Adaptive and Distributed Coordination:

Behaviors, Sensors, and Geometric Optimization

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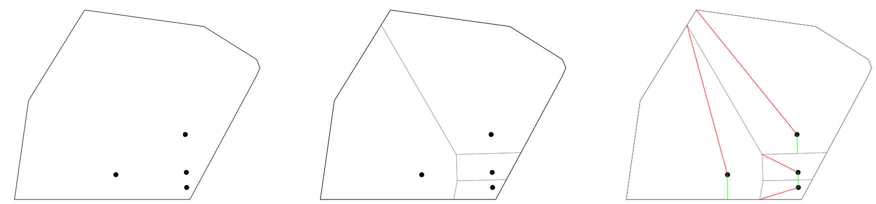
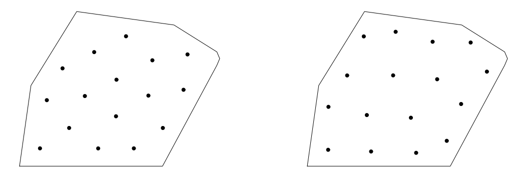
Scalable robotic coordination and nonsmooth dynamical systems



Basic behaviors
"move away from closest"
"move towards furthest"



Conjectures: critical points? stop? optimize? local minima? equidistant?

Outline

(i) graphical proof
(ii) geometric nonsmooth tools
(iii) 1-center problems
(iv) multi-center problems

References

http://motion.csl.uiuc.edu/~bullo

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- J. Cortés and F. Bullo. From geometric optimization and nonsmooth analysis to distributed coordination algorithms. In *IEEE Conf. on Decision and Control*, Maui, Hawaii, December 2003. To appear
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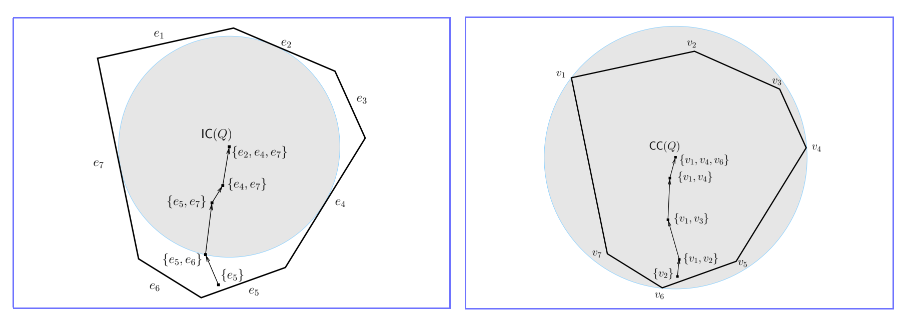
1-center optimization problems

$$\mathrm{sm}_Q(p) = \min\{\|p-q\| \mid q \in \partial Q\} \qquad \mathrm{lg}_Q(p) = \max\{\|p-q\| \mid q \in \partial Q\}$$

"move away from closest edge" converges to incenter of Q = argmax sm_Q

"move toward furthest vertex" converges to circumcenter of Q = argmin lg_Q

Geometric "proof"



Multi-center optimization problems:

cost functions for sphere packing and disk covering

$$\mathcal{H}_{\mathsf{SP}}(P) = \text{minimum } \tfrac{1}{2} \text{ distance between sites and to wall}$$

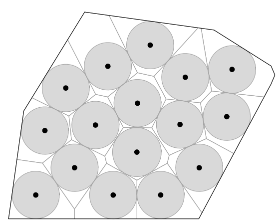
$$= \min\left\{\tfrac{1}{2}\|p_i - p_j\| \mid i \neq j \in \{1,\dots,n\}\right\} \qquad = \min_i \ \mathrm{sm}_{V_i(P)}(p_i)$$

$$\mathcal{H}_{\mathsf{DC}}(P) = \text{maximum distance between sites and Q}$$

$$= \max_{q\in Q} \ \min\{\|q - p_i\| \mid i \in \{1,\dots,n\}\} \qquad = \max_i \ \mathrm{lg}_{V_i(P)}(p_i)$$

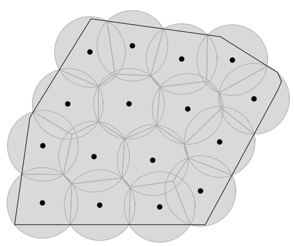
"move away from closest"

"move toward incenter"



"move towards furthest"

"move toward circumcenter"



Aggregate/network performance measures

Locational optimization

$$\min_P \mathcal{H}(P,\mathcal{V}) = \min_P E\left[\min_i \|q - p_i\|\right]$$

Locational optimization — worst case scenario

$$\min_P \mathcal{H}_{\mathsf{DC}}(P,\mathcal{V}) = \min_P \max_{q\in Q}\left[\min_i \|q - p_i\|\right]$$

Locational optimization — non-interference scenario

$$\max_P \mathcal{H}_{\mathsf{SP}}(P,\mathcal{V}) = \max_P \min_{i\neq j}\left[\tfrac{1}{2}\|p_i - p_j\|\right]$$

top-down: from cost function to gradient descent

bottom-up: given flow, does it optimize some appropriate cost?

Nonsmooth analysis

Clarke, Paden, Sastry, Shewitz, Bacciotti, Ceragioli

(i) Take locally Lipschitz and regular (LL&R) map $f\colon \mathbb{R}^N \to \mathbb{R}$.

(ii) Rademacher's Theorem: LL functions are differentiable a.e.

$$\partial f(x) = \mathrm{co}\left\{\lim_{i\to\infty} df(x_i) \mid x_i \to x\,,\ x_i \notin \Omega_f \cup S\right\}$$

(iii) Useful Theorem: $f(x) = \min\{f_1(x),\dots,f_m(x)\}$,

$$\partial f(x) = \mathrm{co}\,\{\partial f_i(x) \mid i \text{ active at } x\}$$

(iv) Ln = least norm operator

(v) if x_0 extremum for f, then $0 \in \partial f(x_0)$
if $0 \notin \partial f(x_0)$, then

$$f(x_0 - \epsilon\,\mathrm{Ln}[\partial f](x_0)) \le f(x_0) - \frac{\epsilon}{2}\|\,\mathrm{Ln}[\partial f](x_0)\|^2$$

$$f(x_0 + \epsilon\,\mathrm{Ln}[\partial f](x_0)) \ge f(x_0) + \frac{\epsilon}{2}\|\,\mathrm{Ln}[\partial f](x_0)\|^2$$

Filippov solutions and set-valued Lie derivatives for inclusions

(i) $\dot{x} \in X(x)$ with X measurable and essentially locally bounded

(ii) Filippov solution is absolutely continuous function $t \in [t_0,t_1] \mapsto x(t)$ s.t.

$$\dot{x} \in K[X](x) = \mathrm{co}\left\{\lim_{i\to\infty} X(x_i) \mid x_i \to x\,,\ x_i \notin S\right\}$$

(iii) set-valued Lie derivative: (closed bounded interval)

$$\widetilde{\mathcal{L}}_X f(x) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ s.t. } \zeta\cdot v = a\,,\ \forall \zeta \in \partial f(x)\}$$

(iv) Useful Theorem: A LL&R function $f:\mathbb{R}^N \to \mathbb{R}$ along the Filippov solution $t \mapsto x(t)$ of differential inclusion X satisfies: $t \mapsto f(x(t))$ is differentiable a.e. and

$$\frac{\mathsf{d}}{\mathsf{d}t} f(x(t)) \in \widetilde{\mathcal{L}}_X f(x(t))$$

Nonsmooth LaSalle Invariance Principle for differential inclusions

If $f^{-1}(\leq f(x_0))$ bounded and $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$

$\Phi_t^X(x_0) \to$ largest weakly invariant in $\overline{\left\{x \in f^{-1}(\leq f(x_0)) \mid 0 \in \widetilde{\mathcal{L}}_X f(x)\right\}}$

Comments:

(i) weakly invariant = there is a solution inside set
strongly invariant = all solutions inside set

(ii) $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$ or empty

(iii) connected component of $f^{-1}(\leq f(x_0))$ containing x_0

(iv) e.g., nonsmooth gradient flow $\dot{x} = -\operatorname{Ln}[\partial f](x)$ converges to critical set

1-center optimization problems – revisited

- $\operatorname{sm}_Q, \operatorname{lg}_Q \colon Q \to \mathbb{R}$ are LL&R and

$$\partial \operatorname{sm}_Q(p) = \operatorname{co}\{\operatorname{vers}(p-e) \mid e \text{ active}\}$$
$$\partial \operatorname{lg}_Q(p) = \operatorname{co}\{\operatorname{vers}(p-v) \mid v \text{ active}\}$$

- "move away from closest edge" = $+$ gradient flow for sm_Q
  "move toward furthest vertex" = $-$ gradient flow for lg_Q

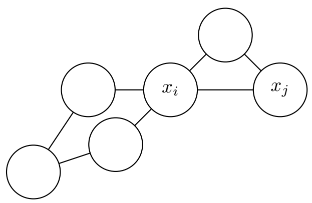
- Analytic proof: By LaSalle, gradient descent converges to $\operatorname{IC}, \operatorname{CC}$ because:

$$0 \in \partial \operatorname{sm}_Q(p) \iff p \in \operatorname{IC}(Q)$$
$$0 \in \partial \operatorname{lg}_Q(p) \iff p = \operatorname{CC}(Q)$$

- unique incenter and convergence in finite time if $0 \in \operatorname{int}(\partial \operatorname{lg}_Q)$, $0 \in \operatorname{int}(\partial \operatorname{sm}_Q)$

Distributed gradients for min/max network optimization

state: $x = (x_1, \dots, x_m)$
performance/penalty of ith agent:
$f_i = f_i(x_i, \mathcal{N}_i)$



$\max_x f(x) = \max_x \min_i f_i(x)$ or $\min_x f(x) = \min_x \max_i f_i(x)$

Generalized gradient descent: $\dot{x} = \pm \operatorname{Ln}[\partial f](x)$
However, **only** active x_i move and **only** active x_i reach optimum
Need to know who are active at **all** times: comparison must be performed
Hence, **centralized** solution with **local** critical points (saddle points)

Distributed generalized gradient descent: $\dot{x}_i = \pm \operatorname{Ln}[\partial_{x_i} f_i](x_i)$
Need to show: $\operatorname{Ln}[\partial f] \cdot \operatorname{Ln}[\partial_{x_i} f_i](x_i)$ sign definite (fewer/no saddle points)

Multi-center problems revisited

Rewrite $\mathcal{H}_{\mathsf{SP}}(P) = \min_i \; F_i(P)$ and $\mathcal{H}_{\mathsf{DC}}(P) = \max_i \; G_i(P)$

$$F_i(P) = \operatorname{sm}_{V_i(P)}(p_i) \qquad G_i(P) = \operatorname{lg}_{V_i(P)}(p_i)$$

Before sm_Q and lg_Q LL&R. Now, **dependence on region** makes analysis much more involved

Useful Theorem: Functions G_i and $F_i : Q \to \mathbb{R}$ are LL&R

- As a consequence, both $\mathcal{H}_{\mathsf{SP}}$ and $\mathcal{H}_{\mathsf{DC}}$ are LL&R.
- Moreover, closed-form expression for $\partial \mathcal{H}_{\mathsf{SP}}$ and $\partial \mathcal{H}_{\mathsf{DC}}$ as convex combinations of $\partial F_i(P)$ and $\partial G_i(P)$

Computation of $\partial G_i(P)$ and $\partial F_i(P)$ is challenging

$$\partial G_i(P) = \mathrm{co}\left\{\partial_v G_i(P) \in (\mathbb{R}^2)^n \mid v \in \mathrm{Ve}(V_i(P)) \text{ such that } G_i(P) = \|p_i - v\|\right\}$$

where if vertex v is **nondegenerate**

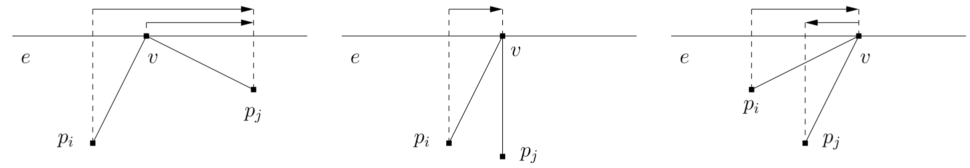
- $\partial_{v(i,j,k)} G_i(P) = \partial_{v(k,i,j)} G_k(P) = \partial_{v(j,k,i)} G_j(P) =$

$$(0, \dots, \underbrace{\mu(i,j,k)\,\mathrm{vers}(p_i - v)}_{i\text{th place}}, \dots, \underbrace{\mu(j,k,i)\,\mathrm{vers}(p_j - v)}_{j\text{th place}}, \dots, \underbrace{\mu(k,i,j)\,\mathrm{vers}(p_k - v)}_{k\text{th place}}, \dots, 0)$$

- $\partial_{v(e,i,j)} G_i(P) = \partial_{v(e,j,i)} G_j(P)$

$$= (0, \dots, \underbrace{\lambda(e,i,j)\,\mathrm{vers}(p_i - v)}_{i\text{th place}}, \dots, \underbrace{\lambda(e,j,i)\,\mathrm{vers}(p_j - v)}_{j\text{th place}}, \dots, 0)$$

- $\partial_{v(e,f,i)} G_i(P) = (0, \dots, 0, \underbrace{\mathrm{vers}(p_i - v)}_{i\text{th place}}, 0, \dots, 0).$



$$\mathrm{proj}_e(p_j - v(e,i,j)) = \lambda(e,i,j)\;\mathrm{proj}_e(p_j - p_i)$$

If vertex v is **degenerate** (i.e., determined by $d > 3$ elements -generators or edges), then there are $\binom{d-1}{2}$ pairs of elements determining v together with the generator p_i.

$$\partial_v G_i(P) = \mathrm{co}\left\{\partial_{v(\alpha,\beta,\gamma)} G_i(P) \mid \forall(\alpha,\beta,\gamma) \text{ determining } v\right\}$$

(i) Analogous expression for $\partial F_i(P)$

(ii) Note relation with $\partial\,\mathrm{sm}_{V_i}(p_i)$ and $\partial\,\mathrm{lg}_{V_i}(p_i)$ at fixed V_i

Critical points

$$\partial \mathcal{H}_{\mathsf{DC}}(P) = \mathrm{co}\{\partial G_i(P) \mid i \in I(P)\}$$
$$\partial \mathcal{H}_{\mathsf{SP}}(P) = \mathrm{co}\{\partial F_i(P) \mid i \in I(P)\}$$

If $0 \in \mathrm{int}\,\partial \mathcal{H}_{\mathsf{DC}}(P)$, then P is a strict local minimum, all generators have same cost, and P is a **circumcenter Voronoi configuration**

If $0 \in \mathrm{int}\,\partial \mathcal{H}_{\mathsf{SP}}(P)$, then P is a strict local maximum, all generators have same cost, and P is a generic **incenter Voronoi configuration**

Dynamical systems -revisited

Nonsmooth gradient descent

$$\dot{P} = -\,\mathrm{Ln}(\partial \mathcal{H}_{\mathsf{DC}})(P) \quad \text{or} \quad \dot{p}_i = -\pi_i(\mathrm{Ln}(\partial \mathcal{H}_{\mathsf{DC}})(p_1, \dots, p_n))$$
$$\dot{P} = +\,\mathrm{Ln}(\partial \mathcal{H}_{\mathsf{SP}})(P) \quad \text{or} \quad \dot{p}_i = +\pi_i(\mathrm{Ln}(\partial \mathcal{H}_{\mathsf{SP}})(p_1, \dots, p_n))$$

- Generators' location $P = (p_1, \dots, p_n)$ **converges** asymptotically to the set of **critical points** of $\mathcal{H}_{\mathsf{DC}}$, respectively $\mathcal{H}_{\mathsf{SP}}$.
- Implementation is **centralized**:
  - (i) all $G_i(P)$, $F_i(P)$ need to be compared in order to determine which generator is active.
  - (ii) $\mathrm{Ln}(\partial \mathcal{H}_{\mathsf{DC}})(P)$, $\mathrm{Ln}(\partial \mathcal{H}_{\mathsf{SP}})(P)$ depend on the relative position of the active generators with respect to each other and to the environment

Nonsmooth dynamical systems based on distributed gradients

$$\dot{p}_i = -\,\mathrm{Ln}(\partial\,\mathrm{lg}_{V_i(P)})(P) \qquad \text{(at fixed } V_i\text{)}$$
$$\dot{p}_i = +\,\mathrm{Ln}(\partial\,\mathrm{sm}_{V_i(P)})(P) \qquad \text{(at fixed } V_i\text{)}$$

- Vector fields are discontinuous, Filippov solutions
- Given $P \in Q^n$, solutions of dynamical systems starting at P are unique
- Relation with behavior-based robotics
  - (i) "**move toward the furthest vertex** in own Voronoi cell"
  - (ii) "**move away from the closest wall** in own Voronoi cell"
- Generators' location P converges asymptotically to largest weakly invariant set in closure of

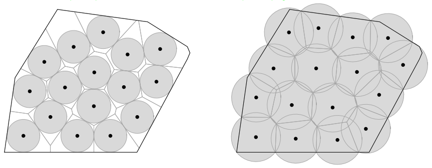
$$A_{\mathrm{DC}}(Q) = \{P \in Q^n \mid i \in I(P) \Rightarrow p_i = \mathrm{CC}(V_i)\}$$
$$A_{\mathrm{SP}}(Q) = \{P \in Q^n \mid i \in I(P) \Rightarrow p_i \in \mathrm{IC}(V_i)\}$$

Dynamical systems based on geometric centering

$$\dot{p}_i = \mathrm{CC}(V_i) - p_i$$
$$\dot{p}_i \in \mathrm{IC}(V_i) - p_i$$

- Circumcenter vector field is continuous. Incenter differential inclusion: existence of solutions can be established
- Q^n is invariant for both vector fields
- P converges asymptotically to largest weakly invariant set in closure of

$$A_{\mathrm{DC}}(Q) = \{P \in Q^n \mid i \in I(P) \Rightarrow p_i = \mathrm{CC}(V_i)\}$$
$$A_{\mathrm{SP}}(Q) = \{P \in Q^n \mid i \in I(P) \Rightarrow p_i \in \mathrm{IC}(V_i)\}$$



Geometric and nonsmooth analysis of multi-center cost functions

	$\mathcal{H}_{\mathrm{C}}$	$\mathcal{H}_{\mathrm{DC}}$	$\mathcal{H}_{\mathrm{SP}}$
Definition	$E\,[\min d(q,p_i)]$	$\max_{q\in Q}\{\min d(q,p_i)\}$	$\min_{i\neq j}\{\frac{1}{2}d(p_i,p_j), d(p_i,\partial Q)\}$
smoothness	C^1	regular, globally Lipschitz	regular, globally Lipschitz
critical pts	Centroidal Voronoi conf	Circumcenter Voronoi conf*	Incenter Voronoi conf*
heuristic	expected distortion	disk covering	sphere packing

Properties of dynamical systems based on geometric centering and nonsmooth gradient

	$\dot{p}_i = -\,\mathrm{Ln}[\partial\,\mathrm{lg}_{V_i(P)}](P)$	$\dot{p}_i = \mathrm{Ln}[\partial\,\mathrm{sm}_{V_i(P)}](P)$	$\dot{p}_i = \mathrm{CM}(V_i(P)) - p_i$	$\dot{p}_i = \mathrm{CC}(V_i(P)) - p_i$	$\dot{p}_i \in \mathrm{IC}(V_i(P)) - p_i$
smoothness	discontinuous	discontinuous	C^0	C^0	upper semicontinuous
distributed character	Voronoi neighbors	closest neighbors	Voronoi neighbors	Voronoi neighbors	Voronoi neighbors
critical pts	Circumcenter Voronoi conf	Incenter Voronoi conf	Centroidal Voronoi conf	Circumcenter Voronoi conf	Incenter Voronoi conf
Lyapunov f.	$\mathcal{H}_{\mathrm{DC}}$	$\mathcal{H}_{\mathrm{SP}}$	$\mathcal{H}_{\mathrm{C}}$	$\mathcal{H}_{\mathrm{DC}}$	$\mathcal{H}_{\mathrm{SP}}$
heuristic	"move toward furthest vertex of own cell"	"move away from closest neighbor"	"move toward centroid of own cell"	"move toward circumcenter of own cell"	"move toward incenter set of own cell"
asymptotic behavior	Active tend to circumcenter of own Voronoi cell	Active tend to incenter of own Voronoi cell	All tend to centroid of own Voronoi cell	Active tend to circumcenter of own Voronoi cell	Active tend to incenter of own Voronoi cell