Stability of Adaptive Delta Modulators with Forgetting Factor and Constant Inputs

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SUMMARY

Motivated by applications to feedback control over communication networks where the actuation and feedback signals are transmitted over communication channels, we study the stability of Adaptive Delta Modulators (ADM) when the coded signal is a constant. The importance of such a setting arises because a common control task is to track a dc input. It is known that a standard accumulator-based adaptive delta modulator (ADM) has the following highly undesirable characteristic: virtually all combinations of the algorithm parameters result in 4-cycles, and the avoidance of 4-cycles requires a nongeneric initialization. Further, the steady state oscillations that generically arise in the course

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of these cycles can have amplitudes that can be arbitrarily close to the initial error. Consequently, we study the use of a forgetting factor in the ADM loop, and provide a detailed stability analysis and design guidelines. Intuitively, adding a forgetting factor to the classical ADM algorithm prevents $4-$periodic cycles from occurring by damping them. In particular we show that for suitably chosen design parameters, the ADM with forgetting factor can track a constant signal arbitrarily closely under mild assumptions. We provide simulations to demonstrate how much better the modified algorithm performs relative to the original ADM algorithm in a remote control setting.

1. Introduction

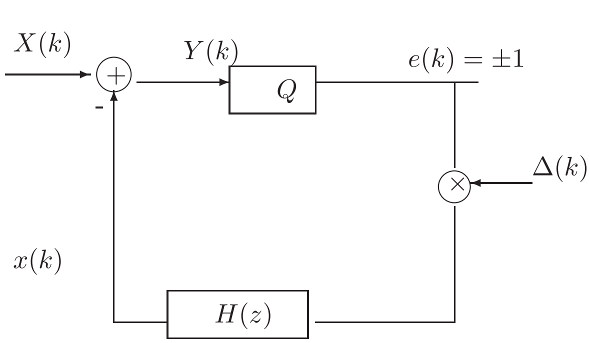

Figure 1. A Delta Modulator at the transmitter

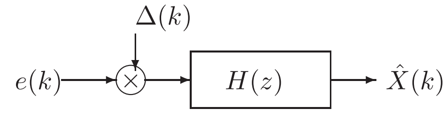

Figure 2. A Delta Modulator at the receiver

Adaptive Delta Modulators (ADM) are a popular device used in signal processing and communications for signal quantization with variable step-size. They seek to increase the dynamic range of the signals that can be tracked while using binary coding.

While several variations of this device exist [2]-[7], the simplest, [1], [2] is depicted in Fig. 1 and Fig 2. This algorithm has the attractive feature that the encoder, housed at a transmitter need only transmit the sign of the decoding error. This alone, at least in principle is enough for the decoder at a receiver to reconstruct the encoded signal, at least in principle. However, as shown in [10] this algorithm generates 4-cycles for generic initializations. As shown by example here these cycles may have oscillations that correspond to errors of magnitude comparable to the initial encoding error. Several papers, e.g. [4, 7, 8, 9] incorporate additional information on the magnitude of the coding error in the transmitted signal. Thus while Jayant's algorithm required only a one bit transmission, these others require a richer transmission protocol. In this paper we show that the 4-cycles can be ameliorated even if one retains the one bit transmission conceived by Jayant, by incorporating a forgetting factor in a predictor that features in Jayant's algorithm.

We now describe Jayant's algorithm and our proposed modification. The structures in Fig. 1 and Fig. 2 are at the encoder and decoder, respectively. The signal $X(k)$ is coded into the binary sequence $e(k)$, taking values from $\{-1, 1\}$. It is $e(k)$ that is actually transmitted. The quantity $\Delta(k)$ represents the variable step size which is increased or decreased according to the sign pattern in $e(k)$. Consequently, if the signal at the receiver input is identical to the transmitted value of $e(k)$, and $\Delta(0)$ is known at the receiver, then for all $k \geq 0$, $\Delta(k)$ is known to the receiver. This also guarantees that the signal $\hat{X}(k)$ at the receiver is identical to $x(k)$, the output of $H(z)$ at the transmitter, if $x(0) = \hat{X}(0)$. Thus should $x(k)$ approach $X(k)$, so also would $\hat{X}(k)$.

A heuristic algorithm for updating $\Delta(k)$ with the goal of forcing $\hat{X}(k)$ to approach $X(k)$, is described in [2]. In Jayant's algorithm $H(z)$ is an *accumulator*: i.e. with $\alpha = 1$,

$$H(z) = \frac{1}{1 - \alpha z^{-1}}. \tag{1.1}$$

Generally, the agreed upon values of $\Delta(0)$ and $x(0)$ (this is generally chosen to be zero) between the transmitter and receiver are part of the communication protocol.

Our goal is to analyze the behavior of $\hat{X}(k)$ and hence $x(k)$ when the signal $X(k) = x$ is constant. The motivation for studying the ability of this ADM to track a constant signal stems from issues connected to networked control systems that are acquiring increasing importance. Specifically, in such a setting the plant and the controller must communicate the actuation signal via a communication channel and must thus quantize it.

It has been noted in [11] that variable step quantization of the actuation signal suffices to achieve acceptable closed loop performance. Thus, it behooves one to understand the effectiveness of ADM's in this setting, where the transmitter and receiver of the actuation signals host the arrangements of Fig. 1 and Fig. 2 respectively.

A typical control problem involves forcing the plant output to track a constant signal. This in turn requires that at steady state both the signals that the ADM's should track should be constants. Thus at the minimum, desirable performance will necessitate that the signal $\hat{X}(k)$ track a constant $X(k)$ in Figs 1 and 2 with reasonable fidelity.

When $\alpha = 1$, $\hat{X}(k)$ either converges to x or enters into a 4-cycle. Further, 4-cycles are avoided only with nongeneric initializations. In view of this conclusion this paper is dedicated to the analysis when $X(k)$ is a constant and when $H(z)$ is as in (1.1) when a forgetting factor is included i.e. when $0 < \alpha < 1$. Intuitively, adding a forgetting factor to the classical ADM algorithm, prevents 4−periodic cycles from occurring by damping them. In accordance with this intuition, we show that in such a case, one can choose the system parameters to make the eventual coding error arbitrarily small provided $x(0) = 0$. The forgetting factor also allows the requirement $\hat{X}(0) = x(0)$ to be relaxed by forcing the initial error to decay.

We note that this paper builds on our conference paper [14], by providing additional insights, expanding on proofs that were terse because of space constraints, and demonstrating the efficacy of the algorithm in a remote control setting through simulations.

2. The detailed algorithm and the motivation for the accumulator-based ADM

The detailed algorithm of [2] is given in (2.2) - (2.5) below with $\Delta(0) > 0$ and $K > 1$.

$$x(k+1) = \alpha x(k) + \Delta(k)e(k) \tag{2.2}$$

$$e(k) = \text{sgn}(X(k) - x(k)) \tag{2.3}$$

$$\Delta(k+1) = \Delta(k)K^{e(k+1)e(k)} \tag{2.4}$$

with

$$\text{sgn}(a) = \begin{cases} 1 & \text{if } a \geq 0 \\ -1 & \text{if } a < 0 \end{cases} \tag{2.5}$$

We now motivate this algorithm in the form proposed in [2], i.e. when $\alpha = 1$. Several features of this algorithm are noteworthy. First observe that as the sequence $\epsilon(k)$ is available at the receiver, so is the sequence $\Delta(k)$, assuming perfect transmission and an agreed upon value for $\Delta(0)$. This is so as $\Delta(k)$ increases by a factor of K if two successive values of $X(k) - x(k)$ have the same sign (i.e. $e(k+1)e(k) = 1$), and decreases by the factor K if two successive values of $X(k) - x(k)$ have opposite signs (i.e. $e(k+1)e(k) = -1$). Thus, the reception of the $\epsilon(k)$ sequence permits reproduction of $\Delta(k)$ at the receiver. Consequently if $\hat{X}(0) = x(0)$ then the accumulation

$$\hat{X}(k+1) = \alpha\hat{X}(k) + \Delta(k)e(k) \tag{2.6}$$

ensures that $\hat{X}(k) = x(k)$.

Second, observe that (2.6) justifies the association of $\Delta(k)$ with variable step-size as at each sample time $\hat{X}(k)$ increases or falls by $\Delta(k)$, depending on whether $x(k)$ and hence $\hat{X}(k)$ is below or above $X(k)$.

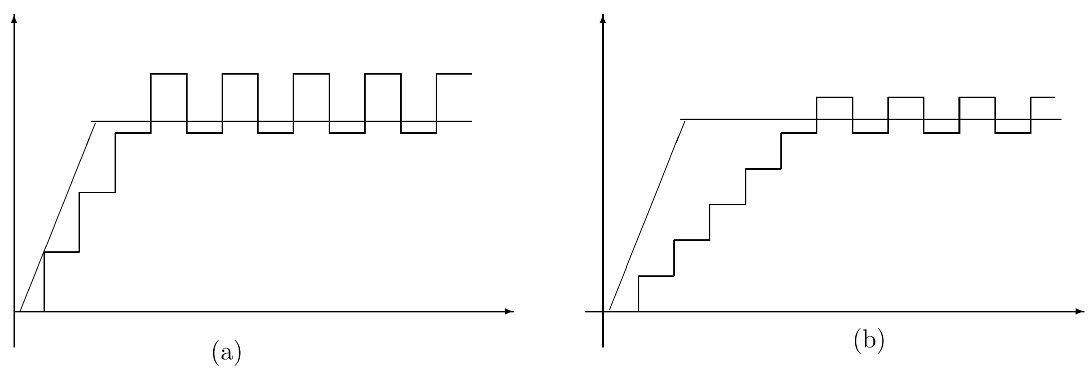

Figure 3. (a) Large Δ. (b) Small Δ.

Third, to understand the role of (2.3, 2.4) consider temporarily a constant Δ replacing $\Delta(k)$, and Fig. 3(a), which simultaneously depicts $X(k)$ and $\hat{X}(k)$: $X(k)$ is the signal that ramps up to a constant value while $\hat{X}(k)$ is the signal that changes in steps. In the ramping stage it is desirable to have a large Δ so that $\hat{X}(k)$ tracks $X(k)$ quicker. The converse applies when $X(k)$ is at a steady state, where a large Δ results in a large granularity in the error between $\hat{X}(k)$ and $X(k)$. Contrast this to Fig. 3(b) where a smaller Δ is used. The result is slower tracking when $X(k)$ is rising rapidly, but smaller steady state error once $X(k)$ has stopped changing. Thus when the signal to be tracked changes quickly, a large Δ is desirable. On the other hand when $X(k)$ is not changing quickly and $\hat{X}(k)$ is close to it, a smaller Δ is desirable. The update laws (2.3, 2.4) judge the quality of tracking by whether or not successive values of $X(k) - \hat{X}(k)$ have the same sign. Their doing so indicates that $\hat{X}(k)$ must approach $X(k)$ at a faster rate requiring a larger Δ. If on the other hand the sign of $X(k) - \hat{X}(k)$ alternates then $\hat{X}(k)$ is likely to be close to $X(k)$ and a decrease in Δ is needed and implemented by the algorithm.

3. An example of 4-cycles when $\alpha = 1$

We demonstrate now the occurrence of 4-cycles when $\alpha = 1$ and the input is constant. Throughout this paper we make the following standing assumption.

Assumption 3.1. *The signal $X(k)$ in Fig. 1 and (2.3) obeys, for some constant $\bar{x}$, $X(k) = \bar{x}$, for all k. Further $\Delta(0) > 0$ and $K > 1$.* ■

Now consider the situation where $x(0) = 0$, and $x > 0$. Define L to be an integer such that

$$\Delta(0)\frac{K^L - 1}{K - 1} < \bar{x} \tag{3.7}$$

but

$$\Delta(0)\frac{K^{L+1} - 1}{K - 1} \geq \bar{x}. \tag{3.8}$$

Thus, because of (2.2-2.5), $x(i) < \bar{x} \quad \forall\, i \leq L$, $x(L+1) \geq \bar{x}$ and

$$\Delta(L) = \Delta(0)K^L. \tag{3.9}$$

Thus as $e(L+1) = -1$ and $e(L) = 1$, $\Delta(L+1) = \Delta(L)/K$. Further now $x(L+2) = x(L) + \Delta(L) - \Delta(L)/K > x(L)$. Thus a combination of $\Delta(0)$ and $\bar{x}$ can always be found such that $x(L+2) \geq \bar{x}$, while the previous equations continue to hold. Then $\Delta(L+2) = \Delta(L)$ and $x(L+3) = x(L) + \Delta(L) - \Delta(L)/K - \Delta(L) < x(L) < \bar{x}$. Thus, $\Delta(L+3) = \Delta(L)/K$ and

$$\begin{aligned} x(L+4) &= x(L) + \Delta(L) - \Delta(L)/K - \Delta(L) + \Delta(L)/K \\ &= x(L) < \bar{x}. \end{aligned}$$

Further, one also has $\Delta(L+4) = \Delta(L)$ connoting the onset of 4-cycles. The swing between the maximum and minimum values of $x(i)$ in this cycle is $\Delta(L)(K+1)/K = \Delta(0)(K^L + K^{L-1})$, which in view of (3.8) and (3.7) has a comparable magnitude to the initial error between x and $x(0)$, being $O(K^L)$. Thus the fidelity of reconstruction is almost as poor as the initial error, approaching it arbitrarily closely for large K.

4. ADM with forgetting factor: Overview of results

As the accumulator based algorithm ($\alpha = 1$), provides poor performance with constant inputs for generic parameter combinations, and initial conditions, we now study the algorithm with

$$0 < \alpha < 1. \tag{4.10}$$

As shown in the sequel good design requires that α be close to 1. In fact, as we will see the following additional assumption will be needed:

$$\alpha K > 1. \tag{4.11}$$

It is highly doubtful whether arguments of the type advanced below would be effective without this assumption. Note (4.10,4.11) imply that $K > 1$. As noted earlier our goal is to study this algorithm for constant $X(i)$, i.e. for all i,

$$X(i) = x. \tag{4.12}$$

In the remainder of this section, we outline the major results when $0 < \alpha < 1$, and contrast them to the case of $\alpha = 1$. First note that a major difficulty with the $\alpha = 1$ case is the necessity of identical initialization of the sequences $x(k)$ and $\hat{X}(k)$. As opposed to this, (4.10) ensures that the effect of the difference $x(0) - \hat{X}(0)$, diminishes over time.

The second important difference relates to the convergence properties even when exact intialization occurs. In particular when $\alpha = 1$, for generic combinations of $\Delta(0)$, K, $x(0)$ and x, for some N and all $k > N$ one has four cycles of the form

$$x(k+4) = x(k), \text{ and } \Delta(k+4) = \Delta(k).$$

Further, the largest $|x(k) - x|$, in the course of these 4-cycles, can be arbitarily close to the initial error $|x(0) - x|$.

When (4.10) holds on the other hand, the following positive parameter plays a pivotal role:

$$\epsilon = \frac{1-\alpha^3}{1-\alpha^2+\frac{\alpha}{K}} \tag{4.13}$$

Indeed we show that under the right conditions, whose enforcement will be discussed in Section 7,

$$\lim_{i\to\infty} \sup \Delta(i) \leq K\epsilon|x|. \tag{4.14}$$

This in turn will be shown to imply that

$$\lim_{i\to\infty} \sup |x(i) - x| \leq \max\{(1-\alpha+K\epsilon)|x|, (\alpha+K\epsilon-1)|x|\} \tag{4.15}$$

Observe that $\alpha + K\epsilon - 1$, can be readily verified to be positive. As will be explained in Section 7 one can make ϵ arbitrarily small by choosing α arbitrarily close to 1. Consequently, one can achieve an error that is an arbitrarily small fraction of x, the value being encoded. We will explain later why (4.14) and (4.15) cannot generically be achieved when $\alpha = 1$. Finally we note that apart from selecting $\alpha \approx 1$, for reasons to be clarified later, it is also desirable to select $\alpha K \approx 1$. This precludes very large values of K.

5. ADM with forgetting factor: Some properties

In this section we present a series of properties of (2.2-2.5) and (4.10-4.12), that will allow us to conduct our stability analysis. For simplicity we will assume

$$x > 0. \tag{5.16}$$

For the moment, we note that the results of this section translate in a rather obvious way to the case where $x < 0$. For example under (5.16), the following set of indices that mark the points at which $x(i)$ transitions from below to above x, will play an important role.

$$\mathcal{I}_+ = \{i | x(i) < x \text{ and } x(i+1) \geq x\}. \tag{5.17}$$

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Henceforth $\Delta(i)$ for $i \in \mathcal{I}_+$, i.e. at a point of transition of $x(i)$ from below to above x, will be referred to as a *transitioning* Δ. For $x < 0$, on the other hand, the corresponding indices are those marking transitions in $x(i)$ from above x to below. In general the results of this section can be applied to the case of $x < 0$, by reversing the relative positions of signal values, i.e. inequalities of the form of $x(k) \geq x$ must be exchanged with $x(k) \leq x$.

The first Lemma shows that $e(i)$ must at some point change sign and that the sign changing persists.

Lemma 5.1. *Consider the system described in (2.2-2.5) and (4.10-4.12), (5.16), and $\mathcal{I}_+$ as in (5.17). Then $\mathcal{I}_+$ is an infinite set.*

Proof: Proof is by contradiction. If $\mathcal{I}_+$ is finite, then for all n exceeding some i, either $x(n) < x$ or $x(n) \geq x$. Suppose the former; then $e(k) > 0$ for all $k \geq i$, and $\Delta(k) = K^{k-i}\Delta(l)$. Thus for all $n > i$

$$\begin{aligned} x(n) &= \alpha^{n-i}x(i) + \Delta(i)\sum_{k=i}^{n-1}\alpha^{n-k-i-1}K^{k-i} \\ &\geq \alpha^{n-i}x(i) + \Delta(i)K^{n-i-1}. \end{aligned}$$

As, $K > 1$, and $\alpha < 1$, at some n, $x(n) > x$ establishing a contradiction. Similarly $x(k) \geq x$ for all k greater than or equal to some i indicates that $e(k) < 0$ for all $k \geq i$. Thus, at some n, $x(n) < x$, and the above argument can be repeated to conclude that $\mathcal{I}_+$ is an infinite set.

■

Recall that the claim of Lemma 5.1 is also valid in the case when $\alpha = 1$. The next Lemma gives a lower bound on the values assumed by Δ when transitions occur. It also provides conditions for $x(i)$ to increase in value, but as we will see there is no corresponding nontrivial result for $\alpha = 1$.

Lemma 5.2. *Consider the system described in (2.2-2.5) and (4.10-4.12), (5.16), and $\mathcal{I}_+$ as in (5.17). If $i \in \mathcal{I}_+$, then*

$$\Delta(i) > (1-\alpha)x. \tag{5.18}$$

Further if for some j, $x(j) < x$ and

$$\Delta(j) > (1-\alpha)x(j), \tag{5.19}$$

then $x(j+1) > x(j)$.

Proof: If $i \in \mathcal{I}_+$, then $x(i) < x$ and $x(i+1) \geq x$. Thus,

$$x \leq x(i+1) = \alpha x(i) + \Delta(i) < \alpha x + \Delta(i)$$

from which (5.18) follows. Further, under (5.19),

$$x(j+1) = \alpha x(j) + \Delta(j) > \alpha x(j) + (1-\alpha)x(j) = x(j).$$

■

Here emerges a key difference with the $\alpha = 1$ case: Namely, the lower bounds in (5.18) and (5.19) are both zero, and thus trivially hold. We now provide a crucial property of this system. Specifically, after the first sign change in $e(i)$, no more than *two successive* values of $x(i)$ may exceed x.

Lemma 5.3. *If $i \in \mathcal{I}_+$ and $x(i+2) \geq x$ then under (2.2-2.5) and (4.10-4.12), (5.16), and $\mathcal{I}_+$ as in (5.17), $x(i+3) < x$ and*

$$x(i+4) = \alpha^4 x(i) + \Delta(i)(1-\alpha^2)(1/K - \alpha) < x. \tag{5.20}$$

Further in this case

$$\Delta(i+4) = \Delta(i). \tag{5.21}$$

Proof: By definition of $\mathcal{I}_+$ $x(i) < x$ and $x(i+1) \geq x$. Thus, from (2.2-2.5), as $x(i+2) \geq x$, one has that $e(i)\Delta(i) = \Delta(i)$, $e(i+1)\Delta(i+1) = -\Delta(i)/K$ and $e(i+2)\Delta(i+2) = -\Delta(i)$. Thus,

$$x(i+3) = \alpha^3 x(i) + \alpha^2 \Delta(i)e(i) + \alpha\Delta(i+1)e(i+1) + \Delta(i+1)e(i+2) = \alpha^3 x(i) + \Delta(i)(\alpha^2 - \frac{\alpha}{K} - 1), \tag{5.22}$$

Observe that $\alpha^2 - \alpha/K - 1 < 0$, from (4.10). Thus if $x(i) \geq 0$, $x(i+3) < x(i) \leq x$. On the other hand $x(i+3)$ is negative and hence less than x, if $x(i) < 0$.

Moreover, in this case $e(i+3)\Delta(i+3) = \Delta(i)/K$. Thus if $x(i) \geq 0$ then because of (4.11),

$$\begin{aligned} x(i+4) &= \alpha^4 x(i) + \alpha^3 \Delta(i) - \alpha^2 \frac{\Delta(i)}{K} - \alpha\Delta(i) + \frac{\Delta(i)}{K} \\ &= \alpha^4 x(i) + \Delta(i)(1-\alpha^2)(1/K - \alpha) < x(i) < x. \end{aligned}$$

If on the other hand, $x(i) < 0$ then

$$x(i+4) = \alpha^4 x(i) + \Delta(i)(1-\alpha^2)(1/K - \alpha) < 0 < x. \tag{5.23}$$

Finally (5.21) occurs because $\Delta(i+3) = \Delta(i)/K$, and $x(i+3)$ and $x(i+4)$ are both less than x.

■

Thus if $i \in \mathcal{I}_+$ then either

$$e(i) = 1,\ e(i+1) = e(i+2) = -1 \text{ and } e(i+3) = 1, \tag{5.24}$$

or

$$e(i) = 1,\ e(i+1) = -1 \text{ and } e(i+2) = 1, \tag{5.25}$$

The fact that no more than two successive $e(i)$ can be negative, after the first transition is also true for the $\alpha = 1$ case. However, when $\alpha = 1$, from (5.20) one sees that $x(i+4) = x(i)$, and $\Delta(i+4) = \Delta(i)$, signaling the onset of 4-cycles. Thus, in the $\alpha = 1$ case *any occurrence of (5.24) will lead to 4-cycles that cannot be arrested.* This will not be the calse when $0 < \alpha < 1$. The next Lemma characterizes conditions under which (5.24) holds.

Lemma 5.4. *Consider the system described in (2.2-2.5) and (4.10-4.12), (5.16), and $\mathcal{I}_+$ as in (5.17). Then $x(i+2) \geq x$ iff*

$$\alpha^2 x(i) \geq (\frac{1}{K} - \alpha)\Delta(i) + x. \tag{5.26}$$

Proof: Follows from noting that $x(i+2) \geq x$ is equivalent to

$$x(i+2) = \alpha^2 x(i) + \alpha\Delta(i) - \frac{\Delta(i)}{K} > x.$$

■

Because of the forgetting factor being smaller than 1, even if $x(i) < x$ and thus $e(i)\Delta(i) > 0$, $x(i+1)$ need not exceed $x(i)$. The following Lemma shows, however that if at any point $x(i)$ does become less than x, then after at most two samples, its value will increase and will continue do so, as long as it remains below x.

Lemma 5.5. *Under (2.2-2.5), (4.10-4.12) and (5.16), suppose $i \in \mathcal{I}_+$. Then the following apply.*

(A) Suppose (5.24) holds and for some $k \geq 4$ and all $n \in \{3, \cdots, k\}$, $e(i+n) = 1$. Then for all $n \in \{4, \cdots, k\}$,

$$x(i+n+1) > x(i+n). \tag{5.27}$$

(B) Suppose (5.25) holds and for some $k \geq 4$ and all $n \in \{2, \cdots, k\}$, $e(i+n) = 1$. Then also (5.27) holds for all $n \in \{4, \cdots, k\}$.

Proof: In case A, from Lemma 5.3, $e(i+3) = e(i+4) = 1$, and $\Delta(i+4) = \Delta(i)$. Thus, as $e(i+n) = 1$, for all $n \in \{3, \cdots, k\}$, $\Delta(i+n) \geq \Delta(i)$ for all $n \in \{4, \cdots, k\}$. Similarly, in case B $e(i+2) = -e(i+1) = 1 = e(i)$. Thus, as $e(i+n) = 1$, for all $n \in \{2, \cdots, k\}$, from (2.4), $\Delta(i+2) = \Delta(i)/K^2$, $\Delta(i+3) = \Delta(i)/K$, and $\Delta(i+n) \geq \Delta(i)$ for all $n \in \{4, \cdots, k\}$. Further as $x > x(i+n)$, for all $n \in \{4, \cdots, k\}$, and $i \in \mathcal{I}_+$, from the first part of Lemma 5.2

$$\Delta(i+n) \geq \Delta(i) > (1-\alpha)x > (1-\alpha)x(i+n).$$

Thus the result follows the from second part of Lemma 5.2. ■

We will use these properties to study the asymptotic behavior of (2.2 - 2.5) in the next section.

6. ADM with forgetting factor: Stability

In this section we provide conditions under which (4.14) and (4.15) hold. These equations embody the stability property. To this end, we will continue to assume that $x > 0$. The translation to the $x < 0$ case will be according to that given at the start of Section 5. The first few lemmas focus on the values of transitioning Δ's. Specifically, the first states that if a given transitioning Δ exceeds ϵx, then the next transitioning Δ can be no greater.

Lemma 6.1. *Suppose (2.2-2.5), (4.10-4.12) and (5.16) hold. Consider i, j two consecutive members of $\mathcal{I}_+$, with $j > i$. Suppose $\Delta(i) > \epsilon x$. Then $\Delta(j) \leq \Delta(i)$ and $j \leq i+4$.*

Proof: From Lemma 5.3 one of the following two cases apply.

Case I: $x(i+2) < x$**.** We will argue now that if $\Delta(j) > \Delta(i)$, then $\Delta(i) < \epsilon x$; this is equivalent to proving the desired result. Now under the case I condition, $\Delta(i+1)e(i+1) = -\Delta(i)/K$, $\Delta(i+2)e(i+2) = \Delta(i)/K^2$, and for all $k \in \{i+2, \cdots, j\}$, $\Delta(k)e(k) = \Delta(i)K^{k-i-4}$. Thus, $\Delta(j) > \Delta(i)$ implies and is implied by $j > i+4$, and so $x(i+4) < x$ and $x(i+5) < x$. Observe that,

$$\begin{aligned} x(i+5) &= \alpha^4 x(i+1) + \alpha^3\Delta(i+1)e(i+1) + \alpha^2\Delta(i+2)e(i+2) + \alpha\Delta(i+3)e(i+3) + \Delta(i+4)e(i+4) \\ &= \alpha^4 x(i+1) - \alpha^3\frac{\Delta(i)}{K} + \alpha^2\frac{\Delta(i)}{K^2} + \alpha\frac{\Delta(i)}{K} + \Delta(i). \end{aligned}$$

Thus $x(i+5) < x$ implies

$$\alpha^4 x(i+1) - \alpha^3\frac{\Delta(i)}{K} + \alpha^2\frac{\Delta(i)}{K^2} + \alpha\frac{\Delta(i)}{K} + \Delta(i) < x$$

Consequently, as $x(i+1) \geq x$,

$$\Delta(i) < \frac{x(1-\alpha^4)}{1 + \frac{\alpha}{K} + \frac{\alpha^2}{K^2} - \frac{\alpha^3}{K}}. \tag{6.28}$$

Thus, to prove that $\Delta(i) \leq \epsilon x$, it is enough to show that the upper bound in (6.28) is smaller than ϵx, or equivalently by (4.13) that $(1+\alpha+\alpha^2)(1+\frac{\alpha}{K}(1-\alpha^2+\frac{\alpha}{K}))$ is greater than

$(1+\alpha+\alpha^2+\alpha^3)(1-\alpha^2+\frac{\alpha}{K})$. Indeed, using the fact that $K>1$, and (4.10) and (4.11), the difference between these two quantities equals and obeys:

$$\begin{aligned}
& (1+\alpha+\alpha^2)(\frac{\alpha^2}{K^2}+\alpha^2(1-\frac{\alpha}{K}))-\alpha^3(1-\alpha^2+\frac{\alpha}{K}) \\
= \; & \alpha^2(1+\alpha+\alpha^2)+(\frac{\alpha^2}{K^2}-\frac{\alpha^3}{K})(1+\alpha+\alpha^2) \\
- \; & \alpha^3(1-\alpha^2+\frac{\alpha}{K}) \\
= \; & \alpha^2(1+\alpha+\alpha^2)-\alpha^3-\alpha^2(\alpha-\frac{1}{K})\left(\frac{1+\alpha+\alpha^2}{K}-\alpha^2\right) \\
> \; & \alpha^2(1+\alpha^2)-\alpha^2(\alpha-\frac{1}{K})\left(1+\alpha+\alpha^2-\alpha^2\right) \\
> \; & \alpha^2(1+\alpha^2)-\alpha^3(1+\alpha)=\alpha^2(1-\alpha)>0,
\end{aligned}$$

where we have used the fact that $\alpha K>1$ and $K>1$. Hence $\Delta(i)>\epsilon x$ implies $\Delta(j)\le\Delta(i)$. Also, as noted in the above argument, $\Delta(j)\le\Delta(i)$ if and only if $j\le(i+4)$, and this yields the second desired conclusion.

Case II: $x(i+2)\ge x$. In this case $\Delta(i+2)e(i+2)=-\Delta(i)$, and as from Lemma 5.3, $x(i+3)$ and $x(i+4)$ are less than x, $\Delta(i+3)e(i+3)=\Delta(i)/K$ and $\Delta(i+4)e(i+4)=\Delta(i)$. As $x(i+2)\ge x$, and $\Delta(i)>\epsilon x$, we have that

$$\begin{aligned}
x(i+5) &= \alpha^4 x(i+1)+\alpha^3\Delta(i+1)e(i+1)+\alpha^2\Delta(i+2)e(i+2) \\
&+ \alpha\Delta(i+3)e(i+3)+\Delta(i+4)e(i+4) \\
&= \alpha^3 x(i+2)+\Delta(i)(1-\alpha^2+\frac{\alpha}{K}) \\
&> x(\alpha^3+1-\alpha^3)=x.
\end{aligned}$$

Hence $j=i+4$ and from Lemma 5.3 $\Delta(j)=\Delta(i)$. ■

When $\alpha=1$, the four-cycles referred to earlier occur, when $j=i+4$ and $x(i+2)\ge x$.

Now, the second step *en route* to the desired stability result is to provide conditions under which the next transitioning Δ is in fact smaller, i.e. if i,j are consecutive members of $\mathcal{I}_+$, then $\Delta(j)<\Delta(i)$. Specifically, recall from Lemma 5.3 that, if $i\in\mathcal{I}_+$, the there are at most two succeeding time instants, $i+1$ and $i+2$ at which $x(.)$ can remain greater than x. The Lemma below shows that if in fact a transitioning Δ exceeds $K\epsilon x$, *and* $x(.)$ stays above x only once, then the next transitioning Δ will be smaller.

Lemma 6.2. *Suppose (2.2-2.5), (4.10-4.12) and (5.16) hold. Consider i,j, two consecutive members of $\mathcal{I}_+$, with $j>i$. Suppose $\Delta(i)>K\epsilon x$ and $x(i+2)<x$. Then $\Delta(j)<\Delta(i)$.*

Proof: Because, $x(i+2)<x$, from the definition of $\mathcal{I}_+$, $\Delta(i+1)e(i+1)=-\Delta(i)/K$, $\Delta(i+2)e(i+2)=\Delta(i)/K^2$, and for all $k\in\{i+2,\cdots,j\}$, $\Delta(k)e(k)=\Delta(i)K^{k-i-4}$. Thus, if $j=i+2$, then $\Delta(j)<\Delta(i)$. Hence, to prove the Lemma we need only show that if $j>i+2$, then $x(i+4) \geq x$. Indeed as $j > i+2$, and $x(i+1) \geq x$,

$$\begin{aligned} x(i+4) &= \alpha^3 x(i+1) + \alpha^2 \Delta(i+1)e(i+1) \\ &+ \alpha\Delta(i+2)e(i+2) + \Delta(i+3)e(i+3) \\ &= \alpha^3 x(i+1) + \frac{\Delta(i)}{K}(1-\alpha^2+\frac{\alpha}{K}) \\ &> x(\alpha^3+1-\alpha^3) = x. \end{aligned}$$

The final equality comes from the lemma hypothesis that $\Delta(i) > K\epsilon x$ and the definition of ϵ in (4.13).

■

The third step is to show in Lemma 6.3 that if a transitioning Δ is less than or equal to ϵx, then the next transitioning Δ cannot exceed $K\epsilon x$. Taken together Lemmas 6.1 and 6.3 show that if at any stage a transitioning Δ becomes less than or equal to $K\epsilon x$, then no future transitioning Δ can exceed $K\epsilon x$.

Lemma 6.3. *Suppose (2.2-2.5), (4.10-4.12) and (5.16) hold. Consider i, j, two consecutive members of $\mathcal{I}_+$, with $j > i$. Suppose $\Delta(i) \leq \epsilon x$. Then $\Delta(j) \leq K\epsilon x$.*

Proof: We need to consider the two cases $x(i+2) < x$ and $x(i+2) \geq x$.
Case I: $x(i+2) < x$**.** In this case $\Delta(i+1)e(i+1) = -\Delta(i)/K$, $\Delta(i+2)e(i+2) = \Delta(i)/K^2$, and for all $k \in \{i+2, \cdots, j\}$, $\Delta(k)e(k) = \Delta(i)K^{k-i-4}$. Suppose $j = i+n$. If $n \leq 5$, then

$$\Delta(j) \leq K\Delta(i) \leq K\epsilon x,$$

proving the result. Now suppose $n \geq 6$, and thus $\Delta(j) > K\epsilon x$. Observe by definition $x(j) = x(i+n) < x$. Then

$$x(i+n) = \alpha^2 x(i+n-2) + \alpha\frac{\Delta(j)}{K^2} + \frac{\Delta(j)}{K} < x.$$

Thus,

$$\begin{aligned} x(i+n-2) &= \frac{x(i+n) - \Delta(j)(\frac{\alpha}{K^2}+\frac{1}{K})}{\alpha^2} \\ &< \frac{x - K\epsilon x(\frac{\alpha}{K^2}+\frac{1}{K})}{\alpha^2} \\ &= \frac{x}{\alpha^2}(1-(\frac{\alpha}{K}+1)\epsilon) \end{aligned} \tag{6.29}$$

On the other hand because of Lemma 5.2, $\Delta(i) > (1-\alpha)x$. Thus,

$$\begin{aligned} x(i+4) &= \alpha^3 x(i+1) + \alpha^2 \Delta(i+1)e(i+1) \\ &+ \alpha\Delta(i+2)e(i+2) + \Delta(i+3)e(i+3) \\ &= \alpha^3 x(i+1) + \frac{\Delta(i)}{K}(1-\alpha^2+\frac{\alpha}{K}) \\ &> \alpha^3 x + x\frac{1-\alpha}{K}(1-\alpha^2+\frac{\alpha}{K}) \end{aligned} \tag{6.30}$$

Further, as $n \geq 6$, from Lemma 5.5, $x(i+n-2) \geq x(i+4)$. Thus to establish a contradiction we need only show that the upper bound in (6.29) is less than the lower bound in (6.30). Indeed, the difference between the upper bound in (6.29) and the lower bound in (6.30), given by (see (4.13))

$$\frac{1}{\alpha^2}[1-\frac{(1-\alpha^3)(1+\frac{\alpha}{K})}{1-\alpha^2+\frac{\alpha}{K}}]-[\alpha^3+\frac{1}{K}(1-\alpha)(1-\alpha^2+\frac{\alpha}{K}]$$

has the same sign as

$$\begin{aligned}
& (1-\alpha^2+\frac{\alpha}{K})-(1-\alpha^3)(1+\frac{\alpha}{K})-\alpha^5(1-\alpha^2+\frac{\alpha}{K}) \\
- & \frac{\alpha^2}{K}(1-\alpha)(1-\alpha^2+\frac{\alpha}{K})^2 \\
= & (1-\alpha^5)(1-\alpha^2+\frac{\alpha}{K})-(1-\alpha^3)(1+\frac{\alpha}{K}) \\
- & \frac{\alpha^2}{K}(1-\alpha)(1-\alpha^2+\frac{\alpha}{K})^2 \\
< & \left[1-\alpha^2+\frac{\alpha}{K}-\alpha^5+\alpha^7-\frac{\alpha^6}{K}\right]-\left[1+\frac{\alpha}{K}-\alpha^3-\frac{\alpha^4}{K}\right] \\
= & \alpha^2(\alpha-1)+\alpha^4(\frac{1}{K}-\alpha)+\alpha^6(\alpha-\frac{1}{K}) \\
< & \alpha^4(\alpha-\frac{1}{K})(\alpha^2-1)<0,
\end{aligned}$$

where the last inequality use (4.10) and (4.11).
Case II: $x(i+2) \geq x$**.** In this case $\Delta(i+2)e(i+2) = -\Delta(i)$, and from Lemma 5.3, $x(i+3)$ and $x(i+4)$ are less than x, and $\Delta(i+4) = \Delta(i)$.

With n defined as in the proof of Case I, suppose, (again to obtain a contradiction) that $\Delta(i+n) > K\epsilon x$. Then in this case $n \geq 6$, as $\Delta(i+5) = K\Delta(i) \leq K\epsilon x$. Now, as

$$x(i+n) = \alpha x(i+n-1) + \frac{\Delta(i+n)}{K}$$

one has

$$x(i+n-1) = \frac{x(i+n)-\frac{\Delta(i+n)}{K}}{\alpha} < x\left(\frac{1-\epsilon}{\alpha}\right), \tag{6.31}$$

where ϵ is defined in (4.13). Further, because of Lemma 5.5 and the fact that $x(i+2) \geq x$, $x(i+5) > x(i+4)$ and $\Delta(i) > (1-\alpha)x$,

$$\begin{aligned}
x(i+5) &= \alpha^3 x(i+2)+\alpha^2\Delta(i+2)e(i+2)+\alpha\Delta(i+3)e(i+3) \\
&+ \Delta(i+4)e(i+4) \\
&> x[\alpha^3+(1-\alpha)(1-\alpha^2+\frac{\alpha}{K})]
\end{aligned} \tag{6.32}$$

As from Lemma 5.4, $\Delta(i+n-1) > \Delta(i+5)$, for all $n \geq 6$, proving that the upper bound in (6.31) is smaller than the lower bound in (6.32), will establish a contradiction. In fact the difference between the upper bound in (6.31) and the lower bound in (6.32), has the same sign as

$$\begin{aligned} & (1-\alpha^2+\frac{\alpha}{K})-(1-\alpha^3)-\alpha^4(1-\alpha^2+\frac{\alpha}{K}) \\ - & \alpha(1-\alpha)(1-\alpha^2+\frac{\alpha}{K})^2 \\ = & (1-\alpha^4)(1-\alpha^2+\frac{\alpha}{K})-\alpha(1-\alpha)(1-\alpha^2+\frac{\alpha}{K})^2 \\ - & (1-\alpha^3) \\ = & (1-\alpha)\Big[(1+\alpha+\alpha^2+\alpha^3)(1-\alpha^2+\frac{\alpha}{K}) \\ - & \alpha(1-\alpha^2+\frac{\alpha}{K})^2-(1+\alpha+\alpha^2)\Big] \\ = & (1-\alpha)\Big[(1+\alpha+\alpha^2)(1-\alpha^2+\frac{\alpha}{K}-1) \\ + & \alpha(1-\alpha^2+\frac{\alpha}{K})\left(\alpha^2-\left(1-\alpha^2+\frac{\alpha}{K}\right)\right)\Big] \\ = & (1-\alpha)\Big[\alpha(\frac{1}{K}-\alpha)(1+\alpha+\alpha^2+\alpha-\alpha^3+\frac{\alpha^2}{K}) \\ + & \alpha(1-\alpha^2+\frac{\alpha}{K})(\alpha^2-1)\Big]<0 \end{aligned}$$

where we have used $K>1$, (4.10) and (4.11) repeatedly. ■

We now establish conditions that ensure (4.14), including naturally $i \notin \mathcal{I}_+$. Our strategy will be to show that under these conditions there exists an N such that for all $i \geq N$ and $i \in \mathcal{I}_+$, there holds:

$$\Delta(i) \leq K\epsilon x. \tag{6.33}$$

We argue that this ensures (4.14). Assume such an N exists and consider any consecutive elements i, j of $\mathcal{I}_+$, obeying $j > i \geq N$. Define k with $i < k \leq j$ to be the unique time instant where $x(k) < x$ and $x(k-1) \geq x$. Then we know that for all $i \leq l \leq k-1$, $\Delta(i) \geq \Delta(l)$. Likewise for all $k \leq l \leq j$, $\Delta(l) \leq \Delta(j)$. This proves that if (6.33) holds for all $i \in \mathcal{I}_+$, and $i \geq N$, then it also holds for all $i \geq N$.

Now we examine how to ensure (6.33) holds for all $i \geq N$ and $i \in \mathcal{I}_+$. Because of Lemmas 6.1 and 6.3, if any transitioning Δ becomes less than or equal to $K\epsilon x$, all future transitioning Δ's must be bounded by $K\epsilon x$. Thus to prove (4.14), it suffices to have the following condition: That for every $i \in \mathcal{I}_+$, at which $\Delta(i) > K\epsilon x$, there exists a $j > i$ and $j \in \mathcal{I}_+$, such that $\Delta(j) < \Delta(i)$. Then as all multiplicative changes in Δ are by factors that are powers of K, (6.33) must hold for all suitably large $i \in \mathcal{I}_+$. Now suppose a given $i \in \mathcal{I}_+$, with $\Delta(i) > K\epsilon x$, has the property that for all $j > i$ and $j \in \mathcal{I}_+$, $\Delta(j) \geq \Delta(i)$. By Lemma 6.1, at all such j, in fact $\Delta(j) = \Delta(i)$. By Lemma 6.2 this implies that for all $j \geq i$ and $j \in \mathcal{I}_+$, $x(j+2) \geq x$. Since, in this case Lemma 5.3 asserts that $x(j+3) < x$, $x(j+4) < x$, and $\Delta(j+4) = \Delta(j)$, this also means that $j+4 \in \mathcal{I}_+$, as failure to transition at this point will result in a large transitioning Δ, thereby violating Lemma 6.1. This argument thus shows the following: if for all N there exists $i \geq N$, such that (6.33) fails, then there must exist an $i \in \mathcal{I}_+$, such that for all nonnegative integer n,

$$i+4n \in \mathcal{I}_+,\ x(i+4n+2) \geq x \text{ and } \Delta(i+4n) = \Delta(i) > K\epsilon x. \tag{6.34}$$

The result is in fact a 2-cycle in Δ (in this case for all $\Delta(i + 4n + 2) = \Delta(i)$ and $\Delta(i+4n+1) = \Delta(i)/K$) with potentially large amplitudes.

It is possible for such cycles to occur. Consider for example the situation where for some i

$$\Delta(i) \geq \Delta^* := \frac{(1+\alpha^2)|x|}{\alpha - \frac{1}{K}}. \tag{6.35}$$

Now select, $x(i) = x^*$, defined as:

$$x^* = -\mathrm{sgn}(x)\frac{\left(\alpha - \frac{1}{K}\right)\Delta(i)}{1+\alpha^2}. \tag{6.36}$$

Suppose $x > 0$. In this case since $x(i) = x^* < 0 < x$,

$$x(i+1) = \alpha x(i) + \Delta(i) = \frac{1+\frac{\alpha}{K}}{1+\alpha^2}\Delta(i) \geq \frac{1+\frac{\alpha}{K}}{1+\alpha^2}\Delta^* = \frac{\alpha+K}{\alpha K - 1}x > x, \tag{6.37}$$

where the last inequality is obtained by using the fact that $K > 1$ and $\alpha < 1$. Thus, $i \in \mathcal{I}_+$, and $\Delta(i+1)e(i+1) = -\Delta(i)/K$. Consequently,

$$x(i+2) = \alpha^2 x(i) + (\alpha - \frac{1}{K})\Delta(i) = \frac{\alpha - \frac{1}{K}}{1+\alpha^2}\Delta(i) > x, \tag{6.38}$$

where the last inequality is obtained by using (6.35). Thus, from Lemma 5.3,

$$\begin{aligned} x(i+4) &= \alpha^4 x(i) - (\alpha - \frac{1}{K})(1-\alpha^2)\Delta(i) \\ &= \alpha^4 x(i) + (1+\alpha^2)(1-\alpha^2)x(i) = x(i). \end{aligned}$$

Evidently, in this case 2-cycles result in Δ and 4-cycles in $x(i)$. Further the resulting $\Delta(i)$ sequence oscillates with bounds of Δ^* and Δ^*/K. Notice two features of this example. First as $\alpha - 1/K$ is to be kept small, Δ^* is a large multiple of x. Second, the $x(i)$ sequence in the course of these cycles changes sign. Below, we show that these features are necessary for such large oscillations in Δ to occur, and in Section 7 provides design guidelines for avoiding them.

Theorem 6.1. *Consider the system described in (2.2-2.5) and (4.10-4.12) and $\mathcal{I}_+$ as in (5.17). Suppose $x > 0$, (respectively, $x < 0$) and at least one of the following two conditions holds: (i) For some $i \in \mathcal{I}_+$, (respectively, $i \in \mathcal{I}_-$), (6.35) is violated. (ii) For all $i \in \mathcal{I}_+$, (respectively, $i \in \mathcal{I}_-$), $x(i) \geq 0$, (respectively, $x(i) \leq 0$). Then under (2.2-2.5) and (4.10-4.12) there exists a finite N, such that for all $i \geq N$, (6.33) holds.*

Proof: We will prove the result when $x > 0$. Suppose for every N, there exists $i \geq N$, such that (6.33) is violated. In view of the argument given after Lemma 6.3, this implies that there exists i such that for all $n \geq 0$, (6.34) holds. Thus, for all $n \geq 0$, one has (see Lemma 5.3),

$$x(i+4(n+1)) = \alpha^4 x(i+4n) - \left(\alpha - \frac{1}{K}\right)(1-\alpha^2)\Delta(i+4n).$$

With x^* defined in (6.36) we thus have

$$\begin{aligned} x(i+4(n+1)) - x^* &= \alpha^4 x(i+4n) - \left(\alpha - \frac{1}{K}\right)(1-\alpha^2)\Delta(i) \\ &+ \frac{\left(\alpha - \frac{1}{K}\right)\Delta(i)}{1+\alpha^2} \\ &= \alpha^4 x(i+4n) + \alpha^4 \frac{\alpha - \frac{1}{K}}{1+\alpha^2}\Delta(i) \\ &= \alpha^4 (x(i+4n) - x^*). \end{aligned}$$

Thus

$$\lim_{n\to\infty} x(i+4n) = x^*, \tag{6.39}$$

and as $x^* < 0$, (ii) must be violated. Further, observe that as $i \in \mathcal{I}_+$, the second equation in (6.34) ensures that for all $n \geq 0$

$$x(i+4n+2) = \alpha^2 x(i+4n) + (\alpha - \frac{1}{K})\Delta(i) \geq x.$$

Because of (6.39) this requires that

$$x \leq \alpha^2 x^* + (\alpha - \frac{1}{K})\Delta(i) = \frac{\alpha - \frac{1}{K}}{1+\alpha^2}\Delta(i)$$

where the last equality follows from (6.36). Thus (6.35) holds for this $i \in \mathcal{I}_+$, and in fact all subsequent transitioning Δ must be no smaller than this $\Delta(i)$. As this $\Delta(i)$ also obeys the last inequality in (6.34), because of Lemma 6.1 no previous transitioning Δ can be less than this $\Delta(i)$ either, i.e. (i) must be violated. ■

The example given before Theorem 6.1 also shows that (i) and (ii) in Theorem 6.1 together constitute sufficient conditions for these potentially large amplitude 4-cycles to be possible, e.g. when $x(0) = x^*$. In the $\alpha = 1$ case, (ii) in Theorem 6.1 is not necessary for such undesirable cycles to occur. In particular, (ii) stems from the requirement of (6.39). Because of the equation before (6.39), (6.39) need not hold if $\alpha = 1$. Further, as noted after Lemma 5.3, when $\alpha = 1$, such cycles in $\Delta(k)$, and indeed 4-cycles in $x(k)$ are guaranteed for $\alpha = 1$, if even once $x(i+2) \geq x$ for $i \in \mathcal{I}_+$. This in general is not true when $\alpha < 1$.

We now examine the error bvehavior in $|x(i) - x|$ when (6.33) is assured.

Theorem 6.2. *Suppose under (2.2-2.5) and (4.10-4.12) there exists a finite N, such that for all $i \geq N$, (6.33) holds. Then (4.15) also holds.*

Proof: Again we will prove the result when $x > 0$. Choose successive members l and j of $\mathcal{I}_+$, $l < j$ and both greater than N. Clearly, from Lemma 5.3 at most $x(l+1)$ and $x(l+2)$ can be greater than or equal to x. Further as $x > 0$, and (4.10) holds, $x(l+2) < x(l+1)$. Thus the maximum value of $x(k)$ for all $k \in \{l, l+1, \cdots, j\}$ is $x(l+1)$. Because of (6.33),

$$x(l+1) = \alpha x(l) + \Delta(l) < (\alpha + K\epsilon)x. \tag{6.40}$$

Consider now the unique i for which $l < i < j$, $x(i-1) \geq x$ and $x(i) < x$. Then from Lemma 5.3 either $i = l+2$ or $i = l+3$, and so from Lemma 5.5 the only candidates for minimum $x(k)$ with $k \in \{l, l+1, \cdots, j\}$ are $x(i)$, $x(i+1)$ and $x(i+2)$. Call $\Delta(i-1) = \Delta \leq K\epsilon x$. Clearly,

$$x(i) = \alpha x(i-1) - \Delta \geq (\alpha - K\epsilon)x. \tag{6.41}$$

We will now show that neither $x(i+1)$, nor $x(i+2)$ can be less than $(\alpha - K\epsilon)x$. If $x(i+1) \geq x(i)$, then of course $x(i+1) \geq (\alpha - K\epsilon)x$. Suppose, $x(i+1) < x(i)$. As $\Delta(i) = \Delta/K$, the second part of Lemma 5.2, leads to the conclusion that $\Delta/K < x(1-\alpha)$. Then,

$$x(i+1) = \alpha^2 x(i-1) + \Delta(\frac{1}{K} - \alpha) > x[(\alpha^2 + (1-\alpha K)(1-\alpha)]. \tag{6.42}$$

We show that $x(i+1) > (\alpha - K\epsilon)x$ by showing that the coefficient of x in (6.42) is no smaller than $(\alpha - K\epsilon)$. Indeed as $K > 1$ and (4.10) holds,

$$\begin{aligned} & \alpha^2 + (1-\alpha K)(1-\alpha) - \alpha + K\epsilon \\ = \; & (1-\alpha)(1-\alpha-\alpha K) + K\epsilon \\ > \; & -(1-\alpha)\alpha K + K\frac{1-\alpha^3}{1-\alpha^2+\frac{\alpha}{K}} \\ > \; & -(1-\alpha)\alpha K + K\frac{1-\alpha^3}{1+\alpha^2+\alpha} \\ = \; & K\frac{(1-\alpha)(1-\alpha^3)}{1+\alpha+\alpha^2} > 0. \end{aligned}$$

Now consider $x(i+2)$. We have

$$x(i+2) = \alpha^3 x(i-1) + \Delta\left[1 - \alpha^2 + \frac{\alpha}{K}\right] > \alpha^3 x.$$

We show that $\alpha^3 > (\alpha - K\epsilon)$. Again as $K > 1$ and (4.10) holds,

$$\alpha^3 - \alpha + K\epsilon > \alpha(\alpha^2 - 1) + \frac{1-\alpha^3}{1-\alpha^2+\alpha}$$

which has the same sign as

$$\begin{aligned} & 1 - \alpha^3 - \alpha(1-\alpha^2)(1-\alpha^2+\alpha) \\ = \; & (1-\alpha)(1-\alpha^2+\alpha^4) > 0. \end{aligned}$$

■

We stress again that this Theorem requires in its hypothesis only that $\Delta(i)$ eventually become no greater than $K\epsilon x$ and nothing else.

7. ADM with forgetting factor: Design guidelines

The parameter ϵ in (4.13) obeys

$$K\epsilon < K^3 \frac{1-\alpha^3}{\alpha}.$$

Evidently, for a given K one can make $K\epsilon$ as small as one pleases by making $\alpha \approx 1$. Thus (4.15) indicates that the error in $x(i) - x$ can be made arbitrarily small by choosing a sufficiently small $K\epsilon$. Of course, a practical limit on how close α can be made to 1 is imposed by the competing role of α as an instrument to diminish the effect of $x(0) - \hat{X}(0)$. Observe that by choosing K to be modest in magnitude, one can achieve the objective of keeping $\alpha K \approx 1$ while still satisfying (4.10), $K > 1$ and $\epsilon \approx 0$.

We now turn to satisfying the requirement to ensure (4.14). The first such design strategy assumes that lower and upper bounds on $|x|$ and its sign are available. Frequently, it is desirable to keep this lower bound greater than zero to permit x to rise above a noise floor. A strategy assuming such a bound, justified in Lemma 7.1 below, requires that

$$\Delta(0) < \frac{1+\alpha^2}{K(\alpha K - 1)}|x| = \frac{\Delta^*}{K^2}. \tag{7.43}$$

The Lemma assumes that $|x(0)| < |x|$ and $x(0)x \geq 0$.

Lemma 7.1. *Consider (2.2-2.5), (4.10-4.12) with (7.43) in force. Suppose* $|x(0)| < |x|$ *and* $x(0)x \geq 0$. *Then (4.14) and hence (4.15) holds if (7.43) holds.*

Proof: Again we treat the case $x > 0$. In view of Theorem 6.2 it suffices to show that (6.33) holds. According to Theorem 6.1 this in turn is satisfied if for some $i \in \mathcal{I}_+$, $\Delta(i) < \Delta^*$. Choose such an i to be the first element in $\mathcal{I}_+$.

If $i \leq 2$ then $\Delta(i) \leq K^2\Delta(0) < \Delta^*$. Thus suppose $i \geq 3$. As by definition $x(k) < x$ for all $k \leq i$ and $i \geq 3$,

$$\begin{aligned} x &> x(i) \geq \alpha^i x(0) + \Delta(0)(K^{i-1} + \alpha K^{i-2} + \alpha^2 K^{i-3}) \\ &= \alpha^i x(0) + \frac{\Delta(i)}{K^2}(K^2 + \alpha K + \alpha^2) \\ &\geq \frac{\Delta(i)}{K^2}(K^2 + \alpha K + \alpha^2). \end{aligned}$$

Thus,

$$\Delta(i) < \frac{K^2}{K^2 + \alpha K + \alpha^2} x < x.$$

As

$$\Delta^* = \frac{1+\alpha^2}{\alpha - 1/K} x > x$$

one thus has $\Delta(i) < \Delta^*$, proving the result. ■

If $|x(0)| \geq |x|$ and $x(0)x > 0$ then replace $|x|$ in (7.43) by $|x - x(0)|$.

Observe, (7.43) is easy to satisfy as long as $x \neq 0$. Simply choose $\Delta(0)$ sufficiently small. Beyond this, all that is required is that $x(0) = 0$. The analysis above indicates that the algorithm will tolerate modest violations of this last requirement.

8. Simulations

In this section we show simulations results that compare the behavior of a remotely controlled network where the ADM algorithm is used to encode and decode a signal. The objective is to remotely control a plant. We compare the behavior of the classical ADM ($\alpha = 1$) to that of the modified algorithm that we propose in this paper. Specifically we have the setting of Fig. 4. Specifically, we have a stable plant with transfer function $1/(z+1)$. The compensator $1/(z-1)$ is used to achieve robust tracking of all constant signals, using the celebrated internal model principle. The controller and the control loop are at the receiver. A command signal, encoded and decoded by the ADM algorithm is transmitted from a remote location. Observe the loop gain here is unstable. The command signal encoded has a constant value of one.

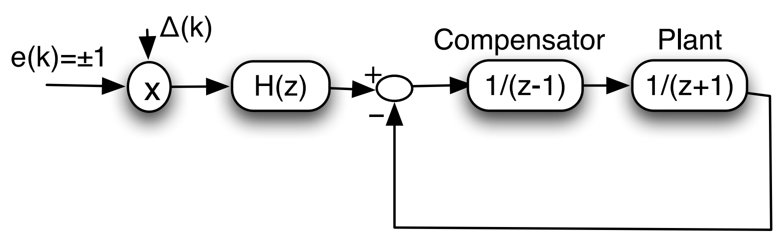

Figure 4. Block diagram showing the receiver and a plant with transfer function $P(z) = 1/(z+1)$ and a compensator with transfer function $C(z) = 1/(z-1)$.

Fig. 5 demonstrates performance of Jayant's ADM with no forgetting factor and $K = 1.01$. Observe the large oscillations that ensue. The error amplitude is in fact as large as the value the plant is supposed to track.

Fig. 6 demonstrates performance of ADM with a forgetting factor of .999 and $K = 1.01$. Observe at steady state the oscillations are negligible, despite the fact that α is so close to one.

These simulations confirm the proved results in this manuscript.

9. Conclusion

Motivated by networked control applications we have studied the behavior of an ADM algorithm with a forgetting factor, when the coded signal is a constant. It is known that, in the absence of a forgetting factor, for generic initializations, convergence is not possible, and 4-cycles must arise. We have shown by example that these 4-cycles could result in large coding errors.

We have analyzed our proposed modification involving the inclusion of a forgetting factor. We have shown that in such a case arbitrarily small coding errors can be achieved under mild assumptions through suitable design selections of the forgetting factor. Areas of further work

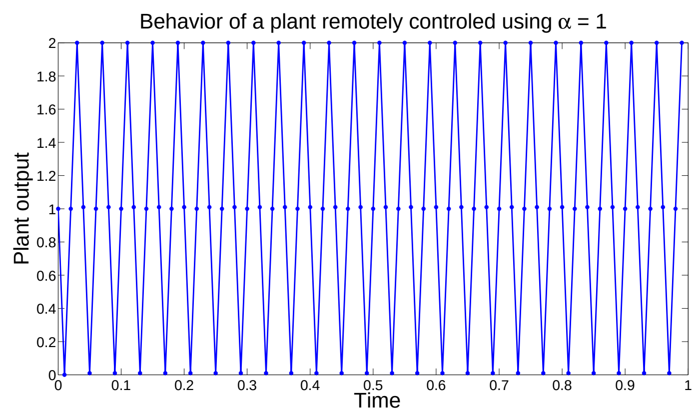

Figure 5. This plot shows the output of a plant with transfer function $P(z) = \frac{1}{z^2-1}$ when the signals are encoded using the modified ADM. Here we choose $\alpha = 1$ and $K = 1.01$. The output of the plant has large oscillations.

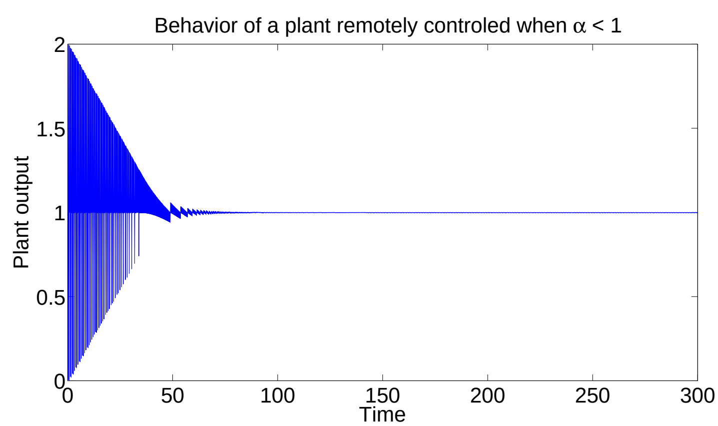

Figure 6. This plot shows the output of a plant with transfer function $P(z) = \frac{1}{z-1}$ when the signals are encoded using the modified ADM. Here we choose $\alpha = 0.999$ and $K = 1.01$, giving $K\epsilon << 1$. The output of the plant approaches the desired output with negligible oscillations.

include studying this ADM with non-constant signals with essential bandwidth well below the sampling rate, by using a singular perturbation method. It is also useful to look directly at stabilizability issues in a remote control setting.

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