Geometric control of Lagrangian systems modeling, analysis, and design

Francesco Bullo
Coordinated Science Lab, General Engineering Dept, ECE, AAE
University of Illinois at Urbana-Champaign
1308 W. Main St, Urbana, IL 61801, USA
bullo@uiuc.edu, http://motion.csl.uiuc.edu

SummerSchool on Modeling and Control of Mechanical Systems
Dutch Institute for Systems and Control (http://www.disc.tudelft.nl/)
Engineering Mechanics (http://www.em.tue.nl/)

July 1st – July 4th 2002
Conference Center Woudschoten, Zeist, the Netherlands
http://www.woudschoten.nl

Acknowledgments

Main coworker:

Andrew D. Lewis
Dept of Mathematics & Statistics, Queen's University
Kingston, Ontario, K7L 3N6, Canada
andrew@mast.queensu.ca

All papers and slides at http://motion.csl.uiuc.edu
and http://penelope.mast.queensu.ca

This work has been supported by NSF grant CMS-0100162.

Collaborators:

- during my years at Caltech: Burdick, Leonard, Marsden, Murray, Žefran
- during my years at University of Illinois: Cerven, Cortés, Frazzoli, Karatas, Lynch, Martínez, Žefran

Table of Contents

A coordinate-free theory for controlled mechanical systems

(i) **modeling**
symmetries, nonholonomic constraints, impacts, kinematic systems

(ii) **analysis**
averaging (series expansions under small-amplitude controls, averaging under highly oscillatory controls), controllability (local controllability, configuration controllability, equilibrium and kinematic controllability), kinematic reductions (decoupling vector fields, fully reducible systems)

(iii) **design**
planning (inverse kinematics for kinematically controllable systems, power series inversion under small amplitude controls), stabilization and tracking

Lecture titles

Lecture #1: From Linear Algebra to Mechanical Control Systems

Lecture #2: Modeling Symmetries and Nonholonomic Constraints

Lecture #3: Perturbation Analyses of Affine Connection Control Systems
http://motion.csl.uiuc.edu/~bullo/papers/1999b-b.html

Lecture #4: Kinematic Reductions and Configuration Controllability
http://motion.csl.uiuc.edu/~bullo/papers/2002a-bll.html

Lecture #5: Stabilization and Tracking for fully actuated systems

Lecture #6: Trajectory Planning via Motion Primitives
http://motion.csl.uiuc.edu/~bullo/papers/1997b-bll.html
http://motion.csl.uiuc.edu/~bullo/papers/2001a-bl.html

1 Geometric Control of Lagrangian Systems

1.1 Scientific Interests

(i) success in linear control theory is unlikely to be repeated for nonlinear systems. In particular, nonlinear system design. no hope for general theory
  ➡ mechanical systems as examples of control systems

(ii) control relevance of tools from geometric mechanics

(iii) geometric control past feedback linearization

1.2 Industrial Trends

autonomous vehicles	➡	new concepts in design
reconfigurable, reactive	➡	implementation on-line
sensing & computation cheap	➡	focus on actuators and algorithm

1.3 Motion planning

Example systems

(i) dexterous manipulation via minimalist robots

(ii) real-time trajectory/path planning for autonomous vehicles

(iii) locomotion systems (walking, swimming, diving, etc)

Application contexts

(i) guidance and control of physical systems

(ii) prototyping and verification

(iii) graphical animation and movie generation

(iv) analysis of animal and human locomotion and prosthesis design in biomechanics

exploit differential geometric structure

Research work reflected in these notes

[1] A. D. Lewis, "Simple mechanical control systems with constraints," *IEEE TAC*, 45(8):1420–1436, 2000.

[2] A. D. Lewis, "When is a mechanical control system kinematic?," in *Proc CDC*, (Phoenix, AZ), pp. 1162–1167, Dec. 1999.

[3] F. Bullo and M. Žefran, "On mechanical control systems with nonholonomic constraints and symmetries," *Syst. & Control L.*, 45(2):133–143, 2002.

[4] F. Bullo and M. Žefran, "Modeling and controllability for a class of hybrid mechanical systems," *IEEE TRA*, Aug 2002. To appear.

[5] A. D. Lewis and R. M. Murray, "Configuration controllability of simple mechanical control systems," *SIAM JCO*, 35(3):766–790, 1997.

[6] F. Bullo and K. M. Lynch, "Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems," *IEEE TRA*, 17(4)L402–412, 2001.

[7] F. Bullo, N. E. Leonard, and A. D. Lewis, "Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups," *IEEE TAC*, 45(8):1437–1454, 2000.

[8] F. Bullo, "Series expansions for mechanical control systems," *SIAM JCO*, 40(1):166–190, 2001.

[9] F. Bullo, "Averaging and vibrational control of mechanical systems," *SIAM JCO*, Submitted 1999. To appear 2002.

[10] J. Cortés, S. Martínez, and F. Bullo, "On nonlinear controllability and series expansions for Lagrangian systems with dissipative forces," *IEEE TAC*, Aug. 2002. To appear.

[11] F. Bullo, "Series expansions for analytic systems linear in controls," *Automatica*, 38(9):1425-1432, 2002.

[12] F. Bullo and A. D. Lewis, "Kinematic controllability and motion planning for the snakeboard," *IEEE TRA*, Jan. 2002. Submitted.

Very incomplete reference on geometric mechanics and geometric control of mechanical systems

[1] S. Arimoto. *Control Theory of Non-linear Mechanical Systems: A Passivity-Based and Circuit-Theoretic Approach*, volume 49 of *OESS*. Oxford University Press, Oxford, UK, 1996.

[2] A. M. Bloch and P. E. Crouch. Nonholonomic control systems on Riemannian manifolds. *SIAM Journal of Control and Optimization*, 33(1):126–148, 1995.

[3] R. W. Brockett. Control theory and analytical mechanics. In R. Hermann and C. Martin, editors, *Geometric Control Theory*, volume 7 of *Lie Groups: History Frontiers and Applications*, pages 1–46. Mathematical Scientific Press, Brookline, MA, 1977.

[4] P. S. Krishnaprasad and D. P. Tsakiris, "Oscillations, $SE(2)$-snakes and motion control: a study of the roller racer," *Dynamics and Stability of Systems*, vol. 16, no. 4, pp. 347–397, 2001.

[5] N. E. Leonard. Mechanics and nonlinear control: Making underwater vehicles ride and glide. In *Nonlinear Control Systems Design (NOLCOS)*, volume 1, pages 1–6, Enschede, The Netherlands, July 1998.

[6] R. M. Murray. Nonlinear control of mechanical systems: a Lagrangian perspective. In *Nonlinear Control Systems Design (NOLCOS)*, pages 378–389, Lake Tahoe, CA, June 1995.

[7] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer Verlag, New York, NY, 1990.

[8] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez. *Passivity-Based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications*. Communications and Control Engineering. Springer Verlag, New York, NY, 1998.

[9] A. J. van der Schaft, "Port-controlled Hamiltonian systems: towards a theory for control and design of nonlinear physical systems," *Journal of the Society of Instrument & Control Engineers*, vol. 39, no. 2, pp. 91–8, 2000.

Lecture #1: From Linear Algebra to Mechanical Control Systems

Francesco Bullo
Coordinated Science Lab, General Engineering Dept, ECE, AAE
University of Illinois at Urbana-Champaign
1308 W. Main St, Urbana, IL 61801, USA
bullo@uiuc.edu, http://motion.csl.uiuc.edu

2 Linear algebra

2.1 Notation

- Linear space V, vectors $v \in V$
- dual space V^* is the space of co-vectors w:

$$\langle w\,,\,v\rangle \in \mathbb{R}$$

- in $\mathbb{R}^n$, think of v as columns (V is space of column vectors), and w as rows (V^* is space of row vectors)
- construction is possible on any vector space!

2.2 Vector versus indicial notation

- $\langle \cdot\,,\,\cdot\rangle$ is natural pairing between dual spaces
- $v \in V = \{$column vectors$\}$, $w \in V^* = \{$row vectors$\}$:

$$w \cdot v = \langle w\,,\,v\rangle \in \mathbb{R}$$

- other example, $f(x_1, \dots, x_n)$ and $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ (column):

$$\langle \frac{\partial f}{\partial x}\,,\,v\rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i$$

that is, we mean

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

2.3 Addendum on linear algebra and multi-variable calculus

(i) vectors: $v = v^i e_i$

(ii) covectors, dual elements

(iii) on $\mathbb{R}^n$, use variables $(q^1, \dots, q^n)$ – notation useful for "summation convention"

(iv) given a function $f \colon \mathbb{R}^n \to \mathbb{R}$, recall its directional derivative

(v) the differential $\mathrm{d}f$ is a covector field with components $\frac{\partial f}{\partial q^1}, \dots \frac{\partial f}{\partial q^n}$ so that

$$\mathrm{d}f = (\frac{\partial f}{\partial q^1}, \dots, \frac{\partial f}{\partial q^n})$$

(vi) X is a vector field, and we can define $\mathscr{L}_X f = \langle \mathrm{d}f\,,\,X\rangle$

(vii) planar body example: V_x, V_y are example vector fields

(viii) infinitesimal work in mechanical system is a pairing (not an inner product)

(ix) a curve $\gamma\colon I \to \mathbb{R}^n$ has a velocity $\gamma\colon I \to \mathbb{R}^n$, which is a vector field along the curve

(x) a vector field X is an ODE and an ODE is a vector field

(xi) vector fields are written in terms of the canonical basis $\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\}$, and co-vector fields in terms of $\{\mathrm{d}q^1, \ldots, \mathrm{d}q^n\}$

$$X(q) = X^i(q)\frac{\partial}{\partial q^i} \qquad \omega = \omega_i(q)\mathrm{d}q^i \qquad \mathrm{d}f = \sum_i \frac{\partial f}{\partial q^i}\mathrm{d}q^i = \frac{\partial f}{\partial q^i}\mathrm{d}q^i$$

"a matrix is a matrix is not a matrix"

(xii) maps between linear space: $A\colon V \to V$ has components A_i^j

$$v = v^i e_i \mapsto Av = A_i^j v^i e_j$$

(xiii) bilinear maps: $B\colon V \times V \to \mathbb{R}$ has components B_{ij}

$$(v, w) = (v^i e_i, w^j e_j) \mapsto B(v, w) = B_{ij} v^i w^j$$

(xiv) associate linear map: $B\colon V \to V^*$ has components B_{ij}

$$v = v^i e_i \mapsto B_{ij} v^i e^j$$

(xv) an inner product $\langle\langle \cdot\,, \cdot \rangle\rangle$ is a bilinear map, need a symbol $\mathbb{G}\colon V \times V \to \mathbb{R}$

(xvi) since $\mathbb{G}\colon V \to V^*$ is non-singular, we can invert it, $\mathbb{G}^{-1}\colon V^* \to V$ is now an inner product on V^*

(xvii) **Lie derivatives do not commute**

(a) $\frac{\partial}{\partial q^i}\frac{\partial}{\partial q^j} f = \frac{\partial}{\partial q^j}\frac{\partial}{\partial q^i} f$

(b) however

$$\mathscr{L}_{X_1}\mathscr{L}_{X_2} f \neq \mathscr{L}_{X_2}\mathscr{L}_{X_1} f$$

(c) correct formula is:

$$\mathscr{L}_{X_1}\mathscr{L}_{X_2} f - \mathscr{L}_{X_2}\mathscr{L}_{X_1} f = \mathscr{L}_{[X_1, X_2]} f$$

where Lie bracket (in indicial notation)

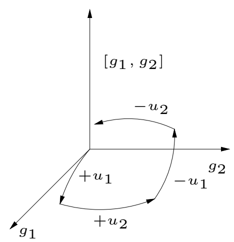
$$[X, Y]^i = \frac{\partial Y^i}{\partial q^j} X^j - \frac{\partial X^i}{\partial q^j} Y^j$$

in vector notation (where now $\partial X/\partial q$ is an $n \times n$ matrix):

$$[X, Y] = \frac{\partial Y}{\partial q} \cdot X - \frac{\partial X}{\partial q} \cdot Y$$

Consider the controlled ODE $\quad \dot{x} = g_1(x)u_1 + g_2(x)u_2$

define **Lie bracket:** $\quad [g_1(x), g_2(x)] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2$



Properties of Lie brackets:

(a) skew symmetry: $[X, Y] = -[Y, X]$

(b) linearity: $[X, Y + Z] = [X, Y] + [X, Z]$

(c) derivation: $[X, fY] = f[X, Y] + (\mathscr{L}_X f)Y$

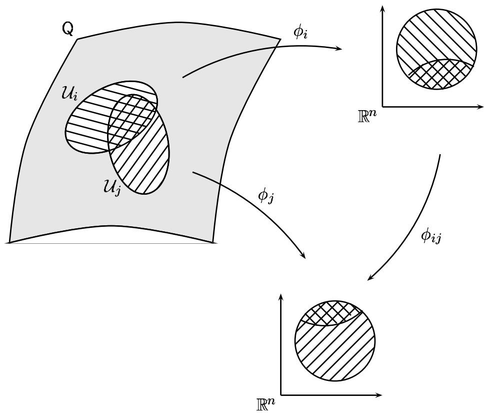
(d) Jacoby identity: $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

3 A primer in Riemannian geometry

3.1 Notation

(i) assume every object is **real analytic**

(ii) Q is a manifold, that is, a locally Euclidean space



(iii) $q \in$ Q is point on manifold, in coordinates $q = (q^1, \ldots, q^n)$

(iv) $q \mapsto f(q) \in \mathbb{R}$ is scalar function

(v) As on $\mathbb{R}^n$, vector fields and covector fields attached to each point on Q:

- the differential df is a covector field with components $\dfrac{\partial f}{\partial q^1}, \ldots \dfrac{\partial f}{\partial q^n}$ so that

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial q^i}\mathrm{d}q^i = \frac{\partial f}{\partial q^i}\mathrm{d}q^i$$

- X is a vector field with components $X^1, \ldots X^n$ so that

$$X = \sum_i X^i \frac{\partial}{\partial q^i} = X^i \frac{\partial}{\partial q^i}$$

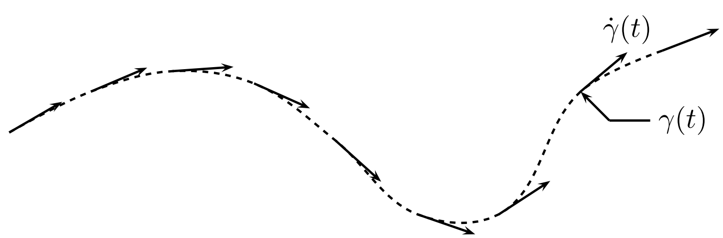
- Lie derivative of a function (X, f are both functions of q):

$$\mathscr{L}_X f := \sum_i \frac{\partial f}{\partial q^i} X^i = \langle \mathrm{d}f \,,\, X \rangle$$

(vi) Last equality is the natural pairing between tangent TQ and cotangent bundle T*Q

(vii) $\gamma \colon I \to$ Q is a curve on Q. Its velocity is a vector field along γ with components

$$\gamma'(t) = \frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}\frac{\partial}{\partial q^i} = \dot\gamma^i(t)\frac{\partial}{\partial q^i}$$



3.2 Affine Connections

- An affine connection ∇ on maps two vector fields X, Y into a third vector field $\nabla_X Y$, satisfying the following properties:
  (i) $\nabla_{fX} Y = f \nabla_X Y$
  (ii) $\nabla_X fY = (\mathscr{L}_X f) Y + f \nabla_X Y$
- Given the basis $\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\}$, ∇ determines and is uniquely determined by the Christoffel symbols:

$$\nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \Gamma^{ij}_k \frac{\partial}{\partial q^k}$$

- In coordinates

$$\nabla_X Y = \left(\mathscr{L}_X Y^k + \Gamma^k_{ij} X^i Y^j\right)\frac{\partial}{\partial q^k}$$
$$= \left(\frac{\partial Y^k}{\partial q^i} X^i + \Gamma^k_{ij} X^i Y^j\right)\frac{\partial}{\partial q^k}$$

3.3 Covariant derivatives of vector fields along curves

- Given a curve $\gamma\colon I \to \mathsf{Q}$, and its velocity $\gamma'\colon I \to \mathsf{TQ}$ is a curve on TQ.
- $\gamma'\colon I \to \mathsf{TQ}$ is an example of a vector field along a curve on Q
- Given a vector field $\eta\colon I \to \mathsf{TQ}$ along γ, define its covariant derivative along γ as

$$\nabla_{\gamma'}\eta = \nabla_{\gamma'}Y$$

where Y is a smooth extensions of η to Q
- In coordinates:

$$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t)) \qquad \gamma'(t) = (\dot\gamma^1(t), \dots, \dot\gamma^n(t))$$
$$\eta(t) = (\eta^1(t), \dots, \eta^n(t))$$
$$\left(\nabla_{\gamma'}\eta\right)^i = \ddot\eta^i + \Gamma^i_{jk}(\gamma)\dot\gamma^j\eta^k$$

3.4 Property of covariant derivatives along curves

Recall: An affine connection ∇ on maps two vector fields X, Y into a third vector field $\nabla_X Y$, satisfying the following properties:

(i) $\nabla_{fX}Y = f\nabla_X Y$

(ii) $\nabla_X fY = (\mathscr{L}_X f)Y + f\nabla_X Y$

Given a function of time f, and a vector field η along γ:

$$\nabla_{\gamma'}f(t)\eta(t) = \left(\frac{\mathsf{d}}{\mathsf{d}t}f(t)\right)\eta(t) + f(t)\left(\nabla_{\gamma'}\eta(t)\right)$$

3.5 Geometric acceleration and geodesic curves

- given a curve γ, the second time derivative $\ddot\gamma^i$ is **not** a vector
- Given a curve γ, define the **geometric acceleration** of γ as the vector field along γ

$$\nabla_{\gamma'(t)}\gamma'(t)$$

- in coordinates (with respect to the respective bases):

$$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t)) \qquad \gamma'(t) = (\dot\gamma^1(t), \dots, \dot\gamma^n(t))$$
$$\nabla_{\gamma'(t)}\gamma'(t) = (\ddot\gamma^1 + \Gamma^1_{ij}\dot\gamma^i\dot\gamma^j \;, \dots, \;\; \ddot\gamma^n + \Gamma^n_{ij}\dot\gamma^i\dot\gamma^j)$$

- **A curve with zero geometric acceleration is a geodesic**
- geodesic curves enjoy various properties: constant point-wise energy, homogeneity, existence and uniqueness.

3.6 Collection of vector fields, distributions, and operations between vector fields

(i) $\mathcal{X} = \{X_1, \dots, X_\ell\}$ a the collection or family of vfs

(ii) $\mathscr{X} = \mathrm{span}_{C(\mathsf{Q})}\{X_1, \dots, X_\ell\}$ is called the distribution, i.e., the point-wise sub-space of $\mathsf{T}_q\mathsf{Q}$. In other words, $\mathscr{X}_q = \mathrm{span}_{\mathbb{R}}\{X_1(q), \dots, X_\ell(q)\}$

(iii) the Lie bracket between X_i and X_j is $[X_i, X_j]$

(iv) The distribution $\mathscr{X}$ is said to be **involutive** if it is closed under operation of Lie bracket, i.e., if for all vector fields X and Y taking values in $\mathscr{X}$, the vector field $[X, Y]$ also takes value in $\mathscr{X}$. The **involutive closure** of the distribution $\mathscr{X}$ is the smallest involutive distribution containing $\mathscr{X}$, and is denoted $\overline{\mathrm{Lie}}\{\mathscr{X}\}$.

(v) the symmetric product between X_i and X_j is the vector field

$$\langle X_i : X_j\rangle = \nabla_{X_i}X_j + \nabla_{X_j}X_i$$

One then can define the notion of **symmetric closure** and **geodesic invariance**.

3.7 Riemannian metric

- Metric is inner product on tangent space

$$\langle\langle \cdot\,,\,\cdot \rangle\rangle : \ \mathsf{TQ} \times \mathsf{TQ} \to \mathbb{R}$$

- inner product is positive definite, symmetric, bilinear form $\mathbb{G}$.
- In coordinates $\mathbb{G}_{ij}$

$$\langle\langle X\,,\,Y \rangle\rangle = \sum_{ij} \mathbb{G}_{ij}(q) X^i(q) Y^j(q)$$

- $\mathbb{G}$ as a matrix (in vector notation): $\langle\langle X\,,\,Y \rangle\rangle = X^T[\mathbb{G}]Y$.

- Summary:
  (i) there is a pairing between functions and vector fields (i.e., $\mathscr{L}_X f$), and similarly between vector fields and co-vector fields (i.e., $\langle \mathrm{d}f\,,\,X \rangle$)
  (ii) $\mathbb{G}$ is a pairing between two vector fields

  where in vector notation "a pairing := combine two vectors to obtain a scalar"

NB: in mechanical systems, metric is usually denoted M. In Riemannian geometry g.

3.8 Associated linear maps between TQ and T*Q

(i) $\mathbb{G} \colon \mathsf{TQ} \to \mathsf{T}^*\mathsf{Q}$:
Given a vector field X, $([\mathbb{G}]X)^T$ is the co-vector field such that

$$\underbrace{([\mathbb{G}]X)^T \cdot Y}_{\langle \mathbb{G}X\,,\,Y \rangle} = \underbrace{X^T[\mathbb{G}]Y}_{\langle\langle X\,,\,Y \rangle\rangle}$$

(ii) $\mathbb{G}^{-1} \colon \mathsf{T}^*\mathsf{Q} \to \mathsf{TQ}$:
Given a co-vector field F, $\mathbb{G}^{-1}F^T$ is the vector field such that

$$\langle\langle M^{-1}F\,,\,Y \rangle\rangle = (\mathbb{G}^{-1}F^T)^T[\mathbb{G}]Y = \langle F\,,\,Y \rangle$$

3.9 Gradient of a function

- Given a function f, its gradient is the vector field

$$\mathrm{grad}\, f = \mathbb{G}^{-1}\mathrm{d}f$$

or alternatively

$$\langle\langle \mathrm{grad}\, f\,,\,X \rangle\rangle \equiv \langle \mathrm{d}f\,,\,X \rangle$$

In indicial notation:

$$(\mathrm{grad}\, f)^i = \sum_{j=1}^{n} (\mathbb{G}^{-1})^{ij} \frac{\partial f}{\partial q^j} = \mathbb{G}^{ij} \frac{\partial f}{\partial q^j}$$

3.10 Levi-Civita (or metric) connection

Theorem 1 (Levi Civita). *A metric $\langle\langle \cdot\,,\,\cdot \rangle\rangle$ induces a unique ${}^{\mathbb{G}}\nabla$ such that*
(i) $\mathscr{L}_X \langle\langle Y\,,\,Z \rangle\rangle = \langle\langle {}^{\mathbb{G}}\nabla_X Y\,,\,Z \rangle\rangle + \langle\langle Y\,,\,{}^{\mathbb{G}}\nabla_X Z \rangle\rangle$
(ii) ${}^{\mathbb{G}}\nabla_X Y - {}^{\mathbb{G}}\nabla_Y X = [X, Y]$

(i) Its symbols are:

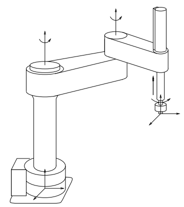
$$\Gamma^k_{ij} = \frac{1}{2}\mathbb{G}^{mk}\left(\frac{\partial \mathbb{G}_{mj}}{\partial q^i} + \frac{\partial \mathbb{G}_{mi}}{\partial q^j} - \frac{\partial \mathbb{G}_{ij}}{\partial q^m}\right)$$

where $\mathbb{G}^{mk}$ is m, k component of $\mathbb{G}^{-1}$

(ii) Proof based on equality:

$$2\langle\langle Z\,,\,{}^{\mathbb{G}}\nabla_X Y \rangle\rangle = X\langle\langle Y\,,\,Z \rangle\rangle + Y\langle\langle X\,,\,Z \rangle\rangle - Z\langle\langle Y\,,\,X \rangle\rangle \\ - \langle\langle [X, Z]\,,\,Y \rangle\rangle - \langle\langle [Y, Z]\,,\,X \rangle\rangle - \langle\langle [X, Y]\,,\,Z \rangle\rangle$$

4 Models of Mechanical Systems



Simple mechanical control system is composed of:

(i) the configuration space Q (manifold)

(ii) the kinetic energy $\mathbb{G}$ (metric)

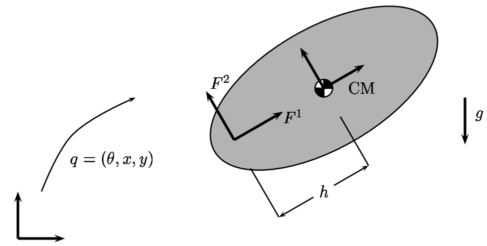
(iii) the potential energy V (function on Q)

(iv) the input forces $F^1, \ldots, F^m$ (co-vectors)

Total energy (Hamiltonian, sum of kinetic and potential) is:

$$\mathcal{E}(q, v_q) = \frac{1}{2}\|v_q\|^2 + V(q)$$

4.1 Planar body example

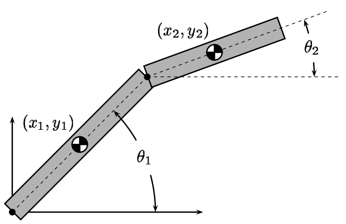


$$q = (\theta, x, y)$$

$$V(q) = mgy \qquad [\mathbb{G}] = \begin{bmatrix} J & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

We shall discuss F^i in a few slides

4.2 Planar two links manipulator example



$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$$

$$K_1(\theta_1, x_1, y_1) = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} m(\dot{x}_1^2 + \dot{y}_1^2)$$

$$V_1(\theta_1, x_1, y_1) = m_1 g\, y_1$$

Therefore easy to write $\mathcal{E}$ as function of **all** variables

4.3 Kinematics

Only necessary variables to describe system are configuration variables, e.g. $q = (\theta_1, \theta_2)$

Write (θ_i, x_i, y_i) in terms of q by means of **kinematic analysis**.

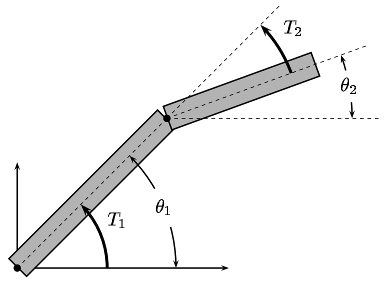
$$\mathcal{E}(q, \dot{q}) := \mathcal{E}(\theta_i, x_i, y_i, \dot{\theta}_i, \dot{x}_i, \dot{y}_i) \quad /. \quad (\theta_i, x_i, y_i) \to (\theta_i, x_i, y_i)(q)$$

After simplification:

$$[\mathbb{G}] = \begin{bmatrix} I_1 + (l_1^2(m_1 + 4m_2))/4 & (l_1 l_2 m_2 \cos[\theta_1 - \theta_2])/2 \\ (l_1 l_2 m_2 \cos[\theta_1 - \theta_2])/2 & I_2 + (l_2^2 m_2)/4 \end{bmatrix}$$

General study of single and multi-body kinematics.

4.4 Forces as co-vectors



Why are forces co-vectors? Assume curve $\gamma\colon I \to \mathsf{Q}$ is solution to controlled equations, then

$$\text{Infinitesimal Work} = \langle F\,,\,\gamma'\rangle$$

where $\gamma' \in \mathsf{T}_\gamma\mathsf{Q}$ and hence $F \in \mathsf{T}^*_\gamma\mathsf{Q}$.

- forces as generalized forces, i.e.,
  both pure forces and pure torques are ok

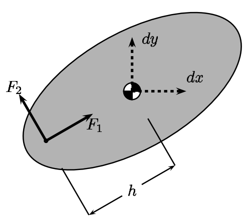
- in this example only pure torques: Joint motor T_1 acts on angle θ_1. Joint motor T_2 acts on angle $\theta_2 - \theta_1$:

$$T_1 = \mathsf{d}\theta_1 = \begin{bmatrix}1 & 0\end{bmatrix}$$
$$T_2 = \mathsf{d}(\theta_1 - \theta_2) = \mathsf{d}\theta_1 - \mathsf{d}\theta_2 = \begin{bmatrix}1 & -1\end{bmatrix}$$

- Note: force is a co-vector, for example, $F = \mathsf{d}f$ for some function f. But not always F is the differential of a function (Poincaré lemma)

4.5 Generalized force = pure force + pure torque

If force is pure torque on angle α, then $F = \mathsf{d}\alpha$. If force is pure force on distance x, then $F = \mathsf{d}x$. Write a generalized force as linear combination of pure force and pure torque.



$$F^1 = \cos\theta\mathsf{d}x + \sin\theta\mathsf{d}y = \begin{bmatrix}0 & \cos\theta & \sin\theta\end{bmatrix}$$
$$F^2 = -h\mathsf{d}\theta - \sin\theta\mathsf{d}x + \cos\theta\mathsf{d}y = \begin{bmatrix}-h & -\sin\theta & \cos\theta\end{bmatrix}$$

4.6 Lagrange-D'Alembert principle

The solution $\gamma\colon I \to \mathsf{Q}$ to the **simple mechanical control system** satisfies the variational principle

$$\delta \int_I \left(\frac{1}{2}\|\gamma'\|^2 - V(\gamma)\right)dt + \int_I \langle F(\gamma,t)\,,\,\delta q\rangle = 0$$

where the variation δq is an arbitrary vector field along γ

- Systems subject to no force follow geodesic flow:

$$\delta \int_I \|\gamma'\|^2\,dt = 0 \quad \rightleftarrows \quad \nabla_{\gamma'}\gamma' = 0$$

- Systems subject to force follow forced geodesic flow:

$$\nabla_{\gamma'}\gamma' = \mathbb{G}^{-1}F$$

5 Simple Mechanical Control Systems (SMCS)

A **simple mechanical control system**:

(i) An n-dimensional **configuration manifold Q**, coordinates $(q^1, \dots, q^n)$

(ii) An **inertia tensor** $\mathbb{G}$ describing the kinetic energy
$\mathbb{G}$ defines an inner product $\langle\langle \cdot , \cdot \rangle\rangle$ between vector fields on Q

(iii) the potential energy V (function on Q)

(iv) m one-forms $F^1, \dots, F^m$, describing m **external control forces**

Given this data, we derive

(i) ${}^{\mathbb{G}}\nabla$ is the Levi-Civita connection associated to $\mathbb{G}$

(ii) we define the input vector fields $Y_a = \mathbb{G}^{-1}F^a$, for $a \in \{1, \dots, m\}$

(iii) **Coordinate-free formulation** of the equations of motion:

$$
{}^{\mathbb{G}}\nabla_{\gamma'}\gamma' = \sum_{a=1}^{m} Y_a(\gamma)u_a
$$

the input functions u_a are assumed Lebesgue measurable

(iv) In coordinates $(q^1, \dots, q^n)$, Christoffel symbols:

$$
\Gamma^k_{ij} = \frac{1}{2}\mathbb{G}^{\ell k}\left(\frac{\partial \mathbb{G}_{\ell j}}{\partial q^i} + \frac{\partial \mathbb{G}_{\ell i}}{\partial q^j} - \frac{\partial \mathbb{G}_{ij}}{\partial q^\ell}\right)
$$

(v) Equations of motion in coordinates for trajectory $\gamma \colon I \to$ Q:

$$
\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = \sum_{a=1}^{m} \mathbb{G}^{kj}(F^a)_j u_a
$$

5.1 Conservative and dissipative forces

(i) potential energy V due to gravity gives rise to a force $F = -\mathrm{d}V$ and a vector field $-\operatorname{grad} V$. More generally, we shall assume an arbitrary vector field $Y_0(q)$ in the equations of motion

(ii) damping or dissipation force is of the form $F = R(v_q)$. R stands for Rayleigh dissipation function (i.e., a linear dissipation function.
The tensor R: TQ $\to$ TQ is dissipative if

$$
\langle\langle R(v_q) \, , \, v_q \rangle\rangle \leq 0
$$

Strict inequality for strictly dissipative forces

In summary, a simple mechanical control system with dissipation and potential energy satisfied

$$
{}^{\mathbb{G}}\nabla_{\gamma'}\gamma' = Y_0(\gamma) + R(\gamma') + Y(\gamma)u(t)
$$

6 Satellites and vehicles - systems on groups

(i) configuration is rotation matrix R
$R \in SO(3) = \{R \in \mathbb{R}^{3\times3} | R^T R = I, \det R = +1\}$

(ii) define $\widehat{\cdot}$ operator as: $\omega \times y = \widehat{\omega} y$

(iii) kinematic equation $\dot{R} = R\widehat{\omega}$
follows from differentiating identity: $R^T R = I_3$
ω body velocity in body-frame

(iv) Kinetic energy: $K = \frac{1}{2}\omega^T \mathbb{J}\omega$
remarkable because R is not present!

(v) no potential, and torques τ expressed body frame

(vi) Euler Poincarè equations of motion:

$$
\dot{R} = R\widehat{\omega}
$$
$$
\dot{\omega} = \mathbb{J}^{-1}(\mathbb{J}\omega \times \omega) + \mathbb{J}^{-1}\tau
$$

(vii) if, for example, $\mathbb{J} = \mathrm{diag}\{J_1, J_2, J_3\}$

$$\dot\omega_1 = ((J_2 - J_3)/J_1)\omega_2\omega_3 + \tau_1/J_1$$
$$\dot\omega_2 = ((J_3 - J_1)/J_2)\omega_1\omega_3 + \tau_2/J_2$$
$$\dot\omega_3 = ((J_1 - J_2)/J_3)\omega_1\omega_2 + \tau_3/J_3$$

these are also called the Euler equations

(viii) $(\omega_1, \omega_2, \omega_3)$ are pseudo-velocities, not the time derivative of any quantity on $SO(3)$

6.1 Mechanical control systems on matrix groups

(i) $g \in$ G is configuration on n-dimensional **matrix group**
local coordinates via $x = \log(g)$

(ii) **kinetic energy** $\mathcal{KE} = \frac{1}{2}v^T \mathbb{I} v$ with $\mathbb{I} > 0$
$v \in \mathbb{R}^n$ velocity in body frame

(iii) body-fixed **forces** $f^1, \ldots, f^m \in (\mathbb{R}^n)^*$.

Example: $\log(R) = \frac{\phi}{2\sin\phi}(R - R^T), \quad 2\cos\phi = \mathrm{tr}(R) - 1$

6.2 Equations of motion, I

Kinematic eqns:

$$\dot g = g\,\widehat{v}$$

where $v \longleftrightarrow \widehat{v}$ is isomorphism $\mathbb{R}^n \longleftrightarrow \mathbb{R}^{n\times n}$.

Lie bracket is matrix commutator: $\widehat{[v, w]} = (\widehat{v}\widehat{w} - \widehat{w}\widehat{v})$

Example: $\dot R = R\widehat{\omega}$
$[\omega, y] = \omega \times y$

6.3 Equations of motion, II

$$\ddot\gamma^i + \Gamma^i_{ab}(\gamma)\dot\gamma^a\dot\gamma^b = (\mathbb{G}^{-1}F^k)^i u_k$$

Euler-Poincaré eqns:

$$\dot g = g\widehat{v}$$
$$\dot v^i + \Gamma^i_{jk}v^j v^k = \sum_a (\mathbb{I}^{-1}f^a)^i u_a(t)$$

where the Γ^i_{jk} are constants determined by G and $\mathbb{I}$.

Symmetric product: $\langle v : w\rangle^i = -\Gamma^i_{ab}(v^a w^b + v^b w^a)$

Example:
$\dot\Omega + \mathbb{J}^{-1}(\Omega \times \mathbb{J}\Omega) = 0$
$\langle \Omega : \Xi\rangle = \mathbb{J}^{-1}(\Omega \times \mathbb{J}\Xi \;+\; \Xi \times \mathbb{J}\Omega)$

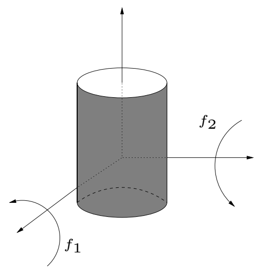
6.4 Satellite with Thrusters

- configuration is rotation matrix R
- kinematic equation:

$$\dot{R} = R\widehat{\Omega}$$

where

$$\Omega \in \mathbb{R}^3 \mapsto \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$



- kinetic energy:

$$KE = \tfrac{1}{2}\Omega^T \mathbb{J}\Omega$$

- two torques: $f_1 = \mathsf{e}_1, \quad f_2 = \mathsf{e}_2$
- **Equations of Motion:**

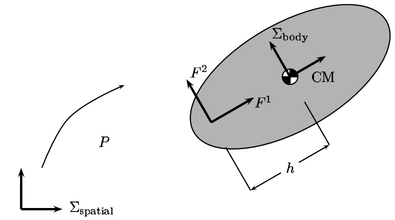
$$\dot{R} = R\widehat{\Omega}$$

$$\mathbb{J}\dot{\Omega} = \mathbb{J}\Omega \times \Omega \quad + \mathsf{e}_1 u_1(t) + \mathsf{e}_2 u_2(t).$$

6.5 Hovercraft

(i) Configuration:

$$P = \begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$



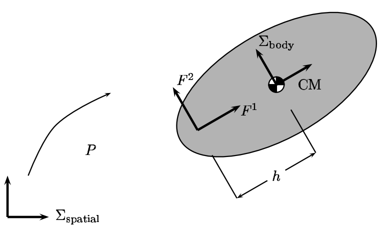
(ii) $KE = \frac{1}{2}(J\omega^2 + mv_x^2 + mv_y^2)$

(iii) $f_1 = \mathsf{e}_2, \quad f_2 = -h\mathsf{e}_1 + \mathsf{e}_3$

Equations of Motion:

$$\dot{P} = P\begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{cases} J\dot{\omega} &= -hu_2 \\ m\dot{v}_x &= m\omega v_y + u_1 \\ m\dot{v}_y &= -m\omega v_x + u_2 \end{cases}$$

6.6 Planar underwater vehicle



Same kinematic description as hovercraft. However, effects of fluid.

6.7 Planar underwater vehicle, cont'd

(i) to model ideal fluid, include added masses into kinetic energy:

$$K = \frac{1}{2}(m_x v_x^2 + m_y v_y^2) + \frac{1}{2}J\omega^2$$

Notice θ, x, y are not present in energy

(ii) generalized forces in body coordinates $F = [f_\theta \quad f_x \quad f_y]$

(iii) Euler Poincarè equation for planar underwater vehicle:

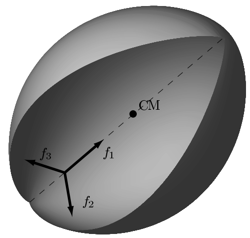
$$\dot{P} = P\begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

$$J\dot{\omega} = (m_x - m_y)v_x v_y + f_\theta$$

$$m_x\dot{v}_x = m_y v_y \omega + f_x$$

$$m_y\dot{v}_y = -m_x v_x \omega + f_y$$

6.8 Underwater Vehicle in Ideal Fluid



3D rigid body with three forces:

(i) $(R,p) \in SE(3), \quad (\Omega, V) \in \mathbb{R}^6$

(ii) $KE = \frac{1}{2}\Omega^T \mathbb{J}\Omega + \frac{1}{2}V^T \mathbb{M} V$,
$\mathbb{M} = \mathrm{diag}\{m_1, m_2, m_3\}$,
$\mathbb{J} = \mathrm{diag}\{J_1, J_2, J_3\}$

(iii) $f_1 = \mathsf{e}_4,\ f_2 = -h\mathsf{e}_3 + \mathsf{e}_5, \quad f_3 = h\mathsf{e}_2 + \mathsf{e}_6$

Equations of Motion:

$$\begin{aligned} \dot{R} &= R\widehat{\Omega} \\ \dot{p} &= RV \end{aligned} \quad , \quad \begin{aligned} \mathbb{J}\dot{\Omega} &= \mathbb{J}\Omega \times \Omega + \mathbb{M}V \times V \\ \mathbb{M}\dot{V} &= \mathbb{M}V \times \Omega. \end{aligned}$$

6.9 Proof of Euler Poincarè equation for satellite, page 1/3

Let us consider geodesic equation without forces:

$$^{\mathbb{G}}\nabla_{\gamma'}\gamma' = 0$$

The geodesic equation is written on a generic manifold. To write it with respect to coordinates $(\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n})$ on TQ, follow the steps:

$$\gamma' = \dot{\gamma}_i \frac{\partial}{\partial q^i}$$

$$^{\mathbb{G}}\nabla_{\gamma'}\left(\dot{\gamma}_i \frac{\partial}{\partial q^i}\right) = \ddot{\gamma}_i \frac{\partial}{\partial q^i} + \dot{\gamma}_k {}^{\mathbb{G}}\nabla_{\gamma'} \frac{\partial}{\partial q^k} = \ddot{\gamma}_i \frac{\partial}{\partial q^i} + \dot{\gamma}_k \dot{\gamma}_j \left({}^{\mathbb{G}}\nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^k}\right)$$

where the last two steps exploit the properties of affine connections.

At this point, the Christoffel symbols are computed by using:

$$\begin{aligned} 2\langle\langle Z\,,\, {}^{\mathbb{G}}\nabla_X Y\rangle\rangle = X\langle\langle Y\,,\, Z\rangle\rangle + Y\langle\langle X\,,\, Z\rangle\rangle - Z\langle\langle Y\,,\, X\rangle\rangle \\ - \langle\langle [X,Z]\,,\, Y\rangle\rangle - \langle\langle [Y,Z]\,,\, X\rangle\rangle - \langle\langle [X,Y]\,,\, Z\rangle\rangle \end{aligned} \tag{1}$$

where X, Y, Z take values in $\{\frac{\partial}{\partial q^i}\}$, and hence all Lie brackets $[\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}]$ vanish.

6.10 Proof of Euler Poincarè equation for satellite, page 2/3

We here perform the same procedure, but with respect a basis of invariant vector fields (i.e., all vector fields are expressed in the body-fixed frame)

Think of γ as a curve on group of matrices, and write

$$\begin{aligned} \gamma'(t) &= \omega_i(t) E_i(\gamma(t)), \\ E_i(R) &= R\widehat{e}_i, \end{aligned}$$

where $R \in SO(3)$ and $e_1 = [1,0,0]$ and accordingly e_2 and e_3. We can do this because of

$$T_R(SO(3)) = \mathrm{span}\{R\widehat{e}_1, R\widehat{e}_2, R\widehat{e}_3\}$$

According to the same steps as above, the geodesic equation is:

$$0 = \dot{\omega}_i E_i(\gamma) + \omega_k \omega_j \left({}^{\mathbb{G}}\nabla_{E_j(\gamma)} E_k(\gamma)\right)$$

6.11 Proof of Euler Poincarè equation for satellite, page 3/3

Assume X, Y, Z take values in the basis $\{E_i\}$, and prove that

$$^{\mathbb{G}}\nabla_{R\widehat{e}_j} R\widehat{e}_k = R\left(e_j \times e_k + \frac{1}{2}\mathbb{J}^{-1}(e_j \times \mathbb{J}e_k) + \frac{1}{2}\mathbb{J}^{-1}(e_k \times \mathbb{J}e_j)\right).$$

This is a consequence of equation (1) and of the fact that the Lie brackets $[E_i(R), E_j(R)] = R(\widehat{e_i \times e_j})$ and that the metric is invariant.

Therefore, the geodesic equation becomes:

$$0 = \gamma\left(\dot{\omega}_i e_i + \omega_k \omega_j \left(e_j \times e_k + \frac{1}{2}\mathbb{J}^{-1}(e_j \times \mathbb{J}e_k) + \frac{1}{2}\mathbb{J}^{-1}(e_k \times \mathbb{J}e_j)\right)\right)$$

and, using the fact that γ is an invertible matrix and a few simplification, we get the right equation:

$$0 = \dot{\omega} + \frac{1}{2}\mathbb{J}^{-1}(\omega \times \mathbb{J}\omega)$$

Lecture #2: Modeling Symmetries and Nonholonomic Constraints

Francesco Bullo
Coordinated Science Lab, General Engineering Dept, ECE, AAE
University of Illinois at Urbana-Champaign
1308 W. Main St, Urbana, IL 61801, USA
bullo@uiuc.edu, http://motion.csl.uiuc.edu

This lecture based on the following references

[1] A. D. Lewis, "Simple mechanical control systems with constraints," *IEEE TAC*, 45(8):1420–1436, 2000.

[2] F. Bullo and M. Žefran, "On mechanical control systems with nonholonomic constraints and symmetries," *Syst. & Control L.*, 45(2):133–143, 2002.

[3] F. Bullo and M. Žefran, "Modeling and controllability for a class of hybrid mechanical systems," *IEEE TRA*, Aug 2002. To appear.

Incomplete List of References on Systems with Constraints

[1] J. L. Synge. Geodesics in nonholonomic geometry. *Mathematische Annalen*, 99:738–751, 1928.

[2] A. M. Vershik. Classical and non-classical dynamics with constraints. In Y. G. Borisovich and Yu. E. Gliklikh, editors, *Global analysis: studies and applications*, volume 1108 of *Lecture Notes in Mathematics*, pages 278–301. Springer Verlag, New York, NY, 1984.

[3] N. Sarkar, X. Yun, and V. Kumar. Control of mechanical systems with rolling constraints: Application to dynamic control of mobile robots. *International Journal of Robotics Research*, 13(1):55–69, 1994.

[4] A. J. van der Schaft and B. M. Maschke. On the Hamiltonian formulation of nonholonomic mechanical systems. *Reports in Mathematical Physics*, 34(2):225–233, 1994.

[5] A. M. Bloch and P. E. Crouch. Nonholonomic control systems on Riemannian manifolds. *SIAM Journal on Control and Optimization*, 33(1):126–148, 1995.

[6] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and R. M. Murray. Nonholonomic mechanical systems with symmetry. *Archive for Rational Mechanics and Analysis*, 136(1):21–99, 1996.

[7] J. P. Ostrowski and J. W. Burdick. The geometric mechanics of undulatory robotic locomotion. *International Journal of Robotics Research*, 17(7):683–701, 1998.

[8] P. S. Krishnaprasad and D. P. Tsakiris. Oscillations, $SE(2)$-snakes and motion control: a study of the roller racer. *Dynamics and Stability of Systems*, 16(4):347–397, 2001.

7 Essential review

7.1 Coordinate-free modelling: I

- manifold Q, metric $\mathbb{G}$
- vector fields are written in terms of the canonical basis $\{\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\}$, and co-vector fields in terms of $\{\mathrm{d}q^1, \dots, \mathrm{d}q^n\}$
- given a function φ:

$$\mathrm{d}\varphi = \frac{\partial \varphi}{\partial q^i}\mathrm{d}q^i$$

$$\operatorname{grad}\varphi = \left(\mathbb{G}^{ij}\frac{\partial \varphi}{\partial q^j}\right)\frac{\partial}{\partial q^i}$$

$$\dot q = -\operatorname{grad}\varphi(q) \qquad \dots \text{ (negative) gradient flow}$$

- metric gives rise to connection with certain properties

7.2 Coordinate-free modelling: II

(i) given functions $\left\{\Gamma^i_{jk}\right\}$, and curve $\gamma\colon I \to \mathbb{R}$

$$\left(\nabla_{\gamma'}\gamma'\right)^i = \ddot\gamma^i + \Gamma^i_{jk}\dot\gamma^j\dot\gamma^k = 0 \qquad \dots \text{ geodesic flow}$$

(ii) Given two vector fields X, Y, the **covariant derivative** of Y with respect to X is the third vector field $\nabla_X Y$ defined via

$$(\nabla_X Y)^i = \frac{\partial Y^i}{\partial q^j}X^j + \Gamma^i_{jk}X^jY^k.$$

(iii) **symmetric product**

$$\langle Y_a : Y_b\rangle = \nabla_{Y_a}Y_b + \nabla_{Y_b}Y_a$$

$$\langle Y_a : Y_b\rangle^i = \frac{\partial Y_a^i}{\partial q^j}Y_b^j + \frac{\partial Y_b^i}{\partial q^j}Y_a^j + \Gamma^i_{jk}\left(Y_a^jY_b^k + Y_a^kY_b^j\right)$$

7.3 Coordinate-free modelling: III

affine connection control system

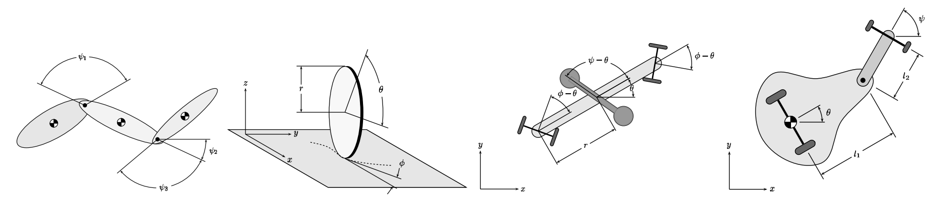
$$\nabla_{\gamma'}\gamma' = Y_0(\gamma) + R(\gamma') + \sum_{a=1}^{m} Y_a(\gamma) u_a(t)$$

Ex #1: robotic manipulators with kinetic energy and forces at joints
simple systems with conservative forces

Ex #2: aerospace and underwater vehicles
invariant systems on Lie groups

Ex #3: systems subject to nonholonomic constraints
locomotion devices with drift, e.g., bicycle, snake-like robots

8 Introduction to systems subject to constraints



Constraints can be of two types:

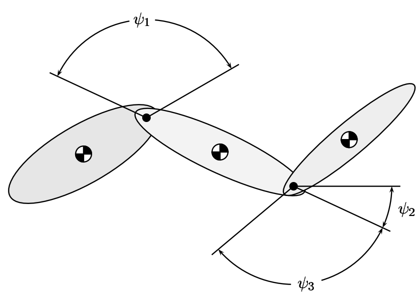
(i) constraints on q are called **integrable**

(ii) constraints on v_q are **sometimes** called **non-integrable**

from the greek roots:

integrable = **holonomic**
nonintegrable = **nonholonomic**

8.1 Integrable constraints



- constraint on the configuration, such as clamping. It is given by

$$\varphi(q) = 0$$

  where $\varphi : \mathsf{Q} \to \mathbb{R}$

- easy case, analyse on smaller space

- Sometimes, an integrable constraints appears as:

$$\langle w\,,\,\gamma'\rangle = 0,$$

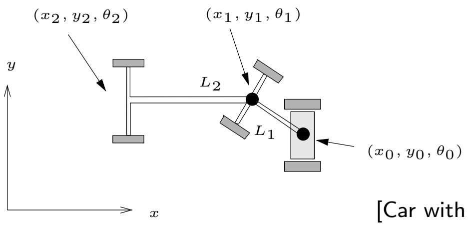
  where, if $w = \mathrm{d}\varphi$, one writes

$$\langle \mathrm{d}\varphi\,,\,\gamma'\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\gamma(t)) \quad \Rightarrow \quad \varphi(\gamma(t)) = \text{constant}$$

- Problem: given an arbitrary co-vector w, when is it $w = \mathrm{d}\varphi$?

  Locally, construct annihilator distribution $\mathscr{D}$. If $\mathscr{D}$ is involutive, then w is a holonomic constraint.

8.2 Nonintegrable constraints I: kinematic systems

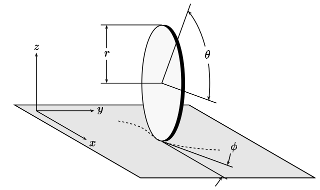


[Car with trailer can be parked anywhere.]

- **nonintegrable constraints are constraints on velocity, that cannot be written as constraints on configurations**
- classic example is rolling without sliding
- If system has full control over all feasible velocities,
  then kinematic analysis suffices

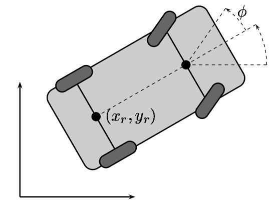
TEST: set all control inputs to zero, does the mechanical systems still move?
➡ driftless systems

Examples of kinematic systems



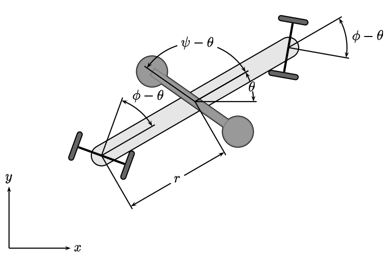
$$\dot{x} = v\cos\phi$$
$$\dot{y} = v\sin\phi$$
$$\dot{\phi} = \omega$$

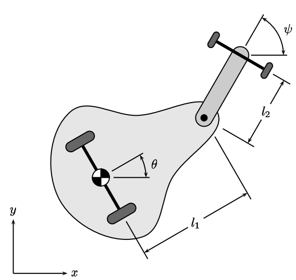
(wheeled robot dynamics)



$$\begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{\ell}\tan\phi \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

8.3 Nonintegrable constraint II: dynamic systems





- general case is a dynamic case, i.e., system can move with input at zero
- basic example: bicycle

9 Simple Mechanical Control Systems with constraints

Nonholonomic constraint described by **constraint one-form** ω

$$\langle \omega \,,\, \gamma' \rangle = 0$$

A simple mechanical control system subject to constraints

(i) A simple mechanical control system $(\mathsf{Q}, \mathbb{G}, V = 0, \mathcal{F} = \{F^1, \dots, F^m\})$

(ii) A collection of constraint one-forms $\{\omega_1, \dots, \omega_p\}$.

The annihilator of $\mathrm{span}\{\omega_1, \dots, \omega_p\}$ is the **constraint distribution** $\mathscr{D}$
i.e., the distribution of feasible velocities

Orthogonal projections:

$$P : \mathsf{TQ} \to \mathscr{D} \subset \mathsf{TQ} \quad \text{and} \quad P^\perp : \mathsf{TQ} \to \mathscr{D}^\perp \subset \mathsf{TQ}$$

9.1 Equations of motion

The solution to the mechanical control system subject to the constraint distribution $\mathscr{D}$ is the curve $\gamma : I \to Q$ solution to

$$^{\mathbb{G}}\nabla_{\gamma'(t)}\gamma'(t) = \lambda(t) + \sum_{a=1}^{m} (\mathbb{G}^{-1}F^a)u_a$$
$$P^{\perp}(\gamma') = 0$$

where $t \mapsto \lambda(t) \in \mathscr{D}^{\perp}$ is the Lagrange multiplier, and $\gamma'(0) \in \mathscr{D}$.

Theorem: Constrained equations of motion (Synge 1928)

$$^{\mathscr{D}}\nabla_{\gamma'}\gamma' = \sum_{a=1}^{m} (P\mathbb{G}^{-1}F^a)u_a$$

with respect to the **constrained affine connection** (Lewis 2000)

$$^{\mathscr{D}}\nabla_X Y = {}^{\mathbb{G}}\nabla_X Y + \left({}^{\mathbb{G}}\nabla_X P^{\perp}\right)(Y)$$

9.2 Expressions in coordinates

(i) design $\mathcal{X} = \{X_1, \dots, X_{n-p}\}$ an **orthogonal** basis for feasible velocities $\mathscr{D}$

(ii) compute $({}^{\mathcal{X}}\Gamma)^k_{ij} = \frac{1}{\|X_k\|^2}\langle\langle {}^{\mathbb{G}}\nabla_{X_i}X_j \, , \, X_k \rangle\rangle$

(iii) compute $Y^k_a = \frac{1}{\|X_k\|^2}\langle F^a \, , \, X_k\rangle$

Then the constrained equations of motion are

$$\gamma'(t) = v^i(t)X_i(\gamma(t))$$
$$\dot{v}^k(t) + ({}^{\mathcal{X}}\Gamma)^k_{ij}v^i(t)v^j(t) = \sum_{a=1}^{m} Y^k_a(\gamma)u_a(t)$$

kinematic + dynamic equations

9.3 Comments

Constrained equations of motion

$$\gamma' = v^i X_i(\gamma)$$
$$\dot{v}^k + ({}^{\mathcal{X}}\Gamma)^k_{ij}v^i v^j = \sum_{a=1}^{m} Y^k_a u_a$$

(i) v^i components of γ' are **pseudo-velocity**

(ii) $({}^{\mathcal{X}}\Gamma)^k_{ij}$ are generalized Christoffel symbols for ${}^{\mathscr{D}}\nabla$ with respect to $\{X_1, \dots, X_n\}$

$$^{\mathscr{D}}\nabla_{X_i}X_j = ({}^{\mathcal{X}}\Gamma)^k_{ij}X_k$$

however, no need to compute the projection P, nor its covariant derivative ${}^{\mathbb{G}}\nabla P^{\perp}$

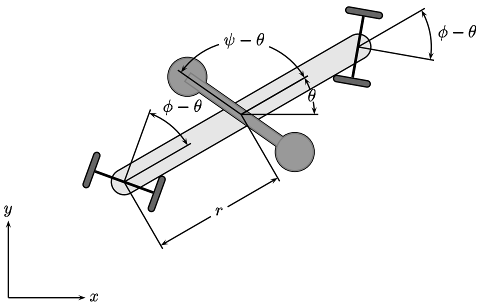
(iii) Y^k_a is the projection of the control vector fields onto X_k. If conservative forces, i.e., $F^a = \mathrm{d}\varphi_a$, then $Y^k_a = \frac{1}{\|X_k\|^2}\mathscr{L}_{X_k}\varphi_a$

Invariance under group action If a system is invariant under a group action and the basis for $\mathscr{D}$ consists of invariant vectors, the generalized Christoffel symbols $({}^{\mathcal{X}}\Gamma)^k_{ij}$ and the coefficients of the control vector fields Y^k_a are invariant.

Key examples easily handled see next pages.

Missing work Still to work out: bicycle, plate-and-ball systems, omni-directional, redundant, variable-geometry vehicles

10 The snakeboard example



Configuration manifold: $SE(2) \times \mathbb{S}^2$

Coordinates: $q = (x, y, \theta, \psi, \phi)$

Input forces: $\mathrm{d}\psi$, $\mathrm{d}\phi$

Inertia tensor:

$$[\mathbb{G}] = \begin{pmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & \ell^2 m & J_r & 0 \\ 0 & 0 & J_r & J_r & 0 \\ 0 & 0 & 0 & 0 & J_w \end{pmatrix}$$

Constraints:

$$\dot{x}_{\text{front}} \sin(\theta - \phi) - \dot{y}_{\text{front}} \cos(\theta - \phi) = 0$$
$$\dot{x}_{\text{back}} \sin(\theta + \psi) - \dot{y}_{\text{back}} \cos(\theta + \psi) = 0$$

Constraint forms:

$$\omega_1 = \sin(\phi - \theta)\mathrm{d}x + \cos(\phi - \theta)\mathrm{d}y + \ell \cos\phi \mathrm{d}\theta$$
$$\omega_2 = -\sin(\phi + \theta)\mathrm{d}x + \cos(\phi + \theta)\mathrm{d}y - \ell \cos\phi \mathrm{d}\theta$$

10.1 Application of the method

Step (i): Choice of basis for $\mathscr{D}$:

$$X_1 = \ell\cos\phi\cos\theta\frac{\partial}{\partial x} + \ell\cos\phi\sin\theta\frac{\partial}{\partial y} - \sin\phi\frac{\partial}{\partial\theta} \, ,$$
$$X_2' = \frac{\partial}{\partial\psi} \, , \quad X_3' = \frac{\partial}{\partial\phi} \, .$$

Using the Gramm-Schmitt procedure we can construct the orthogonal basis:

$$X_2 = \frac{J_r}{m\ell} \cos\phi\sin\phi V_x - \frac{J_r}{m\ell^2} \sin^2\phi \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\psi} \, , \quad X_3 = X_3'$$

Step (ii): compute generalized Christoffel symbols

$$({}^{\mathcal{X}}\Gamma)^1_{32} = \frac{J_r}{m\ell^2}\cos\phi \, , \quad ({}^{\mathcal{X}}\Gamma)^2_{31} = -\frac{m\ell^2\cos\phi}{m\ell^2 + J_r\sin^2\phi} \, , \quad ({}^{\mathcal{X}}\Gamma)^2_{32} = -\frac{J_r\cos\phi\sin\phi}{m\ell^2 + J_r\sin^2\phi}$$

Step (iii): input coefficients: $\mathscr{L}_{X_2}\psi = 1, \quad \mathscr{L}_{X_3}\phi = 1$

10.2 Kinematic and dynamic equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \ell\cos\phi\cos\theta \\ \ell\cos\phi\sin\theta \\ -\sin\phi \\ 0 \\ 0 \end{pmatrix} v^1 + \begin{pmatrix} \frac{J_r}{m\ell}\cos\phi\sin\phi\cos\theta \\ \frac{J_r}{m\ell}\cos\phi\sin\phi\sin\theta \\ -\frac{J_r}{m\ell^2}(\sin\phi)^2 \\ 1 \\ 0 \end{pmatrix} v^2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v^3$$

$$\dot{v}^1 + \frac{J_r}{m\ell^2}(\cos\phi)v^2v^3 = 0$$
$$\dot{v}^2 - \frac{m\ell^2\cos\phi}{m\ell^2 + J_r(\sin\phi)^2}v^1v^3 - \frac{J_r\cos\phi\sin\phi}{m\ell^2 + J_r(\sin\phi)^2}v^2v^3 = \frac{m\ell^2}{m\ell^2 J_r + J_r^2(\sin\phi)^2}u_\psi$$
$$\dot{v}^3 = \frac{1}{J_w}u_\phi \, .$$

10.3 Kinematic and dynamic equations

the kinematic equations are

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\ell\cos\phi\cos\theta\\ \ell\cos\phi\sin\theta\\ -\sin\phi\end{pmatrix} v + \begin{pmatrix}\frac{J_r}{m\ell}\cos\phi\sin\phi\cos\theta\\ \frac{J_r}{m\ell}\cos\phi\sin\phi\sin\theta\\ -\frac{J_r}{m\ell^2}(\sin\phi)^2\end{pmatrix}\dot{\psi}$$

and the dynamic equations are

$$\begin{aligned}
&\dot{v} + \frac{J_r}{m\ell^2}(\cos\phi)\dot{\phi}\dot{\psi} = 0\\
&\ddot{\psi} - \frac{m\ell^2\cos\phi}{m\ell^2 + J_r(\sin\phi)^2} v\dot{\phi} - \frac{J_r\cos\phi\sin\phi}{m\ell^2 + J_r(\sin\phi)^2}\dot{\phi}\dot{\psi}\\
&\qquad = \frac{m\ell^2}{m\ell^2 J_r + J_r^2(\sin\phi)^2} u_\psi\\
&\ddot{\phi} = \frac{1}{J_w} u_\phi\,.
\end{aligned}$$

10.4 Software implementation

Mathematica implementation; FullSimplify commands erased for readability

```
(* CONNECTIONS AND OTHER OPERATIONS *)

LieDer[X_,h_,x_] := Sum[D[h,x[[i]]]X[[i]],{i,Length[x]}];

LieBracket[X_,Y_,x_]:=Module[{i,j,N=Length[x]},
   Table[Sum[D[Y[[i]],x[[j]]]X[[j]]-D[X[[i]],x[[j]]]Y[[j]],{j,N}],{i,N}]];

LeviCivita[\metric_,x_]:=Module[{Minv=Inverse[M],i,j,k,h,
   N=Length[x]},Table[Sum[Minv[[h,k]](D[M[[h,j]],x[[i]]]+
   D[M[[i,h]],x[[j]]] - D[M[[i,j]],x[[h]]])/2,{h,N}],{k,N},{j,N},{i,N}]];

CovariantDer[X_,Y_,Nabla_,x_]:=Module[{i,j,k,N=Length[x]},
   Table[Sum[D[Y[[i]],x[[j]]]X[[j]]+
    Sum[Nabla[[i,j,k]]X[[j]]Y[[k]],{k,N}],{j,N}],{i,N}]];
```

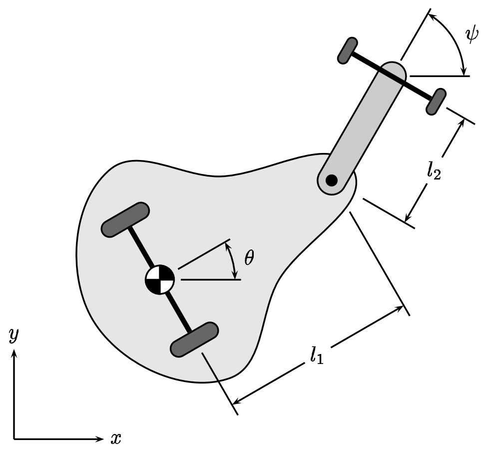


```
(* SNAKEBOARD EXAMPLE *)

q = {x,y,th,psi,phi};  M ={{m,0,0,0,0},{0,m,0,0,0},
{0,0,m ell^2,Jr,0},{0,0,Jr,Jr,0},{0,0,0,0,Jw}};
nabla = LeviCivita[M, q];

(* FEASIBLE VELOCITIES *)

Vx  = {Cos[th],Sin[th],0,0,0}; Vth = {0,0,1,0,0};
X1  = ell Cos[phi] Vx - Sin[phi] Vth;
X2p = {0,0,0,1,0}; X3 = {0,0,0,0,1};

(* ORTHOGONALIZE VECTORS VIA GRAMM-SCHMITT *)

X1X1 = X1.M.X1; X2pX2p = X2p.M.X2p; X3X3 = X3.M.X3;
X1X3 = X1.M.X3; X1X2p = X1.M.X2p; X2pX3 = X2p.M.X3;
X2 = X2p-X1(X1X2p/X1X1); X2X2 = X2.M.X2;
```



```
(* CHRISTOFFEL SYMBOLS *)

X = {X1, X2, X3}; norms = {X1X1, X2X2, X3X3};
Tnabla = Table[ CovariantDer [X[[i]],X[[j]],nabla,q].M.X[[k]]/norms[[k]]
                ,{k,1,3} ,{i,1,3}, {j,1,3}];

(* INPUTS *)

F = Table[ LieDer[X[[k]],psi,q]/norms[[k]]u1
           + LieDer[X[[k]],phi,q]/norms[[k]]u2 ,{k,3}];

(* EQUATIONS OF MOTION *)

v={vel[t], psi'[t], phi'[t]};   EqMotion = Table[
D[v[[k]],t]+Sum[Tnabla[[k,i,j]] v[[i]] v[[j]],{i,3},{j,3}]==F[[k]], {k,3}];

(* CONTROLLABILITY ANALYSIS *)

X13 = LieBracket[X1,X3,q]; X113 = LieBracket[X1,X13,q];
Det[AppendColumns[{X1},{X2},{X3},{X13},{X113}]];
```

11 The roller racer example



Configuration manifold:	$SE(2) \times \mathbb{S}$
Coordinates:	$q = (x, y, \theta, \psi)$
Input force:	$\mathrm{d}\psi$

Inertia tensor:

$$[\mathbb{G}] = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & I_1 + I_2 & I_2 \\ 0 & 0 & I_2 & I_2 \end{pmatrix}.$$

Constraint one-forms:

$$\omega_1 = \sin\theta \mathrm{d}x - \cos\theta \mathrm{d}y$$
$$\omega_2 = \sin(\theta+\psi)\mathrm{d}x - \cos(\theta+\psi)\mathrm{d}y - (\ell_2 + \ell_1\cos\psi)\mathrm{d}\theta - \ell_2 \mathrm{d}\psi\,.$$

11.1 Application of the method

Step (i): Choice of basis for $\mathscr{D}$:

$$X_1 = \cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y} + \left(\frac{\sin\psi}{\ell_2 + \ell_1\cos\psi}\right)\frac{\partial}{\partial\theta}$$
$$X_2' = -\left(\frac{\ell_2}{\ell_2 + \ell_1\cos\psi}\right)\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\psi}$$

Using the Gramm-Schmitt procedure we can construct the orthogonal basis:

$$X_2 = \frac{(\ell_2 I_1 - \ell_1 I_2\cos\psi)\sin\psi}{f_1(\psi)} V_x - \frac{m\ell_2(\ell_2+\ell_1\cos\psi) + I_2\sin^2\psi}{f_1(\psi)}\,\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\psi}\,.$$

where
$$V_x = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$
$$f_1(\psi) = m(\ell_2 + \ell_1\cos\psi)^2 + (I_1 + I_2)\sin^2\psi$$
$$f_2(\psi) = m\ell_2^2 I_1 + \ell_1^2 I_2 m(\cos\psi)^2 + I_1 I_2 \sin^2\psi\,.$$

Step (ii): compute generalized Christoffel symbols

$$({}^{\mathcal{X}}\Gamma)^1_{21} = \left(\frac{\ell_1 + \ell_2\cos\psi}{\ell_2 + \ell_1\cos\psi}\right)\frac{(I_1 + I_2)\sin\psi}{f_1(\psi)}$$
$$({}^{\mathcal{X}}\Gamma)^1_{22} = \frac{m(\ell_1 + \ell_2\cos\psi)(\ell_2 + \ell_1\cos\psi)(\ell_1 I_2\cos\psi - \ell_2 I_1)}{f_1(\psi)^2}$$
$$({}^{\mathcal{X}}\Gamma)^2_{21} = \left(\frac{\ell_1 + \ell_2\cos\psi}{\ell_2 + \ell_1\cos\psi}\right)\frac{m(\ell_1 I_2\cos\psi - \ell_2 I_1)}{f_2(\psi)}$$
$$({}^{\mathcal{X}}\Gamma)^2_{22} = \frac{-m(\ell_1 I_2\cos\psi - \ell_2 I_1)(\sin\psi) f_3(\psi)}{f_1(\psi) f_2(\psi)}$$

where $f_3(\psi) = (\ell_1 I_2 - \ell_2 I_1\cos\psi) + m\ell_1\ell_2(\ell_2 + \ell_1\cos\psi)$.

Step (iii): input coefficients: $\mathscr{L}_{X_1}\psi = 0, \quad \frac{1}{\|X_2\|^2}\mathscr{L}_{X_2}\psi = \frac{f_1(\psi)}{f_2(\psi)}$

11.2 Kinematic and dynamic equations

the kinematic equations are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \frac{\sin\psi}{\ell_2+\ell_1\cos\psi} \end{pmatrix} v + \begin{pmatrix} \frac{(\ell_2 I_1 - \ell_1 I_2 \cos\psi)\sin\psi}{f_1(\psi)}\cos\theta \\ \frac{(\ell_2 I_1 - \ell_1 I_2 \cos\psi)\sin\psi}{f_1(\psi)}\sin\theta \\ \frac{m\ell_2(\ell_2+\ell_1\cos\psi)+I_2(\sin\psi)^2}{-f_1(\psi)} \end{pmatrix} \dot{\psi}$$

and the dynamic equations are

$$\dot{v} + ({}^{\mathcal{X}}\Gamma)^1_{21}(\psi)\dot{\psi}v + ({}^{\mathcal{X}}\Gamma)^1_{22}(\psi)\dot{\psi}^2 = 0$$
$$\ddot{\psi} + ({}^{\mathcal{X}}\Gamma)^2_{21}(\psi)\dot{\psi}v + ({}^{\mathcal{X}}\Gamma)^2_{22}(\psi)\dot{\psi}^2 = \frac{f_1(\psi)}{f_2(\psi)}u_\psi \, .$$

12 Proofs

12.1 Constrained affine connection

Consider

$$^{\mathbb{G}}\nabla_{\gamma'}\gamma' = \lambda(t) + \mathbb{G}^{-1}F \quad (2)$$
$$P^\perp(\gamma') = 0. \quad (3)$$

Project equation (2) onto $\mathscr{D}^\perp$, and covariantly differentiate equation (3):

$$P^\perp({}^{\mathbb{G}}\nabla_{\gamma'}\gamma') = \lambda(t) + P^\perp(\mathbb{G}^{-1}F)$$
$$^{\mathbb{G}}\nabla_{\gamma'}\left(P^\perp(\gamma')\right) = 0 \quad \Rightarrow \quad P^\perp({}^{\mathbb{G}}\nabla_{\gamma'}\gamma') = -\left({}^{\mathbb{G}}\nabla_{\gamma'}P^\perp\right)(\gamma').$$

Hence:

$$\lambda(t) = -\left({}^{\mathbb{G}}\nabla_{\gamma'}P^\perp\right)(\gamma') - P^\perp(\mathbb{G}^{-1}F)$$

and

$$^{\mathbb{G}}\nabla_{\gamma'}\gamma' + \left({}^{\mathbb{G}}\nabla_{\gamma'}P^\perp\right)(\gamma') = P(\mathbb{G}^{-1}F),$$

Define:

$$^{\mathscr{D}}\nabla_X Y = {}^{\mathbb{G}}\nabla_X Y + \left({}^{\mathbb{G}}\nabla_X P^\perp\right)(Y)$$

Summarizing:

$$^{\mathbb{G}}\nabla_{\gamma'}\gamma' + \left({}^{\mathbb{G}}\nabla_{\gamma'}P^\perp\right)(\gamma') = P(\mathbb{G}^{-1}F),$$

can be written as

$$^{\mathscr{D}}\nabla_{\gamma'}\gamma' = P(\mathbb{G}^{-1}F)$$

where $^{\mathscr{D}}\nabla$ is the constrained connection.

The Christoffel symbols of the constrained connection with respect to the basis $\{\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\}$ are

$$({}^{\mathscr{D}}\Gamma)^k_{ij} = \Gamma^k_{ij} + \frac{P_{kj}}{\partial q^i} + \Gamma^k_{im}P_{mj} - \Gamma^m_{ij}P_{km}$$

12.2 Constrained equations in coordinates

Definition 1 $\left({}^{\mathbb{G}}\nabla_X P^\perp\right)(Y) = {}^{\mathbb{G}}\nabla_X\left(P^\perp(Y)\right) - P^\perp({}^{\mathbb{G}}\nabla_X Y)$.

Lemma 2 For $Y \in \mathscr{D}$, $\;{}^{\mathscr{D}}\nabla_X Y = P({}^{\mathbb{G}}\nabla_X Y)$

Lemma 3 Expression for $^{\mathscr{D}}\nabla_{\gamma'}\gamma'$, where $\{X_i\}$ orthogonal family spanning $\mathscr{D}$:

$$^{\mathscr{D}}\nabla_{\gamma'}\gamma' = {}^{\mathscr{D}}\nabla_{\gamma'}(v^i X_i) = \dot{v}^i X_i + v^i\left({}^{\mathscr{D}}\nabla_{\gamma'}X_i\right)$$
$$= \dot{v}^i X_i + v^i v^j\, {}^{\mathscr{D}}\nabla_{X_j}X_i$$

Inner product with X_k:

$$\langle\langle X_k \, , \, {}^{\mathscr{D}}\nabla_{\gamma'}\gamma'\rangle\rangle = \dot{v}^i\langle\langle X_k \, , \, X_i\rangle\rangle + v^i v^j\langle\langle X_k \, , \, {}^{\mathscr{D}}\nabla_{X_j}X_i\rangle\rangle$$
$$= \dot{v}^k\|X_k\|^2 + v^i v^j\langle\langle X_k \, , \, {}^{\mathscr{D}}\nabla_{X_j}X_i\rangle\rangle$$

Final simplification:

$$\langle\langle {}^{\mathscr{D}}\nabla_{X_i}X_j \, , \, X_k\rangle\rangle = \langle\langle P\,{}^{\mathbb{G}}\nabla_{X_i}X_j \, , \, X_k\rangle\rangle = \langle\langle {}^{\mathbb{G}}\nabla_{X_i}X_j \, , \, X_k\rangle\rangle$$

13 Ideal impact models

- here only ideal case: no friction, plastic/elastic, holonomic/nonholonomic impact
- impact entails
  (i) impulsive force that causes a jump in γ'
  (ii) switch in equations of motions

Reference on impact models

[1] B. Brogliato. *Nonsmooth Impact Mechanics: Models, Dynamics, and Control*, volume 220 of *Lecture Notes in Control and Information Sciences*. Springer Verlag, New York, NY, 1996.

13.1 Definition of impact

- $(\mathsf{Q}, \mathbb{G}, \mathcal{F} = \mathrm{span}\{F^1, \ldots, F^m\})$ is a simple mechanical system
- $\mathscr{D}^-$ and $\mathscr{D}^+$ are two set of feasible velocities (right before, right after impact)
- $(\nabla^-, P^-\mathcal{F})$ and $(\nabla^+, P^+\mathcal{F})$ give eqns of motion,
  (P is orthogonal projection onto feasible velocities)

The system undergoes an **impact** at time t if

(i) the dynamics switch from $(\nabla^-, P^-\mathcal{F})$ to $(\nabla^+, P^+\mathcal{F})$,

(ii) there exists a tensor field $J_q : \mathsf{T}_q\mathsf{Q} \to \mathsf{T}_q\mathsf{Q}$ such that

$$q(t^+) = q(t^-)$$
$$\gamma'(t^+) = J_q\left(\gamma'(t^-)\right).$$

13.2 Classic impacts

Plastic impact from large to smaller space: The two sets of feasible velocities $\mathscr{D}^-$ and $\mathscr{D}^+$ are distinct (for example $\mathscr{D}^- = \mathsf{TQ}$ and $\mathscr{D}^+ = \mathsf{TR}$ is the tangent space of a submanifold $\mathsf{R} \subset \mathsf{Q}$). The operator

$$J_q = P_{\mathscr{D}^+}$$

is the orthogonal projection onto $\mathscr{D}^+$.

Elastic impact against surface: The equations of motion do not change, as connection and input forces do not change. There exist a submanifold R such that

$$J_q = P_{\mathsf{TR}} + (-e)P_{\mathsf{TR}}^{\perp}$$

where P_{TR} is the orthogonal projection onto the tangent space to R and where $0 < e < 1$ is the coefficient of restitution.

13.3 Hybrid mechanical control systems

given a mechanical control system $(\mathsf{Q}, \mathbb{G}, \mathcal{F})$ with a given set of constraint distributions $\mathscr{D}_i$, where i belongs to an index set I.

For each constraint $\mathscr{D}_i$, we consider the constrained mechanical control system $\Sigma_i = [\mathsf{Q}, \mathbb{G}, \mathcal{F}, \mathscr{D}_i, U]$, with associated ∇_i and $\mathcal{Y}_i$.

We define the **hybrid mechanical control system** as

$$\mathsf{HMCS} = [I, \mathsf{Q}, \Sigma_{\mathsf{Q}}, \mathbf{V}, \Delta] \quad (4)$$

where I index set, Q, Σ_{Q} collection of constrained mech. sys., $\mathbf{V} = \{v_{ij}\}_{i,j \in I}$ discrete controls and Δ jump transition maps (linear operators in γ' parametrized by v_{ij}).

Summary of Modeling Methods (lectures #1 and #2)

Simple mechanical control systems with constraints

A **simple mechanical control system with constraints** is a quintuple $(\mathsf{Q}, \mathbb{G}, V, \mathscr{D}, \mathscr{F})$ comprised of the following objects:

(i) an n-dimensional configuration manifold Q,

(ii) a Riemannian metric $\mathbb{G}$ on Q describing the kinetic energy,

(iii) a function V on Q describing the potential energy,

(iv) a distribution $\mathscr{D}$ of feasible velocities describing the linear velocity constraints, and

(v) a collection of m covector fields $\mathcal{F} = \{F^1, \dots, F^m\}$, linearly independent at each $q \in \mathsf{Q}$, defining the control forces.

Given the metric $\mathbb{G}$ and the distribution $\mathscr{D}$, we define the following objects. We let $P : T\mathsf{Q} \to T\mathsf{Q}$ be the orthogonal projection onto the distribution $\mathscr{D}$ with respect to the metric $\mathbb{G}$. We let ${}^{\mathbb{G}}\nabla$ be the Levi-Civita connection on Q induced by the metric $\mathbb{G}$. We let ∇ be the **constrained affine connection** defined by the metric $\mathbb{G}$ and the constraint distribution $\mathscr{D}$ according to

$$\nabla_X Y = {}^{\mathbb{G}}\nabla_X Y - \left({}^{\mathbb{G}}\nabla_X P\right)(Y),$$

for any vector fields X and Y. When the vector field Y takes value in $\mathscr{D}$, we have

$$\nabla_X Y = P({}^{\mathbb{G}}\nabla_X Y),$$

Given the Riemannian metric $\mathbb{G}$, we let $\mathbb{G}\colon T\mathsf{Q} \to T^*\mathsf{Q}$ and $\mathbb{G}^{-1}\colon T^*\mathsf{Q} \to T\mathsf{Q}$ denote the musical isomorphisms associated with $\mathbb{G}$. For $a \in \{1, \dots, m\}$, we define the input vector fields $Y_a = P(\mathbb{G}^{-1}(F^a))$, the family of **input vector fields** $\mathcal{Y} = \{Y_1, \dots, Y_m\}$, and the **input distribution** $\mathscr{Y}$ with $\mathscr{Y}_q = \mathrm{span}_{\mathbb{R}}\{Y_1(q), \dots, Y_m(q)\}$. Let $\mathscr{L}_X f$ be the Lie derivative of a scalar function f with respect to the vector field X. The **gradient** of the function V is the vector field $\operatorname{grad} V$ defined implicitly by

$$\mathbb{G}(\operatorname{grad} V, X) = \mathscr{L}_X V.$$

A **controlled trajectory** for the mechanical control system with constraints $(\mathsf{Q}, \mathbb{G}, V, \mathscr{D}, \mathscr{F})$ is a pair (γ, u) with $\gamma\colon [0, T] \to \mathsf{Q}$ and $u = (u_1, \dots, u_m)\colon [0, T] \to \mathbb{R}^m$ satisfying the **controlled geodesic equations**

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = -P(\operatorname{grad} V(\gamma(t))) + \sum_{a=1}^{m} Y_a(\gamma(t)) u_a(t). \tag{5}$$

Here we assume that $\dot\gamma(0) \in \mathscr{D}_{\gamma(0)}$ and comment that this implies that $\dot\gamma(t) \in \mathscr{D}_{\gamma(t)}$ for all $t \in [0, T]$. Furthermore, we assume the input functions $u = (u_1, \dots, u_m)\colon [0, T] \to \mathbb{R}^m$ to be Lebesgue measurable functions, and we write $u \in \mathscr{U}^m_{\mathsf{dyn}}$.

Coordinate representation #1

On an open subset $U \subset \mathsf{Q}$ let $\mathcal{X} = \{X_1, \dots, X_n\}$ be a basis of vector fields, and set

$$\nabla_{X_i} X_j = ({}^{\mathcal{X}}\Gamma)^k_{ij} X_k, \tag{6}$$

where the n^3 functions $\{({}^{\mathcal{X}}\Gamma)^k_{ij} |\ i,j,k \in \{1,\dots,n\}\}$ are called the **generalized Christoffel symbols** with respect to $\mathcal{X}$. Given vector fields Y and Z on U, we can write $Y = Y^i X_i$ and $Z = Z^i X_i$. Accordingly,

$$\nabla_Y Z = \left((\mathscr{L}_{X_i} Z^k) Y^i + ({}^{\mathcal{X}}\Gamma)^k_{ij} Z^i Y^j \right) X_k.$$

Let the velocity curve $\dot\gamma \colon I \to \mathsf{T}U$ have components $(v^1, \dots, v^n)$ with respect to $\mathcal{X}$, i.e.,

$$\dot\gamma(t) = v^i(t) X_i(\gamma(t)).$$

The pair (γ, u) is a controlled trajectory for the controlled geodesic equations (5) if and only if it solves the **controlled Poincaré equations**

$$\dot v^k + ({}^{\mathcal{X}}\Gamma)^k_{ij}(\gamma) v^i v^j = -\left(P \operatorname{grad} V\right)^k(\gamma) + \sum_{a=1}^m Y^k_a(\gamma) u_a. \tag{7}$$

Coordinate representation #2

Let $(q^1, \dots, q^n)$ be a coordinate system for the open subset $U \subset \mathsf{Q}$. The curve $\gamma \colon I \to U$ has therefore components $(\gamma^1, \dots, \gamma^n)$. The coordinate system on U induces the natural coordinate basis $\{\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\}$ for the tangent bundle $\mathsf{T}U$. With respect to this basis, we write the velocity curve $\dot\gamma \colon I \to \mathsf{T}U$ as

$$\dot\gamma(t) = \dot\gamma^i(t) \frac{\partial}{\partial q^i}(\gamma).$$

In the coordinate system $(q^1, \dots, q^n)$, we write $\gamma = (\gamma^1, \dots, \gamma^n)$, $\dot\gamma = (\dot\gamma^1, \dots, \dot\gamma^n)$, and the equations of motion read

$$\ddot\gamma^k + \Gamma^k_{ij} \dot\gamma^i \dot\gamma^j = -\left(P \operatorname{grad} V\right)^k(\gamma) + \sum_{a=1}^m Y^k_a u_a. \tag{8}$$

Here, the Christoffel symbols $\{\Gamma^k_{ij} |\ i,j,k \in \{1,\dots,n\}\}$ and the terms in the right-hand side are computed with respect to the natural coordinate basis. We refer to these equations as the **controlled Euler-Lagrange equations**.

Remarks

(i) If the distribution $\mathscr{D}$ has rank $p < n$, it is useful to construct a local basis for $\mathsf{T}\mathsf{Q}$ by selecting the first p vector fields to generate $\mathscr{D}$, and the remaining $n - p$ to generate $\mathscr{D}^\perp$. In this case, one can see that $v^k(t) = 0$ for all time t and all $k \in \{p+1, \dots, n\}$.

(ii) Assume a Lie group G acts on the manifold Q, and assume the metric $\mathbb{G}$, and the distribution $\mathscr{D}$ are invariant. Then the constrained connection ∇ is invariant, and, selecting invariant vector fields $\{X_1, \dots, X_n\}$, the generalized Christoffel symbols are invariant functions.

(iii) simple mechanical control systems can be modeled under the general framework of **affine connection control systems**

$$\nabla_{\gamma'}\gamma' = Y_0(\gamma) + R(\gamma') + \sum_{a=1}^m Y_a(\gamma) u_a(t)$$

Lecture #3: Perturbation Analyses of Affine Connection Control Systems

Francesco Bullo
Coordinated Science Lab, General Engineering Dept, ECE, AAE
University of Illinois at Urbana-Champaign
1308 W. Main St, Urbana, IL 61801, USA
bullo@uiuc.edu, http://motion.csl.uiuc.edu

This lecture based on the following references

[1] F. Bullo, "Series expansions for mechanical control systems," *SIAM JCO*, 40(1):166–190, 2001.

[2] F. Bullo, "Averaging and vibrational control of mechanical systems," *SIAM JCO*, Submitted 1999. To appear 2002.

[3] F. Bullo, "Series expansions for analytic systems linear in controls," *Automatica*, 38(9):1425-1432, 2002.

13.4 Intro: Perturbation methods for mechanical control systems

Before design, analyse **forced response of Lagrangian system from rest**

I) High magnitude high frequency
"oscillatory control &
vibrational stabilization"

$$H = H(q,p) + \frac{1}{\epsilon}\varphi\left(q,p,u\left(\frac{t}{\epsilon}\right)\right)$$
$$p(0) = p_0$$

II) Small input from rest
"small-time local controllability"

$$H = H(q,p) + \epsilon\varphi(q,p,u(t))$$
$$p(0) = 0$$

III) Classical formulation
integrable Hamiltonian systems

$$H = H(q,p) + \epsilon\varphi(q,p)$$
$$p(0) = p_0$$

13.5 Intro: oscillatory control

Known: Oscillatory controls generate motion in Lie bracket directions

$$\dot{x} = f(x) + g_1(x)\left(\frac{1}{\sqrt{\epsilon}}\sin\frac{t}{\epsilon}\right) + g_2(x)\left(\frac{1}{\sqrt{\epsilon}}\cos\frac{t}{\epsilon}\right)$$

⬇

$$\dot{x} = f(x) + \frac{1}{2}[g_1, g_2](x)$$

Today's objective: **oscillatory controls in mechanical systems**

$$\nabla_{\gamma'}\gamma' = Y(q,t)$$
$$\gamma'(0) = 0, \quad \int_0^T Y(q,t)\mathrm{d}t = 0$$

Incomplete List of References on Series Expansion and Averaging related to Mechanical Systems

[1] A. A. Agračhev and R. V. Gamkrelidze, *The exponential representation of flows and the chronological calculus*, Math. USSR Sbornik, 35 (1978), pp. 727–785.

[2] J. Baillieul, *Stable average motions of mechanical systems subject to periodic forcing*, in Dynamics and Control of Mechanical Systems: The Falling Cat and Related Problems, M. J. Enos, ed., vol. 1, Field Institute Communications, 1993, pp. 1–23.

[3] R. E. Bellman, J. Bentsman, and S. M. Meerkov, *Vibrational control of nonlinear systems: Vibrational controllability and transient behavior*, IEEE TAC, 31 (1986), pp. 717–724.

[4] P. E. Crouch, *Geometric structures in systems theory*, IEE Proceedings, 128 (1981), pp. 242–252.

[5] M. Kawski. Geometric homogeneity and applications to stabilization. In *Nonlinear Control Systems Design Symposium (NOLCOS)*, pages 251–256, Tahoe City, CA, July 1995.

[6] M. Kawski and H. J. Sussmann, *Noncommutative power series and formal Lie-algebraic techniques in nonlinear control theory*, in Operators, Systems, and Linear Algebra, U. Helmke, D. Pratzel-Wolters, and E. Zerz, eds., Teubner, Stuttgart, Germany, 1997, pp. 111–128.

[7] M. Levi, "Geometry of Kapitsa's potentials," *Nonlinearity*, vol. 11, no. 5, pp. 1365–8, 1998.

[8] W. Liu and H. J. Sussmann, "Continuous dependence with respect to the input of trajectories of control-affine systems," *SIAM Journal on Control and Optimization*, vol. 37, no. 3, pp. 777–803, 1999.

[9] J. P. Ostrowski and J. W. Burdick, "The geometric mechanics of undulatory robotic locomotion," *International Journal of Robotics Research*, vol. 17, no. 7, pp. 683–701, 1998.

[10] J. A. Sanders and F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems*, Springer Verlag, New York, NY, 1985.

13.6 Coordinate-free modelling: I

- manifold Q, metric $\mathbb{G}$
- vector fields are written in terms of the canonical basis $\{\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\}$, and co-vector fields in terms of $\{\mathrm{d}q^1, \dots, \mathrm{d}q^n\}$
- given a function φ:

$$\mathrm{d}\varphi = \frac{\partial\varphi}{\partial q^i}\mathrm{d}q^i$$
$$\operatorname{grad}\varphi = \left(\mathbb{G}^{ij}\frac{\partial\varphi}{\partial q^j}\right)\frac{\partial}{\partial q^i}$$
$$\dot{q} = -\operatorname{grad}\varphi(q) \qquad \dots \text{ (negative) gradient flow}$$

13.7 Coordinate-free modelling: II

(i) given functions $\left\{\Gamma^i_{jk}\right\}$, and curve $\gamma \colon I \to \mathbb{R}$

$$\left(\nabla_{\gamma'}\gamma'\right)^i = \ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0 \qquad \ldots \text{ geodesic flow}$$

(ii) Given two vector fields X, Y, the **covariant derivative** of Y with respect to X is the third vector field $\nabla_X Y$ defined via

$$\left(\nabla_X Y\right)^i = \frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k .$$

(iii) **symmetric product**

$$\langle Y_a : Y_b \rangle = \nabla_{Y_a} Y_b + \nabla_{Y_b} Y_a$$

$$\langle Y_a : Y_b \rangle^i = \frac{\partial Y_a^i}{\partial q^j} Y_b^j + \frac{\partial Y_b^i}{\partial q^j} Y_a^j + \Gamma^i_{jk}\left(Y_a^j Y_b^k + Y_a^k Y_b^j\right)$$

13.8 Coordinate-free modelling: III

affine connection control system

$$\nabla_{\gamma'}\gamma' = Y_0(\gamma) + R(\gamma') + \sum_{a=1}^{m} Y_a(\gamma) u_a(t)$$

Ex #1: robotic manipulators with kinetic energy and forces at joints
simple systems with conservative forces

Ex #2: aerospace and underwater vehicles
invariant systems on Lie groups

Ex #3: systems subject to nonholonomic constraints
locomotion devices with drift, e.g., bicycle, snake-like robots

14 Perturbation Analysis I:

the "oscillatory control & vibrational stabilization" setting

(Bentsman et al, '86 – present) vibrational stabilization
(Baillieul '93 – present) discovery, study, apps of **averaged potential**

$$\nabla_{\gamma'}\gamma' = Y_0(\gamma) + R(\gamma') + Y(\gamma)u(t)$$

$$u(t) = \frac{1}{\epsilon} v\left(\frac{t}{\epsilon}\right)$$

where forcing v is T-periodic

$$\int_0^T v(s_1)ds_1 = \int_0^T \int_0^{s_1} v(s_2)ds_1 ds_2 = 0$$

and let

$$\lambda = \frac{1}{2T}\int_0^T \left(\int_0^{s_1} v(s_2)ds_2\right)^2 ds_1$$

14.1 Averaging for general mechanical systems

$$\nabla_{\gamma'}\gamma' = Y_0(\gamma) + R(\gamma') \;+\; \frac{1}{\epsilon} v\left(\frac{t}{\epsilon}\right) Y(\gamma)$$

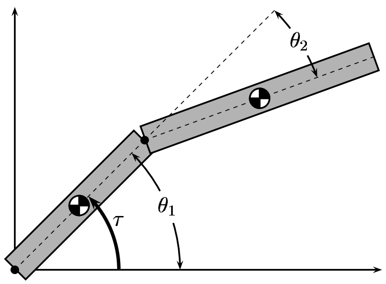
⬇

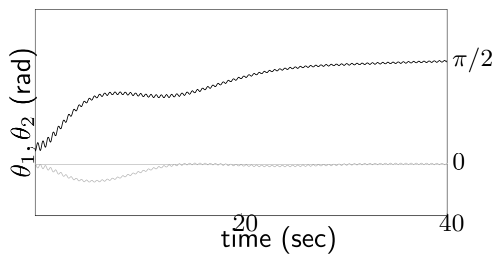
$$\nabla_{\gamma'}\gamma' = Y_0(\gamma) + R(\gamma') \;+\; \lambda \langle Y : Y \rangle(\gamma)$$

(i) approximation valid as $\epsilon \to 0$ on the time scale $t \in [0,1]$

(ii) approximation valid as $\epsilon \to 0$ on the time scale $t \in [0,\infty)$, if $(\gamma,\gamma') = (0,0)$ is an hyperbolically stable critical point

14.2 Ex #1: a 2-link manipulator

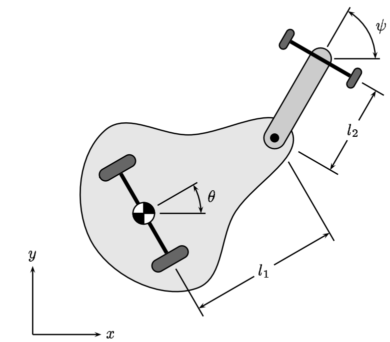




$$u = -\theta_1 + \frac{1}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)$$

Two-link damped manipulator with oscillatory control at first joint. The averaging analysis predicts the behavior. (the gray line is θ_1, the black line is θ_2). See later explanation for stability of $(0, \pi/2)$.

14.3 Ex #2: the roller racer



(i) recall X_1, X_2 two vector fields describing feasible velocities of racer

(ii) racer has single input $Y = X_2$

(iii) symmetric product $\langle Y : Y\rangle$ has component along X_1

(iv) hence, racer moves forward (or backward?) using zero mean input

14.4 Extension: Two-time scales result

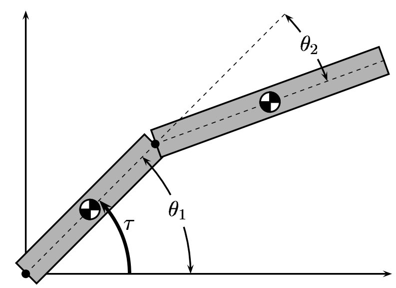
$$\nabla_{\gamma'}\gamma' = \text{Gravity} + \text{Damping} \quad + \quad \frac{1}{\epsilon} v\left(\frac{t}{\epsilon}\right) Y(t,\gamma)$$

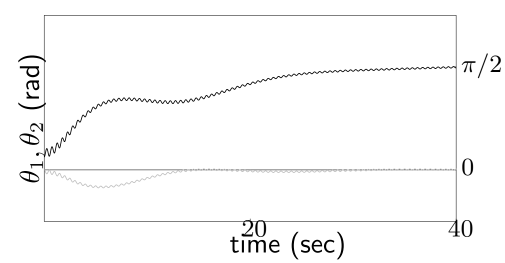
$v(t)$ is T-periodic and cyclic

$$\nabla_{\gamma'}\gamma' = \text{Gravity} + \text{Damping} \quad + \quad \lambda \langle Y : Y\rangle(t,\gamma)$$

$$\lambda = \frac{1}{2T}\int_0^T \left(\int_0^{s_1} v(s_2)ds_2\right)^2 ds_1$$

as $\epsilon \to 0$ on appropriate time scale





$$u = -\theta_1 + \frac{1}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)$$

why stable?

15 Simplified averaging analyses for SMCS with conservative forces

integrable forces in the sense of conservative forces:

$$Y(q,t) = \operatorname{grad}\varphi(q,t), \qquad (\operatorname{grad}\varphi)^i = \mathbb{G}^{ij}\frac{\partial\varphi}{\partial q^j}$$

Symmetric product restricts

$$\langle \operatorname{grad}\varphi_a : \operatorname{grad}\varphi_b\rangle \equiv \operatorname{grad}\langle\varphi_a : \varphi_b\rangle$$

where **Beltrami bracket (Crouch '81)**:

$$\langle\varphi_i : \varphi_j\rangle = \langle\langle \mathsf{d}\varphi_i\,,\, \mathsf{d}\varphi_j\rangle\rangle = \mathbb{G}^{ab}\frac{\partial\varphi_i}{\partial q^a}\frac{\partial\varphi_j}{\partial q^b}$$

Relationship between: (i) certain Lie brackets between vector fields on TQ, (ii) symmetric products of vector fields on Q, Beltrami bracket of functions (and, averaged potential)

15.1 Analysis I: averaging energy

In the open loop,

$$\mathcal{E}(q,v_q) = \frac{1}{2}\|v_q\|^2 + V(q)$$

but for controlled geodesic equations with input vector field

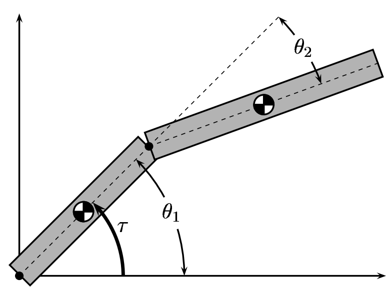
$$\sum_{a=1}^{m}\frac{1}{\epsilon}v^a\left(\frac{t}{\epsilon}\right)\operatorname{grad}\varphi_a(q)$$

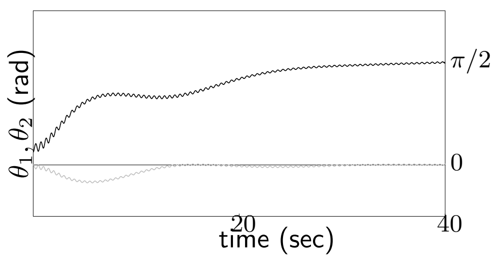
Averaged potential and energy

$$\mathcal{E}_{\text{averaged}}(q,p) = \frac{1}{2}\|v_q\|^2 + V_{\text{averaged}}(q)$$

$$V_{\text{averaged}}(q) = V(q) + \Lambda^{ab}\langle\varphi_a : \varphi_b\rangle(q)$$

$$\Lambda^{ab} = \frac{1}{2T}\int_0^T\left(\int_0^{s_1} v^a(s_2)ds_2\right)\left(\int_0^{s_1} v^b(s_2)ds_2\right)ds_1$$





$$u = -\theta_1 + \frac{1}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)$$

Two-link damped manipulator with oscillatory control at first joint. (the gray line is θ_1, the black line is θ_2).

Despite the superimposed oscillatory behavior the variables (θ_1, θ_2) converge to the global minimum of the averaged controlled potential energy.

16 Proofs

16.1 Theorem statement

Consider a control system described by an affine connection

$$\nabla_{\gamma'}\gamma' = Y_0(q) + R(\gamma') \;+\; Y_a(\gamma)\frac{1}{\epsilon}v^a(t/\epsilon) \tag{9}$$

where $\gamma'(0) = v_0$, and where $\{v^1,\dots,v^m\}$ are T-periodic functions st:

$$\int_0^T v^a(s_1)ds_1 = 0 = \int_0^T\int_0^{s_2} v^a(s_1)ds_1ds_2 = 0$$

Define the matrix Λ according to:

$$\Lambda^{ab} = \frac{1}{2T}\int_0^T\left(\int_0^{s_1} v^a(s_2)ds_2\right)\left(\int_0^{s_1} v^b(s_2)ds_2\right)ds_1.$$

Define the time-varying vector field

$$\Xi(t,q) = \left(\int_0^t v^a(s)ds\right)Y_a(q),$$

Theorem 2 (Averaging under oscillatory control). *Let $\gamma\colon I \to Q$ be the solution to the initial value problem in equation (9) and let $r\colon I \to Q$ be the solution to*

$$\nabla_{r'} r' = Y_0(r) + R(r)\dot{r} - \Lambda^{ab}\langle Y_a : Y_b\rangle(r)$$
$$r(0) = q_0, \quad \dot{r}(0) = v_0.$$

There exist a positive ϵ_0, such that for all $0 < \epsilon \leq \epsilon_0$

$$\gamma(t) = r(t) + O(\epsilon)$$
$$\gamma'(t) = r'(t) + \Xi(t/\epsilon, \gamma(t)) + O(\epsilon)$$

as $\epsilon \to 0$ on the time scale 1.

- F. Bullo, "Averaging and vibrational control of mechanical systems," *SIAM JCO*, Submitted November 1999. Appeared, Jul 2002.

16.2 Fact #1: Coordinate-free Averaging

Let $x, y, x_0 \in \mathbb{R}^n$, let $\epsilon \in (0, \epsilon_0]$ with $\epsilon_0 \ll 1$. Let $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be smooth time-varying vector fields. Consider the initial value problem in **standard form**:

$$\frac{dx}{dt} = \epsilon f(t, x), \qquad x(0) = x_0.$$

Assume $f(t, x)$ is a T-periodic function in t, and define **the averaged system**:

$$\frac{dy}{dt} = \epsilon f^0(y), \qquad y(0) = x_0,$$
$$f^0(y) = \frac{1}{T}\int_0^T f(t, y)dt.$$

Theorem 3 (First order averaging). *There exists ϵ_0, such that for $0 < \epsilon \leq \epsilon_0$,*

$$x(t) - y(t) = O(\epsilon)$$

as $\epsilon \to 0$ on the time scale $1/\epsilon$.

Recall: an estimate is **on the time scale $\delta(\epsilon)$**, if it holds for all t such that $0 < \delta^{-1}(\epsilon)t < L$ with L independent of ϵ.

Fact #1: Coordinate-free Averaging – continued

$$\frac{dx}{dt} = f(x) + \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}, x\right), \qquad x(0) = x_0,$$

where $g(t, x)$ is a T-periodic function in t. Define

$$F(t, x) = \left((\Phi^g_{0,t})^* f\right)(x) \qquad F^0(x) = \frac{1}{T}\int_0^T F(\tau, x)d\tau.$$

Finally, let z and y be solutions to the initial value problems

$$\dot{z} = F(t/\epsilon, z), \quad z(0) = x_0,$$
$$\dot{y} = F^0(y), \quad y(0) = x_0.$$

Theorem 4 (First order averaging for oscillatory controls). *Let F be a T-periodic function in t. For $t \in \mathbb{R}_+$, we have*

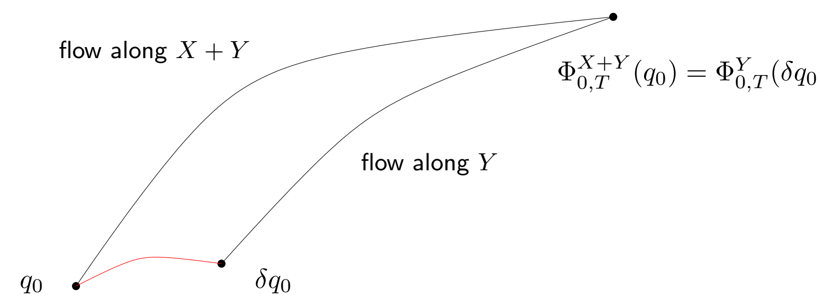
$$x(t) = \Phi^g_{0,t/\epsilon}(z(t)).$$

As $\epsilon \to 0$ on the time scale 1, we have

$$x(t) = \Phi^g_{0,t/\epsilon}(y(t)) + O(\epsilon)$$

Fact #1: Coordinate-free averaging – the variation of constants formula

$$\frac{dx}{dt} = f(x) + g(x) \qquad g \text{ is nominal, } f \text{ is perturbation}$$



$$\Phi^{f+g}_{0,t}(q_0) = \Phi^g_{0,t}(\delta q_0), \qquad \delta q_0 = \Phi^{\triangle}_{0,t}(q_0), \qquad \triangle = \left((\Phi^g_{0,t})^* f\right)$$

$$\triangle = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathrm{ad}^k_g f = f + \sum_{n=1}^{\infty} \int_0^t \ldots \int_0^{s_{n-1}} \left(\mathrm{ad}_{g_{s_n}} \ldots \mathrm{ad}_{g_{s_1}} f\right) ds_n \ldots ds_1$$

16.3 Fact #3: Homogeneity properties and Lie algebraic structure of affine connection control systems

Given $\gamma = (\gamma^1, \ldots, \gamma^n)$, write second order ODE on Q as first order ODE on TQ:

$$\begin{pmatrix} \dot\gamma^i \\ \ddot\gamma^i \end{pmatrix} = \underbrace{\begin{pmatrix} \dot\gamma^i \\ -\Gamma^i_{jk}(\gamma)\dot\gamma^j\dot\gamma^k \end{pmatrix}}_{Z} + \underbrace{\begin{pmatrix} 0 \\ Y^i_t(\gamma) \end{pmatrix}}_{Y^{\text{lift}}}$$

Lie algebraic & homogeneous structure

$$\mathcal{P}_i = \left\{ \begin{bmatrix} \text{homogeneous polynomial of degree } i \text{ in } \dot\gamma^1, \ldots, \dot\gamma^n \\ \text{homogeneous polynomial of degree } (i+1) \text{ in } \dot\gamma^1, \ldots, \dot\gamma^n \end{bmatrix} \right\}$$

$$Z \in \mathcal{P}_1 \quad \ldots \quad Y^{\text{lift}} \in \mathcal{P}_{-1}$$

Lie algebraic & homogeneous structure: cont'd

The sets $\mathcal{P}_j$ enjoy various interesting properties.

(i) $[\mathcal{P}_i, \mathcal{P}_j] \subset \mathcal{P}_{i+j}$, that is, the Lie bracket between a vector field in $\mathcal{P}_i$ and a vector field in $\mathcal{P}_j$ belongs to $\mathcal{P}_{i+j}$.

(ii) $\mathcal{P}_k = \{0\}$ for all $k \leq -2$,

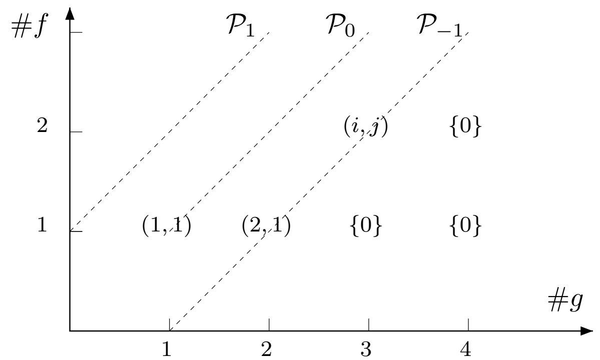
(iii) for all $X \in \mathcal{P}_k$ with $k \geq 1$, $X(0_q) = 0$,

(iv) every $X \in \mathcal{P}_{-1}$ is the lift of a vector field on Q, i.e.,

$$X = Y^{\text{lift}} = \begin{pmatrix} 0 \\ Y \end{pmatrix}$$

where X is vector field on TQ and Y is vector field on Q

Lie bracket diagram



$$[Y_1^{\text{lift}}, [Z, Y_2^{\text{lift}}]] \in \mathcal{P}_{-1}$$

$$[Y_1^{\text{lift}}, [Z, Y_2^{\text{lift}}]] = \begin{pmatrix} 0 \\ \langle Y_1 : Y_2 \rangle \end{pmatrix} = \langle Y_1 : Y_2 \rangle^{\text{lift}}$$

Coordinated independent treatment

(i) **Geometric homogeneity**, Kawski '95:
given a Euler v.f. X_E, Y is homogeneous of degree ν if $[X_E, Y] = \nu Y$

(ii) **Liouville vector field** $X_E(q, v) = v^i \frac{\partial}{\partial v^i}$; key identities on TQ:

$$[X_E, Z] = (+1)Z$$
$$[X_E, Y^{\text{lift}}] = (-1)Y^{\text{lift}}.$$

Hence, degree of Z is $+1$, degree of Y^{lift} is -1

16.4 Fact #4: putting it all together

Write second order equation (9) as first order —let $x = (q, \dot{q})$ and

$$f(x) = Z(x) + Y_0^{\text{lift}}(x) + R^{\text{lift}}(x),$$
$$g(t,x) = \sum_{a=1}^{m} Y_a^{\text{lift}}(x) v^a(t).$$

Define the vector field F

$$F(t,y) = \left((\Phi^g_{0,t})^* f\right)(y) = \left(\Phi_{0,t}^{\sum Y_a^{\text{lift}}(y)v^a(t)}\right)^* \left(Z(y) + Y_0^{\text{lift}}(y) + R^{\text{lift}}(y)\right).$$

and compute it according to the series expansion

$$(\Phi^g_{0,t})^* f = f + \sum_{k=1}^{\infty} \int_0^t \cdots \int_0^{s_{k-1}} \left(\text{ad}_{g(s_k)} \ldots \text{ad}_{g(s_1)} f\right) ds_k \ldots ds_1.$$

The Lie algebraic structure implies

$$\text{ad}^k_{Y_a^{\text{lift}}}(Z(y) + Y_0^{\text{lift}}(y) + R^{\text{lift}}(y)) = 0, \qquad \forall k \geq 3,$$
$$\text{ad}_{Y_b^{\text{lift}}} \text{ad}_{Y_a^{\text{lift}}}(Z(y) + Y_0^{\text{lift}}(y) + R^{\text{lift}}(y)) = -\langle Y_a : Y_b\rangle^{\text{lift}}.$$

Some bookkeeping:

$$\begin{aligned}
&\left(\Phi_{0,t}^{\sum Y_a^{\text{lift}}(y)v^a(t)}\right)^* \left(Z(y) + Y_0^{\text{lift}}(y) + R^{\text{lift}}(y)\right) \\
&\quad = \left(Z + Y_0^{\text{lift}} + R^{\text{lift}}\right) + \left(\int_0^t v^a(s_1) ds_1\right) [Y_a^{\text{lift}}, (Z + Y_0^{\text{lift}} + R^{\text{lift}})] \\
&\qquad + \left(\int_0^t \int_0^{s_b} v^b(s_b) v^a(s_a) ds_a ds_b\right) [Y_b^{\text{lift}}, [Y_a^{\text{lift}}, (Z + Y_0^{\text{lift}} + R^{\text{lift}})]] \\
&\quad = \left(Z + Y_0^{\text{lift}} + R^{\text{lift}}\right) + \left(\int_0^t v^a(s_1) ds_1\right) [Y_a^{\text{lift}}, (Z + R^{\text{lift}})] \\
&\qquad - \left(\int_0^t \int_0^{s_b} v^b(s_b) v^a(s_a) ds_a ds_b\right) \langle Y_a : Y_b\rangle^{\text{lift}}.
\end{aligned}$$

An integration by parts and the symmetry of the symmetric product:

$$\begin{aligned}
&\left(\int_0^t \int_0^{s_b} v^b(s_b) v^a(s_a) ds_a ds_b\right) \langle Y_a : Y_b\rangle \\
&\qquad = \frac{1}{2}\left(\int_0^t v^b(s_b) ds_b \int_0^t v^a(s_a) ds_a\right) \langle Y_a : Y_b\rangle,
\end{aligned}$$

In summary

$$\begin{aligned}
F(t,y) &= \left(Z + Y_0^{\text{lift}} + R^{\text{lift}}\right) + \left(\int_0^t v^a(s_1) ds_1\right) [Y_a^{\text{lift}}, (Z + R^{\text{lift}})] \\
&\quad - \frac{1}{2}\left(\int_0^t v^b(s_b) ds_b \int_0^t v^a(s_a) ds_a\right) \langle Y_a : Y_b\rangle^{\text{lift}}.
\end{aligned}$$

F is T-periodic —compute its average F^0 as

$$F^0(y) = \left(Z + Y_0^{\text{lift}} + R^{\text{lift}}\right) - \Lambda^{ab}\langle Y_a : Y_b\rangle^{\text{lift}}.$$

This is what we wished to show.

17 Perturbation Analysis II:

the "small-time local controllability" setting

Small input from rest

$$H = H(q,p) + \epsilon\varphi(q, u(t))$$
$$p(0) = 0$$

$$\nabla_{\gamma'}\gamma' = \sum_{a=1}^{m} Y_a(\gamma) u_a(t)$$

Objective: characterize forced flow via series expansion

17.1 Series expansions for polynomial systems

$$\dot{x} = P(x,x) + Ax + Bu(t)$$
$$x(0) = 0$$

⬇

$$x = \sum_{k=1}^{+\infty} x_k$$

$$x_1(t) = \int_0^t \mathrm{e}^{A(t-\tau)} Bu(\tau) d\tau$$
$$x_k(t) = \sum_{j=1}^{k-1} \int_0^t \mathrm{e}^{A(t-\tau)} P(x_j(\tau), x_{k-j}(\tau)) d\tau, \quad k \geq 2.$$

convergence radius: $\beta^2 \|u\|_{\mathcal{L}_\infty} < 1$, where $\beta = 2 \left\|\mathrm{e}^{At}\right\|_{\mathcal{L}_1} \|P\|_\infty$

17.2 Series expansion for affine connection control systems

$$\nabla_{\gamma'}\gamma' = -k\gamma' + Y(\gamma, t)$$
$$\gamma'(0) = 0$$

⬇

$$\gamma' = \sum_{k=1}^{+\infty} V_k(\gamma, t)$$

absolute, uniform convergence

$$V_1(q,t) = \int_0^t \mathrm{e}^{k(s-t)} Y(q,s) ds$$
$$V_k(q,t) = -\frac{1}{2} \sum_{j=1}^{k-1} \int_0^t \mathrm{e}^{k(s-t)} \langle V_j(q,s) : V_{k-j}(q,s) \rangle ds$$

17.3 Series: comments

$$\gamma' = \sum_{k=1}^{+\infty} V_k(\gamma, t)$$

$$\begin{cases} V_1(q,t) = \int_0^t \mathrm{e}^{k(s-t)} Y(q,s) ds, \\ V_{k+1}(q,t) = -\frac{1}{2} \sum \int_0^t \mathrm{e}^{k(s-t)} \langle V_a : V_{k-a} \rangle ds \end{cases}$$

Error bounds:

$$\|V_k(q,t)\| = O(\|Y\|^k t^{2k-1}).$$

In abbreviated notation

$$V_1 = \overline{Y} \qquad V_2 = -\frac{1}{2}\overline{\langle \overline{Y} : \overline{Y} \rangle}$$
$$V_3 = \frac{1}{2}\overline{\langle \overline{\langle \overline{Y} : \overline{Y} \rangle} : \overline{Y} \rangle}$$

so that

$$\gamma'(t) = \overline{Y}(q,t) - \frac{1}{2}\overline{\langle \overline{Y} : \overline{Y} \rangle}(q,t) + \frac{1}{2}\overline{\langle \overline{\langle \overline{Y} : \overline{Y} \rangle} : \overline{Y} \rangle}(q,t) + O(\|Y\|^4 t^7).$$

17.4 Analysis II: a forces geodesic flow written as gradient flow

$$\nabla_{\gamma'}\gamma' = \operatorname{grad} \varphi(\gamma, t)$$
$$\gamma'(0) = 0_{q_0}$$

⬇

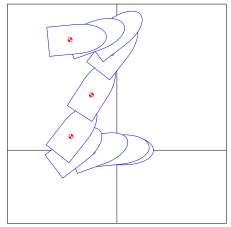
$$\gamma'(t) = \operatorname{grad} \sum_{k=1}^{+\infty} \varphi_k(\gamma(t), t) \qquad \gamma(0) = q_0$$

$$\varphi_1(q,t) = \int_0^t \varphi(q,s) ds$$
$$\varphi_k(q,t) = -\frac{1}{2} \sum_{j=1}^{k-1} \int_0^t \langle \varphi_j(q,s) : \varphi_{k-j}(q,s) \rangle ds$$

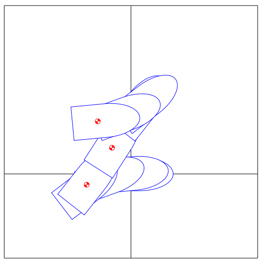
17.5 Example of open-loop response: planar body

simple example: body with one force through center of mass and one torque.

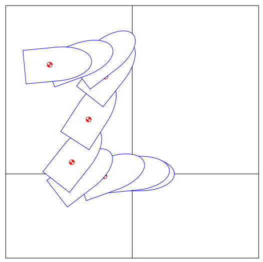
$$q(0) = (0,0,0), \quad T = 2\pi$$
$$u_1 = .5(\sin t - 2\sin 2t), \quad u_2 = .5\cos t$$



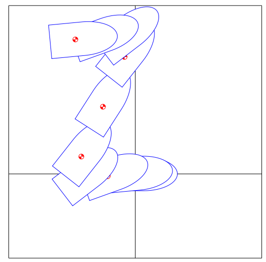
exact solution



first order



second order



third order

17.6 Conjecture

$$\nabla_{\gamma'}\gamma' = R(\gamma') + Y(\gamma, t)$$
$$\gamma'(0) = 0$$

$$\gamma' = \sum_{k=1}^{+\infty} V_k(\gamma, t)$$

$$V_1(q,t) = \int_0^t \mathrm{e}^{R(q)(t-s)} Y(q,s) ds$$
$$V_k(q,t) = -\frac{1}{2}\sum_{j=1}^{k-1}\int_0^t \mathrm{e}^{R(q)(t-s)} \langle V_j(q,s) : V_{k-j}(q,s)\rangle ds$$

Positive answer for isotropic damping: $R = kI_n$.

18 Summary

(i) innovative approach towards control of mechanical systems
(homogeneity vs passivity)
(perturbation methods vs energy and Lyapunov functions)

(ii) challenges: **convergence & complexity**

(iii) applications to **controllability, vibrational stabilization, analysis of locomotion gaits, motion planning, optimal control, normal forms, etc**

Lecture #4: Kinematic Reductions and Configuration Controllability

Francesco Bullo
Coordinated Science Lab, General Engineering Dept, ECE, AAE
University of Illinois at Urbana-Champaign
1308 W. Main St, Urbana, IL 61801, USA
bullo@uiuc.edu, http://motion.csl.uiuc.edu

This lecture based on the following references

[1] A. D. Lewis, "When is a mechanical control system kinematic?," in *Proc CDC*, (Phoenix, AZ), pp. 1162–1167, Dec. 1999.

[2] A. D. Lewis and R. M. Murray, "Configuration controllability of simple mechanical control systems," *SIAM JCO*, 35(3):766–790, 1997.

[3] F. Bullo and K. M. Lynch, "Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems," *IEEE TRA*, 17(4)L402–412, 2001.

[4] F. Bullo and A. D. Lewis, "Kinematic controllability and motion planning for the snakeboard," *IEEE TRA*, Jan. 2002. Submitted.

Incomplete List of References on Kinematic Modeling and Controllability

[1] J. T. Betts, "Survey of numerical methods for trajectory optimization," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 21, no. 2, pp. 193–207, 1998.

[2] R. W. Brockett, "Hybrid models for motion control systems," in *Essays in Control: Perspectives in the Theory and its Applications*, pp. 29–53, Boston, MA: Birkhäuser, 1993.

[3] E. Frazzoli, M. A. Daleh, and E. Feron, "Real-time motion planning for agile autonomous vehicles," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 25, no. 1, pp. 116–129, 2002.

[4] V. Manikonda, P. S. Krishnaprasad, and J. Hendler, "Languages, behaviors, hybrid architectures and motion control," in *Mathematical Control Theory* (J. Baillieul and J. C. Willems, eds.), New York, NY: Springer Verlag, 1998.

[5] K. McIsaac, and J. P. Ostrowski, "Steering algorithms for dynamic robotic locomotion systems," in *Algorithmic and Computational Robotics: New Directions* (B. R. Donald and K. M. Lynch and D. Rus, eds.), Natick, MA: A. K. Peters, pp. 221-231, 2001.

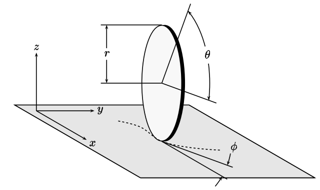
[6] A. D. Lewis, "Affine connections and distributions with applications to nonholonomic mechanics," in *Reports on Mathematical Physics*, vol. 42, no. 1/2. pp. 135-164, 1998.

[7] K. M. Lynch, N. Shiroma, H. Arai, and K. Tanie, "Collision-free trajectory planning for a 3-DOF robot with a passive joint," *International Journal of Robotics Research*, vol. 19, no. 12, pp. 1171–1184, 2000.

[8] A. J. van der Schaft, "Linearization of Hamiltonian and gradient systems," *IMA Journal of Mathematical Control & Information*, vol. 1, pages 185-198, 1984

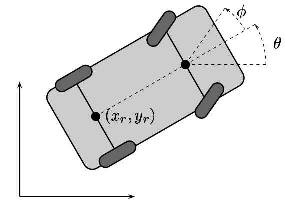
[9] H. J. Sussmann, "A general theorem on local controllability," *SIAM Journal on Control and Optimization*, vol. 25, no. 1, pp. 158–194, 1987.

18.1 Preliminaries: Kinematic modeling



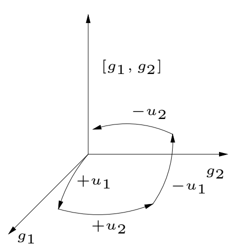
$$\dot{x} = v\cos\phi$$
$$\dot{y} = v\sin\phi$$
$$\dot{\phi} = \omega$$

(wheeled robot dynamics)



$$\begin{bmatrix}\dot{x}_r\\ \dot{y}_r\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\\ \frac{1}{\ell}\tan\phi\\ 0\end{bmatrix} v + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\omega$$

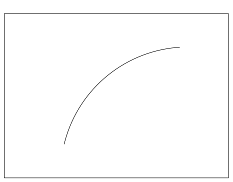
18.2 Preliminaries: Controllability theory



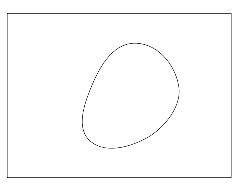
Given a driftless system $\dot{x} = g_1(x)u_1 + g_2(x)u_2$

define **Lie bracket:** $[g_1(x), g_2(x)] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2$

system is controllable iff LARC



not full rank



full rank

Example: car parking problem

19 Kinematic reductions for simple mechanical control systems with constraints

(i) Objective: relationships between the given mechanical control system and an appropriate low-complexity kinematic representation

(ii) treatment for simple mechanical control systems subject to no potential energy

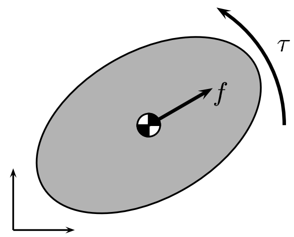
(iii) we relate controlled trajectories for the (second-order) controlled geodesic equation

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = \sum_{a=1}^{m} Y_a(\gamma(t))u_a(t).$$

to controlled trajectories for driftless control systems on Q.

when can a second order system follow the solution of a first order?

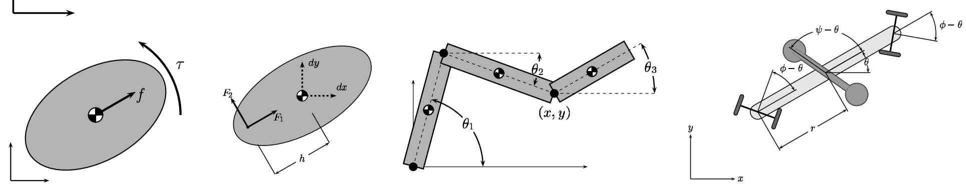
19.1 Motivating example



simple example: body with
one force through center of mass and one torque

(i) Can follow any straight line and can turn (2 preferred velocity fields)

(ii) Controllable via these two motions (hence, interesting for planning problems)



Ok ? ? ?

search for **decoupling** vector fields describing 1st order ODEs whose time-scaled flow is solutions to (forced) 2nd order ODEs

Nomenclature:

(i) The controlled geodesic equation is a **dynamic models** of mechanical systems:

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = \sum_{a=1}^{m} Y_a(\gamma(t))u_a(t).$$

In dynamic models the control inputs u are accelerations, and assumed Lebesgue measurable functions: $u \in \mathscr{U}^m_{\text{dyn}}$.

(ii) In contrast to this, we refer to first-order differential equations on Q as **kinematic models** of mechanical systems. Let $\mathcal{V} = \{V_1, \dots, V_\ell\}$ be a family of vector fields. For curves $\gamma\colon [0,T] \to$ Q and $w\colon [0,T] \to \mathbb{R}^\ell$, consider the kinematic model induced by $\mathcal{V}$

$$\dot\gamma(t) = \sum_{b=1}^{\ell} V_b(\gamma(t))w_b(t).$$

In kinematic models, the control inputs are velocity variables, and are assumed absolutely continuous: $w \in \mathscr{U}^\ell_{\text{kin}}$.

19.2 Kinematic reductions and decoupling vector fields

In short, $\mathcal{V}$ is a **kinematic reduction** if any curve $\gamma\colon I \to$ Q solving the (controlled) kinematic model can be lifted to a solution to a solution of the (controlled) dynamic model.

More accurately, the kinematic model induced by $\mathcal{V} = \{V_1, \dots, V_\ell\}$ is a kinematic reduction of the dynamic model, if, for any control input $w \in \mathscr{U}^\ell_{\text{kin}}$ and corresponding controlled trajectory (γ, w) for the kinematic model, there exists a control input $u \in \mathscr{U}^m_{\text{dyn}}$ such that (γ, u) is a controlled trajectory for the dynamic model.

- The **rank** of a kinematic reduction is the rank of the distribution generated by the vector fields $\mathcal{V}$.
- Rank-one kinematic reductions are particularly interesting. We shall call a vector field V **decoupling** if the rank-one kinematic system induced by $\mathcal{V} = \{V\}$ is a kinematic reduction. Hence, the second-order control system can be steered along any time-scaled integral curve of a decoupling vector field.

19.3 Kinematic reductions and decoupling vector fields: cont'd

The kinematic model induced by $\{V_1, \dots, V_\ell\}$ is a kinematic reduction of the mechanical control system $(\mathsf{Q}, \mathbb{G}, V = 0, \mathcal{F})$
if and only if
the distribution $\operatorname{span}\{V_i, \langle V_j : V_k\rangle \mid i, j, k \in \{1, \dots, \ell\}\}$ is a subdistribution of the input distribution $\mathscr{Y}$.

The vector field V is decoupling
if and only if
$V \in \mathscr{Y}$ and $\langle V : V\rangle \in \mathscr{Y}$.

19.4 Mechanical systems fully reducible to kinematic systems

when is a mechanical system kinematic?

That is, when will the **largest possible** kinematic reduction, i.e., $\mathcal{Y}$ will be attained?

The dynamic model for the system $(\mathsf{Q}, \mathbb{G}, V = 0, \mathcal{F})$ is **fully reducible to the kinematic system induced by $\mathcal{V}$** if, $\mathcal{V}$ is a kinematic reduction of $(\mathsf{Q}, \mathbb{G}, V = 0, \mathcal{F})$ and if, for any control input $u \in \mathscr{U}^m_{\text{dyn}}$, initial condition $\dot\gamma(0) \in \operatorname{span}(\mathcal{V})$, and corresponding controlled trajectory (γ, u) for the dynamic model, there exists a control input $w \in \mathscr{U}^\ell_{\text{kin}}$ such that (γ, w) is a controlled trajectory for the kinematic model induced by $\mathcal{V}$.

A dynamic system is **fully reducible to a kinematic system** is there exists one such collection of vector fields $\mathcal{V}$.

19.5 Mechanical systems fully reducible to kinematic systems: cont'd

A distribution $\mathscr{X}$ is said to be **geodesically invariant** if it is closed under operation of symmetric product, i.e., if for all vector fields X and Y taking values in $\mathscr{X}$, the vector field $\langle X : Y \rangle$ also takes value in $\mathscr{X}$. The **symmetric closure** of the distribution $\mathscr{X}$ is the smallest geodesically invariant distribution containing $\mathscr{X}$.

Theorem 5. *A mechanical control system is fully reducible to a kinematic system if and only if*

(i) the kinematic system is induced by the input distribution $\mathscr{Y}$ and

(ii) the input distribution $\mathscr{Y}$ is geodesically invariant.

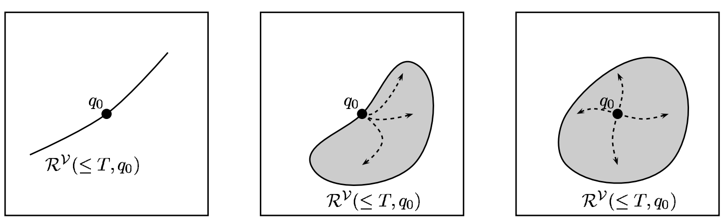
20 Accessibility and controllability notions

20.1 Controllable kinematic systems

Here we consider the family $\mathcal{V} = \{V_1, \dots, V_\ell\}$ giving rise to the driftless / kinematic control system. For $q_0 \in \mathsf{Q}$ we denote

$$\mathcal{R}^{\mathcal{V}}(q_0, T) = \{\gamma(T) \mid \ (\gamma, u) \text{ is a controlled trajectory for kinematic model defined on } [0, T] \text{ with } \gamma(0) = q_0\},$$

and $\mathcal{R}^{\mathcal{V}}(q_0, \leq T) = \bigcup_{t \in [0,T]} \mathcal{R}^{\mathcal{V}}(q_0, t)$.



Definition 6. *The kinematic system induced by $\mathcal{V}$ is*

(i) **locally accessible** *from q_0 if there exists $T > 0$ so that $\operatorname{int}(\mathcal{R}^{\mathcal{V}}(q_0, \leq t)) \neq \emptyset$ for $t \in (0, T]$, is*

(ii) **small-time locally controllable (STLC)** *from q_0 if there exists $T > 0$ so that $q_0 \in \operatorname{int}(\mathcal{R}^{\mathcal{V}}(q_0, \leq t))$ for $t \in (0, T]$, and is*

(iii) **controllable** *if for every $q_1, q_2 \in \mathsf{Q}$ there exists a controlled trajectory (γ, u) defined on $[0, T]$ for some $T > 0$ with the property that $\gamma(0) = q_1$ and $\gamma(T) = q_2$.*

Theorem 7. *The kinematic system is STLC (and therefore accessible) from q_0 if and only if $\overline{\operatorname{Lie}}\{\operatorname{span}(\mathcal{V})\}_{q_0} = T_{q_0}\mathsf{Q}$. Furthermore, if Q is connected and if $\overline{\operatorname{Lie}}\{\operatorname{span}(\mathcal{V})\}_q = T_q\mathsf{Q}$ for each $q \in \mathsf{Q}$, then the kinematic mode is controllable.*

20.2 Kinematically controllable dynamic systems

(i) A dynamic mechanical system described by $(\mathsf{Q}, \mathbb{G}, V, \mathscr{D}, \mathscr{F})$ is **kinematically controllable** if there exists a sequence of kinematic reductions $\{\mathcal{V}_i |\ i \in \{1, \dots, k\}, \operatorname{rank} \mathcal{V}_i = \ell_i\}$ so that for every $q_1, q_2 \in \mathsf{Q}$ there are corresponding controlled trajectories $\{(\gamma_i, w_i) |\ \gamma_i : [T_{i-1}, T_i] \to \mathsf{Q}, w_i : [T_{i-1}, T_i] \to \mathbb{R}^{\ell_i}, i \in \{1, \dots, k\}\}$ such that $\gamma_1(T_0) = q_1$, $\gamma_k(T_k) = q_2$, and $\gamma_i(T_i) = \gamma_{i+1}(T_i)$ for all $i \in \{1, \dots, k-1\}$.

(ii) In other words, any $q_2 \in \mathsf{Q}$ is reachable from any $q_1 \in \mathsf{Q}$ by concatenating motions on Q corresponding to kinematic reductions of the dynamic system

(iii) The dynamic system is **locally kinematically controllable** from q_0 if, for any neighborhood of q_0 on Q, the set of reachable configurations by trajectories remaining in the neighborhood and following motions of its kinematic reductions contains q_0 in its interior.

Theorem 8. *Consider a dynamic mechanical system. The system is locally kinematically controllable if and only if it possesses a collection of decoupling vector fields (i.e., rank-one kinematic reductions) whose involutive closure has maximal rank everywhere in Q.*

20.3 Controllable dynamic systems

Consider a dynamic mechanical system $(\mathsf{Q}, \mathbb{G}, V, \mathscr{D}, \mathscr{F})$. For $q_0 \in \mathsf{Q}$, denote

$$\mathcal{R}_{\mathsf{TQ}}(q_0, T) = \{\dot\gamma(T) \mid \ (\gamma, u) \text{ is a controlled trajectory of the dynamic model defined on } [0, T] \text{ and satisfying } \dot\gamma(0) = 0_{q_0}\}.$$

Here $0_{q_0} \in \mathsf{T}_{q_0}\mathsf{Q}$ is the zero vector. Also, $\mathcal{R}_{\mathsf{TQ}}(q_0, \leq T) = \bigcup_{t \in [0,T]} \mathcal{R}_{\mathsf{TQ}}(q_0, t)$.

Definition 9. *Consider a dynamic mechanical system $(Q, \mathbb{G}, V, \mathscr{D}, \mathscr{F})$ and let $q_0 \in Q$. Suppose that the controls for the dynamic system are restricted to take their values in a compact set of $\mathbb{R}^m$ which contains 0 in the interior of its convex hull. The dynamic system is*

(i) **locally accessible** *from q_0 if there exists $T > 0$ so that $\operatorname{int}(\mathcal{R}_{TQ}(q_0, \leq t)) \neq \emptyset$ for $t \in (0, T]$, and is*

(ii) **small-time locally controllable** *(***STLC***) from q_0 if there exists $T > 0$ so that $0_{q_0} \in \operatorname{int}(\mathcal{R}_{TQ}(q_0, \leq t))$ for all $t \in (0, T]$.*

Notation: Consider iterated symmetric products in the vector fields $\{Y_1, \dots, Y_m\}$.

(i) A symmetric product is **bad** if it contains an even number of each of the vector fields $Y_1, \dots, Y_m$, and otherwise is **good**. Thus, for example, $\langle\langle Y_a : Y_b\rangle : \langle Y_a : Y_b\rangle\rangle$ is bad for all $a, b \in \{1, \dots, m\}$ and $\langle Y_a : \langle Y_b : Y_c\rangle\rangle$ is good for any $a, b, c \in \{1, \dots, m\}$.

(ii) The **degree** of a symmetric product is the total number of input vector fields comprising the symmetric product. For example, our given bad symmetric product has degree 4 and the given good symmetric product has degree 3.

(iii) If P is a symmetric product in the vector fields $\{Y_1, \dots, Y_m\}$ and if $\sigma \in S_m$ is an element of the permutation group on $\{1, \dots, m\}$, $\sigma(P)$ denotes the symmetric product obtained by replacing each occurrence of Y_a with $Y_{\sigma(a)}$.

Theorem 10. *Consider a dynamic mechanical system described by* $(Q, \mathbb{G}, V, \mathscr{D}, \mathscr{F})$ *and let* $q_0 \in Q$*. The dynamic mechanical system is*

(i) locally accessible from q_0 *if and only if* $\overline{\mathrm{Sym}}\{\mathscr{Y}\}_{q_0} = T_{q_0}Q$*, and is*

(ii) STLC from q_0 *if* $\overline{\mathrm{Sym}}\{\mathscr{Y}\}_{q_0} = T_{q_0}Q$ *and if for every bad symmetric product* P *we have*

$$\sum_{\sigma \in S_m} \sigma(P)(q_0) \in \mathrm{span}_{\mathbb{R}}\{P_1(q_0), \dots, P_k(q_0)\},$$

where $P_1, \dots, P_k$ *are good symmetric products of degree less than* P.

The condition stated for STLC is derived from a result of Sussmann '87.

20.4 Configuration controllable dynamic systems

The preceding discussion concerned the set of reachable **states** for a dynamic mechanical system. Let us now restrict to descriptions of the set of reachable configurations. We define

$$\mathcal{R}_{\mathsf{Q}}(q_0, T) = \tau(\mathcal{R}_{\mathsf{TQ}}(q_0, T)), \quad \mathcal{R}_{\mathsf{Q}}(q_0, \leq T) = \bigcup_{t \in [0,T]} \mathcal{R}_{\mathsf{Q}}(q_0, t).$$

This gives the following notions of controllability relative to configurations.

Definition 11. *Consider a dynamic mechanical system described by* $(Q, \mathbb{G}, V, \mathscr{D}, \mathscr{F})$ *and let* $q_0 \in Q$*. The dynamic mechanical system is*

(i) **locally configuration accessible** *from* q_0 *if there exists* $T > 0$ *so that* $\mathrm{int}(\mathcal{R}_Q(q_0, \leq t)) \neq \emptyset$ *for all* $t \in (0, T]$*, and is*

(ii) **small-time locally configuration controllable** *(***STLCC***) from* q_0 *if there exists* $T > 0$ *so that* $q_0 \in \mathrm{int}(\mathcal{R}_Q(q_0, \leq t))$ *for all* $t \in (0, T]$ *with the controls restricted to take their values in a compact subset of* $\mathbb{R}^m$ *that contains the origin in its convex hull.*

Theorem 12. *Consider an analytic dynamic mechanical system described by* $(Q, \mathbb{G}, V, \mathscr{D}, \mathscr{F})$ *and let* $q_0 \in Q$*. The dynamic mechanical system is*

(i) locally configuration accessible from q_0 *if and only if* $\overline{\mathrm{Lie}}\{\overline{\mathrm{Sym}}\{\mathscr{Y}\}\}_{q_0} = T_{q_0}Q$*, and is*

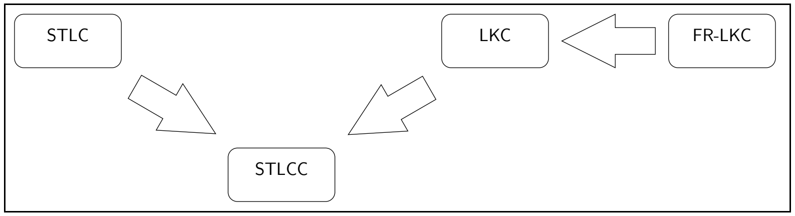
(ii) STLCC from q_0 *if* $\overline{\mathrm{Lie}}\{\overline{\mathrm{Sym}}\{\mathscr{Y}\}\}_{q_0} = T_{q_0}Q$ *and if for every bad symmetric product* P *we have*

$$\sum_{\sigma \in S_m} \sigma(P)(q_0) \in \mathrm{span}_{\mathbb{R}}\{P_1(q_0), \dots, P_k(q_0)\},$$

where $P_1, \dots, P_k$ *are good symmetric products of degree less than* P.

20.5 Controllability inferences

STLC	=	small-time locally controllable
STLCC	=	small-time locally configuration controllable
LKC	=	locally kinematically controllable
FR-LKC	=	fully reducible, locally kinematically controllable



There exist counter-examples for each missing implication sign.

20.6 Controllability and Configuration Controllability

$\left\{\begin{array}{l} \mathrm{rank}(\overline{\mathrm{Sym}}\{\mathscr{Y}\}_{q_0}) = n \\ \text{bad symmetric products are} \\ \text{linear combination of lower} \\ \text{order good products} \end{array}\right.$ ⇨ **Locally controllable**: $(q_0, 0) \xrightarrow{u} (a_f, v_f)$ can reach open set of velocities

$\left\{\begin{array}{l} \mathrm{rank}(\overline{\mathrm{Lie}}\{\overline{\mathrm{Sym}}\{\mathscr{Y}\}\}_{q_0}) = n \\ \text{good/bad as above} \end{array}\right.$ ⇨ **Configuration controllable**: $(q_0, 0) \xrightarrow{u} (q_f, 0)$ can reach open set of configurations

Simplifications:

(i) for systems on group: algebraic tests on the Lie algebra

(ii) for systems with integrable forces: Beltrami brackets between functions

20.7 Graphical illustration

$$\nabla_{\gamma'}\gamma' = -k\gamma' + \sum_{a=1}^{m} Y_a(q)u_a$$
$$\gamma'(0) = 0$$

given forces	$\{F^a\}$
	⬇
accessible accelerations	$\{Y_a = \mathbb{G}^{-1}F^a\}$
	⬇
accessible velocities	$\{Y_a, \langle Y_b : Y_c\rangle, \ldots\}$
	⬇
accessible configurations	$\{Y_a, \langle Y_b : Y_c\rangle, [Y_b, Y_c], [\langle Y_a : Y_b\rangle, Y_c], \ldots\}_{\mathcal{M}}$

20.8 An example controllability analysis: the snakeboard

Symmetric products:

$$\langle X_2 : X_2\rangle = 0, \quad \langle X_3 : X_3\rangle = 0,$$
$$\langle X_2 : X_3\rangle = \frac{J_r}{m\ell^2}(\cos\phi)X_1 - \frac{J_r(\cos\phi\sin\phi)}{m\ell^2 + J_r(\sin\phi)^2}X_2,$$

$\mathrm{span}\{X_2, X_3, \langle X_2 : X_3\rangle\} = \mathscr{D}$ if $\cos\phi \neq 0.$

Lie brackets:

$$[X_1, X_3] = \ell(\sin\phi)V_x + (\cos\phi)\frac{\partial}{\partial\theta}$$
$$[X_1, [X_1, X_3]] = -\ell(\sin\phi)V_y,$$

$\mathrm{span}\{X_1, X_2, X_3, [X_1, X_3], [X_1, [X_1, X_3]]\} =$ **TQ** ➡ **System is STLCC**

20.9 An example controllability analysis: the roller racer

Symmetric products:

$$\langle X_2 : X_2\rangle = 2(^{\mathcal{X}}\Gamma)^1_{22}(\psi)X_1 + 2(^{\mathcal{X}}\Gamma)^2_{22}(\psi)X_2$$

$\mathrm{span}\{X_2, \langle X_2 : X_2\rangle\} = \mathscr{D}$ if $(^{\mathcal{X}}\Gamma)^1_{22}(\psi) \neq 0$

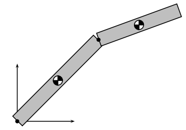
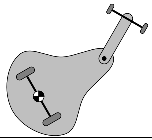
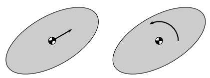
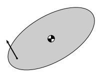
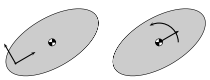
Lie brackets:

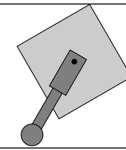
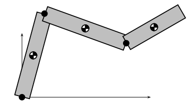
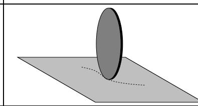
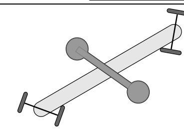
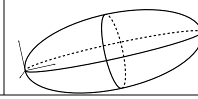
$$[X_1, X_2] = \frac{\ell_2}{\ell_2 + \ell_1\cos\psi}V_y - \frac{\ell_1 + \ell_2\cos\psi}{(\ell_2 + \ell_1\cos\psi)^2}\frac{\partial}{\partial\theta}$$
$$[X_1, [X_1, X_2]] = \frac{-\ell_2\sin\psi}{(\ell_2 + \ell_1\cos\psi)^2}V_x + \frac{\ell_1 + \ell_2\cos\psi}{(\ell_2 + \ell_1\cos\psi)^2}V_y,$$

$\mathrm{span}\{X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]\} =$ **TQ** everywhere $\ell_2 I_1\cos\psi \neq \ell_1 I_2.$

System is locally configuration accessible

20.10 A catalog of affine connection control systems

System	Picture	Reducibility & Controllability
planar 2R robot single torque at either joint: $(1,0),(0,1)$ $n=2, m=1$		$(1,0)$: no reductions, accessible $(0,1)$: decoupling v.f., fully reducible, not accessible or STLCC
roller racer single torque at joint $n=4, m=1$		no kinematic reductions, accessible, not STLCC
planar body with single force or torque $n=3, m=1$		decoupling v.f., reducible, not accessible
planar body with single generalized force $n=3, m=1$		no kinematic reductions, accessible, not STLCC
planar body with two forces $n=3, m=2$		two decoupling v.f., LKC, STLC

robotic leg $n=3, m=2$		two decoupling v.f., fully reducible and LKC
planar 3R robot, two torques: $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ $n=3, m=2$		$(1,0,1)$ and $(1,1,0)$: two decoupling v.f., LKC and STLC $(0,1,1)$: two decoupling v.f., fully reducible and LKC
rolling penny $n=4, m=2$		fully reducible and LKC
snakeboard $n=5, m=2$		two decoupling v.f., LKC, STLCC
3D vehicle with 3 generalized forces $n=6, m=3$		three decoupling v.f., LKC, STLC

Summary of Analysis Methods (lectures #3 and #4)

Comprehensive, coherent body of work encompassing results on

(i) perturbation methods

(ii) kinematic reductions

(iii) controllability properties

Open directions

Averaging higher order, 2-time scales, gait analysis

Controllability gravity or generic dissipation

Lecture #5: Stabilization and Tracking for fully actuated systems

Francesco Bullo
Coordinated Science Lab, General Engineering Dept, ECE, AAE
University of Illinois at Urbana-Champaign
1308 W. Main St, Urbana, IL 61801, USA
bullo@uiuc.edu, http://motion.csl.uiuc.edu

This lecture based on the following references

[1] F. Bullo and R. M. Murray, "Tracking for fully actuated mechanical systems: A geometric framework," *Automatica*, vol. 35, no. 1, pp. 17–34, 1999.

Incomplete List of References on Lyapunov and passivity methods for stabilization and tracking

[1] S. Arimoto. *Control Theory of Non-linear Mechanical Systems: A Passivity-Based and Circuit-Theoretic Approach*, volume 49 of *OESS*. Oxford University Press, Oxford, UK, 1996.

[2] A. M. Bloch, N. E. Leonard, and J. E. Marsden, "Controlled Lagrangians and the stabilization of mechanical systems. I. the first matching theorem," *IEEE TAC*, vol. 45, no. 12, pp. 2253–2270, 2000.

[3] D. E. Koditschek, "The application of total energy as a Lyapunov function for mechanical control systems," in *Dynamics and Control of Multibody Systems* (J. E. Marsden, P. S. Krishnaprasad, and J. C. Simo, eds.), vol. 97, pp. 131–157, AMS, 1989.

[4] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer Verlag, New York, NY, 1990.

[5] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez. *Passivity-Based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications*. Communications and Control Engineering. Springer Verlag, New York, NY, 1998.

[6] R. Ortega, A. J. van der Schaft, I. Mareels, and B. Maschke, "Putting energy back in control," *IEEE Control Systems Magazine*, vol. 21, no. 2, pp. 18–33, 2001.

[7] S. Stramigioli, *Modeling and IPC Control of Interactive Mechanical Systems- A Coordinate - Free Approach*, vol. 266 of *Lecture Notes in Control and Information Sciences*. Springer Verlag, 2001.

[8] A. J. van der Schaft, "Stabilization of Hamiltonian systems," *Nonl. Analysis, Theory, Methods & App.s*, vol. 10, no. 10, pp. 1021–1035, 1986.

[9] A. J. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*, vol. 218 of *Lecture Notes in Control and Information Sciences*. New York, NY: Springer Verlag, 1996.

20.11 Stabilization via the total energy as Lyapunov function

Consider a simple mechanical control system $(\mathsf{Q}, \mathbb{G}, V = 0, \mathcal{F})$ with equations

$$\nabla_{\gamma'}\gamma' = \mathbb{G}^{-1}F$$

Goal: Stabilize $q_0 \in \mathsf{Q}$

(i) **fully actuated system**: $\operatorname{span}(\mathcal{F}) = \mathsf{T}^*\mathsf{Q}$

(ii) $\varphi\colon \mathsf{Q} \to \mathbb{R}$ with critical zero and positive definite Hessian

$$\varphi(q_0) = 0, \quad \mathsf{d}\varphi(q_0) = 0, \quad \operatorname{Hess}\varphi(q_0) > 0$$

(iii) Rayleigh dissipation function $K_d : \mathsf{T}\mathsf{Q} \to \mathsf{T}^*Q$

Classic PD control: $F_{\mathsf{PD}}(v_q) = -\mathsf{d}\varphi(q) - K_d v_q$

Stability, local exponential stability, global convergence to critical points of φ (assuming existence compact and invariant set)

(i) Lyapunov function is

$$\begin{aligned}\frac{\mathsf{d}}{\mathsf{d}t}(\varphi + \frac{1}{2}\|\gamma'\|^2) &= \nabla_{\gamma'}\varphi + \frac{1}{2}\nabla_{\gamma'}\|\gamma'\|^2 \\ &= \langle \mathsf{d}\varphi\,,\,\gamma'\rangle + \langle\!\langle \nabla_{\gamma'}\gamma'\,,\,\gamma'\rangle\!\rangle \\ &= \langle \mathsf{d}\varphi\,,\,\gamma'\rangle + \langle -\mathsf{d}\varphi(q) - K_d\gamma'\,,\,\gamma'\rangle = -\langle K_d\gamma'\,,\,\gamma'\rangle.\end{aligned}$$

(ii) Proof of **exponential** convergence rates: modify Lyapunov function with $\epsilon\dot{\varphi}$ term, or perform linearized analysis

21 Tracking for Fully Actuated Systems

Objective: track reference trajectory γ_{ref}

Configuration and velocity errors:

(i) "distance" between q and r .... **error function**

- positive definite, symmetric, quadratic $\Psi\colon \mathsf{Q} \times \mathsf{Q} \to \mathbb{R}$

(ii) "distance" between γ' and γ'_{ref} .... **transport map**

- linear map $\mathcal{T}_{(q,r)} : \mathsf{T}_rQ \to \mathsf{T}_qQ$
- velocity error is $\dot{e} = \gamma' - \mathcal{T}_{(\gamma,\gamma_{\mathsf{ref}})}\gamma'_{\mathsf{ref}}$
  $\Rightarrow \quad \dot{\Psi} = \langle \mathsf{d}_1\Psi\,,\,\dot{e}\rangle$
- "compatibility:" $\mathsf{d}_2\Psi(q,r) = -\mathcal{T}^*_{(q,r)}\mathsf{d}_1\Psi(q,r)$

Examples: joint or Euler angle rates, body-fixed angular velocities

21.1 Tracking on Manifolds

Goal: Track a reference $\gamma_{\text{ref}}\colon I \to Q$ for $\nabla_{\gamma'}\gamma' = \mathbb{G}^{-1}F$

PD + Feedforward: Let $F = F_{\text{PD}} + F_{\text{FF}}$ with

$$F_{\text{PD}}(\gamma', t) = -\mathrm{d}_1\Psi(\gamma, \gamma_{\text{ref}}) - K_d\,\dot{e}$$

$$F_{\text{FF}}(\gamma', t) = \mathbb{G}\left((\nabla_{\gamma'} \mathcal{T}_{(\gamma,r)} w_r)\Big|_{w_r=\gamma'_{\text{ref}}} + \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{T}_{(q,\gamma_{\text{ref}})}\gamma'_{\text{ref}})\Big|_{q=\gamma(t)} \right)$$

(i) Lyapunov stability with exponential convergence rates.

(ii) time-varying Lyapunov function
$t \mapsto \Psi(\gamma(t), \gamma_{\text{ref}}(t)) + \frac{1}{2}\|\gamma'(t) - \mathcal{T}_{(\gamma(t),\gamma_{\text{ref}}(t))}\gamma'_{\text{ref}}(t)\|^2$

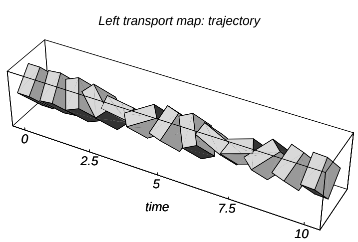
(iii) F_{FF} has two terms: "curvature" and acceleration of γ_{ref}

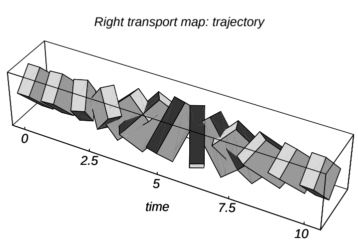
21.2 Table of Examples

Device	configuration space	error function	transport map/ velocity error
Rob. manipulator	$\mathbb{R}^n$	$\|q-r\|^2$	I_n
Pointing device	$\mathbb{S}^2 \subset \mathbb{R}^3$	$1 - q^T r$	$(q^T r)I_3 + (r \times q)\widehat{}$
Satellite	$SO(3)$	$\operatorname{tr}\left(K(I_3 - RR_d^T)\right)$ $\operatorname{tr}\left(K(I_3 - R_d^T R)\right)$	$\Omega - \Omega_d$ $\Omega - R^T R_d \Omega_d$
Submersible	$SE(3)$	[combination of $\mathbb{R}^3$ and $SO(3)$]	[change of reference frame]
Riemannian mfld	Q	geodesic distance	parallel transport

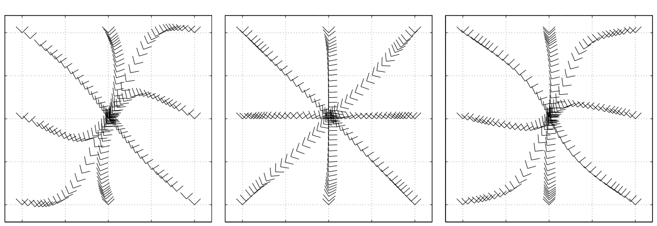
21.3 Effects of Different Choices of Error Computations

Closed-loop trajectory on $SO(3)$ with different feedforward





Closed-loop trajectory on $SE(2)$ with different feedback



Lecture #6: Trajectory Planning via Motion Primitives

Francesco Bullo
Coordinated Science Lab, General Engineering Dept, ECE, AAE
University of Illinois at Urbana-Champaign
1308 W. Main St, Urbana, IL 61801, USA
bullo@uiuc.edu, http://motion.csl.uiuc.edu

This lecture based on the following references

[1] F. Bullo and K. M. Lynch, "Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems," *IEEE TRA*, 17(4)L402–412, 2001.

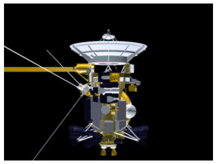
[2] F. Bullo and A. D. Lewis, "Kinematic controllability and motion planning for the snakeboard," *IEEE TRA*, Jan. 2002. Submitted.

[3] F. Bullo, N. E. Leonard, and A. D. Lewis, "Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups," *IEEE TAC*, 45(8):1437–1454, 2000.

[4] W. T. Cerven and F. Bullo, "Constructive controllability algorithms for motion planning and optimization," *IEEE TAC*, Nov. 2001. Submitted.

22 Motion planning for underactuated vehicles

SCAMP project
SSL, U. Maryland



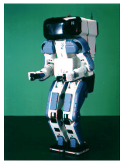
Cassini probe

blimp

hovercraft

tail-less aircraft

helicopter



Honda biped

(i) vehicles, robotic manipulators, locomotion devices

(ii) nonlinearities (kinetic energy, forces, configurations/velocities)

(iii) limited actuation (under-actuation, mag. & rate limits, ...)

22.1 Limited actuation provides for challenges

Real time motion planning

- feedforward and 2 degree-of-freedom design for aggressive tracking
- can compute feasible trajectory only via 2 pt. boundary value optimal control: iterative, off-line algorithms, convergence
- loss of controllability along minimum-time trajectories

Stabilization

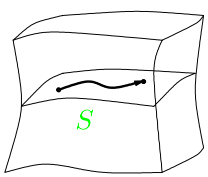
- accurate hovering/station keeping (➡ exponential stab.)
- reconfiguration after actuator failure (not linearly controllable)

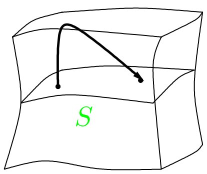
Locomotion

- analysis of gaits and of novel propulsion mechanisms
- system design

22.2 Motion Planning Scenarios

S is submanifold of trim conditions, helices, rel. equilibria, hover





(i) Classic Point-to-Point Setting: on manifold and linearly controllable

(ii) Point-to-Point remaining on manifold and system is not linearly controllable (low velocity regime, internal actuation, actuator failure, ill conditioned linearization)

(iii) Fast Point-to-Point via minimum-time trajectory and system is not linearly controllable

(iv) Harder: Point-to-Point away from S

22.3 Preliminaries: Numerical Optimal Control

Optimal Control

$$\min \int_0^T \|u(t)\|^2 dt$$
$$\text{subj } x(0) = 0, \quad x(T) = x_{\text{desired}}$$
$$\dot{x} = f(x,u)$$
$$|u| \le 1, \quad |\dot{u}| \le 1$$

➡

Transcription

$$u(t) = \sum_{i=1}^{N_u} d_i \psi_i(t)$$
$$x(t) \rightsquigarrow x_j = x(t_j)$$
$$j = 1, \dots, N_x$$

⬇

Finite dimensional NLP

$$\min \|d\|^2$$
$$\text{subj } x_1 = 0, \quad x_{N_x} = x_{\text{desired}}$$
$$x_{j+1} = f(x_j, d)$$
$$|g(d)| \le 1$$

⬅

Numerical implementation

Your favorite solver:
Sequence of QP
Feasible SQP
Dimension: $N_x + N_u$

22.4 Motion planning via primitives

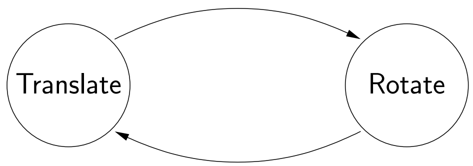
Goal: reduce complexity & abstract dynamics

(i) quantize system dynamics into finite set of primitives $\{P_1, \ldots, P_n\}$
system can evolve on primitive for arbitrary time

(ii) characterize switches/transitions between primitives
transition requires a fix duration and displacement

Wheeled robot example

restrict search / abstract dynamics to straight lines and circles



switch / transition

Incomplete List of References on Motion planning via low complexity models and via series expansions

[1] E. G. Al'brekht, "On the optimal stabilization of nonlinear systems," *PMM - Journal of Applied Mathematics and Mechanics*, vol. 25, pp. 1254–1266, 1961.

[2] E. Frazzoli, M. A. Daleh, and E. Feron, "Real-time motion planning for agile autonomous vehicles," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 25, no. 1, pp. 116–129, 2002.

[3] A. Halme and J. Orava, "Generalized polynomial operators for nonlinear systems analysis," *IEEE Transactions on Automatic Control*, vol. 17, no. 2, pp. 226–8, 1972.

[4] W. Magnus. On the exponential solution of differential equations for a linear operator. *Communications on Pure and Applied Mathematics*, VII: 649–673, 1954.

[5] G. Lafferriere and H. J. Sussmann. A differential geometric approach to motion planning. In Z. Li and J. F. Canny, editors, *Nonholonomic Motion Planning*, pages 235–270. Kluwer Academic Publishers, Boston, MA, 1993.

[6] N. E. Leonard and P. S. Krishnaprasad, *Motion control of drift-free, left-invariant systems on Lie groups*, IEEE Transactions on Automatic Control, 40 (1995), pp. 1539–1554.

[7] V. Manikonda, P. S. Krishnaprasad, and J. Hendler, "Languages, behaviors, hybrid architectures and motion control," in *Mathematical Control Theory* (J. Baillieul and J. C. Willems, eds.), New York, NY: Springer Verlag, 1998.

[8] R. M. Murray and S. S. Sastry, *Nonholonomic motion planning: Steering using sinusoids*, IEEE Transactions on Automatic Control, 5 (1993), pp. 700–726.

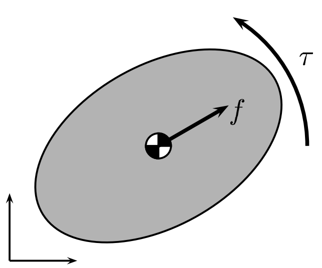
[9] H. J. Sussmann, "New differential geometric methods in nonholonomic path finding," in *Systems, Models, and Feedback: Theory and Applications* (A. Isidori and T. J. Tarn, eds.), pp. 365–384, Boston, MA: Birkhäuser, 1992.

23 Decoupled motion planning via kinematic controllability

Motion planning for underactuated robot system

(i) actuator failure

(ii) lighter design with no actuators



controllable kinematic reduction:

(i) Can follow any straight line
and can turn (2 preferred velocity fields)

(ii) Controllable via these two motions

(iii) Planning via inverse kinematic

23.1 Decoupling vector fields and kinematic controllability

Data structure

(i) given inertia tensor $\mathbb{G}$, Christoffel symbols Γ^i_{jk}

and covariant derivative $$(\nabla_X Y)^i = \frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k$$

(ii) given force co-vectors $\{F^1, \ldots, F^m\}$,

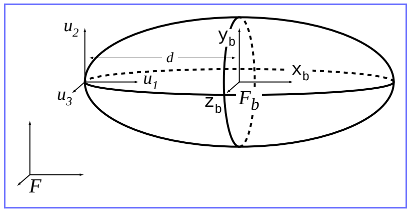
and input distribution $$\mathscr{Y} = \operatorname{span}\{Y_a = \mathbb{G}^{-1} F^a, a = 1, \ldots, m\}$$

Theorems

The vector field V is decoupling if and only if $V \in \mathscr{Y}$ and $\nabla_V V \in \mathscr{Y}$.

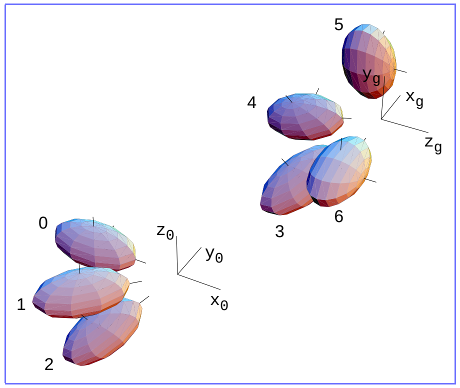
System is **kinematically controllable** if LARC on decoupling v. fields

23.2 Ex #1: A three-dimensional aerospace vehicle with three forces

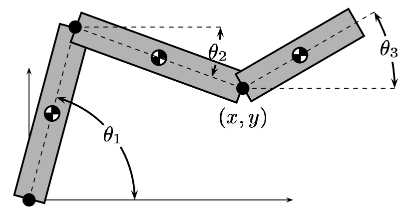


kinematically controllable via
body-fixed constant velocity fields

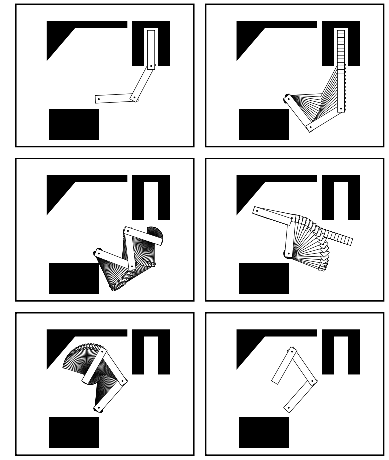
since invariant vector fields
decoupled trajectory planning
via inverse kinematic



23.3 Ex #2: Three link planar manipulator with passive link

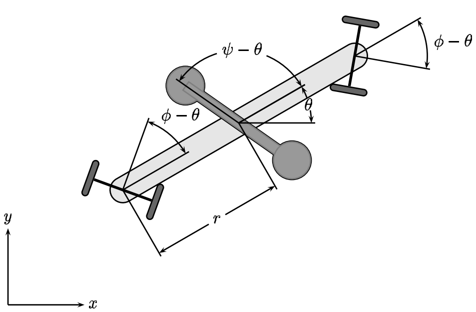


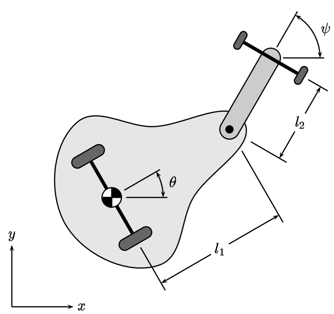
Actuator configuration	Decoupling vector fields	Kinematically controllable
(0,1,1)	2	yes
(1,0,1)	2	yes
(1,1,0)	2	yes



Lynch, Shiroma, Arai, Tanie. "Collision-free trajectory planning for a 3-DOF robot with a passive joint" IJRR, 19(12):1171-1184, 2000

23.4 Ex #3: The snakeboard and the roller racer





(i) snakeboard is kinematically controllable

(ii) roller racer is not:

(a) single input Y such that $\nabla_Y Y \notin \operatorname{span}\{Y\}$

(b) moves forward using zero mean (cyclic) input

24 Motion planning via series expansions

Linear Control Systems

$$\dot{x} = Ax + bu(t)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$.

1) Solution from $x(0) = 0$ is

$$x(t) = \int_0^t e^{A(t-s)} bu(s) ds.$$

2) Iff the system is controllable

$$W_T = \int_0^T e^{A(T-s)} bb' e^{A'(T-s)} ds.$$

3) Open-loop control to reach x_d

$$u(t) = b' e^{A'(T-t)} W_T^{-1} x_d.$$

Nonlinear Mechanical Systems

$$\dot{x} = f_0(x) + \sum f_i(x) u_i(t)$$

1) Evolution is a series expansion, with iterated integrals of u and iterated *Lie brackets* between f_j.

2) Controllability: sufficient tests include a full rank question.

3) Local *constructive* planning procedure: truncate the series, find an inverse (local **motion primitives**), combine in iterative fashion.

24.1 Mechanical control systems on matrix groups

(i) $g \in \mathsf{G}$ is configuration on n-dimensional **matrix group**
local coordinates via $x = \log(g)$

(ii) **kinetic energy** $\mathcal{KE} = \frac{1}{2} v^T \mathbb{I} v$ with $\mathbb{I} > 0$
$v \in \mathbb{R}^n$ velocity in body frame

(iii) body-fixed **forces** $f^1, \ldots, f^m \in (\mathbb{R}^n)^*$.

Generalized Christoffel symbols written with respect to a basis of left invariant vector fields are constant.

24.2 Reviewing various concepts

- rewrite:

$$\dot{v}^i + \Gamma^i_{jk} v^j v^k = \dot{v}^i + \frac{1}{2}\langle v : v \rangle$$
$$\sum (\mathbb{I}^{-1} f_k) u_k(t) = \sum b_k u_k(t) =: \beta(t)$$

- Given the family of input vectors $\{b_1, \ldots, b_m\}$, define
$\overline{\mathrm{Sym}}\{b_1, \ldots, b_m\}$
- a symmetric product in $\overline{\mathrm{Sym}}\{b_1, \ldots, b_m\}$ is **bad** if it contains even number of each b_i. Otherwise **good**.

bad: $\langle b_1 : b_1 \rangle, \ \langle b_1 : \langle b_2 : \langle b_2 : b_1 \rangle\rangle\rangle$

good: $b_1, \ \langle b_1 : b_2 \rangle$

- definite time integral: $\overline{\beta}(t) = \int_0^t \beta(\tau) d\tau$

24.3 Computing the "Force to Displacement" Map

$$\dot{g} = g \cdot \hat{v}$$
$$\dot{v} + \tfrac{1}{2}\langle v : v \rangle = \beta(t)$$

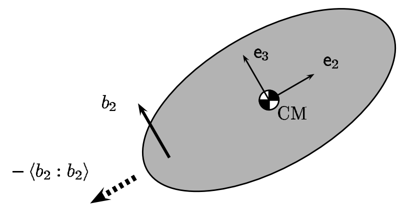
With $\epsilon \ll 1$, let

$$\beta(t, \epsilon) = \epsilon\, \beta^1(t) + \epsilon^2\, \beta^2(t) = \sum_{k=1}^{m} b_k \left(\epsilon u_k^1(t) + \epsilon^2 u_k^2(t)\right)$$

If $x(0) = 0$, $v(0) = 0$, then over **finite** interval

$$v(t) = \epsilon \overline{\beta^1}(t) + \epsilon^2 \left(\overline{\beta^2 - \tfrac{1}{2}\langle \overline{\beta^1} : \overline{\beta^1} \rangle}\right)(t) + O(\epsilon^3)$$
$$x(t) = \epsilon \overline{\overline{\beta^1}}(t) + \epsilon^2 \left(\overline{\overline{\beta^2}}(t) - \tfrac{1}{2}\overline{\overline{\langle \overline{\beta^1} : \overline{\beta^1} \rangle}}(t) + \tfrac{1}{2}\overline{[\overline{\overline{\beta^1}}, \overline{\beta^1}]}(t)\right) + O(\epsilon^3)$$

24.4 Example 1: single input systems

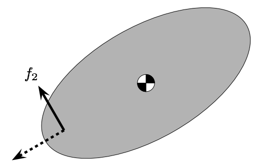


- planar rigid body with only
$b_2 = \mathbb{I}^{-1} f_2$
- set $(\beta^1(t), \beta^2(t)) = (\pm\epsilon\psi(t) b_2, 0)$
- provided $\overline{\psi}(2\pi) = 0$, we have:

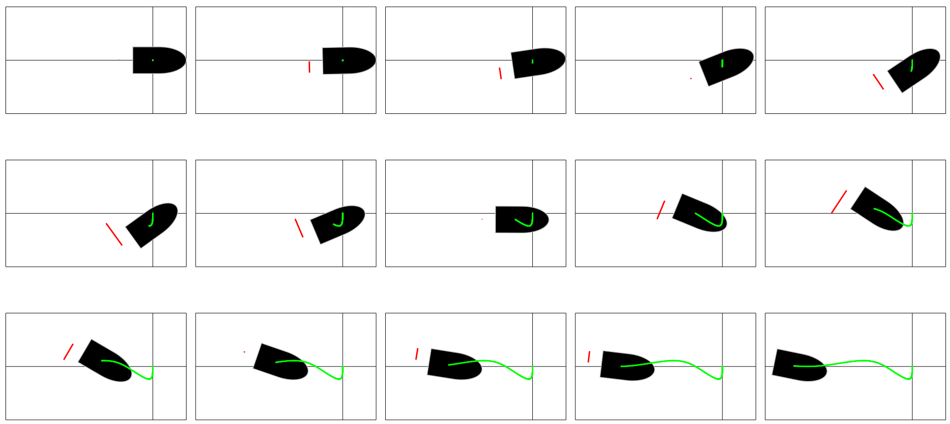
$$v(2\pi) \approx -\frac{\epsilon^2}{2}\overline{\langle \overline{\beta^1} : \overline{\beta^1} \rangle}(2\pi) = \tfrac{1}{2}\epsilon^2 \Big(\int_0^{2\pi} \overline{\psi}^2 dt\Big)\Big(-\langle b_2 : b_2 \rangle\Big)$$

- independent of sign of $\psi(t)$ ("energy integral" always positive)
- $x(2\pi)$ behaves similarly

24.5 Simulation with "uni-directional" motion



red is force   green is center of mass



24.6 Example 2: systems with two inputs

- With $\beta^2 = 0$, and with $\overline{\beta^1}(2\pi) = \overline{\beta^1}(2\pi) = 0$

$$v(2\pi) \approx -\frac{\epsilon^2}{2}\overline{\langle \overline{\beta^1} : \overline{\beta^1} \rangle}(2\pi)$$

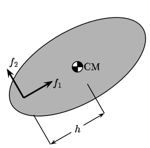
- **Satellite with two thrusters**
  - $\{b_1, b_2\}$ torques about first two axes
    $\langle b_k : b_k \rangle = 0$, $\langle b_1 : b_2 \rangle$ torque about third axis
  - If $\beta^1(t) = \psi(t)(b_1 + b_2)$ then
    $\overline{\langle \overline{\beta^1} : \overline{\beta^1} \rangle}(2\pi) = 2\langle b_1 : b_2 \rangle \left(\int_0^{2\pi} \overline{\psi}^2 dt \right)$

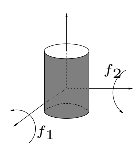
energy integral	➡	in-phase
(classic area integral	➡	out-of-phase)

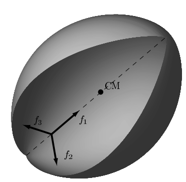
24.7 Interpretation

given accelerations	$\{b_i = \mathbb{I}^{-1} f_i\}$
	$\Downarrow$
"reachable" velocities	$\{b_i, \langle b_j : b_k \rangle, \dots\}$
	$\Downarrow$
"reachable" configurations	$\{b_i, \langle b_j : b_k \rangle, [b_j, b_k], \dots\}$

24.8 Examples of systems with "fully reachable" velocities

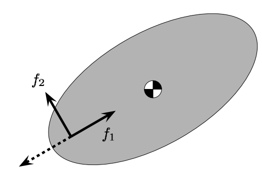






24.9 Inverting the Approximate Map

- recall $v(2\pi) \approx \epsilon^2 \left(\overline{\beta^2 - \tfrac{1}{2}\langle \overline{\beta^1} : \overline{\beta^1} \rangle} \right)(2\pi)$
  $\beta^i = \sum u_k^i(t) b_k$



- assume **"controllable"**
  $\mathrm{rank}\{b_i, \langle b_j : b_k \rangle\} = n$,
  $\langle b_i : b_i \rangle \in \mathrm{span}\{b_1, \dots, b_m\}$

- `Inverse`(v_{desired}) : can design $(\beta_1(t), \beta_2(t))$

$$\left(\overline{\beta^2 - \tfrac{1}{2}\langle \overline{\beta^1} : \overline{\beta^1} \rangle} \right)(2\pi) = v_{\text{desired}}$$

(i) in-phase inputs generate motion along good symmetric product

(ii) uni-directional contribution due to bad symmetric products can be compensated for by lower order, good products

(iii) u_k^i sinusoids (cyclic, in-phase or orthogonal)

24.10 Primitives of Motion

use Inverse as building block for motion planning

Change-Vel $(\epsilon, v_{\text{final}})$ .... steer velocity $v(t)$ to $\epsilon v_{\text{final}}$

Initial state: $v(0) = \epsilon v_0$

Final state: $v(2\pi) \approx \epsilon v_{\text{final}}$

Maintain-Vel $(\epsilon, v_{\text{nom}})$ .... keeps velocity $v(t)$ at ϵv_{ref}

Initial state: $v(0) = \epsilon v_{\text{ref}}$

Final state: $v(2\pi) \approx \epsilon v_{\text{ref}}$

(i) can compute change in g

(ii) expansions with low initial speed: $v(0) = \epsilon v_0^1 + \epsilon^2 v_0^2$

(iii) "sum" contributions over finite and $O(1/\epsilon)$ intervals

24.11 Point to point problem via constant velocity algorithm

GOAL	drive system from $(\text{Id}, 0)$ to $(g_1, 0)$
ARGUMENTS	(g_1, σ)
REQUIRE	$\log(g_1)$ well defined

```
N ⇐ Floor(‖log(g_1)‖/(2πσ))
v_nom ⇐ log(g_1)/(2πσN)
Change-Vel(σ, v_nom)                        {start maneuver}
for k = 1 to (N − 1) do
   Maintain-Vel(σ, v_nom)                   {keep nominal velocity}
end for
Change-Vel(σ, 0)                            {stop maneuver}
```

N intervals $\times$ σv_{nom} $=$ total displacement

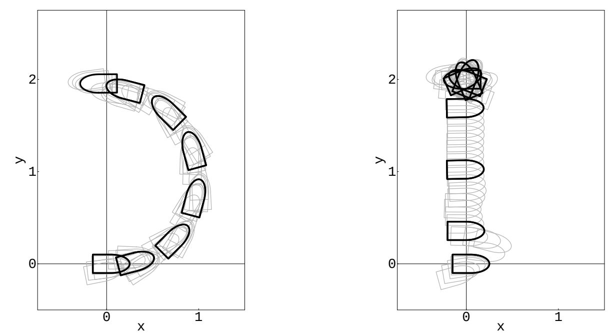
24.12 Stabilization problem via iterative steering

GOAL	drive system to the state $(\text{Id}, 0)$ exponentially as $t \to \infty$
ARGUMENTS	σ
REQUIRE	$\|(\log(g(0)), v(0))\| \leq \sigma.$

```
for k = 1 to +∞ do
   t_k ⇐ 4kπ                                {t_k is the current time}
   σ_k ⇐ ‖(log(g(t_k), v(t_k)))‖
   Change-Vel(σ_k, −(log(g(t_k)) + πv(t_k))/(2πσ_k))
   Change-Vel(σ_k, 0)
end for
```

two primitives force final configuration and velocity to vanish

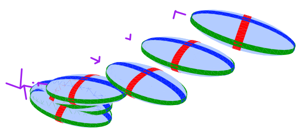
24.13 Simulation of Point-to-Point Problem



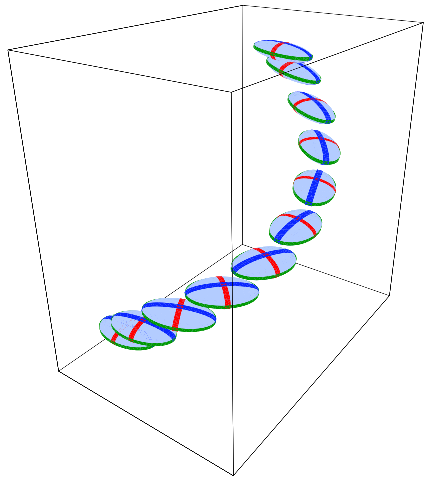
Properties of algorithms

- closed form, negligible computational load
- asymptotic behavior: time $O(\epsilon^{-1})$, final error $(\epsilon^{3/2})$
- series expansion approach leads to complete algorithms

24.14 Simulations for 3D vehicle



motion primitive
based on local inversion



global planning

25 Motion planning for polynomial systems

Linear Control Systems

$$\dot{x} = Ax + Bu(t)$$

1) Solution from $x(0) = 0$ is

$$x(t) = \int_0^t e^{A(t-s)} Bu(s) ds$$

2) Iff the system is controllable

$$W = \int_0^T e^{A(T-s)} BB' e^{A'(T-s)} ds > 0$$

3) Open-loop control to reach x_d

$$u(t) = B' e^{A'(T-t)} W^{-1} x_d$$

Nonlinear Mechanical Systems

$$\dot{x} = f_0(x) + \sum f_i(x) u_i(t)$$

1) Characterize flow map

$$x(T) = \Phi(u)$$

2) Controllability: range Φ

3) Local planning:

$$u = \Phi^{\dagger}(x_d)$$

25.1 Series for polynomial systems

For low-dimensional models of aerospace and underwater vehicles, trigonometric dependencies can be turned into polynomial:

$$\dot{x} = Ax + f^{[2]}(x, x) + Bu, \qquad x(0) = x_0,$$

$$f^{[2]} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \quad \text{is a symmetric tensor}$$

evolution via (Volterra) series $\quad x(t) = \Phi(u) = \sum_{k=1}^{+\infty} x_k(t)$

$$x_1(t) = \mathrm{e}^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$x_k(t) = \int_0^t e^{A(t-\tau)} \left(\sum_{a=1}^{k-1} f^{[2]}(x_a(\tau), x_{k-a}(\tau)) \right) d\tau$$

25.2 Constructive controllability

Let $x(0) = 0$, choose base functions:

$$u(t) = \sum_{i=1}^{n} \psi^i(t) c_i \qquad c \in \mathbb{R}^n$$

then $x(T) = \Phi(u) = \Phi(c)$

$$x_k(T) = \Phi_k(c, \ldots, c)$$

$$\|x_k\| = O(\|c\|^k)$$

To have $x(T) = x_d$, solve

$$x_d = \Phi_1 c + \sum_{k=2}^{+\infty} \Phi_k(c, \ldots, c)$$

25.3 Minimum energy control

Set up Hamilton's equations:

$$\dot{x} = Ax + f^{[2]}(x, x) - BB'\lambda$$

$$\dot{\lambda} = -A'\lambda - 2f^{[2]}(x)'\lambda$$

This time no input, $\lambda(0) = \lambda_0 \in \mathbb{R}^n$

$$x_k = \overline{\Phi}_k(\lambda_0, \ldots, \lambda_0)$$

For boundaries conditions, solve

$$x_d = \overline{\Phi}_1 \lambda_0 + \sum_{k=2}^{+\infty} \overline{\Phi}_k(\lambda_0, \ldots, \lambda_0)$$

25.4 Expression for Φ tensors

In constructive controllability

$$\Phi_1^i(t) = \int_0^t \mathrm{e}^{A(t-\tau)} B\psi^i(\tau) d\tau$$

$$\Phi_2^{i_1 i_2}(t) = \int_0^t \mathrm{e}^{A(t-\tau)} f^{[2]}\left(\Phi_1^{i_1}(\tau), \Phi_1^{i_2}(\tau)\right) d\tau,$$

$$\Phi_3^{i_1 i_2 i_3}(t) = \int_0^t \mathrm{e}^{A(t-\tau)} \Big(f^{[2]}(\Phi_1^{i_1}(\tau), \Phi_2^{i_2 i_3}(\tau)) + f^{[2]}(\Phi_2^{i_1 i_2}(\tau), \Phi_1^{i_3}(\tau)) \Big) d\tau$$

$$\vdots$$

$$\Phi_k^{i_1 \dots i_k}(t) = \int_0^t \mathrm{e}^{A(t-\tau)} \left(\sum_{a=1}^{k-1} f^{[2]}(\Phi_a^{i_1 \dots i_a}(\tau), \Phi_{k-a}^{i_1 \dots i_{k-1}}(\tau)) \right) d\tau.$$

To evaluate at $t = T$

25.5 Inversion for linearly controllable systems

To solve

$$x_d = \Phi_1 c + \sum_{k=2}^{\infty} \Phi_k(c, \dots, c)$$

Φ_1 is full rank iff system is linearly controllable, and appropriate $\{\psi^i(t)\}$

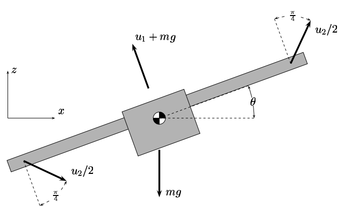
1: iterative numerical scheme $\lim_{k\to\infty} c_k \to c_{goal}$

$$c_1 = \Phi_1^{-1} x_d, \qquad c_{k+1} = \Phi_1^{-1} x_d - \sum_{k=2}^{\infty} \Phi_1^{-1} \Phi_k(c_k, \dots, c_k)$$

2: inverse Taylor expansion $c_{goal} = \sum_{k=1}^{\infty} c_k$

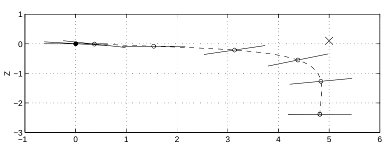
$$c_1 = \Phi_1^{-1} x_d, \qquad c_k = -\Phi_1^{-1} \sum_{\substack{i_1 + \dots + i_m = k \\ i_1, \cdots, i_m < k}} \Phi_m\Big(c_{i_1}, \cdots, c_{i_m}\Big)$$

25.6 Simulations for linearly controllable systems

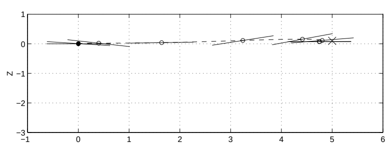


planar vertical takeoff and landing aircraft model (PVTOL)

Desired motion is horizontal translation from left to right without any vertical or rotational displacement.



first order



second order

25.7 Inversion for nonlinearly controllable systems

Solve

$$x_d = \Phi_2(c, c) + \sum_{k=3}^{\infty} \Phi_k(c, \dots, c)$$

for not linearly controllable system such as

$$\dot{x} = f^{[2]}(x, x) + Bu, \qquad x(0) = 0$$

Assume

$$A = 0, \qquad \mathrm{rank}\{B_i, f^{[2]}(B_j, B_k)\} = n$$

$$f^{[2]}(B_i, B_i) \in \mathrm{span}\{B_1, \dots, B_m\}, \ \forall i$$

Can invert $x_d = \Phi_2(c, c)$ via "quadratic inversion"

$$u : [0, 2\pi] \to \mathbb{R}^m \quad = \mathtt{Inverse}(x_d)$$

25.8 Quadratic inversion (compare with linear case)

(i) Let $N = m(m-1)/2$, $P = \{(j,k) \mid 1 \le j < k \le m\}$, $1 \le \alpha \le N$, and

$$\psi_\alpha(t) = \frac{1}{\sqrt{2\pi}}\Big(\alpha \sin(\alpha t) - (\alpha + N)\sin\big((\alpha+N)t\big)\Big).$$

(ii) Compute $(m+N)$ real numbers z_i and z_{jk} such that

$$x_d = \sum_{1 \le i \le m} z_i B_i + \sum_{1 \le j < k \le m} z_{jk} f^{[2]}(B_j, B_k).$$

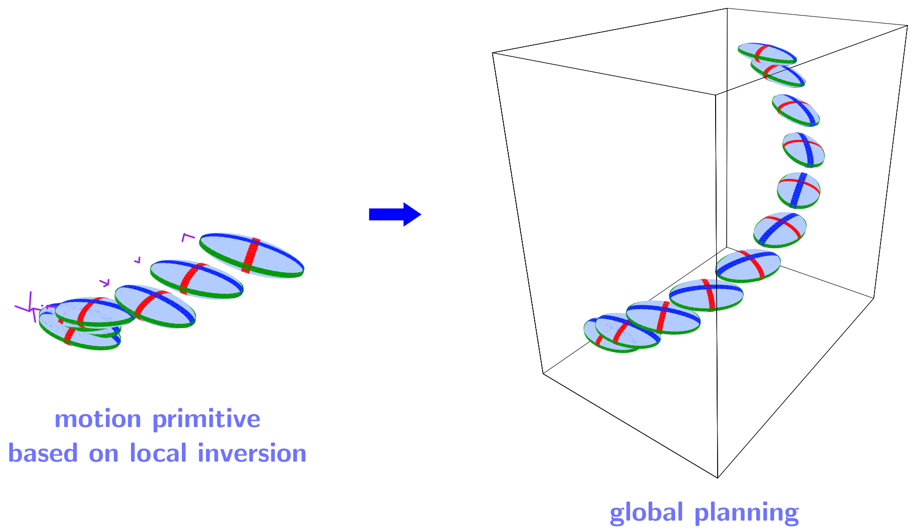
(iii) Let $a : P \mapsto \{1, \dots, N\}$ be an enumeration of P, and set

$$b^1(t) = \sum_{1 \le j < k \le m} \sqrt{|z_{jk}|}\Big(B_j - \operatorname{sign}(z_{jk}) B_k\Big)\psi_{a(j,k)}(t)$$

$$b^2(t) = \frac{1}{2\pi}\sum_{1 \le i \le m} z_i b_i + \frac{1}{4\pi}\sum_{1 \le j < k \le m} |z_{jk}|\Big(f^{[2]}(B_j, B_j) + f^{[2]}(B_k, B_k)\Big)$$

$$Bu(t) = b^1(t) + b^2(t) \quad = \texttt{Inverse}(x_d)$$

25.9 Simulations for nonlinearly controllable systems



Summary of Design Methods (lectures #5 and #6)

Body of work encompassing results on

(i) stabilization via energy methods for fully actuated systems

(ii) motion planning via kinematic reductions

(iii) motion planning via low amplitude oscillations

(iv) talk by Jorge Cortés on motion planning via high amplitude oscillations

Open directions

Motion control via low amplitude oscillations general manifold case

Motion control via kinematic reductions numerical methods for inverse kinematics, time-varying feedback stabilizers