Synchronization in Oscillator Networks and Smart Grids

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References and Acknowledgments

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1 Coupled oscillators and synchronization problems

2 Main results: synchronization tests

3 Case study: predicting transition to instability

4 Detailed treatment of homogeneous case

5 Conclusions
Power Generation and Transmission Network

Extra High Voltage
265 to 275 kV
(mostly AC, some HVDC)

High Voltage
110 kV and up

Transmission Grid

Distribution Grid

Low Voltage
50 kV

Industrial Power Plant
≈ 30 MW

Medium Sized
Power Plant
≈ 150 MW

Coal Plant
600 MW

Nuclear Plant
600 - 1700 MW

Hydro-Electric Plant
≈ 200 MW

Factory

City Network
≈ 3 MW substations

City Power Plant
≈ 150 MW

Industrial Customers
≈ 2 MW

Solar Farm

Rural Network

Farm
≈ 400 kW

Wind Farm

Western US
(WECC 16-m, 25-b)

New England
(10-m, 13-b)
Central task: generators provide power for loads

Problems: stability in face of disturbances, security from cyber attacks
1. power transfer on line $i \rightarrow j$:
   \[ |V_i||V_j||Y_{ij}| \cdot \sin(\theta_i - \theta_j) \]
   
   $a_{ij} = \text{max power transfer}$

2. power balance at node $i$:
   \[ P_i = \sum_j a_{ij} \sin(\theta_i - \theta_j) \]
   
   power injection
Mathematical Model of a Power Transmission Network

1. power transfer on line $i \leadsto j$:

$$a_{ij} = \text{max power transfer}$$

$$|V_i||V_j||Y_{ij}| \cdot \sin(\theta_i - \theta_j)$$

2. power balance at node $i$:

$$P_i = \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

Structure-Preserving Model [Bergen & Hill ’81]

for $\square$, swing eq with $P_i > 0$

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

for $\bullet$, const $P_i < 0$ and $D_i \geq 0$

$$D_i \dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$
islanded microgrid = autonomously-managed low-voltage network with sources, loads, and storage

1. inverter in microgrid
   = DC source + PWM
   = controllable AC source

2. physics: \( P_{i,\ell} = a_{i,\ell} \sin(\theta_i - \theta_\ell) \)

3. Droop-control [Chandorkar et. al., '93]: \( \dot{\theta}_i = \omega_i - \omega^* = n_i (P_i^* - P_{i,\ell}) \)

Droop-controlled inverters are Kuramoto oscillators

for inverter \( i \)
\[
D_i \dot{\theta}_i = P_i^* - a_{i,\ell} \sin(\theta_i - \theta_\ell)
\]

for load \( \ell \)
\[
0 = P_\ell - \sum_{j=1}^{n} a_{\ell,j} \sin(\theta_\ell - \theta_i)
\]
islanded microgrid = autonomously-managed low-voltage network with sources, loads, and storage

1. Inverter in microgrid
   = DC source + PWM
   = controllable AC source

2. Physics:
   \[ P_{i \sim \ell} = a_{i\ell} \sin(\theta_i - \theta_\ell) \]

3. Droop-control [Chandorkar et al., '93]:
   \[ \dot{\theta}_i = \omega_i - \omega^* = n_i(P_i^* - P_{i \sim \ell}) \]

Droop-controlled inverters are Kuramoto oscillators

- For inverter \( i \)
  \[ D_i \dot{\theta}_i = P_i^* - a_{i\ell} \sin(\theta_i - \theta_\ell) \]

- For load \( \ell \)
  \[ 0 = P_\ell - \sum_{j=1}^{n} a_{\ell j} \sin(\theta_\ell - \theta_i) \]
1. Power networks are coupled oscillators

\[ M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]

\[ D_i \dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]

2. Synchronization: coupling strength vs. frequency non-uniformity

3. Graph theory provides notions of

   “coupling/connectivity” and “non-uniformity”

---

Power networks should synchronize for large “coupling/connectivity” and small “non-uniformity”
The Synchronization Problem

Determine conditions on the power injections \((P_1, \ldots, P_{n+m})\), network admittance \(Y\), and node parameters \((M_i, D_i)\), such that:

\[ |\theta_i - \theta_j| \text{ bounded and } \dot{\theta}_i = \dot{\theta}_j \]

Literature

1. **Classic security analysis:** load flow Jacobian & network theory
   [S. Sastry et al. '80, A. Araposthatis et al. '81, F. Wu et al '82, M. Ilić '92, …]

2. **Broad interest for Complex Networks, Network Science** [Ilić '92, Hill & Chen '06] stability, performance, and robustness of power network \(\leftrightarrow\) underlying graph properties (topological, algebraic, spectral, etc.)
Kuramoto model of coupled oscillators:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

- Sync in Josephson junctions [S. Watanabe et. al '97, K. Wiesenfeld et al. '98]
- Sync in a population of fireflies [G.B. Ermentrout '90, Y. Zhou et al. '06]
- Coordination of particle models [R. Sepulchre et al. '07, D. Klein et al. '09]
- Deep-brain stimulation and neuroscience [P.A. Tass '03, E. Brown et al. '04]
- Countless other sync phenomena [A. Winfree '67, S.H. Strogatz '00, J. Acebrón '01]
Synchronization Notions

\[ \dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

1. phase cohesive: \( |\theta_i(t) - \theta_j(t)| < \gamma \) for small \( \gamma < \pi/2 \) ... arc invariance
2. frequency synchrony: \( \dot{\theta}_i(t) = \dot{\theta}_j(t) \)
3. phase synchrony: \( \theta_i(t) = \theta_j(t) \)

- \( \{ a_{ij}\}_{i,j} \in E \) small & \( |\omega_i - \omega_j| \) large \( \implies \) no synchronization
- \( \{ a_{ij}\}_{i,j} \in E \) large & \( |\omega_i - \omega_j| \) small \( \implies \) cohesive + freq sync

Challenge: proper notions of sync, coupling & phase transition
1. Coupled oscillators and synchronization problems

2. Main results: synchronization tests

3. Case study: predicting transition to instability

4. Detailed treatment of homogeneous case

5. Conclusions
Graph: weights $a_{ij} > 0$ on edges $\{i, j\}$, values $x_i$ at nodes $i$

- adjacency matrix $A = (a_{ij})$
- degree matrix $D$ is diagonal with $d_{ii} = \sum_{j=1}^{n} a_{ij}$
- Laplacian matrix $L = L^T = D - A \geq 0$

Notions of Connectivity

topological: connectivity, average and worst-case path lengths
spectral: second smallest eigenvalue $\lambda_2$ of $L$ is “algebraic connectivity”

Notions of Dissimilarity

$$\|x\|_{\infty, \text{edges}} = \max\{i, j\} |x_i - x_j|,$$

$$\|x\|_{2, \text{edges}} = \left( \sum_{\{i, j\} \in \mathcal{E}} |x_i - x_j|^2 \right)^{1/2}$$

(graph edges $\{i, j\} \in \mathcal{E}$) or (all edges $\{i, j\}$ satisfy $i < j$)
Sync Tests: Coupling vs. Power Imbalance

\[ M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]
\[ D_i \dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \]

\[ \sum_j a_{ij} \leq |P_i| \implies \text{no sync} \]
\[ \lambda_2(L) > \|P\|_2, \text{all edges} \implies \text{sync} \]

Valid for: completely arbitrary weighted connected graphs

\[ \|L^\dagger P\|_{\infty, \text{graph edges}} < 1 \iff \text{sync} \]

Sharp for: trees, graphs with disjoint 3- and 4-cycles
Sharp for: graphs with \( L^\dagger P \) bipolar or symmetric
Sharp for:* homogeneous graphs (\( a_{ij} = K > 0 \))

best general conditions known to date
A Nearly Exact Synchronization Condition – Accuracy

Randomized power network test cases
with 50 % randomized loads and 33 % randomized generation

| Randomized test case (1000 instances) | Correctness of condition: \( \|L^\dagger P\|_\infty, \text{graph edges} \leq \sin(\gamma) \) | Accuracy of condition: \( \max_{\{i,j\}\in\mathcal{E}} |\theta_i^* - \theta_j^*| \leq \gamma - \arcsin(\|B^T L^\dagger P\|_\infty) \) | Phase cohesiveness: \( \max_{\{i,j\}\in\mathcal{E}} |\theta_i^* - \theta_j^*| \) |
|--------------------------------------|---------------------------------|-------------------------------------------------|---------------------------------|
| 9 bus system                         | always true                     | 4.1218 \cdot 10^{-5} rad                        | 0.12889 rad                     |
| IEEE 14 bus system                   | always true                     | 2.7995 \cdot 10^{-4} rad                        | 0.16622 rad                     |
| IEEE RTS 24                          | always true                     | 1.7089 \cdot 10^{-3} rad                        | 0.22309 rad                     |
| IEEE 30 bus system                   | always true                     | 2.6140 \cdot 10^{-4} rad                        | 0.1643 rad                      |
| New England 39                       | always true                     | 6.6355 \cdot 10^{-5} rad                        | 0.16821 rad                     |
| IEEE 57 bus system                   | always true                     | 2.0630 \cdot 10^{-2} rad                        | 0.20295 rad                     |
| IEEE RTS 96                          | always true                     | 2.6076 \cdot 10^{-3} rad                        | 0.24593 rad                     |
| IEEE 118 bus system                  | always true                     | 5.9959 \cdot 10^{-4} rad                        | 0.23524 rad                     |
| IEEE 300 bus system                  | always true                     | 5.2618 \cdot 10^{-4} rad                        | 0.43204 rad                     |
| Polish 2383 bus system (winter peak 1999/2000) | always true | 4.2183 \cdot 10^{-3} rad                        | 0.25144 rad                     |

Condition \( \|L^\dagger P\|_\infty, \text{graph edges} \leq \sin(\gamma) \) is extremely accurate for \( \gamma \leq 25^\circ \)
AC power flow, DC power flow and our new condition

Parameters: \( P, \{a_{ij}\}_{i,j} \in \mathcal{E}, \{\gamma_{ij}\}_{i,j} \in \mathcal{E} \)

Variables: \( \theta = (\theta_1, \ldots, \theta_n) \)

**AC power flow**

\[
P_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \quad |\theta_i - \theta_j| < \gamma_{ij}
\]

**DC power flow approximation**

\[
P_i = \sum_{j=1}^{n} a_{ij} (\delta_i - \delta_j), \quad |\delta_i - \delta_j| < \gamma_{ij}
\]

**Novel test**

\[
P_i = \sum_{j=1}^{n} a_{ij} (\delta_i - \delta_j), \quad |\delta_i - \delta_j| < \sin(\gamma_{ij})
\]
1. Coupled oscillators and synchronization problems

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Optimal power dispatch

\[
\text{minimize } \sum (\text{cost})_{i,\text{gen}} P_{i,\text{gen}}
\]

\[
P_i = \sum_j a_{ij} \sin(\theta_i - \theta_j)
\]

\[
|\theta_i - \theta_j| \leq (\text{thermal limit})_{ij}
\]

\[
P_{i,\text{gen}} \in (\text{feasible range})_{i,\text{gen}}
\]

Power flow: periodically, solve optimal power dispatch problem, & real-time perturbations handled via generation adjustments
Two contingencies:

1) generator 323 is tripped
2) increase loads & generation
Increase loads & generation:
⇒ condition \( \left\| B^T L^\dagger P \right\|_\infty \leq \sin(\gamma) \) predicts that thermal limit \( \gamma^* \) of line \( \{121, 325\} \) is violated at 22.23 % of additional loading
⇒ line \( \{121, 325\} \) is tripped at 22.24% of additional loading
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Synchronization in a All-to-All Homogeneous Graph

\[
\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j)
\]

all-to-all homogeneous graph

Explicit, necessary, and sufficient condition [F. Dörrler & F. Bullo '10]

Following statements are equivalent:

1. Coupling dominates non-uniformity, i.e.,
   \[
   K > K_{\text{critical}} \triangleq \omega_{\text{max}} - \omega_{\text{min}}
   \]

2. Kuramoto models with \( \{\omega_1, \ldots, \omega_n\} \subseteq [\omega_{\text{min}}, \omega_{\text{max}}] \) achieve phase cohesiveness & exponential frequency synchronization

Florian Dörrler & FB (UCSB)
Synchronization in an All-to-All Homogeneous Graph

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Define \( \gamma_{\text{min}} \) & \( \gamma_{\text{max}} \) by \( K_{\text{critical}}/K = \sin(\gamma_{\text{min}}) = \sin(\gamma_{\text{max}}) \), then

1) **phase cohesiveness** for all arc-lengths \( \gamma \in [\gamma_{\text{min}}, \gamma_{\text{max}}] \)

2) **practical phase synchronization**: from \( \gamma_{\text{max}} \) arc \( \rightarrow \) \( \gamma_{\text{min}} \) arc

3) exponential **frequency synchronization** in the interior of \( \gamma_{\text{max}} \) arc
Synchronization in a All-to-All Homogeneous Graph

all-to-all homogeneous graph

\[ \dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j) \]

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2. Kuramoto models with \( \{\omega_1, \ldots, \omega_n\} \subseteq [\omega_{\text{min}}, \omega_{\text{max}}] \) achieve phase cohesiveness & exponential frequency synchronization

- **improves** existing sufficient bounds [F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, S.J. Chung et al. '10, J.L. van Hemmen et al. '93, A. Franci et al. '10, S.Y. Ha et al. '10]

- **tight** w.r.t. continuum-limit [G.B. Ermentrout '85, A. Acebron et al. '00]

- **tight** w.r.t. implicit conditions for particular configurations [R.E. Mirollo et al. '05, D. Aeyels et al. '04, M. Verwoerd et al. '08]
Main proof ideas

1. **Cohesiveness:**
   - for $\theta(0)$ in arc of length $\gamma \in [\gamma_{\text{min}}, \gamma_{\text{max}}]$, define arc-length cost function
     \[ V(\theta(t)) = \max\{|\theta_i(t) - \theta_j(t)|\}_{i,j \in \{1,\ldots,n\}} \]
   - $t \mapsto V(\theta(t))$ is non-increasing because
     \[ D^+ V(\theta(t)) < 0 \]
   - $t \mapsto \theta(t)$ remains in (possibly-rotating) arc of length $\gamma$ and, moreover, $\gamma < \pi/2$ in finite time

2. **Frequency synchronization:** once in arc of length $\pi/2$
   \[
   \frac{d}{dt} \dot{\theta}_i = -\sum_{j \neq i} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j)
   \]
   where $a_{ij}(t) = \frac{K}{n} \cos(\theta_i(t) - \theta_j(t)) > 0$. result follows from time-varying consensus theorem
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Conclusions

Summary:
1. connection between power networks and coupled Kuramoto oscillators
2. necessary and sufficient sync conditions

Ongoing and future work:
1. sharp condition: tests and proofs
2. region of attraction
3. more realistic models (reactive power, stochastics etc)
4. smart-grid applications = quick algorithms for security assessment, prediction of cascading failures, remedial action design, etc

IFAC NecSys ’12, Sep 14, 15: Workshop on Networks & Controls
10 invited presentations, 4 interactive sessions with 55 papers

IEEE CDC ’12: Tutorial Session on Coupled Oscillators