 exercises with solutions

Linear Systems:

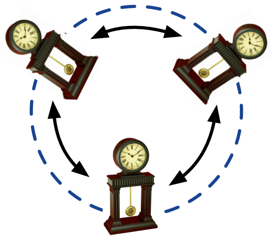
1. social, sensor, robotic & compartmental examples,
2. matrix and graph theory, with an emphasis on Perron–Frobenius theory and algebraic graph theory,
3. averaging algorithms in discrete and continuous time, described by static and time-varying matrices, and
4. positive & compartmental systems, dynamical flow systems, Metzler matrices.

Nonlinear Systems:

5. nonlinear consensus models,
6. population dynamic models in multi-species systems,
7. coupled oscillators, with an emphasis on the Kuramoto model and models of power networks

Synchronization in Networks of Coupled Oscillators

Pendulum clocks & *"an odd kind of sympathy"*
[C. Huygens, Horologium Oscillatorium, 1673]



Today's canonical coupled oscillator model
[A. Winfree '67, Y. Kuramoto '75]

Coupled oscillator model:

$$\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

- n **oscillators** with phase $\theta_i \in \mathbb{S}^1$
- **non-identical** natural frequencies $\omega_i \in \mathbb{R}^1$
- **coupling** with strength $a_{ij} = a_{ji}$

Synchronization in Networks of Coupled Oscillators

Coupled oscillator model:

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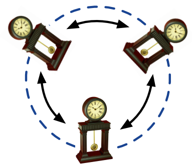
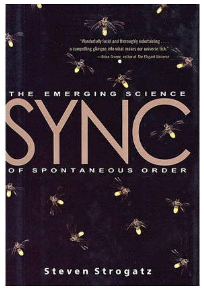


- Sync in Josephson junctions
  [S. Watanabe et. al '97, K. Wiesenfeld et al. '98]
- Sync in a population of fireflies
  [G.B. Ermentrout '90, Y. Zhou et al. '06]
- Canonical model of coupled limit-cycle oscillators
  [F.C. Hoppensteadt et al. '97, E. Brown et al. '04]
- Countless sync phenomena in sciences/bio/tech.
  [S. Strogatz '00, J. Acebrón '01, Arenas '08]

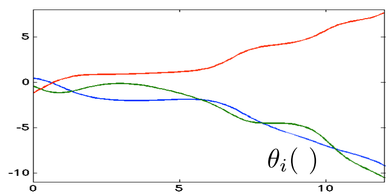


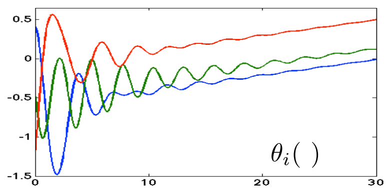
citations on scholar.google: Winfree 1.5K, Kuramoto 1.1K + 6.8K, surveys Strogatz, Acebron, Arenas: 2K, survey by Boccaletti 8K

Synchronization in Networks of Coupled Oscillators
phenomenology and challenges

Function = synchronization

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$





coupling small & $|\omega_i - \omega_j|$ large
$\Rightarrow$ incoherence = no sync

coupling large & $|\omega_i - \omega_j|$ small
$\Rightarrow$ coherence = frequency sync

Central question:

[S. Strogatz '01, A. Arenas et al. '08, S. Boccaletti et al. '06]

- loss of sync due to bifurcation
- trade-off "coupling" vs. "heterogeneity"
- how to quantify this trade-off

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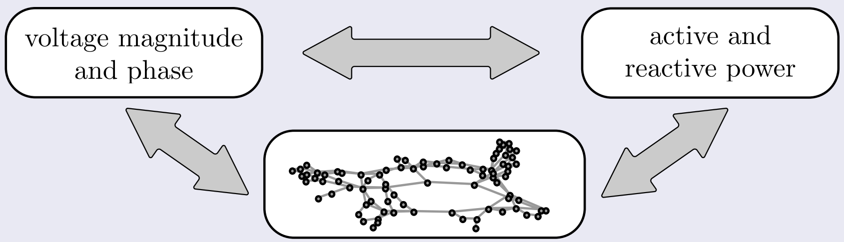
Power flow equations

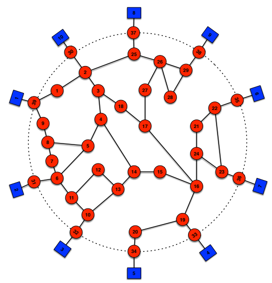


1. secure operating conditions
2. feedback control
3. economic optimization

network structure $\Longleftrightarrow$ function = power transmission

Power networks as quasi-synchronous AC circuits

1. **generators** ■ and **loads** ●
2. **physics:** Kirchoff and Ohm laws
3. today's simplifying assumptions:
   1. **quasi-sync:** voltage and phase V_i, θ_i
      active and reactive power p_i, q_i
   2. lossless lines
   3. approximated decoupled equations



Decoupled power flow equations

$$\text{active:} \quad p_i = \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

$$\text{reactive:} \quad q_i = -\sum_j b_{ij} V_i V_j$$

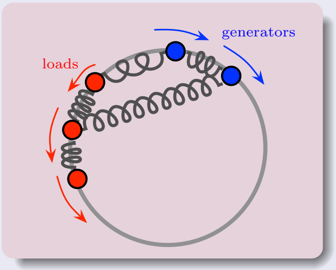
Power Flow Equilibria

$$p_i = \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

As function of network structure/parameters

1. do equations admit solutions / operating points?
2. how much active power can network transmit / flow?
3. how to quantify stability margins?

Active power dynamics and mechanical/spring analogy



Coupled swing equations

$$m_i \ddot{\theta}_i + d_i \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

Kuramoto coupled oscillators

$$\dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

Synchronization in Power Networks

Sync is crucial for the functionality and operation of the AC power grid. Generators have to swing in sync despite fluctuations/faults/contingencies.

Given: network parameters & topology and load & generation profile

Q: "∃ an optimal, stable, and robust synchronous operating point ?"

1. Security analysis [Araposthatis et al. '81, Wu et al. '80 & '82, Ilić '92, ...]
2. Load flow feasibility [Chiang et al. '90, Dobson '92, Lesieutre et al. '99, ...]
3. Optimal generation dispatch [Lavaei et al. '12, Bose et al. '12, ...]
4. Transient stability [Sastry et al. '80, Bergen et al. '81, Hill et al. '86, ...]
5. Inverters in microgrids [Chandorkar et. al. '93, Guerrero et al. '09, Zhong '11, ...]

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Primer on algebraic graph theory

Weighted undirected graph with n nodes and m edges:

Incidence matrix: $n \times m$ matrix B s.t. $(B^\top p_{\text{active}})_{(ij)} = p_i - p_j$

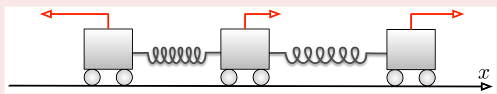
Weight matrix: $m \times m$ diagonal matrix $\mathcal{A}$

Laplacian stiffness: $L = B\mathcal{A}B^\top \geq 0$

Kuramoto eq points: $p_{\text{active}} = B\mathcal{A}\sin(B^\top \theta)$

Algebraic connectivity: $\lambda_2(L)$ = second smallest eig of L

Linear spring networks and $L^\dagger$

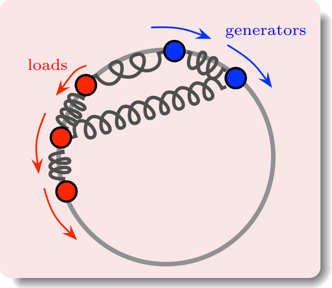


balance equation : $f = Lx$

for balanced $f \perp \mathbb{1}_n$: $\mathbf{x} = \mathbf{L}^\dagger \mathbf{f}$

DC flow approx: $p_{\text{active}} \approx B\mathcal{A}(B^\top \theta) \implies B^\top \theta = B^\top L^\dagger p_{\text{active}}$

Known tests



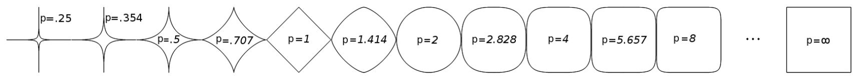
Given balanced p_{active}, do angles exist?

$$p_{\text{active}} = B\mathcal{A}\sin(B^\top\theta)$$

synchronization arises if
power transmission < connectivity strength

Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$$\|B^\top p_{\text{active}}\|_2 < \lambda_2(L) \quad \text{for unweighted graphs} \quad \text{(Old 2-norm T)}$$
$$\|B^\top L^\dagger p_{\text{active}}\|_\infty < 1 \quad \text{for trees, complete} \quad \text{(Old } \infty\text{-norm T)}$$



at fixed radius and 2-norm, volume of ball $\to 0^+$ as $d \to +\infty$

A standing conjecture

IEEE test cases: randomized and increase generation/loads
$\|B^\top L^\dagger p_{\text{active}}\|_\infty < 1$ appears to imply:

1. $\exists$ solution θ^*
2. $|\theta_i^* - \theta_j^*| \leq \arcsin(\|B^\top L^\dagger p_{\text{active}}\|_\infty)$ for all $\{i,j\} \in \mathcal{E}$

Randomized test case (1000 instances)	Numerical worst-case angle differences: $\max_{\{i,j\}\in\mathcal{E}} \lvert\theta_i^* - \theta_j^*\rvert$	Analytic prediction of angle differences: $\arcsin(\|B^\top L^\dagger p_{\text{active}}\|_\infty)$	Accuracy of condition: $\max_{\{i,j\}\in\mathcal{E}} \lvert\theta_i^* - \theta_j^*\rvert - \arcsin(\|B^\top L^\dagger p_{\text{active}}\|_\infty)$
9 bus system	0.12889 rad	0.12885 rad	$4.1218 \cdot 10^{-5}$ rad
IEEE 14 bus system	0.16622 rad	0.16594 rad	$2.7995 \cdot 10^{-4}$ rad
IEEE 30 bus system	0.1643 rad	0.16404 rad	$2.6140 \cdot 10^{-4}$ rad
New England 39	0.16821 rad	0.16815 rad	$6.6355 \cdot 10^{-5}$ rad
IEEE 57 bus system	0.20295 rad	0.18232 rad	$2.0630 \cdot 10^{-2}$ rad
IEEE 118 bus system	0.23524 rad	0.23464 rad	$5.9959 \cdot 10^{-4}$ rad
IEEE 300 bus system	0.43204 rad	0.43151 rad	$5.2618 \cdot 10^{-4}$ rad
Polish 2383 bus system	0.25144 rad	0.24723 rad	$4.2183 \cdot 10^{-3}$ rad

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Novel today

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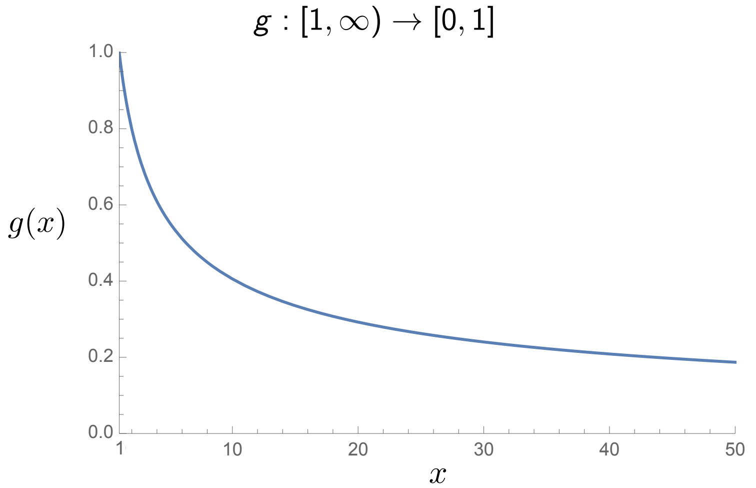
Equilibrium angles (neighbors within $\pi/2$ arc) exist if

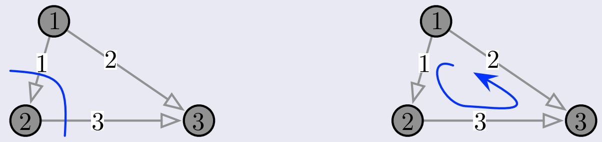
$$\|B^\top L^\dagger p_{\text{active}}\|_2 < 1 \quad \text{for unweighted graphs} \quad \text{(New 2-norm T)}$$
$$\|B^\top L^\dagger p_{\text{active}}\|_\infty < g(\|\mathcal{P}\|_\infty) \quad \text{for all graphs} \quad \text{(New } \infty\text{-norm T)}$$

where g is monotonically decreasing



$$g(x) = \frac{y(x) + \sin(y(x))}{2} - x\,\frac{y(x) - \sin(y(x))}{2}\Big|_{y(x) = \arccos(\frac{x-1}{x+1})}$$

and where $\mathcal{P}$ is a projection



$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\text{Im}(B^\top)}_{\substack{\text{cutset space}\\ \text{flow vectors}}} \oplus \underbrace{\text{Ker}(B\mathcal{A})}_{\substack{\text{weighted cycle space}\\ \text{cycle vectors}}}$$

$\mathcal{P} = B^\top L^\dagger B\mathcal{A}$ = oblique projection onto $\text{Im}(B^\top)$ parallel to $\text{Ker}(B\mathcal{A})$

(recall: orthogonal projector onto $\text{Im}(C)$ is $C(C^\top C)^{-1}C^\top$ for full rank C)

1. if G unweighted, then $\mathcal{P}$ is orthogonal and $\|\mathcal{P}\|_2 = 1$
2. if G acyclic, then $\mathcal{P} = I_m$ and $\|\mathcal{P}\|_p = 1$
3. if G uniform complete or ring, then $\|\mathcal{P}\|_\infty = 2(n-1)/n \leq 2$

Unifying Theorem

Equilibrium angles (neighbors within γ arc) exist if, in some p-norm,

$\|B^\top L^\dagger p_{\text{active}}\|_p \leq \gamma\alpha_p(\gamma)$ for all graphs (New p-norm T)

$\alpha_p(\gamma) :=$ min amplification factor of $\mathcal{P}\,\text{diag}[\text{sinc}(x)]$

For unweighted $p = 2$, new test sharper than old

$\|B^\top L^\dagger p_{\text{active}}\|_2 \leq \sin(\gamma)$ (New 2-norm T)

For $p = \infty$, new test is for all graphs

$\|B^\top L^\dagger p_{\text{active}}\|_\infty \leq g(\|\mathcal{P}\|_\infty)$ (New ∞-norm T)

Comparison of sufficient and approximate sync tests

K_c = critical coupling of Kuramoto model, computed via MATLAB *fsolve*
K_T = smallest value of scaling factor for which test T fails

Test Case	Critical ratio K_T/K_c				
	old 2-norm conjectured	new 2-norm conjectured	new ∞-norm	old ∞-norm approximate	α_∞ test *fmincon*
IEEE 9	16.54 %	59.06 %	73.74 %	92.13 %	85.06 %†
IEEE 14	8.33 %	42.27 %	59.42 %	83.09 %	81.32 %†
IEEE RTS 24	3.86 %	35.62 %	53.44 %	89.48 %	89.48 %†
IEEE 30	2.70 %	40.98 %	55.70 %	85.54 %	85.54 %†
IEEE 39	2.97 %	37.32 %	67.57 %	100 %	100 %†
IEEE 57	0.36 %	31.93 %	40.69 %	84.67 %	—*
IEEE 118	0.29 %	24.61 %	43.70 %	85.95 %	—*
IEEE 300	0.20 %	24.13 %	40.33 %	99.80 %	—*
Polish 2383	0.11 %	13.93 %	29.08 %	82.85 %	—*

† *fmincon* has been run for 100 randomized initial phase angles.
* *fmincon* does not converge.

Proof sketch 1/2: Rewriting the equilibrium equation

For what $B, \mathcal{A}, p_{\text{active}}$ does there exist θ solution to:

$$p_{\text{active}} = B\mathcal{A}\sin(B^\top\theta)$$

For what flow z and projection $\mathcal{P}$ onto cutset/flow space, does there exist a flow x that solves

$$\mathcal{P}\sin(x) = z$$

$$\iff \quad \mathcal{P}\,\text{diag}[\text{sinc}(x)]x = z$$

$$\iff \quad x = (\mathcal{P}\,\text{diag}[\text{sinc}(x)])^{-1}z =: h(x)$$

Proof sketch 2/2: Amplification factor & Brouwer

1. look for x solving

$$x = h(x) = (\mathcal{P}\,\text{diag}[\text{sinc}(x)])^{-1}z$$

2. take p norm, define min amplification factor of $\mathcal{P}\,\text{diag}[\text{sinc}(x)]$:

$$\alpha_p(\gamma) := \min_{\|x\|_p\leq\gamma}\ \min_{\|y\|_p=1} \|\mathcal{P}\,\text{diag}[\text{sinc}(x)]y\|_p$$

If $\|z\|_p \leq \gamma\alpha_p(\gamma)$ and $x \in \mathcal{B}_p(\gamma) = \{x \mid \|x\|_p \leq \gamma\}$, then

$$\|h(x)\|_p \leq \max_x \max_y \|(\mathcal{P}\,\text{diag}[\text{sinc}(x)])^{-1}y\|_p \cdot \|z\|_p$$

$$= \left(\min_x \min_y \|\mathcal{P}\,\text{diag}[\text{sinc}(x)]y\|_p\right)^{-1}\|z\|_p \quad \leq \frac{\|z\|_p}{\alpha_p(\gamma)} \leq \gamma$$

hence $h(x) \in \mathcal{B}_p(\gamma)$ and h satisfies Brouwer on $\mathcal{B}_p(\gamma)$

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Computational method via power series

Given z, compute x solution to

$$z = \mathcal{P}\sin(x)$$

Assume $x = \sum_{i=0}^{\infty} A_{2i+1}(z)$, where $A_{2i+1}(z)$ is homogeneous degree $2i+1$

$$z = \mathcal{P}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{\circ 2k+1} = \mathcal{P}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\Big(\sum_{i=0}^{\infty}A_{2i+1}(z)\Big)^{\circ 2k+1}$$

Equate left-hand and right-hand side at order $1, 3, \dots, 2j+1$:

$$A_1(z) = z$$

$$A_{2j+1}(z) = \mathcal{P}\left(\sum_{k=1}^{j}\frac{(-1)^{k+1}}{(2k+1)!}\sum_{\substack{\text{odd } \alpha_1,\dots,\alpha_{2k+1} \text{ s.t.}\\ \alpha_1+\dots+\alpha_{2k+1}=2j+1}} A_{\alpha_1}(z)\circ\dots\circ A_{\alpha_{2k+1}}(z)\right)$$

Step 3: Series expansion for inverse Kuramoto map

Unique solution to $z = \mathcal{P}\sin(x)$ is

$$x = \sum_{i=0}^{\infty} A_{2i+1}(z)$$

$$A_1(z) = z \qquad = B^\top L^\dagger p_{\text{active}}$$

$$A_3(z) = \mathcal{P}\left(\frac{1}{3!} z^{\circ 3}\right)$$

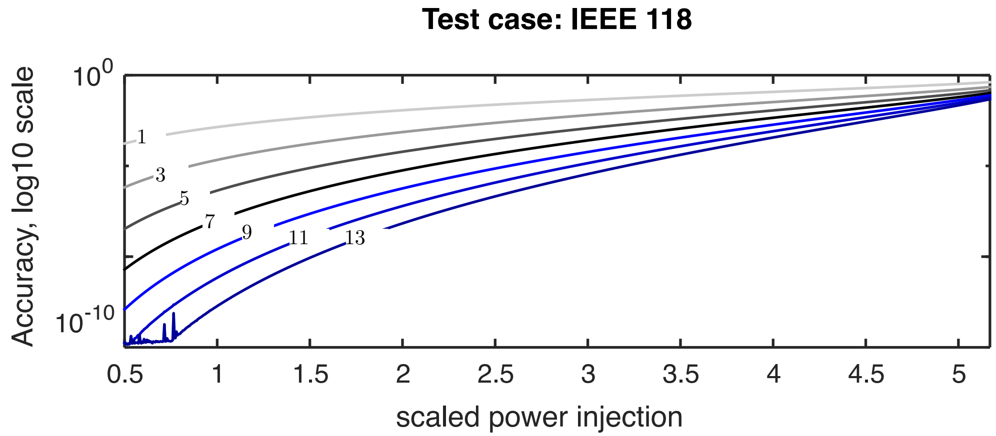
$$A_5(z) = \mathcal{P}\left(\frac{3}{3!} A_3(z) \circ z^{\circ 2} - \frac{1}{5!} z^{\circ 5}\right)$$

$$A_7(z) = \mathcal{P}\left(\frac{3}{3!} A_5(z) \circ z^{\circ 2} + \frac{3}{3!} A_3(z)^{\circ 2} \circ z - \frac{5}{5!} A_3(z) \circ z^{\circ 4} + \frac{1}{7!} z^{\circ 7}\right)$$

arbitrary higher-order terms can be computed symbolically

For sufficiently small $\|z\|_p$, series converges uniformly absolutely

Numerical examples



Kuramoto Oscillators and Power Flow

1. improved sufficient conditions for equilibria
2. upper bounds on transmission capacity
3. stability margins as notions of distance from bifurcations ¡

Applications

1. secure operating conditions: **(Dörfler et al, PNAS '13)**
2. feedback control: **(Simpson-Porco et al, TIE '15)**
3. economic optimization: **(Todescato et al, TCNS '17)**

Future research

1. close the gap between sufficient and necessary conditions
2. consider increasingly realistic power flow equations
3. apply methods to other flow networks (water, gas, ...)