Synchronization in Power Networks and in Non-uniform Kuramoto Oscillators

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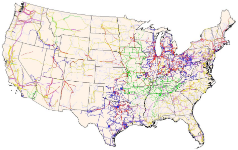
Intro: North American power grid



". . . the largest and most complex machine engineered by humankind."
[P. Kundur '94, V. Vittal '03, . . .]

". . . the greatest engineering achievement of the 20th century."
[National Academy of Engineering '10]

1. large-scale, complex, nonlinear, and rich dynamic behavior
2. 100 years old and operating at its capacity limits

$\Rightarrow$ recent blackouts: New England '03 + Italy '03, Brazil '09

Intro: Transient Stability in Power Networks

The New York Times

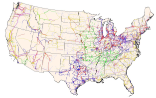
THE BLACKOUT OF 2003: Failure Reveals Creaky System, Experts Believe 8/15/2003

Energy is one of the top three national priorities

Expected additional synergetic effects in future "**smart grid**":

$\Rightarrow$ increasing complexity and renewable stochastic power sources

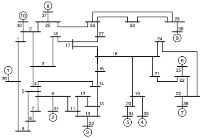
$\Rightarrow$ increasingly many transient disturbances to be detected and rejected

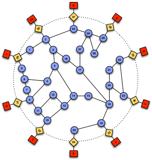


Transient Stability: Generators have to maintain synchronism in presence of large transient disturbances such as faults or loss of

- transmission lines and components,
- generation or load.

Intro: New England power grid





Power network topology:

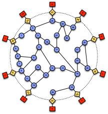
1. n generators ■, each connected to a generator terminal bus ◊
2. n generators terminal buses ◊ and m load buses ● form connected graph
3. admittance matrix $Y_{\text{network}} \in \mathbb{C}^{(2n+m)\times(2n+m)}$ characterizes the network

Intro: Mathematical Model of a Power Network

- generator nodes ■: swing equation for generator i

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + P_{mi} - P_{ei}$$

$\theta_i(t)$ is measured w.r.t. a 60Hz rotating frame



- network-preserving model: power flow equations for passive nodes ◇ & ● ⇒ DAE system

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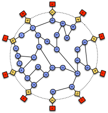

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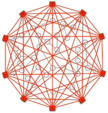

- network-reduced model: reduction to ■ nodes with all-to-all reduced (transfer) admittance matrix Y_{ij}

$$P_{ei} = E_i^2 G_{ii} + \sum\nolimits_{j \neq i} E_i E_j |Y_{ij}| \sin(\theta_i - \theta_j + \varphi_{ij})$$

Classic model

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

Intro: Transient Stability Analysis in Power Networks

Classic Model

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

Transient stability and synchronization:

- frequency equilibrium: $(\dot{\theta}_i, \ddot{\theta}_i) = (0, 0)$ for all i
- synchronous equilibrium: $|\theta_i - \theta_j|$ bounded & $\dot{\theta}_i - \dot{\theta}_j = 0$ for all $\{i, j\}$

Classic problem setup in transient stability analysis:

1. power network in stable frequency equilibrium
2. → transient network disturbance and fault clearance
3. stability analysis of a new frequency equilibrium in post-fault network

Intro: Transient Stability Analysis in Power Networks

Classic Model

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Transient stability and synchronization:

Classic analysis methods: Hamiltonian arguments

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i - \nabla_i U(\theta)^T$$

Energy function analysis, (extended) invariance principle, analysis of reduced gradient flow [N. Kakimoto et al. '78, H.-D. Chiang et al. '94]

$$\dot{\theta}_i = -\nabla_i U(\theta)^T$$

Key objective: compute domain of attraction via numerical methods

Intro: Transient Stability Analysis in Power Networks

Classic Model

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Transient stability and synchronization:

Classic analysis methods: Hamiltonian arguments

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i - \nabla U_i(\theta)^T \quad \rightsquigarrow \quad \dot{\theta}_i = -\nabla_i U(\theta)^T$$

$\Rightarrow$ **Open problem** [D. Hill and G. Chen '06]: power sys $\overset{?}{\rightsquigarrow}$ network:

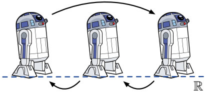
transient stability, performance, and robustness of a power network
$\overset{?}{\rightsquigarrow}$ state, parameters, and topology of underlying network

Detour – Consensus Protocols & Kuramoto Oscillators

Consensus protocol in $\mathbb{R}^n$:

$$\dot{x}_i = -\sum\nolimits_{j\neq i} a_{ij}(x_i - x_j)$$

- n **agents** with state $x_i \in \mathbb{R}$ and connected **graph** with weights $a_{ij} > 0$
- **objective** is state agreement: $x_i(t) - x_j(t) \to 0$
- **application:** social networks, computer science, systems theory
  robotic rendezvous, distributed computing, filtering and control, ...
- **some references:** [M. DeGroot '74, J. Tsitsiklis '84, ...]

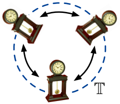


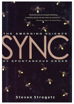
Detour – Consensus Protocols & Kuramoto Oscillators

Kuramoto model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i - \theta_j)$$

- **oscillators** with phase $\theta_i \in \mathbb{T}$, frequency $\omega_i \in \mathbb{R}$, **complete** coupling
- **objective** is synchronization: $\theta_i(t) - \theta_j(t)$ bounded, $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
- **application** in physics, biology, engineering:
  coupled neurons, Josephson junctions, motion coordination, ...
- **some references:** [Y. Kuramoto '75, A. Winfree '80, ...]





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Kuramoto model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i - \theta_j)$$

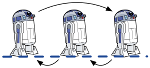
- degrees of synchronization:
  1. phase locking: $|\theta_i - \theta_j|$ bounded
  2. frequency entrainment: $\dot{\theta}_i = \dot{\theta}_j$
  3. phase synchronization: $\theta_i = \theta_j$
- known that
  1. K large & $|\omega_i - \omega_j|$ small $\Rightarrow$ frequency entrainment & phase locking
  2. additionally, for $\omega_i = \omega_j$ $\Rightarrow$ phase synchronization

Intro: The Big Picture

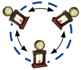
Open problem in synchronization and transient stability in **power networks:** relation to underlying network state, parameters, and topology

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i-\theta_j+\varphi_{ij})$$

Consensus Protocols:



Kuramoto Oscillators:



$$\dot{x}_i = -\sum\nolimits_{j\neq i} a_{ij}(x_i-x_j)$$

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i-\theta_j)$$

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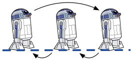
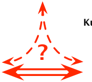
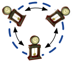




Kuramoto Oscillators:



$$\dot{x}_i = -\sum\nolimits_{j\neq i} a_{ij}(x_i-x_j)$$

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i-\theta_j)$$

Possible connection has often been hinted at in the literature!

Power systems: [D. Subbarao et al., '01, G. Filatrella et al., '08, V. Fioriti et al., '09]
Networked control: [D. Hill et al., '06, M. Arcak, '07]
Dynamical systems: [H. Tanaka et al., '97]

Outline

1. Introduction
   1. Power network model
   2. Synchronization and transient stability
   3. Consensus protocol and Kuramoto oscillators
2. Singular perturbation analysis
   (to relate power network and Kuramoto model)
3. Synchronization analysis (of non-uniform Kuramoto model)
   1. Main synchronization result
   2. Sufficient condition (based on weakest lossless coupling)
   3. Sufficient condition (based on lossless algebraic connectivity)
4. Structure-preserving power network models
   1. Kron-reduction of graphs
   2. Sufficient conditions for synchronization
5. Conclusions

From the swing equations to the Kuramoto model

$$\boxed{\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i-\theta_j+\varphi_{ij})}$$

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i-\theta_j) \implies \boxed{D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i-\theta_j+\varphi_{ij})}$$

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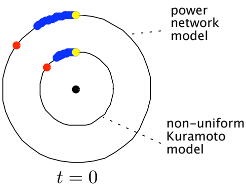
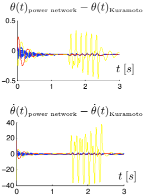
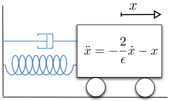


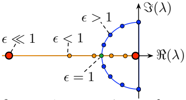


Singular Perturbation Analysis

Time-scale separation in power network model:

- **Motivation:** harmonic oscillator





for $\epsilon \ll 1 \Rightarrow$ two time-scales

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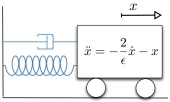
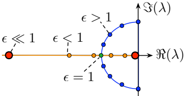


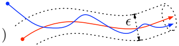


for $\epsilon \ll 1 \Rightarrow$ two time-scales

- **Singular perturbation analysis:**

full system $\begin{cases}\dot{x} = f(x,z) \\ \epsilon\dot{z} = g(x,z)\end{cases} \xrightarrow{\epsilon = 0} \begin{cases}\dot{x} = f(x,h(x)) \\ z = h(x)\end{cases}$ reduced (slow) system quasi–steady state

initial error: $z(0) \neq h(x(0))$



approximation error in slow time–scale: $\mathcal{O}(\epsilon)$

error exp. stable in fast time–scale

Singular Perturbation Analysis

Time-scale separation in power network model:

- **power network model:**

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

- singular perturbation parameter: $\epsilon = \frac{M_{\max}}{\pi f_0 D_{\min}}$
- reduced system for $\epsilon = 0$ is a **non-uniform Kuramoto model**:

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Tikhonov's Theorem:
Assume the non-uniform Kuramoto model synchronizes exponentially.
Then $\forall\,(\theta(0), \dot{\theta}(0))$ there exists $\epsilon^* > 0$ such that $\forall\,\epsilon < \epsilon^*$ and $\forall\, t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon)\,.$$

Singular Perturbation Analysis

Discussion of the assumption $\epsilon = \frac{M_{\text{max}}}{\pi f_0 D_{\text{min}}}$ sufficiently small:

1. physical interpretation: damping and sync on separate time-scales
2. classic assumption in literature on coupled oscillators: over-damped mechanical pendula and Josephson junctions
3. physical reality: with generator internal control effects $\epsilon \in \mathcal{O}(0.1)$
4. simulation studies show accurate approximation even for large ϵ
5. first-order and second-order models have the same equilibria with the same stability properties, and the regions of attractions are bounded by the same separatrices (independent of ϵ)
6. non-uniform Kuramoto model corresponds to reduced gradient system $\dot{\theta}_i = -\nabla_i U(\theta)^T$ used successfully in academia and industry since 1978

Outline

Main Synchronization Result

Condition on network parameters:

network connectivity > network's non-uniformity + network's losses,

1. **Non-Uniform Kuramoto Model:**

$\Rightarrow$ exponential synchronization: phase locking & frequency entrainment

$\Rightarrow$ guaranteed region of attraction: $|\theta_i(t_0) - \theta_j(t_0)| < \pi/2 - \varphi_{\text{max}}$

$\Rightarrow$ gap in condition determines ultimate phase locking

$\Rightarrow$ further conditions on φ_{ij} and ω_i: explicit synchronization frequency, synchronization rates, exponential phase synchronization

2. **Power Network Model:**

$\Rightarrow$ there exists ϵ sufficiently small such that for all $t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon).$$

$\Rightarrow$ for ϵ and network losses φ_{ij} sufficiently small, $\mathcal{O}(\epsilon)$ error converges

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Outline

Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$:

$$D_i \dot{\theta}_i = \omega_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

- **Non-uniformity** in network: D_i, ω_i, P_{ij}, φ_{ij}
- **Directed coupling** between oscillator i and j
- **Phase shift** φ_{ij} induces lossless and lossy couling:
  $P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij}) = P_{ij} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + P_{ij} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$
- **Synchronization analysis** in multiple steps:
  1. phase locking: $|\theta_i(t) - \theta_j(t)|$ becomes bounded
  2. frequency entrainment: $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
  3. phase synchronization: $|\theta_i(t) - \theta_j(t)| \to 0$

Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum\nolimits_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

Condition (1) for synchronization:
Assume the graph induced by $P = P^T$ is complete and

$$\underbrace{n \frac{P_{\min}}{D_{\max}} \cos(\varphi_{\max})}_{\text{worst lossless coupling}} > \underbrace{\max_{\{i,j\}} \left(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right)}_{\text{worst non-uniformity}} + \underbrace{\max_i \sum\nolimits_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij})}_{\text{worst lossy coupling}}.$$

Gap determines the ultimate lack of phase locking in a $\frac{\pi}{2}$ interval.

Synchronization of Non-Uniform Kuramoto Oscillators

Classic (uniform) Kuramoto Model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

Condition (1) for synchronization:

$$K > \omega_{\max} - \omega_{\min}$$

Gap determines the ultimate lack of phase locking in a $\frac{\pi}{2}$ interval.

Condition (1) strictly improves existing bounds on Kuramoto model: [F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, J.L. van Hemmen et al. '93].

Necessary condition for sync of n oscillators: $K > \frac{n}{2(n-1)}(\omega_{\max} - \omega_{\min})$ [J.L. van Hemmen et al. '93, A. Jadbabaie et al. '04, N. Chopra et al. '09]

Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: Phase locking and frequency entrainment (1)

Non-uniform Kuramoto with complete $P = P^T$
Assume minimal coupling larger than a critical value, i.e.,

$$P_{\min} > P_{\text{critical}} := \frac{D_{\max}}{n \cos(\varphi_{\max})} \left(\max_{\{i,j\}} \left(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right) + \max_i \sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \right)$$

Define $\gamma_{\min} = \arcsin\left(\cos(\varphi_{\max}) \frac{P_{\text{critical}}}{P_{\min}}\right)$ and set of locked phases

$$\Delta(\gamma) := \{\theta \in \mathbb{T}^n | \max_{\{i,j\}} |\theta_i - \theta_j| \leq \gamma\}$$

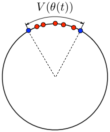
Then $\forall\, \gamma \in [\gamma_{\min}, \frac{\pi}{2} - \varphi_{\max})$

1) **phase locking**: the set $\Delta(\gamma)$ is positively invariant
2) **frequency entrainment:** $\forall\, \theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$

Synchronization of Non-Uniform Kuramoto Oscillators

Main proof ideas:

1. **Phase locking** in $\Delta(\gamma) \Leftrightarrow$ arc-length $V(\theta(t))$ is non-increasing



$$\Leftrightarrow \begin{cases} V(\theta(t)) = \max\{|\theta_i(t) - \theta_j(t)| \mid i,j \in \{1,\dots,n\}\} \\ D^+ V(\theta(t)) \overset{!}{\leq} 0 \end{cases}$$

$\sim$ **contraction property** from consensus literature: [D. Bertsekas et al. '94, L. Moreau '04 & '05, Z. Lin et al. '08, ...]

2. **Frequency entrainment** in $\Delta(\gamma) \Leftrightarrow$ consensus protocol in $\mathbb{R}^n$

$$\frac{d}{dt}\dot{\theta}_i = -\sum_{j \neq i} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j)\,,$$

where $a_{ij}(t) = \frac{P_{ij}}{D_i} \cos(\theta_i(t) - \theta_j(t) + \varphi_{ij}) > 0$ for all $t \geq 0$

Outline

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum\nolimits_{j\neq i} \frac{P_{ij}}{D_i}\cos(\varphi_{ij})\sin(\theta_i-\theta_j) + \frac{P_{ij}}{D_i}\sin(\varphi_{ij})\cos(\theta_i-\theta_j)$$

Condition (2) for synchronization:
Assume the graph induced by $P = P^T$ is connected with unweighted Laplacian L and weighted Laplacian $L(P_{ij}\cos(\varphi_{ij}))$ and

$$\underbrace{\lambda_2(L(P_{ij}\cos(\varphi_{ij})))}_{\text{lossless connectivity}} > \underbrace{f(D_i)}_{\text{non-uniform } D_i\text{s}} \cdot \underbrace{(1/\cos(\varphi_{\max}))}_{\text{necessary phase locking}} \times$$
$$\Big(\underbrace{\Big|\Big|\Big[\dots,\frac{\omega_i}{D_i}-\frac{\omega_j}{D_j},\dots\Big]\Big|\Big|_2}_{\text{non-uniformity}} + \underbrace{\sqrt{\lambda_{\max}(L)}\,\Big|\Big|\Big[\dots,\sum\nolimits_j \frac{P_{ij}}{D_i}\sin(\varphi_{ij}),\dots\Big]\Big|\Big|_2}_{\text{lossy coupling}}\Big)$$

Gap determines the admissible initial lack of phase locking in a π interval.

Classic (uniform) Kuramoto Model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i-\theta_j)$$

Condition (2) for synchronization:

$$K > ||[\dots,\omega_i-\omega_j,\dots]||_2$$

Gap determines the admissible initial lack of phase locking in a π interval.

Condition (2) corresponds to the bound in [N. Chopra et al. '09].

Theorem: Phase locking and frequency entrainment (2)

Graph induced by $P = P^T$ is connected with unweighted Laplacian L, incidence matrix H, and weighted Laplacian $L(P_{ij}\cos(\varphi_{ij}))$.
Assume algebraic connectivity is larger than a critical value, i.e.,

$$\lambda_2(L(P_{ij}\cos(\varphi_{ij}))) > \lambda_{\text{critical}} := \frac{\|HD^{-1}\omega\|_2 + \sqrt{\lambda_{\max}(L)}\,||[\dots,\sum_j \frac{P_{ij}}{D_i}\sin(\varphi_{ij}),\dots]||_2}{\cos(\varphi_{\max})(\kappa/n)\mu\min_{\{i,j\}}\{D_{\neq\{i,j\}}\}},$$

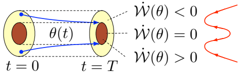
$$\text{where}\quad \kappa := \sum_{k=1}^{n}\frac{1}{D_{\neq k}},\quad \mu := \sqrt{\min_{i\neq j}\{D_iD_j\}/\max_{i\neq j}\{D_iD_j\}}$$

Define $\rho_{\max} \in (\frac{\pi}{2}-\varphi_{\max},\pi)$ by $\text{sinc}(\rho_{\max}) = \frac{\lambda_{\text{critical}}\cos(\varphi_{\max})}{\lambda_2(L(P_{ij}\cos(\varphi_{ij})))(\pi/2-\varphi_{\max})}$.

1) **phase locking:** $\forall\,\rho \in (\pi/2-\varphi_{\max},\rho_{\max})$, $\forall\,\|H\theta(0)\|_2 \le \mu\rho$, there is $T \ge 0$ such that $\|H\theta(t)\|_2 < \pi/2-\varphi_{\max}$ for all $t > T$
2) **frequency entrainment:** if $\|H\theta(0)\|_2 \le \mu\rho$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0),\dot{\theta}_{\max}(0)]$

Main proof ideas:

1. **Phase locking** via ultimate boundedness arguments



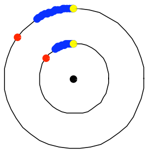
$$\mathcal{W}(\theta) = \sum\nolimits_{\{i,j\}} \frac{1}{2\prod_{k\neq i,j}^n D_k}|\theta_i-\theta_j|^2$$

2. **Frequency entrainment** for $t > T \Leftrightarrow$ consensus protocol in $\mathbb{R}^n$

$$\frac{d}{dt}\dot{\theta}_i = -\sum\nolimits_{j\neq i} a_{ij}(t)(\dot{\theta}_i-\dot{\theta}_j),$$

where $a_{ij}(t) = \frac{P_{ij}}{D_i}\cos(\theta_i(t)-\theta_j(t)+\varphi_{ij}) > 0$ for all $t > T$

Simulation Studies



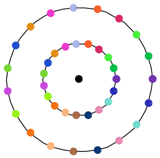
Simulation data:

- initial phases mostly clustered besides red phasor
- disturbance in yellow phasor for $\in [1.5\text{s}, 2.5\text{s}]$
- $\epsilon = 0.3\text{s}$ & network is non-uniform

$\Rightarrow$ sufficient conditions for synchronization are **satisfied**

Result: singular perturbation analysis is accurate ✓
both models synchronize ✓

Simulation Studies



Simulation data:

- worst-case initial phase-differences: $\theta_i(0)$ in **splay state**
- $\epsilon = 0.12\text{s}$ is small
- **strongly** non-uniform network

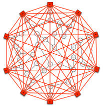
$\Rightarrow$ sufficient conditions for synchronization are **not satisfied**

Result: singular perturbation analysis is accurate ✓
both models synchronize ✓

Outline

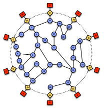
Structure-preserving power network models

So far we considered a **network-reduced** power system model:



- network reduced to active nodes (generators)
- synchronization conditions on $\lambda_2(P)$ and $P_{\min}$
- all-to-all reduced admittance matrix $Y_{\text{reduced}} \sim P$ (for uniform voltage levels)

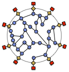
Topological non-reduced **network-preserving** power system model:



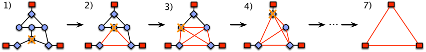
- boundary nodes (generators) & interior nodes (buses)
- topological bus admittance matrix $\mathbf{Y}_{\text{network}}$
- Schur-complement relationship:
  $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$
  c.f. "Kron reduction", "Dirichlet-to-Neumann map"

Detour – Kron reduction of graphs

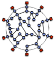
Kron reduction of a graph with Laplacian matrix $\mathbf{Y}_{\text{network}}$, boundary nodes ■, and interior nodes ●



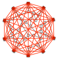
1. Subsequent one-step removal of a single interior node ●:
   - Topological evolution of the graph:



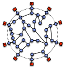
   - Algebraic evolution of Laplacian matrix: $\mathbf{Y}^{k+1}_{\text{reduced}} = \mathbf{Y}^{k}_{\text{reduced}}/$ ●
2. Fully reduced Laplacian $Y_{\textbf{reduced}}$ given by Schur complement:



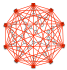
$\mathbf{Y}_{\text{network}}$ $\quad$ $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}}/Y_{\text{interior}}$



Detour – Kron reduction of graphs



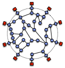
$\mathbf{Y}_{\text{network}}$ $\quad$ $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}}/Y_{\text{interior}}$



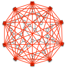
Graph-theoretic and algebraic properties of **Kron reduction** process:

1. Symmetric & irreducible Laplacians closed under Schur complement
2. interior network connected $\Rightarrow$ reduced network complete
3. Spectral interlacing property: $\lambda_2(Y_{\text{reduced}}) \geq \lambda_2(\mathbf{Y}_{\text{network}})$
   $\Rightarrow$ algebraic connectivity λ_2 is non-decreasing
4. Effective resistance among boundary nodes ■ is invariant
5. For boundary nodes ■: effective resistance $R(i,j)$ uniform
   $\Leftrightarrow$ coupling $Y_{\text{reduced}}(i,j)$ uniform $\Leftrightarrow$ $1/R(i,j) = \frac{n}{2}|Y_{\text{reduced}}(i,j)|$

Structure-preserving power network models



$\mathbf{Y}_{\text{network}}$ $\quad$ $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}}/Y_{\text{interior}}$



Assumption I: lossless network, zero shunt admittances (no self loops)

1. **Spectral condition for synchronization:** $\lambda_2(P) \geq \dots$ becomes

$$\lambda_2(\Im(-\mathbf{Y}_{\text{network}})) > \left\| \left(\frac{\omega_2}{D_2} - \frac{\omega_1}{D_1}, \dots\right)\right\|_2 \cdot \frac{f(D_i)}{E^2}$$

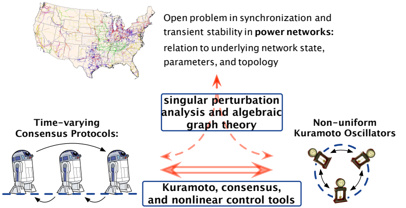
Assumption II: effective resistance R among boundary nodes is uniform

2. **Resistance-based condition for synchronization:** $nP_{\min} \geq \dots$ becomes

$$\frac{1}{R} > \max_{\{i,j\}}\left\{\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}\right\} \cdot \frac{D_{\max}}{2E^2}$$

Conclusions

Summary:



Ongoing and Future Work:

- relation to network topology, clustering, and scalability
- synchronization in optimal power flow problems

Further Results

Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: A refined result on frequency entrainment

Assume there exists $\gamma \in (0, \pi/2)$ such that the phases are locked in the set $\Delta(\gamma)$ and the graph induced by P has globally reachable node.

1) $\forall\, \theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$.

2) If $P = P^T$ & $\varphi_{ij} = 0$ for all $i, j \in \{1, \dots, n\}$, then $\forall\, \theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ synchronize exp. to the weighted mean frequency

$$\Omega = \frac{1}{\sum_i D_i} \sum\nolimits_i D_i \omega_i$$

and the exponential synchronization rate is no worse than

$$\lambda_{\text{fe}} = -\underbrace{\lambda_2(L(P_{ij}))}_{\text{connectivity}}\ \underbrace{\cos(\gamma)}_{\Delta(\gamma)}\underbrace{\cos(\angle(D\mathbf{1}, \mathbf{1}))^2}_{\mathbf{1} \not\parallel D\mathbf{1}} /\ \underbrace{D_{\max}}_{\text{slowest}}$$

Result can be reduced to [N. Chopra et al. '09].

Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: A result on phase synchronization

Assume the graph induced by P has a globally reachable node, $\varphi_{ij} = 0$, and $\omega_i / D_i = \bar{\omega}$ for all $i \in \{1, \dots, n\}$.

1) $\forall\, \theta(0) \in \{\theta \in \mathbb{T}^n : \max_{\{i,j\}} |\theta_i - \theta_j| < \pi\}$ the phases $\theta_i(t)$ synchronize exponentially to $\theta_\infty(t) \in [\theta_{\min}(0), \theta_{\max}(0)] + \bar{\omega} t$; and

2) if $P = P^T$ and $\forall\, \|H\theta(0)\|_2 \leq \mu\rho$ with $\rho \in [0, \pi)$, then

$$\theta_\infty(t) = \frac{\sum_i D_i \theta_i(0)}{\sum_i D_i} + \bar{\omega} t$$

and the exponential sync. rate is no worse than

$$\lambda_{\text{ps}} = -\underbrace{(\kappa/n) \min_{\{i,j\}} \{D_{\neq\{i,j\}}\}}_{\text{weighting of } D_i}\ \underbrace{\text{sinc}(\rho)}_{\theta(0)}\ \underbrace{\lambda_2(L(P_{ij}))}_{\text{connectivity}}$$

Results can be reduced to [Z. Lin et al. '07] and [A. Jadbabaie et al. '04].