Coordination in multi-agent systems

What kind of systems?
- each agent senses its immediate environment,
- communicates with others,
- processes information gathered, and
- takes local action in response

Cooperation in multi-agent systems

Cooperative robotics: technologies and applications

What scenarios?
Security systems, disaster recovery, environmental monitoring, study of natural phenomena and biological species, science imaging

What kind of tasks?
1. coordinated motion: rendezvous, flocking, formation
2. cooperative sensing: surveillance, exploration, search and rescue
3. cooperative material handling and transportation

Model: customers appear randomly in space/time
robotic network knows locations and provides service

Goal: minimize customer delay

Approach: assign customers to robots by partitioning the space

1. robot coordination via territory partitioning
2. gossip algorithms: mathematical setup
3. gossip algorithms: technological advances

Territory partitioning is ... art

abstract expressionism
“Ocean Park No. 27” and “Ocean Park No. 129”
by Richard Diebenkorn (1922-1993), inspired by aerial landscapes

Territory partitioning ... centralized district design

California Voting Districts: 2008 Obama/McCain votes

Territory partitioning is ... animal territory dynamics

Tilapia mossambica, “Hexagonal Territories,” Barlow et al, ’74
Sage sparrows, “Territory dynamics in a sage sparrows population,” Petersen et al ’87
Territory partitioning: behaviors and optimality

ANALYSIS of cooperative distributed behaviors

- how do animals share territory?
- how do they decide foraging ranges?
- how do they decide nest locations?
- what if each robot goes to “center” of own dominance region?
- what if each robot moves away from closest robot?

DESIGN of performance metrics

- how to cover a region with \( n \) minimum-radius overlapping disks?
- how to design a minimum-distortion (fixed-rate) vector quantizer?
- where to place mailboxes in a city / cache servers on the internet?

Expected wait time

\[
H(p, \nu) = \int_{V_1} \|q - p_1\| \phi(q) dq + \cdots + \int_{V_n} \|q - p_n\| \phi(q) dq
\]

- \( n \) robots at \( p = \{p_1, \ldots, p_n\} \)
- environment is partitioned into \( \nu = \{\nu_1, \ldots, \nu_n\} \)

Optimal partitioning

The Voronoi partition \( \{V_1, \ldots, V_n\} \) generated by points \( (p_1, \ldots, p_n) \)

\[
V_i = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|, \forall j \neq i\}
= Q \bigcap_j \text{(half plane between } i \text{ and } j, \text{ containing } i)\]

Optimal centering (for region \( \nu \) with density \( \phi \))

function of \( p \)  
minimizer = center

\[
p \mapsto \int_\nu \|q - p\|^2 \phi(q) dq \quad \text{centroid (or center of mass)}
\]

\[
p \mapsto \int_\nu \|q - p\| \phi(q) dq \quad \text{Fermat–Weber point (or median)}
\]

\[
p \mapsto \text{area}(\nu \cap \text{disk}(p, r)) \quad \text{r-area center}
\]

\[
p \mapsto \text{radius of largest disk centered at } p \text{ enclosed inside } \nu
\]

\[
p \mapsto \text{radius of smallest disk centered at } p \text{ enclosing } \nu
\]

From online Encyclopedia of Triangle Centers
From optimality conditions to algorithms

\[ H(p, v) = \int_{v_1} f(\|q - p_1\|) \phi(q) dq + \cdots + \int_{v_n} f(\|q - p_n\|) \phi(q) dq \]

- at fixed positions, optimal partition is Voronoi
- at fixed partition, optimal positions are "generalized centers"

Voronoi+centering algorithm for robots

Voronoi+centering law
At each comm round:
1: acquire neighbors’ positions
2: compute own dominance region
3: move towards center of own dominance region

Incomplete literature
Gossip partitioning policy

At random comm instants, between two random regions:

1. compute two centers
2. compute bisector of centers
3. partition two regions by bisector

Lyapunov function for gossip territory partitioning

\[ H(v) = \sum_{i=1}^{n} \int_{v_i} f(\parallel \text{center}(v_i) - q\parallel) \phi(q) dq \]

- state space is not finite-dimensional
- non-convex disconnected polygons
- arbitrary number of vertices
- gossip map is not deterministic, ill-defined and discontinuous
- two regions could have same centers

Convergence with persistent switches (proof sketch 2/4)

- \( X \) is metric space
- finite collection of maps \( T_i : X \rightarrow X \) for \( i \in I \)
- consider sequences \( \{x_\ell\}_{\ell \geq 0} \subset X \) with
  \[ x_{\ell+1} = T_{i(\ell)}(x_\ell) \]

Assume:

- \( W \subset X \) compact and positively invariant for each \( T_i \)
- \( U : W \rightarrow \mathbb{R} \) decreasing along each \( T_i \)
- \( U \) and \( T_i \) are continuous on \( W \)
- there exists probability \( p \in ]0, 1[ \) such that, for all indices \( i \in I \) and times \( \ell \), we have
  \[ \text{Prob} \{ x_{\ell+1} = T_i(x_\ell) \mid \text{past} \} \geq p \]

If \( x_0 \in W \), then almost surely
\[ x_\ell \rightarrow \text{ (intersection of sets of fixed points of all } T_i) \]
The space of partitions (proof sketch 3/4)

Let $C$ be set of closed subsets of $Q$ — is it compact?

Hausdorff metric

$$d_H(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(a, b) \right\}$$

1. $(C, d_H)$ is compact metric space
2. Dynamical system and Lyapunov function are not continuous wrt $d_H$!
3. Hausdorff metric sensitive to sets of measure zero

The space of partitions (proof sketch 4/4)

Let $C$ be set of closed subsets of $Q$ — is it compact?

Symmetric difference metric

$$d_{\text{symm}}(A, B) = \text{measure}(A \setminus B) + \text{measure}(B \setminus A)$$

1. redefine $C \leftarrow C/\sim$ where $A \sim B$ whenever $d_{\text{symm}}(A, B) = 0$
2. Dynamical system and Lyapunov function are continuous in $(C, d_{\text{symm}})$
3. No compactness result is available for $(C, d_{\text{symm}})$!

Theorem: for any $k$, $(C(k), d_{\text{symm}})$ is compact.

$C(k)$ is set of $k$-convex subsets (union of $k$ convex sets)
Incomplete literature


Outline

1. robot coordination via territory partitioning
2. gossip algorithms: mathematical setup
3. gossip algorithms: technological advances

Gossip algorithms: technological advances

- non-convex environments
- motion protocols (for communication persistency)
- hardware and large-scale implementations

Nonconvex environments as graphs

Revised setup

environment: weighted graph partitioned in connected subgraphs
multi-center cost function: 
\[ H(p, v) = H_1(p_1, v_1) + \cdots + H_1(p_n, v_n) \]
single-region cost function: 
\[ H_1(p, v) = \sum_{q \in v} \text{dist}(p, q) \phi(q) \]
center of subgraph \( v \): minimizer of \( p \mapsto H_1(p, v) \)

Range-dependent stochastic comm

Two robots communicate at the sample times of a Poisson process with distance-dependent intensity
Nonconvex environments as graphs

Revised setup

environment: weighted graph partitioned in connected subgraphs

multi-center cost function:
\[ H(p, v) = H_1(p_1, v_1) + \cdots + H_1(p_n, v_n) \]

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Range-dependent stochastic comm

Two robots communicate at the sample times of a Poisson process with distance-dependent intensity

Discretized gossip algorithm

1. Ensure that neighbors meet frequently enough:
   \( \Rightarrow \) Random Destination & Wait Motion Protocol

2. Update partition when two robots meet:
   \( \Rightarrow \) Pairwise Partitioning Rule

Random Destination & Wait Motion Protocol

Each robot continuously executes:
1. select sample destination \( q_i \in v_i \)
2. move to \( q_i \)
3. wait at \( q_i \) for time \( \tau > 0 \)

Pairwise Partitioning Rule

Whenever robots \( i \) and \( j \) communicate:
1. \( w \leftarrow v_i \cup v_j \)
2. while (computation time is available) do
3. \( (q_i, q_j) \leftarrow \text{sample vertices in } w \)
4. \( (w_i, w_j) \leftarrow \text{Voronoi of } w \) by \( (q_i, q_j) \)
5. if \( (H_1(q_i, w_i) + H_1(q_j, w_j)) \text{ improves} \) then
6. centroids \( - (q_i, q_j) \)
7. \( (v_i, v_j) \leftarrow (w_i, w_j) \)
8. end if
9. end while

Discretized gossip algorithm

Pairwise Partitioning Rule

Whenever robots $i$ and $j$ communicate:
1: $w \leftarrow v_i \cup v_j$
2: while (computation time is available) do
3: $(q_i, q_j) \leftarrow$ sample vertices in $w$
4: $(w_i, w_j) \leftarrow$ Voronoi of $w$ by $(q_i, q_j)$
5: if $(H_1(q_i, w_i) + H_1(q_j, w_j)$ improves) then
6: centroids $\leftarrow (q_i, q_j)$
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Discretized gossip algorithm

Pairwise Partitioning Rule

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6: centroids $\leftarrow (q_i, q_j)$
7: $(v_i, v_j) \leftarrow (w_i, w_j)$
8: end if
9: end while

(Combinatorial optimization) – interruptible anytime algorithm

**Harware-in-the-loop experiment**

Player/Stage robot simulation & control system: realistic robot models with integrated wireless network model & obstacle-avoidance planner. Hardware-in-the-loop experiment: 3 physical and 6 simulated robots

**Larger-scale simulation experiment**

Player/Stage robot simulation & control system: realistic robot models with integrated wireless network model & obstacle-avoidance planner. Simulation experiment: 30 robots; UCSB campus.

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**Conclusions**

**Summary**

1. gossip algorithms: mathematical setup
2. gossip algorithms: technological advances

**Open problems**

1. topology and comp geometry of power sets
2. coordination: resource allocation, weak comm protocols
3. ecology of territory partitioning

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**Robot hardware**

**Localization:**
Adaptive Monte Carlo Localization particle filter method for matching scans to a map
(Thrun et al., 2001)

**Navigation:**
Smooth Nearestness Diagram navigation for local obstacle avoidance using sensor data
(Durham et al., 2008)
**Convergence Result**

**Theorem**  
Convergence almost surely to a pairwise-optimal partition in finite time.

**Proof sketch**

1. Algorithm maintains a connected $n$-partition
2. Probability neighbors communicate in any interval
3. $H$ decreases with every pairwise update
4. Pairwise-optimal partitions are equilibrium set


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**Pairwise Optimal Partitions**

**Definition**

A partition is pairwise-optimal if every pair of neighboring robots $(i, j)$ has reached lowest possible coverage cost of $v_i \cup v_j$, i.e. that

$$H_1(c_i; v_i) + H_1(c_j; v_j) = \min_{a, b \in w} \left\{ \sum_{k \in w} \min \{d_w(a, k), d_w(b, k)\} \right\}$$

where $w = v_i \cup v_j$ and $(c_i, c_j)$ are the centers of $(v_i, v_j)$

$\Rightarrow$ Every pairwise-optimal partition is also centroidal Voronoi

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**Subset Example**

Cost: 12 hops  
Cost: 11 hops  
Cost: 10 hops

- All are centroidal Voronoi partitions  
- Only lowest cost is pairwise-optimal (by definition)

$\Rightarrow$ Avoid all pairwise local minima

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**Monte Carlo Results I**

Initial cost: 5.48 m  
Optimum cost: 2.17 m  
116 sequences of random pairwise exchanges

Black - Pairwise-optimal Algorithm  
Gray - Gossip Lloyd Algorithm  
Red - Lloyd Algorithm

$\Rightarrow$ 99% confidence that with at least 80% probability the Pairwise-optimal algorithm gets within 4% of the global optimum
Monte Carlo Results II

Tests shown:
- 8 random initial conditions
- 116 sequences of pairwise exchanges

Black - Pairwise-optimal Algorithm
Gray - Gossip Lloyd Algorithm
Red - Lloyd Algorithm
Green - Starting cost

Motivational Scenario

One-to-Base Communication Model
For each robot, there exists a finite upper bound \( \Delta \) on the time between its communications with the base station.

Changes from Gossip Case

New Approach:
- One-sided territory & centroid updates
- Evolve overlapping territories

New cost function

\[
H_{\text{max}}(c, v) = \sum_{q \in Q} \max_{i \in \{1, \ldots, n\}} \{\text{dist}(c_i, a) \mid q \in v_i\} \phi(q)
\]

⇒ Split centering and partitioning

One-to-Base Partitioning

Base station holds local copy of robot territories

When robot \( i \) talks to base:
1. Update robot \( i \)'s centroid
2. Transmit local copy of \( v_i \) to robot \( i \)
3. for every other robot \( j \) do
4. Add vertices to \( v_j \) which are in \( v_i \) but closer to \( j \)
5. Remove vertices from \( v_j \) which are in both but closer to \( i \)
6. end for
**One-to-Base Partitioning**

Base station holds local copy of robot territories.

When robot $i$ talks to base:
1. Update robot $i$’s centroid
2. Transmit local copy of $v_i$ to robot $i$
3. for every other robot $j$ do
   4. Add vertices to $v_j$ which are in $v_i$ but closer to $j$
   5. Remove vertices from $v_j$ which are in both but closer to $i$
4. end for

⇒ Split centering and partitioning

**Convergence Results**

Theorem (Durham et al., 2011)

Convergence to a centroidal Voronoi partition in finite time.

Let $M(P)$ be set of vertices owned by multiple robots and

$$H_{\min}(c, v) = \sum_{q \in Q} \min_{i \in \{1, \ldots, n\}} \{\text{dist}(c_i, q) \mid q \in v_i\} \phi(q)$$

**Proof of Decreasing Cost:** One of these conditions holds

1. $H_{\max}(c^+, v^+) < H_{\max}(c, v)$
2. $H_{\max}(c^+, v^+) = H_{\max}(c, v)$ and $H_{\min}(c^+, v^+) < H_{\min}(c, P)$
3. $H_{\max}(c^+, v^+) = H_{\max}(c, v)$, $H_{\min}(c^+, v^+) = H_{\min}(c, v)$, and $|M(v^+)| < |M(v)|$

**Simulation Movie**

Four robots each initially own the entire environment, but then settle on a centroidal Voronoi partition.