Network Systems and Kuramoto Oscillators

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A kind invitation to participate in CSS activities

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Outline
1 Intro to Network Systems and Power Flow
2 Known tests and a conjecture
F. Dörfler, M. Chertkov, and FB. Synchronization in complex oscillator networks and smart grids.
doi:10.1073/pnas.1212134110

A new approach and new tests
S. Jafarpour and FB. Synchronization of Kuramoto oscillators via cutset projections.

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Example network systems
- Smart grid
- Amazon robotic warehouse
- Portland water network
- Industrial chemical plant

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systems of interest
asymptotic behavior
tools

\[ x(k + 1) = Ax(k) + b \quad \text{or} \quad \dot{x}(t) = Ax(t) + b \]

network structure \iff \text{function} = \text{asymptotic behavior}

Synchronization in Networks of Coupled Oscillators

Pendulum clocks & “an odd kind of sympathy”
[C. Huygens, Horologium Oscillatorum, 1673]

Today’s canonical coupled oscillator model
[A. Winfree ’67, Y. Kuramoto ’75]

Coupled oscillator model:
\[ \dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

- \text{n oscillators} with phase \( \theta_i \in S^1 \)
- \text{non-identical} natural frequencies \( \omega_i \in \mathbb{R}^1 \)
- \text{coupling} with strength \( a_{ij} = a_{ji} \)

A few related applications:
- Sync in Josephson junctions
- Sync in a population of fireflies
  [G.B. Ermentrout ’90, Y. Zhou et al. ’06]
- Canonical model of coupled limit-cycle oscillators
- Countless sync phenomena in sciences/bio/tech.
  [A. Winfree ’67, S.H. Strogatz ’00, J. Acebrón ’01]

citations on scholar.google:
Kuramoto oscillators 1.4K, synchronzation complex 2.2M
Synchronization in Networks of Coupled Oscillators
phenomenology and challenges

Function = synchronization

\[ \dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

- coupling small & \(|\omega_i - \omega_j| \) large \( \Rightarrow \) incoherence = no sync
- coupling large & \(|\omega_i - \omega_j| \) small \( \Rightarrow \) coherence = frequency sync

Central question:
[S. Strogatz '01, A. Arenas et al. '08, S. Boccaletti et al. '06]

- loss of sync due to bifurcation
- trade-off “coupling” vs. “heterogeneity”
- how to quantify this trade-off

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Power flow equations

voltage magnitude and phase
active and reactive power

1 secure operating conditions
2 feedback control
3 economic optimization

network structure \( \leftrightarrow \) function = power transmission

Power networks as quasi-synchronous AC circuits

1 generators ■ and loads ●
2 physics: Kirchoff and Ohm laws
3 today’s simplifying assumptions:
   • quasi-sync: voltage and phase \( V_i, \theta_i \)
     active and reactive power \( p_i, q_i \)
   • lossless lines
   • approximated decoupled equations

Decoupled power flow equations
active: \( p_i = \sum_j a_{ij} \sin(\theta_i - \theta_j) \)
reactive: \( q_i = -\sum_j b_{ij} V_i V_j \)
Power Flow Equilibria

\[ p_i = \sum_j a_{ij} \sin(\theta_i - \theta_j) \]

As function of network structure/parameters

- do equations admit solutions / operating points?
- how much active power can network transmit / flow?
- how to quantify stability margins?

Active power dynamics and mechanical/spring analogy

Coupled swing equations

\[ m_i \ddot{\theta}_i + d_i \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]

Kuramoto coupled oscillators

\[ \dot{\theta}_j = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]

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Synchronization in Power Networks

Sync is crucial for the functionality and operation of the AC power grid. Generators have to swing in sync despite fluctuations/faults/contingencies.

Given: network parameters & topology and load & generation profile

Q: “∃ an optimal, stable, and robust synchronous operating point?”

- Load flow feasibility [Chiang et al. ’90, Dobson ’92, Lesieutre et al. ’99, …]
- Optimal generation dispatch [Lavaei et al. ’12, Bose et al. ’12, …]
- Transient stability [Sastry et al. ’80, Bergen et al. ’81, Hill et al. ’86, …]
- Inverters in microgrids [Chandorkar et. al. ’93, Guerrero et al. ’09, Zhong ’11, …]

Primer on algebraic graph theory

Weighted undirected graph with \( n \) nodes and \( m \) edges:

- **Incidence matrix**: \( n \times m \) matrix \( B \) s.t.
  \[ (B^\top p_{\text{active}})(j) = p_i - p_j \]
- **Weight matrix**: \( m \times m \) diagonal matrix \( A \)
- **Laplacian stiffness**: \( L = B A B^\top \)

Kuramoto eq points: \( p_{\text{active}} = B A \sin(B^\top \theta) \)

Algebraic connectivity: \( \lambda_2(L) = \) second smallest eig of \( L \)

Linear spring networks and \( L^\dagger \)

\[ f_i = \sum_j a_{ij}(x_j - x_i) = -(Lx)_i \]

\[ x = L^\dagger f \quad \text{for balanced } f \perp 1_n \]
Given balanced $p_{\text{active}}$, do angles exist?

$p_{\text{active}} = B A \sin(B^T \theta)$

synchronization arises if

power transmission < connectivity strength

Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\|B^T p_{\text{active}}\|_2 < \lambda_2(L)$ for unweighted graphs (Old 2-norm T)

$\|B^T L^T p_{\text{active}}\|_\infty < 1$ for trees, complete (Old $\infty$-norm T)

at fixed radius and 2-norm, volume of ball $\to 0^+$ as $d \to +\infty$

Novel today

Known tests and conjectures

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A standing conjecture

$\|B^T L^T p_{\text{active}}\|_\infty < 1$ appears to imply:

1. $\exists$ solution $\theta^*$

2. $|\theta_i^* - \theta_j^*| \leq \arcsin(\|B^T L^T p_{\text{active}}\|_\infty)$ for all $(i, j) \in \mathcal{E}$

| Randomized test case (1000 instances) | Numerical worst-case angle differences: $\max_{(i,j) \in E} |\theta_i^* - \theta_j^*|$ | Analytic prediction of angle differences: $\arcsin(\|B^T L^T p_{\text{active}}\|_\infty)$ | Accuracy of condition: $\max_{(i,j) \in \mathcal{E}} |\theta_i^* - \theta_j^*| - \arcsin(\|B^T L^T p_{\text{active}}\|_\infty)$ |
|--------------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 9 bus system                         | 0.12889 rad                     | 0.12885 rad                     | 4.1218 · 10^{-5} rad           |
| IEEE 14 bus system                  | 0.16022 rad                     | 0.16594 rad                     | 2.7995 · 10^{-4} rad           |
| IEEE RTS 24                         | 0.22309 rad                     | 0.22139 rad                     | 1.7089 · 10^{-3} rad           |
| IEEE 30 bus system                  | 0.1643 rad                      | 0.16404 rad                     | 2.6140 · 10^{-4} rad           |
| New England 39                      | 0.16821 rad                     | 0.16815 rad                     | 6.6355 · 10^{-5} rad           |
| IEEE 57 bus system                  | 0.20295 rad                     | 0.19232 rad                     | 2.0630 · 10^{-2} rad           |
| IEEE RTS 96                         | 0.24593 rad                     | 0.24532 rad                     | 2.6076 · 10^{-3} rad           |
| IEEE 118 bus system                 | 0.23524 rad                     | 0.23464 rad                     | 5.9959 · 10^{-4} rad           |
| IEEE 300 bus system                 | 0.43204 rad                     | 0.43151 rad                     | 5.2618 · 10^{-4} rad           |
| Polish 2383 bus system              | 0.25144 rad                     | 0.24723 rad                     | 4.2183 · 10^{-3} rad           |

IEEE test cases: 50 % randomized loads and 33 % randomized generation
where $g$ is monotonically decreasing

$$g : [1, \infty) \rightarrow [0, 1]$$

Equilibrium angles (neighbors within $\gamma$ arc) exist if, in some $p$-norm,

$$\|B^T L^\dagger \rho_{\text{active}}\|_p \leq \gamma \alpha_p(\gamma)$$

for all graphs  \quad (\text{New } p\text{-norm } T)

$$\alpha_p(\gamma) := \min \text{ amplification factor of } P[\text{sinc}(x)]$$

For unweighted $p = 2$, new test sharper than old

$$\|B^T L^\dagger \rho_{\text{active}}\|_2 \leq \sin(\gamma)$$

(New 2-norm $T$)

For $p = \infty$, new test is for all graphs

$$\|B^T L^\dagger \rho_{\text{active}}\|_\infty \leq g(\|P\|_\infty)$$

(New $\infty$-norm $T$)

Comparison of sufficient and approximate sync tests

$K_c$ = critical coupling of Kuramoto model, computed via MATLAB $fsolve$

$K_T$ = smallest value of scaling factor for which test $T$ fails

<table>
<thead>
<tr>
<th>Test Case</th>
<th>old 2-norm conjectured</th>
<th>new 2-norm conjectured</th>
<th>new $\infty$-norm approximate</th>
<th>old $\infty$-norm approximate</th>
<th>$\alpha_\infty$ test</th>
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<tbody>
<tr>
<td>IEEE 9</td>
<td>16.54 %</td>
<td>59.06 %</td>
<td>73.74 %</td>
<td>92.13 %</td>
<td>85.06 %†</td>
</tr>
<tr>
<td>IEEE 14</td>
<td>8.33 %</td>
<td>42.27 %</td>
<td>59.42 %</td>
<td>83.09 %</td>
<td>81.32 %†</td>
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<tr>
<td>IEEE RTS 24</td>
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<td>35.62 %</td>
<td>53.44 %</td>
<td>89.48 %</td>
<td>89.48 %†</td>
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<tr>
<td>IEEE 30</td>
<td>2.70 %</td>
<td>40.98 %</td>
<td>55.70 %</td>
<td>85.54 %</td>
<td>85.54 %†</td>
</tr>
<tr>
<td>IEEE 39</td>
<td>2.97 %</td>
<td>37.32 %</td>
<td>67.57 %</td>
<td>100 %</td>
<td>100 %†</td>
</tr>
<tr>
<td>IEEE 57</td>
<td>0.36 %</td>
<td>31.93 %</td>
<td>40.69 %</td>
<td>84.67 %</td>
<td>—</td>
</tr>
<tr>
<td>IEEE 118</td>
<td>0.29 %</td>
<td>24.61 %</td>
<td>43.70 %</td>
<td>85.95 %</td>
<td>—</td>
</tr>
<tr>
<td>IEEE 300</td>
<td>0.20 %</td>
<td>24.13 %</td>
<td>40.33 %</td>
<td>99.80 %</td>
<td>—</td>
</tr>
<tr>
<td>Polish 2383</td>
<td>0.11 %</td>
<td>13.93 %</td>
<td>29.08 %</td>
<td>82.85 %</td>
<td>—</td>
</tr>
</tbody>
</table>

† $fmincon$ has been run for 100 randomized initial phase angles.  
* $fmincon$ does not converge.
Proof sketch 1/2: Rewriting the equilibrium equation

For what projection $\mathcal{P}$ and flow $z$ in cutset space, does there exist $x$ in cutset space solution to:

$$z = \mathcal{P} \sin(x) \iff z = \mathcal{P}[\text{sinc}(x)]x$$

$$\iff x = (\mathcal{P}[\text{sinc}(x)])^{-1}z =: h(x)$$

Proof sketch 2/2: Amplification factor & Brouwer fixed point thm

1. Look for $x$ solving
   \[ x = h(x) = (\mathcal{P}[\text{sinc}(x)])^{-1}z \]

2. Take $p$ norm, define min amplification factor of $\mathcal{P}[\text{sinc}(x)]$:
   \[ \alpha_p(\gamma) := \min \min_{\|x\|_p \leq \gamma} \|\mathcal{P}[\text{sinc}(x)]y\|_p \]

If $\|z\|_p \leq \gamma \alpha_p(\gamma)$ and $x \in B_p(\gamma) = \{x \mid \|x\|_p \leq \gamma\}$, then

\[ \|h(x)\|_p \leq \max_{x} \max_{y} \|\mathcal{P}[\text{sinc}(x)]^{-1}y\|_p \cdot \|z\|_p \leq \frac{\|z\|_p}{\alpha_p(\gamma)} \leq \gamma \]

Hence $h(x) \in B_p(\gamma)$ and $h$ satisfies Brouwer on $B_p(\gamma)$

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Computational method via power series

Given $z$, compute $x$ solution to

$$z = \mathcal{P} \sin(x)$$

Assume $x = \sum_{i=0}^{\infty} A_{2i+1}(z)$, where $A_{2i+1}(z)$ is homogeneous degree $2i + 1$

$$z = \mathcal{P} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = \mathcal{P} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left( \sum_{i=0}^{\infty} A_{2i+1}(z) \right)^{2k+1}$$

Equate left-hand and right-hand side at order 1, 3, \ldots, $2j + 1$:

$$A_1(z) = z$$

$$A_{2j+1}(z) = \mathcal{P} \left( \sum_{k=1}^{j} \frac{(-1)^{k+1}}{(2k+1)!} \sum_{\text{odd } \alpha_1, \ldots, \alpha_{2k+1} \text{ s.t. } \alpha_1 + \cdots + \alpha_{2k+1} = 2j+1} A_{\alpha_1}(z) \circ \cdots \circ A_{\alpha_{2k+1}}(z) \right)$$
Step 3: Series expansion for inverse Kuramoto map

Unique solution to \( z = P \sin(x) \) is

\[
x = \sum_{i=0}^{\infty} A_{2i+1}(z)
\]

\[
A_1(z) = z = B^\top L^\dagger \rho_{\text{active}}
\]

\[
A_3(z) = P \left( \frac{1}{3!} z^{o3} \right)
\]

\[
A_5(z) = P \left( \frac{3}{3!} A_3(z) \circ z^{o2} - \frac{1}{5!} z^{o5} \right)
\]

\[
A_7(z) = P \left( \frac{3}{3!} A_5(z) \circ z^{o2} + \frac{3}{3!} A_3(z)^{o2} \circ z - \frac{5}{5!} A_3(z) \circ z^{o4} + \frac{1}{7!} z^{o7} \right)
\]

arbitrary higher-order terms can be computed symbolically

For sufficiently small \( \|z\|_p \), series converges uniformly absolutely

Kuramoto Oscillators and Power Flow

New physical insight

- sharp sufficient conditions for equilibria
  upper bounds on transmission capacity
  stability margins as notions of distance from bifurcations

- new computational methods via power series

Applications

- secure operating conditions: \((\text{Dörfler et al, PNAS '13})\)
- feedback control: \((\text{Simpson-Porco et al, TIE '15})\)
- economic optimization: \((\text{Todescato et al, TCNS '17})\)

Future research

- close the gap between sufficient and necessary conditions
- more realistic coupled power flow equations
- applications to other flow networks (water, gas ...)

Numerical examples

Test case: IEEE 118

Accuracy, log10 scale