Network Systems and Kuramoto Oscillators

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A kind invitation to participate in CSS activities

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Intro to Network Systems and Power Flow

Known tests and a conjecture

A new approach and new tests

Example network systems

Smart grid

Amazon robotic warehouse

Portland water network

Industrial chemical plant
Linear network systems

\[ x(k + 1) = Ax(k) + b \quad \text{or} \quad \dot{x}(t) = Ax(t) + b \]

network structure \iff function = asymptotic behavior

1. systems of interest
2. asymptotic behavior
3. tools
Lectures on Network Systems

Francesco Bullo

With contributions by
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These lecture notes provide a mathematical introduction to multi-agent dynamical systems, including their analysis via algebraic graph theory and their application to engineering design problems. The focus is on fundamental dynamical phenomena over interconnected network systems, including consensus and disagreement in averaging systems, stable equilibria in compartmental flow networks, and synchronization in coupled oscillators and networked control systems. The theoretical results are complemented by numerous examples arising from the analysis of physical and natural systems and from the design of network estimation, control, and optimization systems.

Francesco Bullo is professor of Mechanical Engineering and member of the Center for Control, Dynamical Systems, and Computation at the University of California at Santa Barbara. His research focuses on modeling, dynamics and control of multi-agent network systems, with applications to robotic coordination, energy systems, and social networks. He is an award-winning mentor and teacher.

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300 pages (plus 200 pages solution manual)
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150 exercises with solutions

Linear Systems:
1. social, sensor, robotic & compartmental examples,
2. matrix and graph theory, with an emphasis on Perron–Frobenius theory and algebraic graph theory,
3. averaging algorithms in discrete and continuous time, described by static and time-varying matrices, and
4. positive & compartmental systems, dynamical flow systems, Metzler matrices.

Nonlinear Systems:
5. nonlinear consensus models,
6. population dynamic models in multi-species systems,
7. coupled oscillators, with an emphasis on the Kuramoto model and models of power networks.
Pendulum clocks & “an odd kind of sympathy”
[C. Huygens, Horologium Oscillatorium, 1673]

Today’s canonical coupled oscillator model
[A. Winfree ’67, Y. Kuramoto ’75]

Coupled oscillator model:
\[
\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)
\]

- \textit{n oscillators} with phase \(\theta_i \in \mathbb{S}^1\)
- \textit{non-identical} natural frequencies \(\omega_i \in \mathbb{R}^1\)
- \textit{coupling} with strength \(a_{ij} = a_{ji}\)
Synchronization in Networks of Coupled Oscillators

Coupled oscillator model:

\[
\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)
\]

A few related applications:

- Sync in Josephson junctions
  [S. Watanabe et al. '97, K. Wiesenfeld et al. '98]

- Sync in a population of fireflies
  [G.B. Ermentrout '90, Y. Zhou et al. '06]

- Canonical model of coupled limit-cycle oscillators
  [F.C. Hoppensteadt et al. '97, E. Brown et al. '04]

- Countless sync phenomena in sciences/bio/tech.
  [A. Winfree '67, S.H. Strogatz '00, J. Acebrón '01]

citations on scholar.google:
  Kuramoto oscillators 1.4K, synchronization 3.2M
Synchronization in Networks of Coupled Oscillators
phenomenology and challenges

Function = synchronization

\[ \dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

Central question:

- loss of sync due to bifurcation
- trade-off “coupling” vs. “heterogeneity”
- how to quantify this trade-off

**Coupling small & \(|\omega_i - \omega_j|\) large**

⇒ incoherence = no sync

**Coupling large & \(|\omega_i - \omega_j|\) small**

⇒ coherence = frequency sync

[S. Strogatz '01, A. Arenas et al. '08, S. Boccaletti et al. '06]
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A new approach and new tests
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Power flow equations

1. secure operating conditions
2. feedback control
3. economic optimization

network structure $\iff$ function $=$ power transmission
Power networks as quasi-synchronous AC circuits

generators ■ and loads ◆

physics: Kirchoff and Ohm laws

today's simplifying assumptions:

quasi-sync: voltage and phase $V_i$, $\theta_i$
active and reactive power $p_i$, $q_i$

lossless lines

approximated decoupled equations

Decoupled power flow equations

active: $p_i = \sum_j a_{ij} \sin(\theta_i - \theta_j)$

reactive: $q_i = -\sum_j b_{ij} V_i V_j$
\[ p_i = \sum_j a_{ij} \sin(\theta_i - \theta_j) \]

**As function of network structure/parameters**

1. do equations admit solutions / operating points?
2. how much active power can network transmit / flow?
3. how to quantify stability margins?

**Active power dynamics and mechanical/spring analogy**

**Coupled swing equations**

\[ m_i \ddot{\theta}_i + d_i \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]

**Kuramoto coupled oscillators**

\[ \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]
Sync is crucial for the functionality and operation of the AC power grid. Generators have to swing in sync despite fluctuations/faults/contingencies.

**Given:** network parameters & topology and load & generation profile

**Q:** “∃ an optimal, stable, and robust synchronous operating point?”

1. Security analysis [Araposthatis et al. '81, Wu et al. '80 & '82, Ilić '92, ...]
2. Load flow feasibility [Chiang et al. '90, Dobson '92, Lesieutre et al. '99, ...]
3. Optimal generation dispatch [Lavaei et al. '12, Bose et al. '12, ...]
4. Transient stability [Sastry et al. '80, Bergen et al. '81, Hill et al. '86, ...]
5. Inverters in microgrids [Chandorkar et al. '93, Guerrero et al. '09, Zhong '11, ...]
Intro to Network Systems and Power Flow

1. Known tests and a conjecture

2. A new approach and new tests

Weighted undirected graph with $n$ nodes and $m$ edges:

**Incidence matrix:** $n \times m$ matrix $B$ s.t. $(B^\top p_{active})_{(ij)} = p_i - p_j$

**Weight matrix:** $m \times m$ diagonal matrix $A$

**Laplacian stiffness:** $L = BAB^\top$

**Kuramoto eq points:** $p_{active} = BA \sin(B^\top \theta)$

**Algebraic connectivity:** $\lambda_2(L) =$ second smallest eig of $L$

**Linear spring networks and** $L^\dagger$

\[ f_i = \sum a_{ij}(x_j - x_i) = -(Lv)_i \]

\[ x = L^\dagger f \quad \text{for balanced} \ f \perp 1_n \]
Known tests

Given balanced $p_{\text{active}}$, do angles exist?

$$p_{\text{active}} = BA \sin(B^T \theta)$$

synchronization arises if

**power transmission < connectivity strength**

Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$$\|B^T p_{\text{active}}\|_2 < \lambda_2(L) \quad \text{for unweighted graphs} \quad \text{(Old 2-norm T)}$$

$$\|B^T L^+ p_{\text{active}}\|_\infty < 1 \quad \text{for trees, complete} \quad \text{(Old } \infty\text{-norm T)}$$

at fixed radius and 2-norm, volume of ball $\to 0^+$ as $d \to +\infty$
A standing conjecture

$$\|B^\top L^\dagger p_{active}\|_\infty < 1$$ appears to imply:

1. \exists solution \(\theta^*\)
2. \(|\theta^*_i - \theta^*_j| \leq \arcsin(\|B^\top L^\dagger p_{active}\|_\infty)\) for all \(\{i, j\} \in \mathcal{E}\)

| Randomized test case (1000 instances) | Numerical worst-case angle differences: \(\max_{\{i,j\} \in \mathcal{E}} |\theta^*_i - \theta^*_j|\) | Analytic prediction of angle differences: \(\arcsin(\|B^\top L^\dagger p_{active}\|_\infty)\) | Accuracy of condition: \(\max_{\{i,j\} \in \mathcal{E}} |\theta^*_i - \theta^*_j| - \arcsin(\|B^\top L^\dagger p_{active}\|_\infty)\) |
|---------------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 9 bus system                          | 0.12889 rad                     | 0.12885 rad                     | 4.1218 \cdot 10^{-5} rad       |
| IEEE 14 bus system                    | 0.16622 rad                     | 0.16594 rad                     | 2.7995 \cdot 10^{-4} rad       |
| IEEE RTS 24                           | 0.22309 rad                     | 0.22139 rad                     | 1.7089 \cdot 10^{-3} rad       |
| IEEE 30 bus system                    | 0.1643 rad                      | 0.16404 rad                     | 2.6140 \cdot 10^{-4} rad       |
| New England 39                        | 0.16821 rad                     | 0.16815 rad                     | 6.6355 \cdot 10^{-5} rad       |
| IEEE 57 bus system                    | 0.20295 rad                     | 0.18232 rad                     | 2.0630 \cdot 10^{-2} rad       |
| IEEE RTS 96                           | 0.24593 rad                     | 0.245332 rad                    | 2.6076 \cdot 10^{-3} rad       |
| IEEE 118 bus system                   | 0.23524 rad                     | 0.23464 rad                     | 5.9959 \cdot 10^{-4} rad       |
| IEEE 300 bus system                   | 0.43204 rad                     | 0.43151 rad                     | 5.2618 \cdot 10^{-4} rad       |
| Polish 2383 bus system                | 0.25144 rad                     | 0.24723 rad                     | 4.2183 \cdot 10^{-3} rad       |

IEEE test cases: 50 % randomized loads and 33 % randomized generation
1 Intro to Network Systems and Power Flow

Known tests and a conjecture


A new approach and new tests


Equilibrium angles (neighbors within $\pi/2$ arc) exist if
\[
\| B^\top p_{active} \|_2 < \lambda_2(L) \quad \text{for unweighted graphs} \quad \text{(Old 2-norm T)}
\]
\[
\| B^\top L^\dagger p_{active} \|_\infty < 1 \quad \text{for trees, complete} \quad \text{(Old } \infty\text{-norm T)}
\]

Equilibrium angles (neighbors within $\pi/2$ arc) exist if
\[
\| B^\top L^\dagger p_{active} \|_2 < 1 \quad \text{for unweighted graphs} \quad \text{(New 2-norm T)}
\]
\[
\| B^\top L^\dagger p_{active} \|_\infty < g(\| P \|_\infty) \quad \text{for all graphs} \quad \text{(New } \infty\text{-norm T)}
\]
where \( g \) is monotonically decreasing

\[ g : [1, \infty) \to [0, 1] \]

\[
g(x) = \frac{y(x) + \sin(y(x))}{2} - x \frac{y(x) - \sin(y(x))}{2} \quad \text{where} \quad y(x) = \arccos\left(\frac{x-1}{x+1}\right)
\]
and where $\mathcal{P}$ is a projection

\[ \mathbb{R}^m \] edge space

\[
\begin{align*}
\text{Im}(B^\top) & \oplus \\
\text{cutset space} & \oplus \\
\text{Ker}(BA) & \oplus \\
\text{weighted cycle space}
\end{align*}
\]

\[ \mathcal{P} = B^\top L^\dagger BA \] = oblique projection onto $\text{Im}(B^\top)$ parallel to $\text{Ker}(BA)$

(recall: orthogonal projector onto $\text{Im}(C)$ is $C(C^\top C)^{-1}C^\top$ for full rank $C$)

1. if $G$ unweighted, then $\mathcal{P}$ is orthogonal and $\|\mathcal{P}\|_2 = 1$
2. if $G$ acyclic, then $\mathcal{P} = I_m$ and $\|\mathcal{P}\|_p = 1$
3. if $G$ uniform complete or ring, then $\|\mathcal{P}\|_\infty = 2(n - 1)/n \leq 2$
Equilibrium angles (neighbors within $\gamma$ arc) exist if, in some $p$-norm,

$$\|B^\top L^\dagger p_{active}\|_p \leq \gamma \alpha_p(\gamma) \quad \text{for all graphs} \quad \text{(New $p$-norm T)}$$

$$\alpha_p(\gamma) := \min \text{ amplification factor of } P[\text{sinc}(x)]$$

For unweighted $p = 2$, new test sharper than old

$$\|B^\top L^\dagger p_{active}\|_2 \leq \sin(\gamma) \quad \text{(New 2-norm T)}$$

For $p = \infty$, new test is for all graphs

$$\|B^\top L^\dagger p_{active}\|_\infty \leq g(\|P\|_\infty) \quad \text{(New $\infty$-norm T)}$$
Comparison of sufficient and approximate sync tests

\[ K_c = \text{critical coupling of Kuramoto model, computed via MATLAB} \ fsolve \]
\[ K_T = \text{smallest value of scaling factor for which test T fails} \]

<table>
<thead>
<tr>
<th>Test Case</th>
<th>Critical ratio ( K_T/K_c )</th>
<th>( \alpha_\infty ) test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>old 2-norm conjectured</td>
<td>new 2-norm conjectured</td>
</tr>
<tr>
<td>IEEE 9</td>
<td>16.54 %</td>
<td>59.06 %</td>
</tr>
<tr>
<td>IEEE 14</td>
<td>8.33 %</td>
<td>42.27 %</td>
</tr>
<tr>
<td>IEEE RTS 24</td>
<td>3.86 %</td>
<td>35.62 %</td>
</tr>
<tr>
<td>IEEE 30</td>
<td>2.70 %</td>
<td>40.98 %</td>
</tr>
<tr>
<td>IEEE 39</td>
<td>2.97 %</td>
<td>37.32 %</td>
</tr>
<tr>
<td>IEEE 57</td>
<td>0.36 %</td>
<td>31.93 %</td>
</tr>
<tr>
<td>IEEE 118</td>
<td>0.29 %</td>
<td>24.61 %</td>
</tr>
<tr>
<td>IEEE 300</td>
<td>0.20 %</td>
<td>24.13 %</td>
</tr>
<tr>
<td>Polish 2383</td>
<td>0.11 %</td>
<td>13.93 %</td>
</tr>
</tbody>
</table>

† \( \text{fmincon} \) has been run for 100 randomized initial phase angles.
* \( \text{fmincon} \) does not converge.
For what $B, A, p_{active}$ does there exist $\theta$ solution to:

$$p_{active} = B A \sin(B^\top \theta)$$

For what projection $P$ and flow $z$ in cutset space, does there exist $x$ in cutset space solution to:

$$z = P \sin(x) \iff z = P [\text{sinc}(x)] x$$

$$\iff x = (P [\text{sinc}(x)])^{-1} z =: h(x)$$
Proof sketch 2/2: Amplification factor & Brouwer fixed point thm

1. Look for \( x \) solving

\[
x = h(x) = (\mathcal{P}[^\text{sinc}(x)])^{-1}z
\]

2. Take \( p \) norm, define \( \text{min amplification factor of } \mathcal{P}[\text{sinc}(x)]: \)

\[
\alpha_p(\gamma) := \min_{\|x\|_p \leq \gamma} \min_{\|y\|_p = 1} \|\mathcal{P}[\text{sinc}(x)]y\|_p
\]

If \( \|z\|_p \leq \gamma \alpha_p(\gamma) \) and \( x \in \mathcal{B}_p(\gamma) = \{ x \mid \|x\|_p \leq \gamma \} \), then

\[
\|h(x)\|_p \leq \max_{x} \max_{y} \|\mathcal{P}[\text{sinc}(x)]^{-1}y\|_p \cdot \|z\|_p
\]

\[
\leq \frac{\|z\|_p}{\alpha_p(\gamma)} \leq \gamma
\]

Hence \( h(x) \in \mathcal{B}_p(\gamma) \) and \( h \) satisfies Brouwer on \( \mathcal{B}_p(\gamma) \)
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S. Jafarpour, E. Y. Huang, and FB. Synchronization of coupled oscillators: The Taylor expansion of the inverse Kuramoto map.
Computational method via power series

Given \( z \), compute \( x \) solution to

\[
z = \mathcal{P} \sin(x)
\]

Assume \( x = \sum_{i=0}^{\infty} A_{2i+1}(z) \), where \( A_{2i+1}(z) \) is homogeneous degree \( 2i + 1 \)

\[
z = \mathcal{P} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^\circ2k+1 = \mathcal{P} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left( \sum_{i=0}^{\infty} A_{2i+1}(z) \right)^\circ2k+1
\]

Equate left-hand and right-hand side at order 1, 3, \ldots, 2j + 1:

\[
A_1(z) = z
\]

\[
A_{2j+1}(z) = \mathcal{P} \left( \sum_{k=1}^{j} \frac{(-1)^{k+1}}{(2k+1)!} \sum \text{odd } \alpha_1, \ldots, \alpha_{2k+1} \text{ s.t. } \alpha_1 + \cdots + \alpha_{2k+1} = 2j+1 A_{\alpha_1}(z) \circ \cdots \circ A_{\alpha_{2k+1}}(z) \right)
\]
Step 3: Series expansion for inverse Kuramoto map

Unique solution to $z = \mathcal{P} \sin(x)$ is

$$x = \sum_{i=0}^{\infty} A_{2i+1}(z)$$

$A_1(z) = z$

$A_3(z) = \mathcal{P}\left(\frac{1}{3!} z^3\right)$

$A_5(z) = \mathcal{P}\left(\frac{3}{3!} A_3(z) \circ z^2 - \frac{1}{5!} z^5\right)$

$A_7(z) = \mathcal{P}\left(\frac{3}{3!} A_5(z) \circ z^2 + \frac{3}{3!} A_3(z)^2 \circ z - \frac{5}{5!} A_3(z) \circ z^4 + \frac{1}{7!} z^7\right)$

arbitrary higher-order terms can be computed symbolically

For sufficiently small $\|z\|_p$, series converges uniformly absolutely
Numerical examples

Test case: IEEE 118

Accuracy, log10 scale

scaled power injection

0.5 1 1.5 2 2.5 3 3.5 4 4.5 5

10^{-10}

10^{0}

1st

3rd

5th

7th

9th

11th

13th
Kuramoto Oscillators and Power Flow

New physical insight
1. sharp sufficient conditions for equilibria
   upper bounds on transmission capacity
   stability margins as notions of distance from bifurcations
2. new computational methods via power series

Applications
1. secure operating conditions: (Dörfler et al, PNAS ’13)
2. feedback control: (Simpson-Porco et al, TIE ’15)
3. economic optimization: (Todescato et al, TCNS ’17)

Future research
1. close the gap between sufficient and necessary conditions
2. more realistic coupled power flow equations
3. applications to other flow networks (water, gas ...)