Kron Reduction of Graphs
with Applications to Electrical Networks

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Los Alamos National Labs, New Mexico, June 8, 2011 
Article available online at: http://arxiv.org/abs/1102.2950

Motivation: the current power grid is . . .

“. . . the greatest engineering achievement of the 20th century.”
[National Academy of Engineering ’10]

“. . . the largest and most complex machine engineered by humankind.”
[P. Kundur ’94, V. Vittal ’03, . . .]

Motivation: the envisioned power grid

Energy is one of the top three national priorities

Expected developments in “smart grid”: 
1 large number of distributed power sources 
2 increasing adoption of renewables 
3 sophisticated cyber-coordination layer

∫ challenges: increasingly complex networks & stochastic disturbances 

∫ opportunity: some smart grid keywords: 
control/sensing/optimization ⊕ distributed/coordinated/decentralized

Today: “reducing the complexity by means of circuit and graph theory”
Kron reduction of a resistive circuit

- Nodal analysis by Kirchhoff’s and Ohm’s laws:

\[ I = Y \cdot V \]

- nodal current injections

\[ V \in \mathbb{C}^n \]

- nodal voltages/potentials

\[ Y \in \mathbb{C}^{n \times n} \]

- nodal conductance matrix

\[ Y = Y^T = \begin{bmatrix} \vdots & \cdots & \vdots & \cdots & \vdots \\ -Y_{11} & \cdots & \sum_{k=1, k \neq i}^{n} Y_{ik} + Y_{k, \text{shunt}} & \cdots & -Y_{in} \\ \vdots & \cdots & \vdots & \cdots & \vdots \end{bmatrix} \]

= \{ \text{weighted Laplacian matrix} \} + \text{diag}(Y_{k, \text{shunt}}) = \text{“loopy Laplacian”}

Kron reduction of graphs

Consider either of the following three equivalent setups:

- a connected electrical network with conductance matrix \( Y \), terminals \( \square \), interior nodes \( \bullet \), & possibly shunt conductances

- a symmetric and irreducible loopy Laplacian matrix \( Y \) with partition \( \square, \bullet \), & possibly diagonally dominance

- an undirected, connected, & weighted graph with boundary nodes \( \square \), interior nodes \( \bullet \), & possibly self-loops

**Kron reduction** via Schur complement: $Y_{\text{red}} = Y / Y_{\text{interior}}$

**Kron reduction** of graphs:

**Purpose:** construct low-dimensional equivalent circuits / graphs / models

Simplest non-trivial case: star-$\Delta$ transformation

[A. E. Kennelly 1899, A. Rosen 1924]

- Engineering applications: smart grid monitoring, circuit theory, model reduction for power and water networks, power electronics, large-scale integration chips, electrical impedance tomography, data-mining, …
- Mathematics applications: sparse matrix algorithms, finite-element methods, sparse multi-grid solvers, Markov chain reduction, stochastic complementation, applied linear algebra & matrix analysis, Dirichlet-to-Neumann map, …

Electrical impedance tomography

to reconstruct spatial conductivity

[E. Curtis and J. Morrow ’94 & ’00]

Smart grid monitoring

through cut-set variables

[I. Dobson ’11]

**Kron reduction** of graphs: applications

**Representations of integration chips**

[J. Rommes and W. H. A. Schilders ’09]

Reduced power network modeling

for stability analysis and control

[F. Dörfler and F. Bullo ’09]
Kron reduction of graphs: properties

Kron reduction of a graph with
- boundary ■, interior ●, non-neg self-loops ○
- loopy Laplacian matrix \( Y \)
- Schur complement: \( Y_{\text{red}} = Y / Y_{\text{interior}} \)

Properties of Kron reduction:
- Well-posedness: set of loopy Laplacian matrices is closed

\[ \text{Kron reduction} \]

Augmentation: replace self-loops ○ by edge to grounded node ◊

\[ \text{Augmentation: replace self-loops} \circ \text{by edge to grounded node} \shades \]

⇒ Equivalence: the following diagram commutes:

\[ Y \xrightarrow{\text{augment}} \hat{Y} \]

\[ Y_{\text{red}} \xrightarrow{\text{augment}} \hat{Y}_{\text{red}} \]

⇒ Iterative 1-dim Kron reduction: \( Y_{\text{red}}^{k+1} = Y_{\text{red}}^k / Y_{\text{interior}} \)

⇒ Topological properties:
- interior network connected ⇒ reduced network complete
- at least one node in interior network features a self-loop ◊
  ⇒ all nodes in reduced network feature self-loops ◊

⇒ Algebraic properties: self-loops in interior network
- decrease mutual coupling in reduced network
- increase self-loops in reduced network
Kron reduction of graphs: properties

- **Spectral properties:**
  - interlacing property: \( \lambda_i(Y) \leq \lambda_i(Y_{\text{red}}) \leq \lambda_{i+n-1}(Y) \)  
  \( \Rightarrow \) algebraic connectivity \( \lambda_2 \) is non-decreasing  
  - effect of self-loops \( \odot \) on loop-less Laplacian matrices:  
    \( \lambda_2(L_{\text{red}}) + \max\{\odot\} \geq \lambda_2(L) + \min\{\odot\} \)  
  \( \Rightarrow \) self-loops weaken the algebraic connectivity \( \lambda_2 \)

**Example:** all mutual edges have unit weight

> \[
\begin{align*}
\text{Kron reduction} & \quad \text{Kron reduction} \\
\begin{array}{c}
\begin{array}{c}
\text{without self-loops: } \lambda_2(L) = 0.39 \leq 0.69 = \lambda_2(L_{\text{red}}) \\
\text{with unit self-loops: } \lambda_2(L) = 0.39 \geq 0.29 = \lambda_2(L_{\text{red}})
\end{array}
\end{array}
\end{align*}
\]

**Effective resistance** \( R_{ij} \):

- **Equivalence and invariance** of \( R_{ij} \) among \( \square \) nodes:

\[
\begin{align*}
\text{Kron reduction of } & \quad \text{Kron reduction of } \\
\begin{array}{c}
\begin{array}{c}
\text{augment} \quad \text{augment}
\end{array}
\end{array}
\end{align*}
\]

- no self-loops: \( R_{ij} \) among \( \square \) uniform \( \Leftrightarrow \) \[ \frac{1}{R_{ij}} = \frac{1}{2} |Y_{\text{red}}(i,j)| \]
- self-loops: \( R_{ij} \) among \( \square \) & \( \blacklozenge \) uniform \( \Leftrightarrow \) \[ \frac{1}{R_{ij}} = \frac{1}{2} |Y_{\text{red}}(i,j)| + \max\{\odot\} \]

**Conclusions**

- Kron reduction is important in various applications
- Analysis of Kron reduction via algebraic graph theory
- Open problem: directed & complex-weighted graphs