Synchronization in Oscillator Networks and Smart Grids

Francesco Bullo

Center for Control,
Dynamical Systems & Computation
University of California at Santa Barbara
http://motion.me.ucsb.edu

20th Int. Symp. on Mathematical Theory of Networks and Systems
Melbourne, Australia, July 10, 2012

References and Acknowledgments

Florian Dörfler

Collaborators: Misha Chertkov (LANL) and John Simpson-Porco (UCSB)

Funding: NSF CyberPhysical Program, CNS-1135819

[2] F. Dörfler and F. Bullo. On the critical coupling for Kuramoto oscillators. *SIAM Journal on Applied Dynamical Systems*, 10(3):1070–1099, 2011

[1] F. Dörfler and F. Bullo. Synchronization and transient stability in power networks and non-uniform Kuramoto oscillators. *SIAM Journal on Control and Optimization*, 50(3):1616–1642, 2012

[3] F. Dörfler, M. Chertkov, and F. Bullo. Synchronization assessment in power networks and coupled oscillators. In *IEEE Conf. on Decision and Control*, Maui, HI, USA, December 2012. Submitted

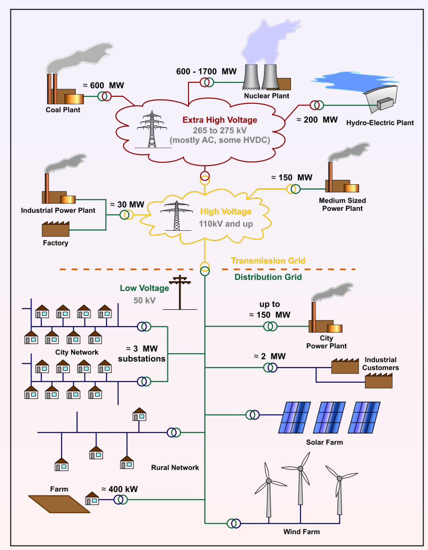
[4] J. W. Simpson-Porco, F. Dörfler, and F. Bullo. Droop-controlled inverters are Kuramoto oscillators. In *IFAC Workshop on Distributed Estimation and Control in Networked Systems*, Santa Barbara, CA, USA, September 2012. To appear (http://arxiv.org/pdf/1206.5033)

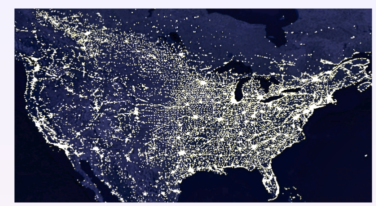
[5] F. Dörfler and F. Bullo. Kron reduction of graphs with applications to electrical networks. *IEEE Transactions on Circuits and Systems*, November 2011. To appear (http://arxiv.org/pdf/1102.2950)

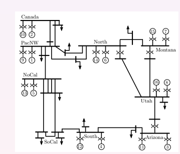
Outline

1. **Coupled oscillators and synchronization problems**
2. Main results: synchronization tests
3. Case study: predicting transition to instability
4. Detailed treatment of homogeneous case
5. Conclusions

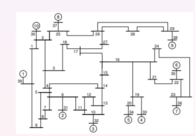
Power Generation and Transmission Network





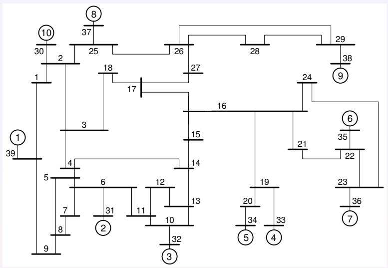


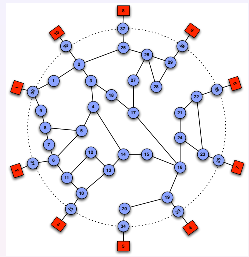
Western US
(WECC 16-m, 25-b)



New England
(10-m, 13-b)

Mathematical Model of a Power Transmission Network





1. n **generators** $\blacksquare$ and m **load buses** $\bullet$
2. **admittance matrix** $Y \in \mathbb{C}^{(n+m)\times(n+m)}$, symmetric, sparse, lossless

Central task: generators provide power for loads

Problems: stability in face of disturbances, security from cyber attacks

Mathematical Model of a Power Transmission Network

1. power transfer on line $i \rightsquigarrow j$: $\underbrace{|V_i||V_j||Y_{ij}|}_{a_{ij}=\text{max power transfer}} \cdot \sin(\theta_i - \theta_j)$
2. power balance at node i: $\underbrace{P_i}_{\text{power injection}} = \sum_j a_{ij} \sin(\theta_i - \theta_j)$

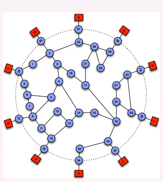
Mathematical Model of a Power Transmission Network

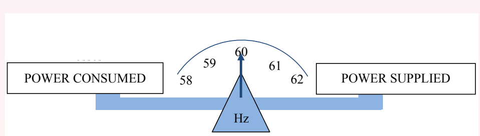
1. power transfer on line $i \rightsquigarrow j$: $\underbrace{|V_i||V_j||Y_{ij}|}_{a_{ij}=\text{max power transfer}} \cdot \sin(\theta_i - \theta_j)$
2. power balance at node i: $\underbrace{P_i}_{\text{power injection}} = \sum_j a_{ij} \sin(\theta_i - \theta_j)$

Structure-Preserving Model [Bergen & Hill '81]

for $\blacksquare$, swing eq with $P_i > 0$ $\quad M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

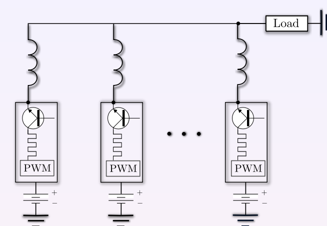
for $\bullet$, const $P_i < 0$ and $D_i \geq 0$ $\quad D_i\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$





Mathematical Model of a Islanded Microgrid

islanded microgrid =
autonomously-managed low-voltage network with sources, loads, and storage



1. inverter in microgrid
   = DC source + PWM
   = controllable AC source
2. physics: $P_{i \rightsquigarrow \ell} = a_{i\ell} \sin(\theta_i - \theta_\ell)$
3. Droop-control [Chandorkar et. al., '93]: $\dot{\theta}_i = \omega_i - \omega^* = n_i(P_i^* - P_{i\rightsquigarrow\ell})$

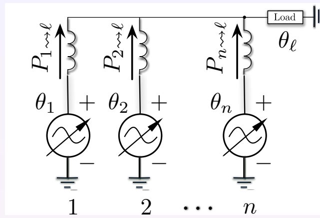
Droop-controlled inverters are Kuramoto oscillators

for inverter i $\quad D_i\dot{\theta}_i = P_i^* - a_{i\ell} \sin(\theta_i - \theta_\ell)$

for load ℓ $\quad 0 = P_\ell - \sum_{j=1}^{n} a_{\ell j} \sin(\theta_\ell - \theta_i)$

Mathematical Model of a Islanded Microgrid

islanded microgrid =
autonomously-managed low-voltage network with sources, loads, and storage



1. inverter in microgrid
   = DC source + PWM
   = controllable AC source
2. physics: $P_{i\rightsquigarrow\ell} = a_{i\ell}\sin(\theta_i - \theta_\ell)$
3. Droop-control [Chandorkar et. al., '93]: $\dot\theta_i = \omega_i - \omega^* = n_i(P_i^* - P_{i\rightsquigarrow\ell})$

Droop-controlled inverters are Kuramoto oscillators

for inverter i $\quad D_i\dot\theta_i = P_i^* - a_{i\ell}\sin(\theta_i - \theta_\ell)$

for load ℓ $\quad 0 = P_\ell - \sum_{j=1}^{n} a_{\ell j}\sin(\theta_\ell - \theta_i)$

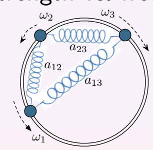
Synchronization in Power Networks

1. power networks are coupled oscillators

$$M_i\ddot\theta_i + D_i\dot\theta_i = P_i - \sum\nolimits_j a_{ij}\sin(\theta_i - \theta_j)$$

$$D_i\dot\theta_i = P_i - \sum\nolimits_j a_{ij}\sin(\theta_i - \theta_j)$$

2. synchronization: coupling strength vs. frequency non-uniformity



3. graph theory provides notions of
   "coupling/connectivity" and "non-uniformity"

power networks should synchronize
for large "coupling/connectivity" and small "non-uniformity"

The Synchronization Problem

Determine conditions on the power injections $(P_1, \dots, P_{n+m})$, network admittance Y, and node parameters (M_i, D_i), such that:

$|\theta_i - \theta_j|$ bounded and $\dot\theta_i = \dot\theta_j$

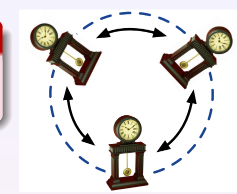
Literature

1. **Classic security analysis:** load flow Jacobian & network theory
   [S. Sastry et al. '80, A. Araposthatis et al. '81, F. Wu et al '82, M. Ilić '92, ...]
2. **Broad interest for Complex Networks, Network Science** [Ilić '92, Hill & Chen '06] stability, performance, and robustness of power network $\overset{?}{\leftrightsquigarrow}$ underlying graph properties (topological, algebraic, spectral, etc.)

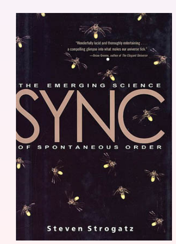
Coupled Oscillators in Science and Technology

Kuramoto model of coupled oscillators:

$$\dot\theta_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)$$



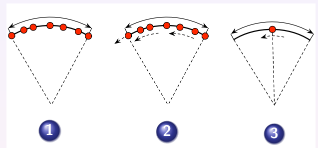
- Sync in Josephson junctions [S. Watanabe et. al '97, K. Wiesenfeld et al. '98]
- Sync in a population of fireflies [G.B. Ermentrout '90, Y. Zhou et al. '06]
- Coordination of particle models
  [R. Sepulchre et al. '07, D. Klein et al. '09]
- Deep-brain stimulation and neuroscience
  [P.A. Tass '03, E. Brown et al. '04]
- Countless other sync phenomena
  [A. Winfree '67, S.H. Strogatz '00, J. Acebrón '01]



Synchronization Notions

$$\dot\theta_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

1. phase cohesive: $|\theta_i(t) - \theta_j(t)| < \gamma$ for small $\gamma < \pi/2$ ... arc invariance
2. frequency synchrony: $\dot\theta_i(t) = \dot\theta_j(t)$
3. phase synchrony: $\theta_i(t) = \theta_j(t)$



- $\{a_{ij}\}_{\{i,j\}\in\mathcal{E}}$ small & $|\omega_i - \omega_j|$ large $\implies$ no synchronization
- $\{a_{ij}\}_{\{i,j\}\in\mathcal{E}}$ large & $|\omega_i - \omega_j|$ small $\implies$ cohesive + freq sync

Challenge: proper notions of sync, coupling & phase transition

[A. Jadbabaie et al. '04, P. Monzon et al. '06, Sepulchre et al. '07, S.J. Chung et al. '10, J.L. van Hemmen et al. '93, F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, F. Dörfler et al. '09 & '11, S.J. Chung et al. '10, A. Franci et al. '10, S.Y. Ha et al. '10, D. Aeyels et al. '04, R.E. Mirollo et al. '05, M. Verwoerd et al. '08, L. DeVille '11, ...]

Outline

Primer on Algebraic Graph Theory

Graph: weights $a_{ij} > 0$ on edges $\{i,j\}$, values x_i at nodes i

- adjacency matrix $A = (a_{ij})$
- degree matrix D is diagonal with $d_{ii} = \sum_{j=1}^{n} a_{ij}$
- Laplacian matrix $L = L^T = D - A \geq 0$

Notions of Connectivity

topological: connectivity, average and worst-case path lengths
spectral: second smallest eigenvalue λ_2 of L is "algebraic connectivity"

Notions of Dissimilarity

$\|x\|_{\infty,\text{edges}} = \max_{\{i,j\}} |x_i - x_j|$, $\quad \|x\|_{2,\text{edges}} = \big(\sum_{\{i,j\}} |x_i - x_j|^2\big)^{1/2}$

(graph edges $\{i,j\} \in \mathcal{E}$) or (all edges $\{i,j\}$ satisfy $i < j$)

Sync Tests: Coupling vs. Power Imbalance

$$M_i\ddot\theta_i + D_i\dot\theta_i = P_i - \sum\nolimits_j a_{ij}\sin(\theta_i - \theta_j)$$
$$D_i\dot\theta_i = P_i - \sum\nolimits_j a_{ij}\sin(\theta_i - \theta_j)$$

$\sum_j a_{ij} \leq |P_i| \implies$ no sync $\qquad \lambda_2(L) > \|P\|_{2,\text{all edges}} \implies$ sync

Valid for: completely arbitrary weighted connected graphs

$$\big\|L^\dagger P\big\|_{\infty,\text{graph edges}} < 1 \iff \text{sync}$$

Sharp for: trees, graphs with disjoint 3- and 4-cycles
Sharp for: graphs with $L^\dagger P$ bipolar or symmetric
Sharp for:* homogeneous graphs ($a_{ij} = K > 0$)

best general conditions known to date

A Nearly Exact Synchronization Condition – Accuracy

Randomized power network test cases

with 50 % randomized loads and 33 % randomized generation

Randomized test case (1000 instances)	Correctness of condition: $\|L^{\dagger}P\|_{\infty,\text{g. edges}} \leq \sin(\gamma)$ $\Rightarrow \max_{\{i,j\}\in\mathcal{E}} \lvert\theta_i^* - \theta_j^*\rvert \leq \gamma$	Accuracy of condition: $\max_{\{i,j\}} \lvert\theta_i^* - \theta_j^*\rvert - \arcsin(\|B^T L^{\dagger} P\|_{\infty})$	Phase cohesiveness: $\max_{\{i,j\}\in\mathcal{E}} \lvert\theta_i^* - \theta_j^*\rvert$
9 bus system	`always true`	$4.1218 \cdot 10^{-5}$ rad	0.12889 rad
IEEE 14 bus system	`always true`	$2.7995 \cdot 10^{-4}$ rad	0.16622 rad
IEEE RTS 24	`always true`	$1.7089 \cdot 10^{-3}$ rad	0.22309 rad
IEEE 30 bus system	`always true`	$2.6140 \cdot 10^{-4}$ rad	0.1643 rad
New England 39	`always true`	$6.6355 \cdot 10^{-5}$ rad	0.16821 rad
IEEE 57 bus system	`always true`	$2.0630 \cdot 10^{-2}$ rad	0.20295 rad
IEEE RTS 96	`always true`	$2.6076 \cdot 10^{-3}$ rad	0.24593 rad
IEEE 118 bus system	`always true`	$5.9959 \cdot 10^{-4}$ rad	0.23524 rad
IEEE 300 bus system	`always true`	$5.2618 \cdot 10^{-4}$ rad	0.43204 rad
Polish 2383 bus system (winter peak 1999/2000)	`always true`	$4.2183 \cdot 10^{-3}$ rad	0.25144 rad

condition $\|L^{\dagger}P\|_{\infty,\text{graph edges}} \leq \sin(\gamma)$ is extremely accurate for $\gamma \leq 25°$

AC power flow, DC power flow and our new condition

Parameters: P, $\{a_{ij}\}_{\{i,j\}\in\mathcal{E}}$, $\{\gamma_{ij}\}_{\{i,j\}\in\mathcal{E}}$ Variables: $\theta = (\theta_1, \dots, \theta_n)$

AC power flow

$$P_i = \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \qquad |\theta_i - \theta_j| < \gamma_{ij}$$

DC power flow approximation

$$P_i = \sum\nolimits_{j=1}^{n} a_{ij} (\delta_i - \delta_j), \qquad |\delta_i - \delta_j| < \gamma_{ij}$$

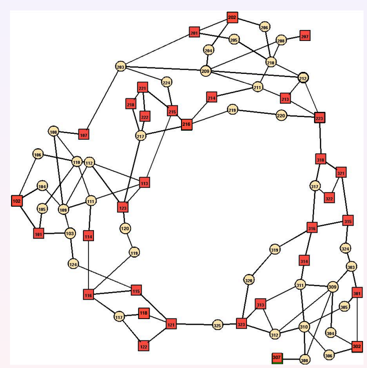
Novel test

$$P_i = \sum\nolimits_{j=1}^{n} a_{ij} (\delta_i - \delta_j), \qquad |\delta_i - \delta_j| < \sin(\gamma_{ij})$$

Outline

Case Study: Predicting Transition to Instability

IEEE Reliability Test System '96 (33-m 44-b)



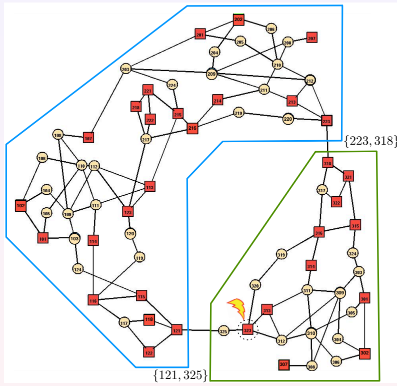
Optimal power dispatch

$$\text{minimize} \quad \sum (\text{cost})_{i,\text{gen}} P_{i,\text{gen}}$$
$$P_i = \sum\nolimits_j a_{ij} \sin(\theta_i - \theta_j)$$
$$|\theta_i - \theta_j| \leq (\text{thermal limit})_{ij}$$
$$P_{i,\text{gen}} \in (\text{feasible range})_{i,\text{gen}}$$

Power flow: periodically, solve optimal power dispatch problem, &
real-time perturbations handled via generation adjustments

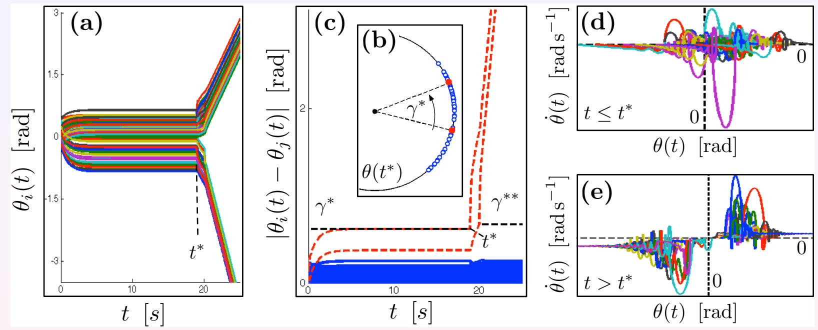
Case Study: Predicting Transition to Instability
IEEE Reliability Test System '96 (33-m 44-b)

Two contingencies:



1) generator 323 is tripped
2) increase loads & generation

Case Study: Predicting Transition to Instability
IEEE Reliability Test System '96 (33-m 44-b)



Increase loads & generation:

$\Rightarrow$ condition $\left\| B^T L^\dagger P \right\|_\infty \le \sin(\gamma)$ predicts that thermal limit γ^* of line $\{121, 325\}$ is violated at 22.23 % of additional loading

$\Rightarrow$ line $\{121, 325\}$ is tripped at 22.24% of additional loading

Outline

Synchronization in a All-to-All Homogeneous Graph

all-to-all homogeneous graph
$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j)$$

Explicit, necessary, and sufficient condition [F. Dörfler & F. Bullo '10]

Following statements are equivalent:

1. Coupling dominates non-uniformity, i.e., $\boxed{K > K_{\text{critical}} \triangleq \omega_{\max} - \omega_{\min}}$
2. Kuramoto models with $\{\omega_1, \dots, \omega_n\} \subseteq [\omega_{\min}, \omega_{\max}]$ achieve phase cohesiveness & exponential frequency synchronization

Synchronization in a All-to-All Homogeneous Graph

all-to-all homogeneous graph $$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j=1}^{n} \sin(\theta_i - \theta_j)$$

Explicit, necessary, and sufficient condition [F. Dörfler & F. Bullo '10]

Following statements are equivalent:

1. Coupling dominates non-uniformity, i.e., $\boxed{K > K_{\text{critical}} \triangleq \omega_{\max} - \omega_{\min}}$
2. Kuramoto models with $\{\omega_1, \dots, \omega_n\} \subseteq [\omega_{\min}, \omega_{\max}]$ achieve phase cohesiveness & exponential frequency synchronization

Define $\gamma_{\min}$ & $\gamma_{\max}$ by $K_{\text{critical}}/K = \sin(\gamma_{\min}) = \sin(\gamma_{\max})$, then

1) **phase cohesiveness** for all arc-lengths $\gamma \in [\gamma_{\min}, \gamma_{\max}]$
2) **practical phase synchronization:** from $\gamma_{\max}$ arc $\to$ $\gamma_{\min}$ arc
3) exponential **frequency synchronization** in the interior of $\gamma_{\max}$ arc

Synchronization in a All-to-All Homogeneous Graph

all-to-all homogeneous graph $$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j=1}^{n} \sin(\theta_i - \theta_j)$$

Explicit, necessary, and sufficient condition [F. Dörfler & F. Bullo '10]

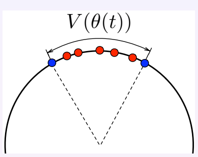
Following statements are equivalent:

1. Coupling dominates non-uniformity, i.e., $\boxed{K > K_{\text{critical}} \triangleq \omega_{\max} - \omega_{\min}}$
2. Kuramoto models with $\{\omega_1, \dots, \omega_n\} \subseteq [\omega_{\min}, \omega_{\max}]$ achieve phase cohesiveness & exponential frequency synchronization

- **improves** existing sufficient bounds [F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, S.J. Chung et al. '10, J.L. van Hemmen et al. '93, A. Franci et al. '10, S.Y. Ha et al. '10]
- **tight** w.r.t. continuum-limit [G.B. Ermentrout '85, A. Acebron et al. '00]
- **tight** w.r.t. implicit conditions for particular configurations [R.E. Mirollo et al. '05, D. Aeyels et al. '04, M. Verwoerd et al. '08]

Main proof ideas

1. **Cohesiveness:**



- for $\theta(0)$ in arc of lenght $\gamma \in [\gamma_{\min}, \gamma_{\max}]$, define arc-lenght cost function

$$V(\theta(t)) = \max\{|\theta_i(t) - \theta_j(t)|\}_{i,j \in \{1,\dots,n\}}$$

- $t \mapsto V(\theta(t))$ is non-increasing because

$$D^+ V(\theta(t)) < 0$$

- $t \mapsto \theta(t)$ remains in (possibly-rotating) arc of length γ and, moreover, $\gamma < \pi/2$ in finite time

2. **Frequency synchronization:** once in arc of length $\pi/2$

$$\frac{d}{dt}\dot{\theta}_i = -\sum\nolimits_{j \neq i} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j)$$

where $a_{ij}(t) = \frac{K}{n}\cos(\theta_i(t) - \theta_j(t)) > 0$. result follows from time-varying consensus theorem

Outline

Conclusions

Summary:

1. connection between power networks and coupled Kuramoto oscillators
2. necessary and sufficient sync conditions

Ongoing and future work:

1. sharp condition: tests and proofs
2. region of attraction
3. more realistic models (reactive power, stochastics etc)
4. smart-grid applications = quick algorithms for security assessment, prediction of cascading failures, remedial action design, etc

IFAC NecSys '12, Sep 14, 15: Workshop on Networks & Controls
10 invited presentations, 4 interactive sessions with 55 papers
IEEE CDC '12: Tutorial Session on Coupled Oscillators
F. Dörfler and F. Bullo. Exploring synchronization in complex oscillator networks. In *IEEE Conf. on Decision and Control*, Maui, HI, USA, December 2012. Invited Tutorial Session