Synchronization in Power Networks and in Non-uniform Kuramoto Oscillators

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Intro: North American power grid

“...the largest and most complex machine engineered by humankind.”
[P. Kundur ’94, V. Vittal ’03, ...]

“...the greatest engineering achievement of the 20th century.”
[National Academy of Engineering ’10]

- large-scale, complex, nonlinear, and rich dynamic behavior
- 100 years old and operating at its capacity limits

⇒ recent blackouts: New England ’03 + Italy ’03, Brazil ’09

Intro: Transient Stability in Power Networks

The New York Times


Energy is one of the top three national priorities

Expected additional synergetic effects in future "smart grid":
⇒ increasing complexity and renewable stochastic power sources
⇒ increasingly many transient disturbances to be detected and rejected

Transient Stability: Generators have to maintain synchronism in presence of large transient disturbances such as faults or loss of
- transmission lines and components,
- generation or load.

Intro: New England power grid

Power network topology:
- n generators, each connected to a generator terminal bus
- n generators terminal buses and m load buses form connected graph
- admittance matrix \( \mathbf{Y}_{\text{network}} \in \mathbb{C}^{(2n+m) \times (2n+m)} \) characterizes the network
Intro: Mathematical Model of a Power Network

- generator nodes ■: swing equation for generator $i$
  \[
  \frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + P_{mi} - P_{ei} 
  \]
  $\theta_i(t)$ is measured w.r.t. a 60Hz rotating frame

- network-preserving model: power flow equations
  for passive nodes ◇ & ● ⇒ DAE system

Intro: Transient Stability Analysis in Power Networks

Classic Model

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

Transient stability and synchronization:

- frequency equilibrium: $(\dot{\theta}_i, \ddot{\theta}_i) = (0, 0)$ for all $i$
- synchronous equilibrium: $|\theta_i - \theta_j|$ bounded & $\dot{\theta}_i - \dot{\theta}_j = 0$ for all $\{i,j\}$

Classic problem setup in transient stability analysis:

◇ power network in stable frequency equilibrium
● → transient network disturbance and fault clearance
◇ stability analysis of a new frequency equilibrium in post-fault network

Intro: Transient Stability Analysis in Power Networks

Classic Model

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

Transient stability and synchronization:

Classic analysis methods: Hamiltonian arguments

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i - \nabla U(\theta)^T
\]


\[
\dot{\theta}_i = -\nabla_i U(\theta)^T
\]

Key objective: compute domain of attraction via numerical methods
Intro: Transient Stability Analysis in Power Networks

Classic Model

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

Transient stability and synchronization:

Classic analysis methods: Hamiltonian arguments

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i - \nabla U_i(\theta)^T \Rightarrow \dot{\theta}_i = -\nabla_i U(\theta)^T
\]

⇒ Open problem [D. Hill and G. Chen '06]: power sys \(\overset{\sim}{\rightarrow}\) network:

- transient stability, performance, and robustness of a power network
  \(\overset{?}{\rightarrow}\) state, parameters, and topology of underlying network

Detour – Consensus Protocols & Kuramoto Oscillators

Consensus protocol in \(\mathbb{R}^n\):

\[
\dot{x}_i = -\sum_{j \neq i} a_{ij} (x_i - x_j)
\]

- \(n\) agents with state \(x_i \in \mathbb{R}\) and connected graph with weights \(a_{ij} > 0\)
- objective is state agreement: \(x_i(t) - x_j(t) \to 0\)
- application: social networks, computer science, systems theory
  robotic rendezvous, distributed computing, filtering and control, . . .
- some references: [M. DeGroot '74, J. Tsitsiklis '84, . . .]

Kuramoto model in \(\mathbb{T}^n\):

\[
\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)
\]

- oscillators with phase \(\theta_i \in \mathbb{T}\), frequency \(\omega_i \in \mathbb{R}\), complete coupling
- objective is synchronization: \(\theta_i(t) - \theta_j(t)\) bounded, \(\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0\)
- application in physics, biology, engineering:
  coupled neurons, Josephson junctions, motion coordination, . . .
- some references: [Y. Kuramoto '75, A. Winfree '80, . . .]

Kuramoto model in \(\mathbb{T}^n\):

\[
\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)
\]

- degrees of synchronization:
  - phase locking: \(|\theta_i - \theta_j|\) bounded
  - frequency entrainment: \(\dot{\theta}_i = \dot{\theta}_j\)
  - phase synchronization: \(\theta_i = \theta_j\)
- known that
  - \(K\) large & \(|\omega_i - \omega_j|\) small ⇒ frequency entrainment & phase locking
  - additionally, for \(\omega_i = \omega_j\) ⇒ phase synchronization

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Outline

1 Introduction
   - Power network model
   - Synchronization and transient stability
   - Consensus protocol and Kuramoto oscillators

2 Singular perturbation analysis
   (to relate power network and Kuramoto model)

3 Synchronization analysis (of non-uniform Kuramoto model)
   - Main synchronization result
   - Sufficient condition (based on weakest lossless coupling)
   - Sufficient condition (based on lossless algebraic connectivity)

4 Structure-preserving power network models
   - Kron-reduction of graphs
   - Sufficient conditions for synchronization

5 Conclusions

From the swing equations to the Kuramoto model

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

\[
\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j) \quad \Rightarrow \quad D_i \dot{\theta}_i = \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

Possible connection has often been hinted at in the literature!

Power systems: [D. Subbarao et al., '01, G. Filatrella et al., '08, V. Fioriti et al., '09]
Networked control: [D. Hill et al., '06, M. Arcak, '07]
From the swing equations to the Kuramoto model

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

Motivation:
- harmonic oscillator

\[
\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j) \quad \Rightarrow \quad D_i \dot{\theta}_i = \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

Time-scale separation in power network model:
- **Motivation:** harmonic oscillator

\[
\dot{x} = \frac{2}{\epsilon} \dot{x} - x
\]

for \( \epsilon < 1 \Rightarrow \) two time-scales

Singular perturbation analysis:
- **Motivation:** harmonic oscillator

\[
\dot{x} = f(x, z)
\]

\[
\dot{z} = g(x, z)
\]

\( \epsilon = 0 \) quasi-steady state

\( \epsilon \rightarrow 0 \) approximation error in slow time-scale: \( O(\epsilon) \)

\( \epsilon = 1 \) error exp. stable in fast time-scale

\[
D_i \dot{\theta}_i = \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

Singular perturbation parameter: \( \epsilon = \frac{M_{\text{max}}}{\frac{\pi f_0}{D_{\text{min}}}} \)

\[
M_i \dot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

**Tikhonov's Theorem:**
Assume the non-uniform Kuramoto model synchronizes exponentially. Then \( \forall (\theta(0), \dot{\theta}(0)) \) there exists \( \epsilon^* > 0 \) such that \( \forall \epsilon < \epsilon^* \) and \( \forall t \geq 0 \)

\[
\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = O(\epsilon).
\]
Singular Perturbation Analysis

Discussion of the assumption $\epsilon = \frac{M_{\text{max}}}{\pi f_0 D_{\text{min}}}$ sufficiently small:

1. physical interpretation: damping and sync on separate time-scales
2. classic assumption in literature on coupled oscillators: over-damped mechanical pendula and Josephson junctions
3. physical reality: with generator internal control effects $\epsilon \in O(0.1)$
4. simulation studies show accurate approximation even for large $\epsilon$
5. first-order and second-order models have the same equilibria with the same stability properties, and the regions of attractions are bounded by the same separatrices (independent of $\epsilon$)
6. non-uniform Kuramoto model corresponds to reduced gradient system $\dot{\theta}_i = -\nabla_i U(\theta)^T$ used successfully in academia and industry since 1978

Main Synchronization Result

Condition on network parameters:

- network connectivity > network’s non-uniformity + network’s losses,

1. Non-Uniform Kuramoto Model:
   - exponential synchronization: phase locking & frequency entrainment
   - guaranteed region of attraction: $|\theta_i(t_0) - \theta_j(t_0)| < \pi/2 - \varphi_{\text{max}}$
   - gap in condition determines ultimate phase locking
   - further conditions on $\varphi_{ij}$ and $\omega_i$: explicit synchronization frequency, synchronization rates, exponential phase synchronization
2. Power Network Model:
   - there exists $\epsilon$ sufficiently small such that for all $t \geq 0$
     $$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = O(\epsilon).$$
   - for $\epsilon$ and network losses $\varphi_{ij}$ sufficiently small, $O(\epsilon)$ error converges
Main Synchronization Result

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1. **Non-Uniform Kuramoto Model:**
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2. **Power Network Model:**
   - there exists $\epsilon$ sufficiently small such that for all $t \geq 0$
   - $\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = O(\epsilon)$.
   - for $\epsilon$ and network losses $\varphi_{ij}$ sufficiently small, $O(\epsilon)$ error converges

### Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$:

$D_i \dot{\theta}_i = \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$

- **Non-uniformity** in network: $D_i, \omega_i, P_{ij}, \varphi_{ij}$
- **Directed coupling** between oscillator $i$ and $j$
- **Phase shift** $\varphi_{ij}$ induces lossless and lossy coupling: $P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij}) = P_{ij} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + P_{ij} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$
- **Synchronization analysis** in multiple steps:
  - phase locking: $|\theta_i(t) - \theta_j(t)|$ becomes bounded
  - frequency entrainment: $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
  - phase synchronization: $|\theta_i(t) - \theta_j(t)| \to 0$

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3. **Synchronization analysis** (of non-uniform Kuramoto model)
   - Main synchronization result
   - **Sufficient condition** (based on weakest lossless coupling)
   - **Sufficient condition** (based on lossless algebraic connectivity)
4. **Structure-preserving power network models**
   - Kron-reduction of graphs
   - **Sufficient conditions** for synchronization
5. **Conclusions**

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$

**Condition (1) for synchronization:**

Assume the graph induced by $P = P^T$ is complete and

$$\frac{P_{\text{min}}}{D_{\text{max}}} \cos(\varphi_{\text{max}}) > \max_{(i,j)} \left( \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right) + \max_i \sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}).$$

Gap determines the ultimate lack of phase locking in a $\frac{\pi}{2}$ interval.
Synchronization of Non-Uniform Kuramoto Oscillators

Main proof ideas:

1. **Phase locking** in $\Delta(\gamma) \iff$ arc-length $V(\theta(t))$ is non-increasing

\[ V(\theta(t)) = \max\{|\dot{\theta}_i(t) - \dot{\theta}_j(t)|, i, j \in \{1, \ldots, n\} \} \]

\[ D^+ V(\theta(t)) \leq 0 \]

$\sim$ contraction property from consensus literature:
[...]

2. **Frequency entrainment** in $\Delta(\gamma) \iff$ consensus protocol in $\mathbb{R}^n$

\[ \frac{d}{dt} \dot{\theta}_i = -\sum_{j \neq i} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j), \]

where $a_{ij}(t) = \frac{P_{ij}}{D_i} \cos(\theta_i(t) - \theta_j(t) + \varphi_{ij}) > 0$ for all $t \geq 0$

Theorem: Phase locking and frequency entrainment (1)

Non-uniform Kuramoto with complete $P = P^T$

Assume minimal coupling larger than a critical value, i.e.,

\[ P_{\min} > P_{\text{critical}} := \frac{D_{\max}}{n \cos(\varphi_{\max})} \left( \max_{\{i,j\}} \left( \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right) + \max_i \sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \right) \]

Define $\gamma_{\min} = \arcsin(\cos(\varphi_{\max}) P_{\text{critical}} / P_{\min})$ and set of locked phases

\[ \Delta(\gamma) := \{ \theta \in \mathbb{T}^n | \max_{\{i,j\}} |\dot{\theta}_i - \dot{\theta}_j| \leq \gamma \} \]

Then $\forall \gamma \in [\gamma_{\min}, \pi - \varphi_{\max}]:$

1. **phase locking**: the set $\Delta(\gamma)$ is positively invariant
2. **frequency entrainment**: $\forall \theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_{\infty} \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$

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Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

Condition (2) for synchronization:
Assume the graph induced by $P = P^T$ is connected with unweighted Laplacian $L$ and weighted Laplacian $L(P_{ij} \cos(\varphi_{ij}))$ and

$$\lambda_2(L(P_{ij} \cos(\varphi_{ij}))) > f(D_i) \cdot \left(\frac{1}{\cos(\varphi_{\text{max}})}\right) \times$$
lossless connectivity non-uniform $D_i$s necessary phase locking

$$\left(\left\|\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}, \ldots, \ldots\right\|_2^2 + \sqrt{\lambda_{\text{max}}(L)} \left\|\ldots, \sum_{j} \frac{P_{ij}}{D_i} \sin(\varphi_{ij}), \ldots\right\|_2\right)$$
non-uniformity lossy coupling

Gap determines the admissible initial lack of phase locking in a $\pi$ interval.

Graph induced by $P = P^T$ is connected with unweighted Laplacian $L$, incidence matrix $H$, and weighted Laplacian $L(P_{ij} \cos(\varphi_{ij}))$. Assume algebraic connectivity is larger than a critical value, i.e.,

$$\lambda_2(L(P_{ij} \cos(\varphi_{ij}))) > \lambda_{\text{critical}} := \frac{\|HD^{-1}\omega\|_2^2 + \sqrt{\lambda_{\text{max}}(L)} \left\|\ldots, \sum_{j} \frac{P_{ij}}{D_i} \sin(\varphi_{ij}), \ldots\right\|_2}{\cos(\varphi_{\text{max}}) \kappa/n \mu \min_{\{i,j\}, \{D_{\neq i,j}\}}},$$

where $\kappa := \sum_{k=1}^n \frac{1}{D_{\neq k}}, \mu := \sqrt{\min_{i,j} \{D_{i,j}\} / \max_{i,j} \{D_{i,j}\}}$

Define $\rho_{\text{max}} \in (\frac{\pi}{2} - \varphi_{\text{max}}, \pi)$ by $\text{sinc}(\rho_{\text{max}}) = \frac{\lambda_{\text{critical}} \cos(\varphi_{\text{max}})}{\lambda_2(c(t(L_{i,j} \cos(\varphi_{ij})))^2)}$

1) **phase locking**: $\forall \rho \in (\pi/2 - \varphi_{\text{max}}, \rho_{\text{max}})$, $\forall \|H0(0)\|_2 \leq \mu \rho$, there is $T \geq 0$ such that $\|H0(t)\|_2 < \pi/2 - \varphi_{\text{max}}$ for all $t > T$

2) **frequency entrainment**: if $\|H0(0)\|_2 \leq \mu \rho$ the frequencies $\hat{\theta}_i(t)$ synchronize exponentially to some frequency $\hat{\theta}_\infty \in [\hat{\theta}_{\text{min}}(0), \hat{\theta}_{\text{max}}(0)]$

Main proof ideas:
- **Phase locking** via ultimate boundedness arguments
- **Frequency entrainment** for $t > T \iff$ consensus protocol in $\mathbb{R}^n$

$$\frac{d}{dt} \theta_i = -\sum_{j \neq i} a_{ij}(t)(\hat{\theta}_i - \hat{\theta}_j),$$
where $a_{ij}(t) = \frac{P_{ij}}{D_i} \cos(\varphi_{ij} - (\theta_i(t) - \theta_j(t) + \varphi_{ij})) > 0$ for all $t > T$
Simulation data:
- initial phases mostly clustered besides red phasor
- disturbance in yellow phasor for $\epsilon \in [1.5s, 2.5s]$
- $\epsilon = 0.3s$ & network is non-uniform

\[ \Rightarrow \text{sufficient conditions for synchronization are satisfied} \]

Result: singular perturbation analysis is accurate ✓
both models synchronize ✓

Structure-preserving power network models

So far we considered a network-reduced power system model:
- network reduced to active nodes (generators)
- synchronization conditions on $\lambda_2(P)$ and $P_{min}$
- all-to-all reduced admittance matrix $Y_{\text{reduced}} \sim P$
  (for uniform voltage levels)

Topological non-reduced network-preserving power system model:
- boundary nodes (generators) & interior nodes (buses)
- topological bus admittance matrix $Y_{\text{network}}$
- Schur-complement relationship:
  $Y_{\text{reduced}} = Y_{\text{network}} / Y_{\text{interior}}$
  c.f. “Kron reduction”, “Dirichlet-to-Neumann map”
**Kron reduction** of a graph with Laplacian matrix $Y_{\text{network}}$, boundary nodes $\blacksquare$, and interior nodes $\bullet$:

1. Subsequent one-step removal of a single interior node $\bullet$: Topological evolution of the graph:

   1) $\rightarrow$ 2) $\rightarrow$ 3) $\rightarrow$ 4) $\rightarrow$ $\cdots$ 7)

2. Algebraic evolution of Laplacian matrix: $Y_{\text{network}}^{k+1} = Y_{\text{network}}^k / \bullet$

3. Fully reduced Laplacian $Y_{\text{reduced}}$ given by Schur complement:

   $Y_{\text{network}} \xrightarrow{\text{Schur complement}} Y_{\text{reduced}} = \frac{Y_{\text{network}}}{Y_{\text{interior}}}$

4. Graph-theoretic and algebraic properties of **Kron reduction** process:
   - Symmetric & irreducible Laplacians closed under Schur complement
   - Interior network connected $\Rightarrow$ reduced network complete
   - Spectral interlacing property: $\lambda_2(Y_{\text{reduced}}) \geq \lambda_2(Y_{\text{network}}) \Rightarrow$ algebraic connectivity $\lambda_2$ is non-decreasing
   - Effective resistance among boundary nodes $\blacksquare$ is invariant
   - For boundary nodes $\blacksquare$: effective resistance $R(i,j)$ uniform $\Leftrightarrow$ coupling $Y_{\text{reduced}}(i,j)$ uniform $\Leftrightarrow 1/R(i,j) = \frac{\alpha}{2} |Y_{\text{reduced}}(i,j)|$

---

**Structure-preserving power network models**

$Y_{\text{network}} \xrightarrow{\text{Kron reduction}} Y_{\text{reduced}} = \frac{Y_{\text{network}}}{Y_{\text{interior}}}$

**Assumption I**: lossless network, zero shunt admittances (no self loops)

- **Spectral condition for synchronization**: $\lambda_2(P) \geq \ldots$ becomes

$$\lambda_2(\Im(-Y_{\text{network}})) > \left| \begin{bmatrix} \omega_2 & -\omega_1 & \cdots \\ D_2 & D_1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \right|_2 \frac{f(D_i)}{E^2}$$

**Assumption II**: effective resistance $R$ among boundary nodes is uniform

- **Resistance-based condition for synchronization**: $nP_{\min} \geq \ldots$ becomes

$$\frac{1}{R} > \max_{(i,j)} \left\{ \frac{\omega_j}{D_i} - \frac{\omega_i}{D_j} \right\} \frac{D_{\max}}{2E^2}$$

---

**Conclusions**

**Summary**:

- Open problem in synchronization and transient stability in power networks: relation to underlying network state, parameters, and topology

- Time-varying Consensus Protocols:
  - Singular perturbation analysis and algebraic graph theory

- Non-uniform Kuramoto Oscillators

- Kuramoto, consensus, and nonlinear control tools

- Ongoing and Future Work:
  - relation to network topology, clustering, and scalability
  - synchronization in optimal power flow problems
**Further Results**

**Synchronization of Non-Uniform Kuramoto Oscillators**

**Theorem: A refined result on frequency entrainment**

Assume there exists $\gamma \in (0, \pi/2)$ such that the phases are locked in the set $\Delta(\gamma)$ and the graph induced by $P$ has globally reachable node.

1) $\forall \theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to $\theta_\infty \in [\theta_{\min}(0), \theta_{\max}(0)]$.

2) If $P = P^T$ and $\varphi_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$, then $\forall \theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ synchronize exp. to the weighted mean frequency

$$\Omega = \frac{1}{\sum_i D_i} \sum_i D_i \omega_i$$

and the exponential synchronization rate is no worse than

$$\lambda_{fe} = -\lambda_2(L(P_{ij})) \cos(\gamma) \cos(\angle(D\mathbf{1}, \mathbf{1}))^2 \bigg/ \frac{D_{\max}}{\text{slowest connectivity}}$$

Result can be reduced to [N. Chopra et al. ’09].

**Theorem: A result on phase synchronization**

Assume the graph induced by $P$ has a globally reachable node, $\varphi_{ij} = 0$, and $\omega_i/D_i = \bar{\omega}$ for all $i \in \{1, \ldots, n\}$.

1) $\forall \theta(0) \in \{ \theta \in \mathbb{T}^n : \max_{i,j} |\theta_i - \theta_j| < \pi \}$ the phases $\theta_i(t)$ synchronize exponentially to $\theta_\infty(t) \in [\theta_{\min}(0), \theta_{\max}(0)]$ + $\bar{\omega} t$; and

2) if $P = P^T$ and $\forall \|H\theta(0)\|_2 \leq \mu \rho$ with $\rho \in [0, \pi)$, then

$$\theta_\infty(t) = \frac{\sum_i D_i \theta_i(0)}{\sum_i D_i} + \bar{\omega} t$$

and the exponential sync. rate is no worse than

$$\lambda_{ps} = -\lambda_2(L(P_{ij})) \frac{\text{sinc}(\rho)}{\theta(0)} \frac{\text{weighting of } D_i}{\text{connectivity}}$$

Results can be reduced to [Z. Lin et al. ’07] and [A. Jadbabaie et al. ’04].