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Modeling and Analysis of Social Network Dynamics: Propagation, Learning and Structural Balance

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by

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Modeling and Analysis of Social Network Dynamics:

Propagation, Learning and Structural Balance

To my family and my friends.

To everyone who ever came into my life.

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[2] W. Mei and F. Bullo, "Modeling and Analysis of Competitive Propagation with Social Conversion," In *IEEE Conference on Decision and Control*, Los Angeles, CA, USA, pp. 6203-6208, December 2014, doi: 10.1109/CDC.2014.7040361.

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Abstract

Modeling and Analysis of Social Network Dynamics:
Propagation, Learning and Structural Balance

by

Wenjun Mei

Network is a natural physical model of social systems and an important tool to understand various dynamical phenomena in human groups and societies. Network dynamics is a powerful theoretical approach to study how local interactions among individuals lead to certain macroscopic phenomena, and the role of network structure in such dynamical processes. In this thesis, we model and analyze the following two aspects of social network dynamics: dynamics on networks and dynamics of networks. The former means the evolution of individual states via social interactions, while the latter refers to the evolution of the social relations themselves.

Regarding the dynamics on social networks, we focus on the modeling and analysis of network propagation processes. Firstly, we review a class of deterministic nonlinear models for the propagation of infectious diseases over contact networks. For each model setting, we provide a comprehensive nonlinear analysis including both known and novel results. Secondly, we propose a class of stochastic propagation models for multiple competing products over a social network, and study their mean-field approximations. Two types of games based on the mean-field competitive propagation models are proposed and the quality-seeding trade-off is investigated. Finally, we apply the general idea of social influence to an engineering sensors system and study the sequential decision aggregation with social pressure.

For the dynamics of social networks, we study the evolution of the interpersonal appraisal networks and its emergent collective behavior. Firstly, we proposes models of learning processes in teams of individuals collectively executing a sequence of tasks. The closely-related proposed models have increasing complexity, starting with a centralized assignment and learning model, and finishing with a social model of interpersonal appraisal, assignment, learning and influence. Theoretical analysis shows how rational optimal behavior arises along the task sequence, while conditions of suboptimality are investigated numerically. Secondly, we propose two discrete-time dynamical systems that explain how an appraisal network evolves towards social balance from an initially unbalanced configuration via homophily and influence mechanisms respectively.

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General Introduction

Overview of research on social networks

Network is a natural model of the inner structure and mutual interactions in social systems, and a powerful tool to characterizes and understand various dynamical phenomena in human groups and societies. A network is a structure made up of a set of nodes and their mutual connections. There are three basic elements in a network: nodes, nodal states, and network links. Individuals or social entities (collectively referred to as individuals in the rest of this thesis unless specified) are modeled as nodes on the social networks, while the nodal states characterize certain individual states or attributes of interest. Individuals on social networks are connected via social links, which define the type and strength of certain social relations or interactions. In social networks, local interactions among individuals via various types of mutual connections often result in rich and complicated global phenomena, such as the consensus of opinions, the propagation of diseases or innovations, the formation of fractions, and the evolution of individuals social powers. Network dynamics is a powerful theoretical framework to study how the microscopic and local interactions among individuals lead to the emergence of certain macroscopic behavior, and to investigate the role of network structure in such dynamical processes.

On account of the recent progress in multi-agent systems, network science, and data mining, the last decades have witnessed a rapid development of the research on social networks, spanning the following topics:

(i. static theory: the global structures, statistical properties and nodal attributes of social networks, e.g., the network diameter, the clustering feature, the degree distribution, the nodal centrality measure, and the detection of communities;

(ii. dynamics on social networks: dynamical processes occurring on social networks, which can be equivalently interpreted as the evolution of individual states, e.g., propagation and information aggregation, averaging systems, network flows;

(iii. dynamics of social networks: the evolution of social network themselves, i.e., the dynamics of the nodes interconnections. Examples are the co-evolution of interpersonal influence and appraisals, the emergence of structural balance in social networks, and the network formation games.

Preliminaries: algebraic graph theory

This thesis focuses on the modeling and analysis of dynamical processes both on and of social networks, collectively referred to as social network dynamics. In this subsection, we briefly introduce the algebraic graph theory as the mathematical formalization of social networks. Notations frequently used in this thesis are listed in Table 0.1.

In algebraic graph theory, networks are modeled as graphs. In the rest of this thesis, we consider these two terms as interchangeable. A graph is a triple $G(V, E, A)$. Here set V denotes the set of nodes and $V = \{1, \dots, n\}$ for any network of n nodes. Let $E \in V \times V$ be the set of links defined as follows: $(i, j) \in E$ if and only if there exists a link in the network from node i to node j. A link from node i to itself is referred to as a *self loop*. Graphs in which the links are all undirected can be considered as the graphs in which all the links are directed but bilateral. Therefore, in this thesis, we assume all the network links to be directed, unless specified. The third element of the triple (V, E, A) is a matrix $A = (a_{ij})_{n\times n}$, referred to as the *adjacency matrix* associated with the graph G. Usually A is assumed to be entry-wise nonnegative. For any $i, j \in V$, $a_{ij} > 0$ if $(i, j) \in E$, and $a_{ij} = 0$ if $(i, j) \notin E$. In the meanwhile, the magnitude of a_{ij} represents the weight of the directed link (i, j).

A *path* from node i_0 to node i_l with length l is an ordered sequence of distinct nodes

Table 0.1: Notations frequently used in this thesis

$\succ$ ($\prec$ resp.)	entry-wise greater than (less than resp.)
$\succeq$ ($\preceq$ resp.)	entry-wise no less than (no greater than resp.)
$\mathbb{1}_n$ ($\mathbb{0}_n$ resp.)	n-dimension column vector with all entries equal to 1 (0 resp.)
$\mathbb{1}_{n\times m}$ ($\mathbb{0}_{n\times m}$ resp.)	$n \times m$ matrix with all entries equal to 1 (0 resp.)
$\mathbb{N}$	set of natural numbers, i.e., $\{0, 1, 2, 3, \dots\}$
$\mathbb{R}^n$	n-dimension Euclidean space
$\lvert\lambda\rvert$	the magnitude of complex number λ
Δ_n	the n-simplex $\{y \in \mathbb{R}^n \,\vert\, y^\top \mathbb{1}_n = 1, y \succeq \mathbb{0}_n\}$
$\mathrm{int}(\Delta_n)$	the relative interior of the n-simplex, i.e., $\mathrm{int}(\Delta_n) = \{y \in \mathbb{R}^n \,\vert\, {}^\top \mathbb{1}_n = 1, y \succ \mathbb{0}_n\}$
$\rho(A)$	the spectral radius of matrix A, i.e., $\rho(A) = \max\{\lvert\lambda\rvert \,\vert\, \lambda \text{ is an eigenvalue of } A\}$
$\boldsymbol{v}_{\mathrm{left}}(A)$	the left dominant eigenvector of the non-negative and irreducible matrix A, i.e., the entry-wise positive left eigenvector associated with the eigenvalue equal to As spectral radius and satisfying $\boldsymbol{v}_{\mathrm{left}}(A)^\top \mathbb{1}_n$
$G(A)$	the directed and weighted graph associated with the adjacency matrix $A \in \mathbb{R}^{n\times n}$.

$\{i_0, i_1, \ldots, i_l\}$ in which $(i_k, i_{k+1}) \in E$ for any $k \in \{0, \ldots, l-1\}$. A graph has a *globally reachable node* if there exists a node i such that, for any $j \in V$, there exists a path from j to i. That is, every node in the graph can reach node i via at least one path in the graph. A path from node i to itself, with no repeating node except node i, is referred to as a *cycle* and the number of nodes involved is called the length of the cycle. A self loop is a cycle with length 1. The greatest common divisor of the lengths of all the cycles in a graph is defined as the *period* of the graph. A graph with period equal to 1 is called aperiodic. Apparently, a graph with self loops is aperiodic. Actually, the associated adjacency matrix A contains all the information of a graph. For simplicity, we adopt the notation $G(A)$ to represent the graph with A as its adjacency matrix. There are some interesting equivalence relations between the connectivity properties of the graph $G(A)$ and the algebraic properties of the adjacency matrix A, stated in the following lemma.

Lemma 0.0.1 (Equivalence relations between graphs and adjacency matrices) *Consider a graph $G(A)$ associated with the adjacency matrix $A \in \mathbb{R}^{n\times n}$. The following statements hold:*

(i. graph $G(A)$ is strongly connected if and only if matrix A is irreducible, that is, matrix A can not be transformed into a block upper-triangular form by any simultaneous row and column permutation, or, equivalently, matrix A satisfies $\sum_{k=0}^{n-1} A^k \succ \mathbb{0}_{n\times n}$;

(ii. graph $G(A)$ is strongly connected and aperiodic if and only if matrix A is primitive, which means that there exists some $k \in \mathbb{N}$ such that $A^k \succ \mathbb{0}_{n\times n}$.

One can easily deduce from the lemma above that, for a strongly connected graph with at least one self loop, its associated adjacency matrix is primitive. Now we present the Perron-Frobenius theorem, which has been widely used in the study of network dynamics. We refer to Section 2.3 of the textbook by Bullo [1] for more detailed discussion.

Theorem 0.0.2 (Perron-Frobenius theorem) *Consider an entry-wise nonnegative matrix $A \in \mathbb{R}^{n \times n}$ with $n \geq 2$. The following statements hold:*

(i. there exists a real eigenvalue $\lambda \geq |\mu| \geq 0$ for all other eigenvalues μ;

(ii. the right and left eigenvectors v and w of λ can be selected non-negative.

If, additionally, A is irreducible, then

(iii. the eigenvalue λ is strictly positive and simple;

(iv. the right and left eigenvectors v and w of λ are unique and positive, up to scaling.

If, additionally, A is primitive, then

(v. the eigenvalue λ satisfies $\lambda > |\mu|$ for all other eigenvalues μ.

The following theorem states the limit property of the powers of primitive matrices.

Theorem 0.0.3 (Powers of primitive matrices) *Consider a primitive matrix $A \in \mathbb{R}^{n \times n}$. Let v and w be respectively the entry-wise positive left and right eigenvalues associated with the largest eigenvalues of A in magnitude, i.e., $\rho(A)$. Suppose v and w are normalized such that $v^\top w = 1$. Then we have*

$$\lim_{k \to \infty} \frac{A^k}{\rho(A)^k} = vw^\top.$$

Any entry-wise non-negative matrix $A \in \mathbb{R}^{n \times n}$ satisfying $A\mathbb{1}_n = \mathbb{1}_n$ ($\mathbb{1}^\top A = \mathbb{1}^\top$ resp.) is referred to as a row-stochastic (column-stochastic resp.) matrix. A matrix that is both row-stochastic and column-stochastic is called doubly-stochastic. For any row-stochastic matrix A, $\rho(A) = 1$ is the eigenvalue of A with the largest magnitude, and is associated with an right eigenvector of the form $\alpha\mathbb{1}_n$, where α is a scalar.

Contributions and Organizations

This thesis consists of three parts. Part I is on the modeling and analysis of dynamics on social networks, to be specific, the propagation dynamics on social networks. Part II discusses dynamics of social networks, particularly, the interpersonal appraisal networks. Part III is a brief discussion of some potentially interesting future research directions in social network dynamics. We state the main contents and contributions of each chapter in Part I and part II as follows.

Chapter 1: We review a class of deterministic nonlinear models for the propagation of infectious diseases over contact networks with strongly-connected topologies. We consider network models for susceptible-infected (SI), susceptible-infected-susceptible (SIS), and susceptible-infected-recovered (SIR) settings. In each setting, we provide a comprehensive nonlinear analysis of equilibria, stability properties, convergence, monotonicity, positivity, and threshold conditions, including both known and novel results.

Chapter 2: We propose a class of propagation models for multiple competing products over a social network. We consider two propagation mechanisms: social conversion and self conversion, corresponding, respectively, to endogenous and exogenous factors. According to the chronological order of social and self conversions, we develop two Markov-chain models and, based on the independence approximation, we approximate them with two corresponding difference equations systems. Both theoretical and numerical study are conducted for both mean-field systems. Finally, we propose two classes of games based on the mean-field competitive propagation models and characterize their Nash equilibria.

Chapter 3: We apply the idea of social influence to an engineering system, in which local sensors perform binary sequential hypothesis testing and a fusion center collects the local decisions and reaches a global decision according to some threshold rule. The local sensors decisions are influenced by those who have already made their decisions, which characterizes the role of social pressure. We establish the accuracy and expected decision time of the fusion center in systems with finite local sensors. In systems with infinitely many local sensors, we analyze the limit accuracy and expected decision time of some specific threshold rules by means of a mean-field analysis.

Chapter 4: We propose models of learning processes in teams of individuals who collectively execute a sequence of tasks and whose actions are determined by individual skill levels and networks of interpersonal appraisals and influence. The closely-related proposed models have increasing complexity, starting with a centralized manager-based assignment and learning model, and finishing with a social model of interpersonal appraisal, assignments, learning, and influences. We show how rational optimal behavior arises along the task sequence for each model, and discuss conditions of suboptimality. Our models are grounded in replicator dynamics from evolutionary games, influence networks from mathematical sociology, and transactive memory systems from organization science.

Chapter 5: We propose two discrete-time dynamical systems that explain how an appraisal network evolves towards social balance from an initially unbalanced configuration. These two models are based on two different socio-psychological mechanisms respectively: the homophily mechanism and the influence mechanism. Our main theoretical contribution is a comprehensive analysis for both models in three steps. First, we establish the well-posedness and bounded evolution of the interpersonal appraisals. Second, we characterize the set of equilibrium points as follows: for both models, each equilibrium network is composed by an arbitrary number of complete subgraphs satisfying structural balance. Third, under a technical condition, we establish convergence of the appraisal network to a final equilibrium network satisfying structural balance. In addition to our theoretical analysis, we provide numerical evidence that our technical condition for convergence holds for generic initial conditions in both models. Finally, adopting the homophily-based model, we present numerical results on the mediation and globalization of local conflicts, the competition for allies, and the asymptotic formation of a single versus two factions.

In addition, at the beginning of each part, a general and brief overview is provided.

Part I

Propagation Dynamics on Social Networks

Overview: dynamics on social networks

The two most widely studied classes of dynamics on social networks are the opinion dynamics and the network propagation dynamics. Part I of this thesis focuses on the latter but the opinion dynamics has some connections with what to be discussed in Part II. Therefore we provide brief overviews for both classes of dynamics.

Opinion dynamics studies the evolution of individuals' opinions driven by the social influence of other individuals in the network. There are mainly two types of opinion dynamics models: the averaging-based models and the voter models. In the averaging-based models, individuals' opinions are denoted by real numbers and change continuously. Individuals update their opinions by computing certain convex combination of their own opinions, the opinions of their social neighbors, and, potentially, their initial opinions or some external inputs. Widely-studied models include the classic DeGroot model [2], the Friedkin-Johnsen model [3], and the bounded-confidence model [4]. The voter models assume that individual opinions take their values from a discrete (usually binary) set. The switching of opinions is driven by the social pressure from individuals' neighbors via some stochastic processes, see [5, 6, 7]. We refer to the surveys [8] and [9] for more detailed literature review.

There are also two main approaches to study the network propagation processes. The first approach is to build the dynamical models on random graphs. This approach is based on the following observation: although it is almost impossible to obtain the detailed and well-quantified information of every local connection in the large-scale social networks, social network as an entirety does exhibit some prominent global characteristics, such as the small-world feature, the scale-free degree distribution, and the clustering property. Therefore, dynamical models based only on those estimable global network characteristics should lead to some theoretical predictions that are also data-verifiable. Examples of network propagation models following this approach include the graph percolation model [10, 11] and the degree-based model [12, Chapter 17]. The second approach is to first assume that the network has an arbitrary topology with all the connections well-quantified and known, and then try to derive the theoretical results that do not depend on all the local details of the network. The advantage of such approach is that, more sophisticated and realistic dynamical processes can be modeled and understood based on the well-established mathematical tools in dynamical systems, control theory, algebraic graph theory, and matrix analysis. Examples include the network epidemic spreading models [13, 14] and the linear threshold model [15, 16]. Regarding the network propagation models, this thesis focuses on the second approach.

Chapter 1

Deterministic Epidemic Propagation over Networks

1.1 Introduction

Problem motivation and description

Propagation phenomena appear in numerous disciplines. Examples include the spread of infectious diseases in contact networks, the transmission of information in communication networks, the diffusion of innovations in competitive economic networks, cascading failures in power grids, and the spreading of wild-fires in forests.

One important class of models of propagation phenomena are scalar deterministic models. These models have been widely studied, e.g., see the survey [17]. These models qualitatively capture some dynamic features, including phase transitions and asymptotic states. However, shortcomings of scalar models are also prominent: for example, scalar models are typically based on the assumption that individuals in the population have the same chances of interacting with each other. This assumption overlooks the internal structure of the network over which the propagation occurs, as well as the heterogeneity of individuals in the network. Both these aspects play critical roles in shaping the dynamical behavior of the propagation processes.

In a general formulation, propagation is a dynamical process on a complex network. Each network node has a state taking value in a discrete set and state changes are influenced by the nodes' neighbors in the network. Many relevant research questions arise naturally, including: how to model the local dynamics at each node, how to identify model parameters, how to estimate the state of such a dynamical system, and how to analyze the system transient and asymptotic properties.

Various types of models have been proposed to describe propagation processes over complex networks; one key distinguishing feature of these models is whether the propagation dynamics is assumed to be stochastic or deterministic. Deterministic network epidemic models were originally proposed in the late 1970's in the seminal works [13, 18]. These models are of great research value, as attested by the large literature focusing on them (see below). Moreover, they can be considered as approximations of certain Markov-chain models, e.g., see [19].

In this chapter, we review three key continuous-time deterministic models for epidemic propagation over networks. Depending upon the nodal dynamics, i.e., the disease propagation behavior, deterministic epidemic propagation models are classified as: the *Susceptible-Infected* (SI) model, the *Susceptible-Infected-Susceptible* (SIS) model and the *Susceptible-Infected-Recovered* (SIR) model; basic representations of these models are illustrated in Figure 1.1. In this work we focus on transient and asymptotic behavior of these three continuous-time dynamical models over networks. It is our key objective to relate the structure of the network to the function of the network (i.e., the transient and asymptotic behavior of the propagation phenomenon).

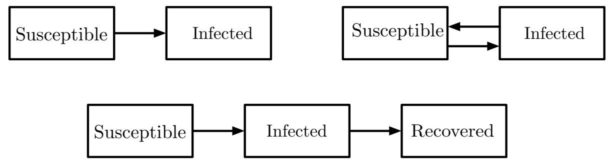


Figure 1.1: Three basic models of infectious diseases: SI, SIS and SIR.

Literature review on deterministic network epidemic models

The literature on epidemic propagation is exceedingly vast. This chapter focuses on deterministic models over networks and on their dynamical behavior. Accordingly, this subsection reviews the literature on deterministic epidemic models. Unless specified, the works and results reviewed in what follows are all for the deterministic models. For readers interested in Markov-chain models and in the mean-field approximation method, we refer to [19, 20, 21, 14] and [1, Chapter 17]. (Note that Markov-chain network epidemic models and their deterministic approximating models are different in some of the dynamical properties, such as the epidemic threshold and the asymptotic behavior.)

The dynamics of several classic scalar epidemic models, i.e., the population models without network structure, are surveyed in detail by Hethcote [17]. Among the different metrics discussed, identifying the *effective reproduction number* R is of particular interest to researchers; R is the expected number of individuals that a randomly infected individual can infect during its infection period. In these scalar models, whether an epidemic outbreak occurs or the disease dies down depends upon whether $R > 1$ or $R < 1$, i.e., upon whether the system is above or below the so-called *epidemic threshold*. Here by epidemic outbreak we mean an exponential growth of the fraction of the infected population for small time. The *basic reproduction number* R_0 is the effective reproduction number in a fully-healthy susceptible population. In what follows we focus our review on deterministic network models.

The earliest work on the (continuous-time heterogeneous) SIS model on networks is [13]. This work proposes an n-dimensional model on a contact network and analyzes the system's asymptotic behavior. This article proposes a rigorous analysis of the threshold for the epidemic outbreak, which depends on both the disease parameters and the spectral radius of the contact network. For the case when the basic reproduction number is above the epidemic threshold, this paper establishes the existence and uniqueness of a nonzero steady-state infection probability, called the endemic state. In what follows we refer to the model by Lajmanovich et al. [13] as the *network SIS model*; it is also known as the multi-group or multi-population SIS model.

Allen [22] proposes and analyzes a discrete-time network SIS model. This work appears to be the first to revisit and formally reproduce, for the discrete-time case, the earlier results proposed in [13]; see also the later work [23]. This work confirms the existence of an epidemic threshold, as a function of the spectral radius of the contact network. Further recent results on the discrete-time model are obtained by Ahn et al. [24] and Azizan Ruhi et al. [20].

Van Mieghem et al. [25] argue that the (continuous-time) network SIS model is in fact the mean-field approximation of the original Markov-chain SIS model of exponential dimension; this claim is rigorously proven in [19]. Van Mieghem et al. [25] refer to this model as the intertwined SIS model and write the endemic state as a continued fraction.

The works [26] and [27] discuss the continuous-time network SIS model in a more modern language. Fall et al. [26] refer to this model as the n-group SIS model and apply Lyapunov techniques and Metzler matrix theory to establish existence, uniqueness, and stability of the equilibrium points below and above the epidemic threshold. Khanafer et al. [27] use positive system theory in their analysis and extend the existence, uniqueness, and stability results to the setting of weakly connected digraphs.

Numerous extensions of these basic results on the network SIS model and other related works have appeared over the years. For example, the estimation of the epidemic threshold in contact networks with power-law degree distributions has been studied both by mathematically rigorous analysis, see [28], and by numerical simulation, see [29]. The deterministic network SIS models without mean-field approximation and with second-order mean-field approximation have been analyzed in [30] and [31], respectively.

An early work by Hethcote [18] proposes a general multi-group SIR model with birth, death, immunization, and de-immunization. The epidemic threshold and the equilibria below/above the threshold are characterized. For the simplified model without birth/death and de-immunization, Hethcote [18] proves that the system converges asymptotically to an all-healthy state. Guo et al. [32] consider a generalized network SIR model with vital dynamics, that is, with birth and death. They characterize the basic reproduction number and, through a careful Lyapunov analysis, show the existence and global asymptotic stability of an endemic state above the threshold. Youssef et al. [33] study a special case of the network SIR model under the name of individual-based SIR model over undirected networks. Through a simulation-based analysis, the epidemic threshold is given as a function of the spectral radius of the network.

There are also some extensions and related studies regarding the network SIR model. Sharkey [34] investigates the deterministic network SIR model without mean-field approximation. Castellano et al. [29] point out that the (mean-field) network SIR predicts a vanishing threshold for a certain class of power-law distributed networks, which is inconsistent with the corresponding stochastic SIR model. Sharkey et al. [35] show that, different from the network SIR model with mean-field approximation, the so-called pair-based approach gives an exact description of the stochastic SIR process for the tree topology.

To the best of our knowledge, no works have comprehensively characterized the properties of the network SI model.

We conclude by mentioning other surveys and textbook treatments. In [36], the stability of equilibria for the SEIR model is reviewed through Lyapunov and graph theory. The additional state E represents the exposed population, i.e., the individuals who are infected but not infectious. Various heterogeneous epidemic models are reviewed in [12, Chapter 17], [37, Chapter 21], and [38, Chapter 9]. The recent survey by Nowzari et al. [14] presents various epidemic models and addresses many solved and open problems in the control of epidemic spreading.

Statement of Contribution

This chapter reviews, in a comprehensive and coherent manner, deterministic models and dynamical behavior of SI, SIS and SIR epidemic phenomena over networks. This review includes known results from the literature as well as several novel results. We discuss SI, SIS and SIR models in three subsequent corresponding sections. Each section starts by reviewing the well-known results for the corresponding scalar models; these are the models in which variables represent an entire ?well-mixed? population or nodes of an all-to-all unweighted graph. The core of each section is a discussion about multi-group network models. We provide a tutorial treatment with comprehensive statements and proofs for the deterministic network SI, SIS and SIR models.

We first analyze the network SI model. We analyze its asymptotic convergence, positivity of infection probabilities, initial and asymptotic growth rates, and the stability of equilibria. We show that in the network SI model, the system does not display a threshold and, with the exception of the trivial no-epidemics equilibrium, all the trajectories converge to the full contagion state. While these results are not technically difficult, they are novel here in the sense that, to the best of our knowledge, the properties of the network SI model have never before been formally characterized.

Next we focus on the network SIS model. Our presentation includes known results from [13] (see also [26, 27]) regarding the epidemic threshold, the system's behavior below the epidemic threshold, the existence and uniqueness of the endemic state for systems above the epidemic threshold, and the asymptotic stability of the endemic state. Moreover, we provide a novel provably-correct iterative algorithm for computing the fraction of infected individuals converging to the endemic state. This algorithm also provides an alternative proof for the existence and uniqueness of the endemic state for systems above the epidemic threshold. We argue that this alternative proof is more concise that the those proposed in the previous works [13, 26, 27]. In addition, we present novel Taylor expansions for the endemic state near the epidemic threshold and in the limit of high infection rates. These novel Taylor expansions shed light on these previously poorly-understood regimes. Finally, we show that the spread of infection takes place instantaneously upon infecting at least one node in the network.

Finally, for the network SIR model, we review some known results on the monotonicity of the individuals' susceptible probabilities and the system's asymptotic behavior from [18]. More importantly, we provide the several novel results: We present novel transient behavior and system properties. First, we propose new threshold conditions above which the epidemic grows initially, and below which it exponentially dies down. The initial rate of growth above the threshold is given in terms of network characteristics, initial conditions, and infection parameters. Moreover, we show that our proposed weighted average of the infected population, obtained by the entries of dominant eigenvector of an irreducible quasi-positive matrix, captures information regarding the distribution of infection in the system. We also establish positivity of the infection probabilities. Finally, we provide a novel iterative algorithm to compute the asymptotic state of the network SIR model, with any arbitrary initial condition. For the iterative algorithm, the existence and uniqueness of the fixed point, and the convergence of the iteration are rigorously proved. Our results are analogous to the scalar SIR model properties and are valid for any arbitrary network topologies. In comparison with [33], our treatment builds on their numerical results but our result is more general in that it does not depend upon specific initial conditions and graph topologies, and establishes numerous properties, including the novel characterization of epidemic threshold.

Organization

Section 1.2 introduces our model set-up and some preliminary notations. The SI, SIS and SIR models are presented, respectively, in Sections 1.3, 1.4, and 1.5. Section 6 is the conclusion.

1.2 Model Set-Up and Notations

For the scalar models, we use the notation $x(t)$ ($s(t)$ and $r(t)$ resp.) for the fraction of infected (susceptible and recovered resp.) individuals in the population at time t. The rest of this section is about the notations and basic model set-up for the network epidemic model.

a) Contact Network: The epidemics are assumed to propagate over a weighted digraph $G = (V, E)$, where $V = \{1, \ldots, n\}$ and E is the set of directed links. Nodes of G can be interpreted as either single individuals in the contact network or as homogeneous populations of individuals at each location/node in the contact network. $A = (a_{ij})_{n\times n}$ denotes the adjacency matrix associated with G. For any i, $j \in V$, a_{ij} characterizes the contact strength from node j to node i. For $(i, j) \in E$, $a_{ij} > 0$ and for $(i, j) \notin E$, $a_{ij} = 0$. In this chapter, G is assumed to be strongly connected.

b) Node States and Probabilities: For different epidemic propagation models, the set of possible node states are distinct. For network SI or SIS models, each node can be in either the "susceptible" or "infected" state, while in the network SIR model, there is an additional possible node state: "recovered." For a graph in which the nodes are single individuals, let $s_i(t)$ ($x_i(t)$ and $r_i(t)$ resp.) be the probability that individual i is in the susceptible (infected and recovered resp.) state at time t. Alternatively, if the nodes are considered to be the populations, then $s_i(t)$ ($x_i(t)$ and $r_i(t)$ resp.) is interpreted as the fraction of susceptible (infected and recovered resp.) individuals in population i. In this chapter, without loss of generality, we adopt the interpretation of nodes as single individuals.

c) Frequently Used Notations: The symbol $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. The symbol ϕ denotes the empty set. For any two vectors $x, y \in \mathbb{R}^n$, we write

$$
\begin{aligned}
x \prec y, &\qquad \text{if } x_i < y_i \text{ for all } i \in \{1, \dots, n\}, \\
x \preceq y, &\qquad \text{if } x_i \leq y_i \text{ for all } i \in \{1, \dots, n\}, \text{ and} \\
x < y, &\qquad \text{if } x \leq y \text{ and } x \neq y.
\end{aligned}
$$

Let I_n denote the $n \times n$ identity matrix. Given $x = [x_1, \dots, x_n]^\top \in \mathbb{R}^n$, let $\operatorname{diag}(x)$ denote the diagonal matrix whose diagonal entries are $x_1, \dots, x_n$. For an irreducible nonnegative matrix A, let $\lambda_{\max}(A)$ denote the dominant eigenvalue of A that is equal to the spectral radius $\rho(A)$. Moreover, we let $v_{\max}(A)$ ($u_{\max}(A)$ resp.) denote the corresponding entry-wise strictly positive left (right resp.) eigenvector associated with $\lambda_{\max}(A)$, normalized to satisfy $\mathbb{1}_n^\top v_{\max}(A) = 1$ (resp. $\mathbb{1}_n^\top u_{\max}(A) = 1$). The Perron-Frobenius Theorem for irreducible matrices guarantees that $\lambda_{\max}(A)$, $v_{\max}(A)$ and $u_{\max}$ are well defined and unique. Where not ambiguous, we will drop the (A) argument and, for example, write

$$
v_{\max}^\top A = \lambda_{\max} v_{\max}^\top \quad \text{and} \quad A u_{\max} = \lambda_{\max} u_{\max},
$$

with $v_{\max} \succ \mathbb{0}_n$ and $\mathbb{1}_n^\top v_{\max} = 1$; $u_{\max} \succ \mathbb{0}_n$ and $\mathbb{1}_n^\top u_{\max} = 1$.

1.3 Susceptible-Infected Model

In this section, we first review the classic scalar susceptible-infected (SI) model, and then present and characterize the network SI model.

1.3.1 Scalar SI model

The scalar SI model assumes that the growth rate of the fraction of the infected individuals is proportional to the fraction of the susceptible individuals, multiplied by a so-called *infection rate* $\beta > 0$. The model is given by

$$\dot{x}(t) = \beta s(t) x(t) = \beta\big(1 - x(t)\big)x(t). \tag{1.1}$$

This is the well-established logistic equation. The following results can be found for example in the textbook [39].

Lemma 1.3.1 (Dynamical behavior of the SI model) *Consider the scalar SI model* (1.1) *with* $\beta > 0$. *The solution from initial condition* $x(0) = x_0 \in [0, 1]$ *is*

$$x(t) = \frac{x_0 e^{\beta t}}{1 - x_0 + x_0 e^{\beta t}}. \tag{1.2}$$

All initial conditions $0 < x_0 < 1$ *result in the solution* $x(t)$ *being monotonically increasing and converging to the unique equilibrium* 1 *as* $t \to \infty$.

Solutions to equation (1.1) with different initial conditions are plotted in Figure 1.2. The SI model (1.1) results in an evolution akin to a logistic curve, and is also called the logistic equation for population growth.

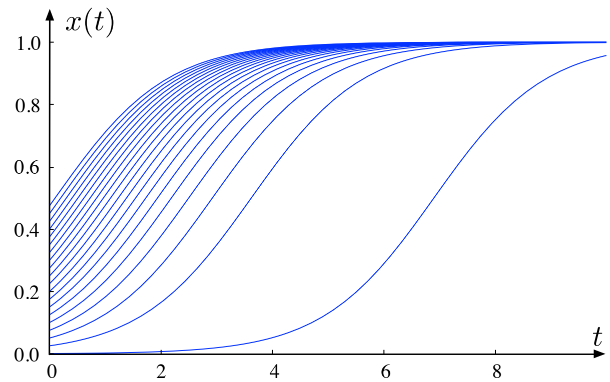


Figure 1.2: Evolution of the (lumped deterministic) SI model ($\beta = 1$) from small initial fraction of infected individuals.

1.3.2 Network SI model

The network SI model on a weighted digraph with the adjacency matrix $A \in \mathbb{R}^{n\times n}_{\geq 0}$ is given by

$$\dot{x}_i(t) = \beta\big(1 - x_i(t)\big)\sum_{j=1}^{n} a_{ij}x_j(t), \tag{1.3}$$

or, in equivalent vector form,

$$\dot{x}(t) = \beta\Big(I_n - \mathrm{diag}\big(x(t)\big)\Big)Ax(t), \tag{1.4}$$

where $\beta > 0$ is the infection rate. Alternatively, in terms of the fractions of susceptible individuals $s(t) = \mathbb{1}_n - x(t)$, the network SI model is

$$\dot{s}(t) = -\beta\,\mathrm{diag}(s(t))A(\mathbb{1}_n - s(t)). \tag{1.5}$$

The network SI model is a particular case of the widely-studied network SIS model, which is to be discussed in the next section. The dynamical properties of the network SI model are not difficult to analyze, but, to the best of our knowledge, have not been formally presented in any previous literature. We present the results on the transient and asymptotic behavior of the network SI model, as well as the proof, in the following theorem.

Theorem 1.3.2 (Dynamical behavior of network SI model) *Consider the network SI model* (1.4) *with* $\beta > 0$. *For strongly connected graph with adjacency matrix* A, *the following statements hold:*

(i. if $x(0), s(0) \in [0,1]^n$, *then* $x(t), s(t) \in [0,1]^n$ *for all* $t > 0$. *Moreover,* $x(t)$ *is monotonically non-decreasing (here by monotonically non-decreasing we mean* $x(t_1) \preceq x(t_2)$ *for all* $t_1 \leq t_2$*). Finally, if* $x(0) > \mathbb{0}_n$, *then* $x(t) \succ \mathbb{0}_n$ *for all* $t > 0$*;*

(ii. the model (1.4) *has two equilibrium points:* $\mathbb{0}_n$ *(no epidemic), and* $\mathbb{1}_n$ *(full contagion);*

(a) the linearization of model (1.4) *about the equilibrium point* $\mathbb{0}_n$ *is* $\dot{x} = \beta A x$ *and it is exponentially unstable;*

(b) let $D = \operatorname{diag}(A\mathbb{1}_n)$ *be the degree matrix. The linearization of model* (1.5) *about the equilibrium* $\mathbb{0}_n$ *is* $\dot{s} = -\beta D s$ *and it is exponentially stable;*

(iii. each trajectory with initial condition $x(0) \neq \mathbb{0}_n$ *converges asymptotically to* $\mathbb{1}_n$, *that is, the epidemic spreads monotonically to the entire network.*

Proof:

(i) The fact that, if $x(0), s(0) \in [0,1]^n$, then $x(t), s(t) \in [0,1]^n$ for all $t > 0$ means that $[0,1]^n$ is an invariant set for the differential equation (1.4). This is the consequence of Nagumo's Theorem (see [40, Theorem 4.7]), since for any x belonging on the boundary of the set $[0,1]^n$, the vector $\beta\Big(I_n - \operatorname{diag}\big(x\big)\Big)Ax$ is either tangent, or points inside the set $[0,1]^n$.

Observe that the invariance of the set $[0,1]^n$ implies that $\dot{x}(t) \succeq \mathbb{0}_n$ and so $x(t_1) \preceq x(t_2)$ for all $t_1 \leq t_2$.

We want to prove now that,if $x(0) > 0_n$, then $x(t) \succ 0_n$ for all $t > 0$. If by contradiction there is $i \in \{1, \dots, n\}$ and $T > 0$ such that $x_i(T) = 0$, then the monotonicity of $x_i(t) = 0$ would imply that $x_i(t) = 0$ for all $t \in [0, T]$, which would yield $\dot{x}_i(t) = 0$ for all $t \in [0, T]$. By (1.3) this would imply that $x_j(t) = 0$ for all $t \in [0, T]$ for all j such that $a_{ij} > 0$. We could iterate this argument and using the irreducibility of A we would get the contradiction that $x(t) = 0$ for all $t \in [0, T]$ concluding in this way the proof of (i.
(ii) Regarding statement (ii, note that $\mathbb{0}_n$ and $\mathbb{1}_n$ are clearly equilibrium points. Let $\bar{x} \in [0, 1]^n$ be an equilibrium and assume that $\bar{x} \neq \mathbb{1}_n$. Then there is i such that $\bar{x}_i \neq 1$. Since $\beta\big(1 - \bar{x}_i\big) \sum_{j=1}^n a_{ij}\bar{x}_j = 0$, then $\sum_{j=1}^n a_{ij}\bar{x}_j = 0$ which implies that $\bar{x}_j = 0$ for all j such that $a_{ij} > 0$. By iterating this argument and using the irreducibility of A we get that $\bar{x} = 0$ concluding only $\mathbb{0}_n$ and $\mathbb{1}_n$ are equilibrium points. Statements (iia and (iib are obvious. Exponential stability of the linearization $\dot{s} = -\beta D s$ is obvious, and the Perron-Frobenius Theorem implies the existence of the unstable positive eigenvalue $\rho(A) > 0$ for the linearization $\dot{x} = \beta A x$.
(iii) Consider the function $V(x) = \mathbb{1}_n^\top(\mathbb{1}_n - x)$; this is a smooth function defined over the compact and forward invariant set $[0, 1]^n$ (see statement (i). Since $\dot{V} = -\beta\mathbb{1}_n^\top\big(I_n - \operatorname{diag}(x)\big)Ax$, we know that $\dot{V} \leq 0$ for all x and $\dot{V}(x) = 0$ if and only if $x \in \{\mathbb{0}_n, \mathbb{1}_n\}$. The LaSalle Invariance Principle implies that all trajectories with $x(0)$ converge asymptotically to either $\mathbb{1}_n$ or $\mathbb{0}_n$. Additionally, note that $0 \leq V(x) \leq n$ for all $x \in [0, 1]^n$, that $V(x) = 0$ if and only if $x = \mathbb{1}_n$ and that $V(x) = n$ if and only if $x = \mathbb{0}_n$. Therefore, all trajectories with $x(0) \neq \mathbb{0}_n$ converge asymptotically to $\mathbb{1}_n$. ■

In the next two paragraphs we present the "initial-time" ("final-time" resp.) approximation of the solution to the network SI model, i.e., the approximated solution to equation (1.4), or equation (1.5) equivalently, when t is sufficiently small (large resp.). These results are novel.

For the adjacency matrix A, there exists a non-singular matrix T such that $A =$

TJT^{-1}, where J is the Jordan normal form of A. Since A is non-negative and irreducible, according to Perron-Frobenius theorem, the first Jordan block $J_1 = (\lambda_{\max})_{1\times 1}$ and $\lambda_{\max} > Re(\lambda_i)$ for any other eigenvalue λ_i of A. Consider now the onset of an epidemic in a large population characterized by a small initial infection $x(0) = x_0$ much smaller than $\mathbb{1}_n$. The system evolution is approximated by $\dot{x} = \beta Ax$. This "initial-times" linear evolution satisfies

$$x(t) = \mathrm{e}^{\beta At}x(0) = T\mathrm{e}^{\beta Jt}T^{-1}x(0) = \mathrm{e}^{\beta\lambda_{\max}t}\big(T\mathbb{e}_1\mathbb{e}_1^\top T^{-1}x(0) + o(1)\big),$$

where $\mathbb{e}_1$ is the first standard basis vector in $\mathbb{R}^n$ and $o(1)$ denotes a time-varying vector that vanishes as $t \to +\infty$. Let u_1 denote the first column of T and let $v_1^\top$ denote the first row of T^{-1}. Since $AT = TJ$ and $T^{-1}A = JT^{-1}$, one can check that u_1 (v_1 resp.) is the right (left resp.) eigenvector of A associated with the eigenvalue $\lambda_{\max}$. Since $T^{-1}T = I_n$, we have $v_1^\top u_1 = 1$. therefore,

$$x(t) = \mathrm{e}^{\beta\lambda_{\max}t}\big(u_1v_1^\top x(0) + o(1)\big) = \mathrm{e}^{\beta\lambda_{\max}t}\Big(\frac{v_{\max}^\top x(0)}{v_{\max}^\top u_{\max}}u_{\max} + o(1)\Big). \tag{1.6}$$

That is, the epidemic initially experiences exponential growth with rate $\beta\lambda_{\max}$ and with distribution among the nodes given by the eigenvector $u_{\max}$.

Now suppose that at some time T, for all i we have that $x_i(T) = 1 - \epsilon_i$, where each ϵ_i is much smaller than 1. Then, for time $t > T$, the approximated system for $s(t)$ is given by:

$$\dot{s}_i(t) = -\beta d_i s_i(t) \quad \Longrightarrow \quad s_i(t) = \epsilon_i \mathrm{e}^{-\beta d_i(t-T)},$$

where, for any $i \in \{1, \ldots, n\}$, $d_i = \sum_{j=1}^n a_{ij}$ denotes the out-degree of node i in the network. From the discussion above, we conclude that the initial infection rate is proportional to the eigenvector centrality, and the final infection rate is proportional to the degree centrality.

1.4 Susceptible-Infected-Susceptible model

In this section we review the Susceptible-Infected-Susceptible (SIS) epidemic model. In addition to the existence of an infection process with rate $\beta > 0$, this model assumes that the infected individuals recover to the susceptible state at so-called *recovery rate* $\gamma > 0$.

1.4.1 Scalar SIS model

In the scalar SIS model, the population is divided into two fractions: the infected $x(t)$ and the susceptible $s(t)$, with $x(t) + s(t) = 1$, obeying the following dynamics:

$$\dot{x}(t) = \beta s(t)x(t) - \gamma x(t) = (\beta - \gamma - \beta x(t))x(t). \tag{1.7}$$

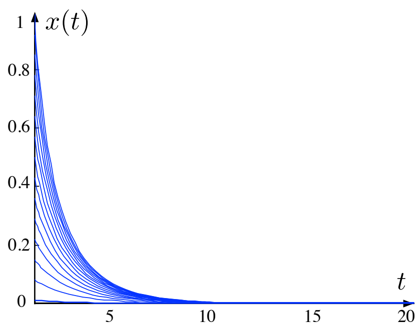
The dynamical behavior of system (1.7) given below can be found in [17].

Lemma 1.4.1 (Dynamical behavior of the SIS model) *For the SIS model* (1.7) *with* $\beta > 0$ *and* $\gamma > 0$*:*

(i. the closed-form solution to equation (1.7) *from initial condition* $x(0) = x_0 \in [0, 1]$*, for* $\beta \neq \gamma$*, is*

$$x(t) = \frac{(\beta - \gamma)x_0}{\beta x_0 - \mathrm{e}^{-(\beta-\gamma)t}(\gamma - \beta(1 - x_0))}; \tag{1.8}$$

(ii. if $\beta \leq \gamma$*, all trajectories converge to the unique equilibrium* $x = 0$ *(i.e., the epidemic disappears);*



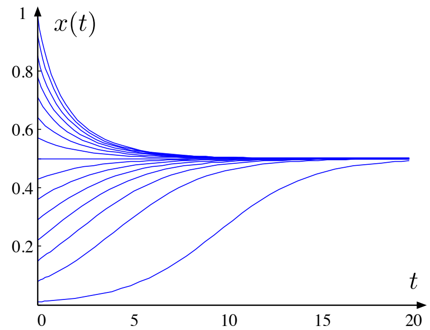


Figure 1.3: Evolution of the scalar SIS model with varying initial fraction of infected individuals. Top figure: $\beta = 0.5 < \gamma = 1$. Bottom figure: $\beta = 0.8 > \gamma = .4$.

(iii. if $\beta > \gamma$, then each trajectory from an initial condition $x(0) > 0$ converges to the exponentially stable equilibrium $x^* = (\beta - \gamma)/\beta$, which is called the endemic state.

Case (iii corresponds to the case in which epidemic outbreaks take place and a steady-state epidemic contagion persists. The basic reproduction number in this deterministic scalar SIS model is given by $R_0 = \beta/\gamma$. Simulations regarding to Lemma 1.4.1(ii and (iii are shown in Figure 1.3.

1.4.2 Network SIS Model

In this section we study the network SIS model which is closely related to the original "multi-group SIS model" proposed by [13]; see also the intertwined SIS model in [25].

The network SIS model with infection rate β and recovery rate γ is given by:

$$\dot{x}_i(t) = \beta(1 - x_i(t)) \sum_{j=1}^{n} a_{ij} x_j(t) - \gamma x_i(t), \tag{1.9}$$

or, in equivalent vector form,

$$\dot{x}(t) = \beta\big(I_n - \operatorname{diag}(x(t))\big) A x(t) - \gamma x(t). \tag{1.10}$$

In the rest of this section we study the dynamical properties of this model. We start by defining the monotonically-increasing functions

$$f_+(y) = y/(1+y), \quad \text{and} \quad f_-(z) = z/(1-z),$$

for $y \in \mathbb{R}_{\geq 0}$ and $z \in [0,1[$. Note that $f_+(f_-(z)) = z$ for all $z \in [0,1)$. For vector variables $y \in \mathbb{R}^n_{\geq 0}$ and $z \in [0,1)^n$, we write $F_+(y) = (f_+(y_1), \dots, f_+(y_n))$, and $F_-(z) = (f_-(z_1), \dots, f_-(z_n))$.

Behavior of System Below the Threshold In this subsection, we characterize the behavior of the network SIS model in a regime we describe as "below the threshold." The results presented in the theorem below can be found in [13, 26, 27]. Historically, it is meaningful to attribute this theorem to [13], even if the language adopted here is more modern.

Theorem 1.4.2 (Dynamical behavior of the network SIS model: Below the threshold) *Consider the network SIS model* (1.9)*, with $\beta > 0$ and $\gamma > 0$, over a strongly connected digraph with adjacency matrix A. Let $\lambda_{\max}$ and $v_{\max}$ be the dominant eigenvalue of A and the corresponding normalized left eigenvector respectively. If $\beta\lambda_{\max}/\gamma < 1$, then*

(i. if $x(0), s(0) \in [0,1]^n$, then $x(t), s(t) \in [0,1]^n$ for all $t > 0$. Moreover, if $x(0) > \mathbb{0}_n$, then $x(t) \succ \mathbb{0}_n$ for all $t > 0$;

(ii. there exists a unique equilibrium point $\mathbb{0}_n$, the linearization of (1.9) *about $\mathbb{0}_n$ is $\dot{x} = (\beta A - \gamma I_n)x$ and it is exponentially stable;*

(iii. from any $x(0) \neq \mathbb{0}_n$, the weighted average $t \mapsto v_{\max}^\top x(t)$ is monotonically and exponentially decreasing, and all the trajectories converge to $\mathbb{0}_n$.

Proof: (i As in Theorem 1.3.2 the first part is the consequence of Nagumo's Theorem. Then define $y(t) := e^{\gamma t}x(t)$. Notice that this variable satisfies the differential equation $\dot{y}(t) = \beta \operatorname{diag}(s(t))Ay(t)$. From the same arguments used in the proof of the point (i of Theorem 1.3.2 we argue that $y(t) \succ \mathbb{0}_n$ for all $t > 0$. From this it follows that also $x(t) \succ \mathbb{0}_n$ for all $t > 0$.

(ii Assume that x^* is an equilibrium point. It is easy to see that $x^* \prec \mathbb{1}_n$. Let $\hat{A} = \beta A/\gamma$. Observe moreover that x^* is an equilibrium point if and only if $\hat{A}x^* = F_-(x^*)$ or, equivalently, if and only if $F_+\big(\hat{A}x^*\big) = x^*$. This means that x^* is an equilibrium if and only if it is a fixed point of $\mathcal{F}$, where $\mathcal{F}(x) := F_+\big(\hat{A}x\big)$. For $x \in [0,1]^n$, note $F_+(\hat{A}x) \preceq \hat{A}x$ because $f_+(z) \preceq z$. Moreover, $\mathbb{0}_n \preceq x \preceq y$ implies that $\mathbb{0}_n \preceq \mathcal{F}(x) \preceq \hat{A}y$. Therefore, if $\mathbb{0}_n \preceq x$, then $\mathcal{F}^k(x) \preceq \hat{A}^k x$, for all k. Since $\hat{A}$ is Schur stable, then $\lim_{k\to\infty} \mathcal{F}^k(x) = 0$. This shows that the only fixed point of $\mathcal{F}$ is zero.

Next, the linearization of equation (1.10) is verified by dropping the second-order terms. The linearized system is exponentially stable at $\mathbb{0}_n$ for $\beta\lambda_{\max} - \gamma < 0$ because $\lambda_{\max}$ is larger, in real part, than any other eigenvalue of A by the Perron-Frobenius Theorem for irreducible matrices.

(iii Finally, regarding statement (iii, define $y(t) = v_{\max}^\top x(t)$ and note that $\big(I_n - \operatorname{diag}(z)\big)v_{\max} \leq v_{\max}$ for any $z \in [0,1]^n$. Therefore,

$$\dot{y}(t) \leq \beta v_{\max}^\top A x(t) - \gamma v_{\max}^\top x(t) = (\beta\lambda_{\max} - \gamma)y(t) < 0.$$

By the Grönwall-Bellman Comparison Lemma, $y(t)$ is monotonically decreasing and satisfies $y(t) \leq y(0)e^{(\beta\lambda_{\max}-\gamma)t}$ from all initial conditions $y(0)$. This concludes our proof of statement (iii. ■

Behavior of System Above the Threshold We present the dynamical behavior of the network SIS model above the threshold as follows. Statement (i of the theorem below is a straightforward result from equation (1.10). Historically, the existence of a unique endemic state and its global attractivity properties, i.e., statements (ii, (iii, (iiia and (iv in the theorem below, are due to [13], and can be found in [26, 27]. To the best our knowledge, the Taylor expansions in parts (iiib and (iiic and the algorithm in part (iiid are novel. In addition, compared with the previous works [13, 26, 27], construction of the algorithm in part (iiid provides an alternative and more concise proof for the existence and uniqueness of the endemic state, and the convergence of any solution starting with $x(0) \in (0,1)^n$ to this endemic state.

Theorem 1.4.3 (Dynamical behavior of the network SIS model: Above the threshold) *Consider the network SIS model* (1.9)*, with $\beta > 0$ and $\gamma > 0$, over a strongly connected digraph with adjacency matrix A. Let $\lambda_{\max}$ be the dominant eigenvalue of A and let $v_{\max}$ and $u_{\max}$ be the corresponding normalized left and right eigenvectors respectively. Let $d = A\mathbb{1}_n$. If $\beta\lambda_{\max}/\gamma > 1$, then*

(i. if $x(0), s(0) \in [0,1]^n$, then $x(t), s(t) \in [0,1]^n$ for all $t > 0$. Moreover, if $x(0) > \mathbb{0}_n$, then $x(t) \succ \mathbb{0}_n$ for all $t > 0$;

(ii. $\mathbb{0}_n$ is an equilibrium point, the linearization of system (1.10) *at $\mathbb{0}_n$ is unstable due to the unstable eigenvalue $\beta\lambda_{\max} - \gamma$ (i.e., there will be an epidemic outbreak);*

(iii. besides the equilibrium $\mathbb{0}_n$, there exists a unique equilibrium point x^*, called the endemic state*, such that*

(a) $x^* \succ \mathbb{0}_n$,

(b) $x^* = \delta a u_{\max} + O(\delta^2)$ *as* $\delta \to 0^+$*, where* $\delta := \beta\lambda_{\max}/\gamma - 1$ *and*

$$a = \frac{v_{\max}^T u_{\max}}{v_{\max}^T \operatorname{diag}(u_{\max}) u_{\max}},$$

(c) $x^* = \mathbb{1}_n - (\gamma/\beta)\operatorname{diag}(d)^{-1}\mathbb{1}_n + O(\gamma^2/\beta^2)$*, at fixed* A*, as* $\gamma/\beta \to 0^+$*, where* $d = A\mathbb{1}_n$*,*

(d) define a sequence $\{y(k)\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ *by*

$$y(k+1) := F_+\left(\frac{\beta}{\gamma} A y(k)\right). \tag{1.11}$$

If $y(0) \geq 0$ *is a scalar multiple of* $u_{\max}$ *and satisfies either* $0 < \max_i y_i(0) \leq 1 - \gamma/(\beta\lambda_{\max})$ *or* $\min_i y_i(0) \geq 1 - \gamma/(\beta\lambda_{\max})$*, then*

$$\lim_{k\to\infty} y(k) = x^*.$$

Moreover, if $\max_i y_i(0) \leq 1 - \gamma/(\beta\lambda_{\max})$*, then* $y(k)$ *is monotonically non-decreasing; if* $\min_i y_i(0) \geq 1 - \gamma/(\beta\lambda_{\max})$*, then* $y(k)$ *is monotonically non-increasing.*

(iv. the endemic state x^* *is locally exponentially stable and its domain of attraction is* $[0,1]^n \setminus \mathbb{0}_n$*.*

Note: statement (ii means that, near the onset of an epidemic outbreak, the exponential growth rate is $\beta\lambda_{\max} - \gamma$ and the outbreak tends to align with the dominant eigenvector $u_{\max}$; for more details see the discussion leading up to the approximate evolution (1.6). The basic reproduction number for this deterministic network SIS model is given by $R_0 = \beta\lambda_{\max}/\gamma$. The network SI model discussed in Section 3 describes the limit behavior of the network SIS model as $\gamma/\beta \to 0^+$. Statement (iiic in Theorem 1.4.3

indicates that $x^* \to \mathbb{1}_n$ as $\gamma/\beta \to 0^+$, which is consistent with statement (iii in Theorem 1.3.2.

Proof: [Proof of selected statements in Theorem 1.4.3]

(i This point can be proved as done in point (i of Theorem 1.3.2.

(ii This follows from the same analysis of the linearized system as in the proof of Theorem 1.4.2(ii.

(iii We begin by establishing two properties of the map $x \mapsto F_+(\hat{A}x)$, for $\hat{A} = \beta A/\gamma$. First, we claim that, $y \succ z \succeq \mathbb{0}_n$ implies $F_+(\hat{A}y) \succ F_+(\hat{A}z)$. Indeed, note that G being connected implies that the adjacency matrix A has at least one strictly positive entry in each row. Hence, $y - z \succ \mathbb{0}_n$ implies $\hat{A}(y-z) \succ \mathbb{0}_n$ and, since f_+ is monotonically increasing, $\hat{A}y \succ \hat{A}z$ implies $F_+(\hat{A}y) \succ F_+(\hat{A}z)$.

Second, we observe that, for any $0 < \alpha < 1$ and $z > 0$, we have $f_+(\alpha z) \geq z$ if and only if $z \leq 1 - 1/\alpha$. Suppose $y(0)$ is a scalar multiple of $u_{\max}$ and $0 < \max_i y_i(0) \leq 1 - \gamma/(\beta\lambda_{\max})$. We have

$$F_+(\hat{A}y(0))_i = f_+\Big(\frac{\beta\lambda_{\max}}{\gamma} y_i(0)\Big) \geq y_i(0).$$

Therefore, the sequence $\{y(k)\}_{k\in\mathbb{N}}$ defined by equation (1.11) satisfies $y(1) \succeq y(0)$, which in turn leads to $y(2) = F_+(\hat{A}y(1)) \succeq F_+(\hat{A}y(0)) = y(1)$, and by induction, $y(k+1) = F_+(\hat{A}y(k)) \succeq y(k)$ for any $k \in \mathbb{N}$. Such sequence $\{y(t)\}$ is monotonically non-decreasing and entry-wise upper bounded by $\mathbb{1}_n$. Therefore, as k diverges, $y(k)$ converges to some $x^* \succ \mathbb{0}_n$ such that $F_+\big(\hat{A}x^*\big) = x^*$. This proves the existence of an equilibrium $x^* = \lim_{k\to\infty} y(k) \succ \mathbb{0}_n$ as claimed in statements (iiia and (iiid.

Similarly, for any $0 < \alpha < 1$ and $z > 0$, $f_+(\alpha z) \leq z$ if and only if $z \geq 1 - 1/\alpha$. Following the same line of argument in the previous paragraph, one can check that the $\{y(k)\}_{k\in\mathbb{N}}$ defined by equation (1.11) is monotonically non-increasing and converges to some x^* , if $y(0)$ is a scalar multiple of $u_{\max}$ and satisfies $\min_i y_i(0) \geq 1 - \gamma/(\beta\lambda_{\max})$.

Now we establish the uniqueness of the equilibrium $x^* \in [0,1]^n \setminus \{\mathbb{0}_n\}$. First, we claim that an equilibrium point with an entry equal to 0 must be $\mathbb{0}_n$. Indeed, assume y^* is an equilibrium point and assume $y_i^* = 0$ for some $i \in \{1,\dots,n\}$. The equality $y_i^* = f_+(\sum_{j=1}^n a_{ij} y_j^*)$ implies that also any node j with $a_{ij} > 0$ must satisfy $y_j^* = 0$. Because G is connected, all entries of y^* must be zero. Second, by contradiction, we assume there exists another equilibrium point $y^* \succ \mathbb{0}_n$ distinct from x^*. Let $\alpha := \min_j\{y_j^*/x_j^*\}$ and let i such that $\alpha = y_i^*/x_i^*$. Then $y^* \succeq \alpha x^* \succ \mathbb{0}_n$ and $y_i^* = \alpha x_i^*$. Notice that we can assume with no loss of generality that $\alpha < 1$ otherwise we exchange x^* and y^*. Observe now that

$$
\begin{aligned}
\big(F_+(\hat{A}y^*) - y^*\big)_i &= f_+\big((\hat{A}y^*)_i\big) - \alpha x_i^* \\
&\geq f_+\big(\alpha(\hat{A}x^*)_i\big) - \alpha x_i^* && (\hat{A} \succeq \mathbb{0}_{n\times n}) \\
&> \alpha f_+\big((\hat{A}x^*)_i\big) - \alpha x_i^* && (0 < \alpha < 1 \text{ and } z > 0) \\
&= \alpha\big(F_+(\hat{A}x^*) - x^*\big)_i = 0. && (x^* \text{ is an equilibrium})
\end{aligned}
$$

Therefore, $\big(F_+(\hat{A}y^*) - y^*\big)_i > 0$, which contradicts the fact that y^* is an equilibrium.

Now we prove (iiib. Observe first that, since taking

$$
y(0) = \left(1 - \frac{\gamma}{\beta\lambda_{\max}}\right) \frac{u_{\max}}{\max_i\{u_{\max,i}\}} = \frac{\delta}{\delta+1} \frac{u_{\max}}{\max_i\{u_{\max,i}\}}
$$

then $y(k)$ is monotonically non-decreasing and converges to x^*, and since taking instead

$$
y(0) = \left(1 - \frac{\gamma}{\beta\lambda_{\max}}\right) \frac{u_{\max}}{\min_i\{u_{\max,i}\}} = \frac{\delta}{\delta+1} \frac{u_{\max}}{\min_i\{u_{\max,i}\}}
$$

then $y(k)$ is monotonically non-increasing and converges to x^*, we can argue that

$$\frac{\delta}{\delta+1}\frac{u_{\max}}{\max_i\{u_{\max,i}\}} \leq x^* \leq \frac{\delta}{\delta+1}\frac{u_{\max}}{\min_i\{u_{\max,i}\}}$$

This implies that x^* is infinitesimal as a function of δ. Consider the expansion $x^*(\delta) = x_1\delta + x_2\delta^2 + O(\delta^3)$. Since the equilibrium x^* satisfies the equation

$$(\delta+1)\big(I_n - \operatorname{diag}(x^*)\big)Ax^* - \lambda_{\max}x^* = 0$$

by substituting the expansion and equating to zero the coefficient of the term δ we obtain the equation

$$Ax_1 - \lambda_{\max}x_1 = 0$$

which proves that x_1 is a multiple of $u_{\max}$, namely $x_1 = au_{\max}$ for some constant a. By equating to zero the coefficient of the term δ^2 we obtain instead the equation

$$Ax_1 + Ax_2 - \operatorname{diag}(x_1)Ax_1 - \lambda_{\max}x_2 = 0$$

Using the fact that $x_1 = au_{\max}$ we argue that

$$a\lambda_{\max}u_{\max} + Ax_2 - a^2\lambda_{\max}\operatorname{diag}(u_{\max})u_{\max} - \lambda_{\max}x_2 = 0$$

By multiplying on the left by $v_{\max}^T$ we obtain

$$a\lambda_{\max}v_{\max}^Tu_{\max} - a^2\lambda_{\max}v_{\max}^T\operatorname{diag}(u_{\max})u_{\max} = 0$$

which proves that

$$a = \frac{v_{\max}^Tu_{\max}}{v_{\max}^T\operatorname{diag}(u_{\max})u_{\max}}$$

Point (iiic can be proved in a similar way. Indeed, define $\epsilon := \gamma/\beta$. Since

$$\left(1-\frac{\epsilon}{\lambda_{\max}}\right)\frac{u_{\max}}{\max_i\{u_{\max,i}\}} \leq x^* \leq \left(1-\frac{\epsilon}{\lambda_{\max}}\right)\frac{u_{\max}}{\min_i\{u_{\max,i}\}}$$

we can argue that the expansion $x^*(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$ as ϵ tends to zero is such that $x_0 \succ \mathbb{0}_n$. Since the equilibrium x^* satisfies the equation

$$\big(I_n - \operatorname{diag}(x^*)\big)Ax^* - \epsilon x^* = 0$$

by substituting the expansion and equating to zero the coefficient of the term ϵ^0 we obtain the equation

$$Ax_0 - \operatorname{diag}(x_0)Ax_0 = 0$$

which proves that $x_0 = vectorones[n]$. By equating to zero the coefficient of the term ϵ^1 we obtain instead the equation

$$Ax_1 - \operatorname{diag}(x_1)Ax_0 - \operatorname{diag}(x_0)Ax_1 - x_0 = 0$$

Using the fact that $x_0 = \mathbb{1}_n$ we argue that

$$\operatorname{diag}(A\mathbb{1}_n)x_1 + \mathbb{1}_n = 0$$

which yieds the thesis.

(iv For this point we refer to [13, 26] or [27, Theorems 1 and 2] in the interest of brevity.

■

Remark 1.4.4 *The network SI model can be regarded as the limit case of the network SIS model with vanishing curing rate $\gamma \to 0^+$. According to Theorem 1.4.2 and 1.4.3, for* *any strongly connected digraph and any fixed infection rate $\beta > 0$, the quantity $\beta\lambda_{\max}/\gamma$ is always above the threshold in the limit $\gamma \to 0^+$. Moreover, statement (iii)(c) indicates that, as $\gamma \to 0^+$, the endemic state x^* satisfies $x^* \to \mathbb{1}_n$. Therefore, the behavior of the network SI model is the same as that for the network SIS model in the limit $\gamma \to 0^+$.*

1.5 Network Susceptible-Infected-Recovered Model

In this section we review the Susceptible-Infected-Susceptible (SIR) epidemic model.

1.5.1 Scalar SIR model

In this model individuals who recover from infection are assumed not susceptible to the epidemic any more. In this case, the population is divided into three distinct groups: $s(t)$, $x(t)$, and $r(t)$, denoting the fraction of susceptible, infected, and recovered individuals, respectively, with $s(t)+x(t)+r(t) = 1$. We write the (Susceptible–Infected–Recovered) SIR model as:

$$\begin{aligned} \dot{s}(t) &= -\beta s(t)x(t), \\ \dot{x}(t) &= \beta s(t)x(t) - \gamma x(t), \\ \dot{r}(t) &= \gamma x(t). \end{aligned} \tag{1.12}$$

The following results on the dynamical behavior of the scalar SIR model can be found in [17].

Lemma 1.5.1 (Dynamical behavior of the SIR model) *Consider the SIR model* (1.12). *From each initial condition $s(0) + x(0) + r(0) = 1$ with $s(0) > 0$, $x(0) > 0$ and $r(0) \geq 0$, the resulting trajectory $t \mapsto (s(t), x(t), r(t))$ has the following properties:*

(i. $s(t) > 0$, $x(t) > 0$, $r(t) \geq 0$, and $s(t) + x(t) + r(t) = 1$ for all $t \geq 0$;

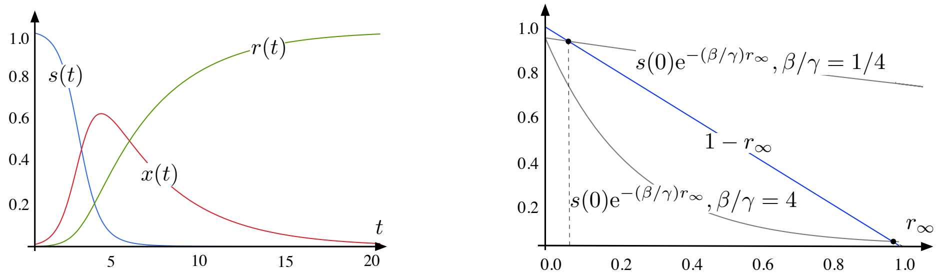


Figure 1.4: Left figure: evolution of the scalar SIR model from small initial fraction of infected individuals (and zero recovered); parameters $\beta = 2$, $\gamma = 1/4$ (case (iv in Lemma 1.5.1). Right figure: intersection between the two curves in equation (1.13) with $s(0) = 0.95$, $r(0) = 0$ and $\beta/\gamma \in \{1/4, 4\}$. If $\beta/\gamma = 1/4$, then $.05 < r_\infty < .1$. If $\beta/\gamma = 4$, then $.95 < r_\infty$.

(ii. $t \mapsto s(t)$ is monotonically decreasing and $t \mapsto r(t)$ is monotonically increasing;

(iii. $\lim_{t\to\infty}(s(t), x(t), r(t)) = (s_\infty, 0, r_\infty)$, where r_∞ is the unique solution to the equality

$$1 - r_\infty = s(0)e^{-\frac{\beta}{\gamma}\left(r_\infty - r(0)\right)}; \tag{1.13}$$

(iv. if $\beta s(0)/\gamma < 1$, then $t \mapsto x(t)$ monotonically and exponentially decreases to zero as $t \to \infty$;

(v. if $\beta s(0)/\gamma > 1$, then $t \mapsto x(t)$ first monotonically increases to a maximum value and then monotonically decreases to 0 as $t \to \infty$; the maximum fraction of infected individuals is given by:

$$x_{\max} = x(0) + s(0) - \frac{\gamma}{\beta}\Big(\log(s(0)) + 1 - \log\Big(\frac{\gamma}{\beta}\Big)\Big).$$

As mentioned before, we describe the behavior in statement (v as an epidemic outbreak, an exponential growth of $t \mapsto x(t)$ for small times.) The effective reproduction number in the deterministic scalar SIR model is $R = \beta s(t)/\gamma$. Note that the basic reproduction number $R_0 = \beta/\gamma$ does not have predict power in this model.

1.5.2 Network SIR model

The network SIR model on a graph with adjacency matrix A is given by

$$\begin{aligned}\dot{s}_i(t) &= -\beta s_i(t) \sum\nolimits_{j=1}^{n} a_{ij} x_j(t),\\ \dot{x}_i(t) &= \beta s_i(t) \sum\nolimits_{j=1}^{n} a_{ij} x_j(t) - \gamma x_i(t),\\ \dot{r}_i(t) &= \gamma x_i(t),\end{aligned}$$

where $\beta > 0$ is the infection rate and $\gamma > 0$ is the recovery rate. Note that the third equation is redundant because of the constraint $s_i(t) + x_i(t) + r_i(t) = 1$. Therefore, we regard the dynamical system in vector form as:

$$\dot{s}(t) = -\beta \operatorname{diag}(s(t)) A x(t), \tag{1.14a}$$

$$\dot{x}(t) = \beta \operatorname{diag}(s(t)) A x(t) - \gamma x(t). \tag{1.14b}$$

We state our main results of this section below. Weaker versions of statements (ia and (ib are due to [18]. To the best of our knowledge, statements (ic, (ii, (iii, (iv and (v are novel.

Theorem 1.5.2 (Dynamical behavior of the network SIR model) *Consider the network SIR model* (1.14)*, with $\beta > 0$ and $\gamma > 0$, over a strongly connected digraph with adjacency matrix A. For $t \geq 0$, let $\lambda_{\max}(t)$ and $v_{\max}(t)$ be the dominant eigenvalue of the non-negative matrix* $\operatorname{diag}(s(t))A$ *and the corresponding normalized left eigenvector, respectively. The following statements hold:*

(i. if $x(0) > \mathbb{0}_n$, and $s(0) \succ \mathbb{0}_n$, then

(a) $t \mapsto s(t)$ and $t \mapsto x(t)$ are strictly positive for all $t > 0$,

(b) $t \mapsto s(t)$ *is monotonically decreasing, and*

(c) $t \mapsto \lambda_{\max}(t)$ *is monotonically decreasing;*

(ii. the set of equilibrium points is the set of pairs $(s^*, \mathbb{0}_n)$*, for any* $s^* \in [0,1]^n$*, and the linearization of model* (1.14) *about* $(s^*, \mathbb{0}_n)$ *is*

$$\begin{aligned} \dot{s}(t) &= -\beta \operatorname{diag}\big(s^*\big) Ax, \\ \dot{x}(t) &= \beta \operatorname{diag}\big(s^*\big) Ax - \gamma x; \end{aligned} \tag{1.15}$$

(iii. (behavior below the threshold) let the time $\tau \geq 0$ *satisfy* $\beta\lambda_{\max}(\tau) < \gamma$*. Then the weighted average* $t \mapsto v_{\max}(\tau)^\top x(t)$ *, for* $t \geq \tau$*, is monotonically and exponentially decreasing to zero;*

(iv. (behavior above the threshold) if $\beta\lambda_{\max}(0) > \gamma$ *and* $x(0) > \mathbb{0}_n$*, then,*

(a) (epidemic outbreak) for small time, the weighted average $t \mapsto v_{\max}(0)^\top x(t)$ *grows exponentially fast with rate* $\beta\lambda_{\max}(0) - \gamma$*, and*

(b) there exists $\tau > 0$ *such that* $\beta\lambda_{\max}(\tau) < \gamma$*;*

(v. each trajectory converges asymptotically to an equilibrium point, that is, $\lim_{t\mapsto\infty} x(t) = \mathbb{0}_n$ *so that the epidemic asymptotically disappears.*

The effective reproduction number in the deterministic network SIR model is $R(t) = \beta\lambda_{\max}(t)/\gamma$. When $R(0) > 1$, we have an epidemic outbreak, i.e., an exponential growth of infected individual for short time. In any case, the theorem guarantees that, after at most finite time, $R(t) < 1$ and the infected population decreases exponentially fast to zero.

Proof: Regarding statement (ia, $s(t) \succ \mathbb{0}_n$ is due to the fact that Ax is bounded and $s(t)$ is continuously differentiable to t. The statement that $x(t) \succ \mathbb{0}_n$ for all $t > 0$ is proved in the same way as Theorem 1.4.2 (i. Statement (ib is the immediate consequence of $\dot{s}_i(t)$ being strictly negative. From statement (ia we know that each $s_i(t)$ is positive, and from A being irreducible and $x(0) \neq \mathbb{0}_n$ we know that $\sum_{j=1}^{n} a_{ij}x_j$ is positive. Therefore, $\dot{s}_i(t) = -\beta s_i(t) \sum_{j=1}^{n} a_{ij}x_j(t) < 0$ for all $i \in V$ and $t \geq 0$.

For statement (ic, we start by recalling the following property from [41, Example 7.10.2]: for B and C nonnegative square matrices, if $B \leq C$, then $\rho(B) \leq \rho(C)$. Now, pick two time instances t_1 and t_2 with $0 < t_1 < t_2$. Let $\alpha = \max_i s_i(t_2)/s_i(t_1)$ and note $0 < \alpha < 1$ because $s(t)$ is strictly positive and monotonically decreasing. Now note that,

$$\mathrm{diag}(s(t_1))A > \alpha \, \mathrm{diag}(s(t_1))A \succeq \mathrm{diag}(s(t_2))A,$$

so that, using the property above, we know

$$\rho(\mathrm{diag}(s(t_1))A) > \alpha\rho(\mathrm{diag}(s(t_1))A) \geq \rho(\mathrm{diag}(s(t_2))A).$$

This concludes the proof of statement (ic.

Regarding statement (ii, note that a point (s^*, x^*) is an equilibrium if and only if:

$$\begin{aligned} \mathbb{0}_n &= -\beta \, \mathrm{diag}\left(s^*\right) A x^*, \quad \text{and} \\ \mathbb{0}_n &= \beta \, \mathrm{diag}\left(s^*\right) A x^* - \gamma x^*. \end{aligned}$$

Therefore, each point of the form $(s^*, \mathbb{0}_n)$ is an equilibrium. On the other hand, summing the last two equalities we obtain $\mathbb{0}_n = \gamma x^*$ and thus x^* must be $\mathbb{0}_n$. As a straightforward result, the linearization of model (1.14) about any equilibrium point $(s^*, \mathbb{0}_n, \mathbb{1}_n - s^*)$ is given by equation (1.15).

Regarding statement (iii, multiplying $v_{\max}(\tau)^\top$ from the left on both sides of equation (1.14b) we obtain:

$$\frac{d}{dt}\big(v_{\max}(\tau)^\top x(t)\big) = v_{\max}(\tau)^\top \Big(\beta \operatorname{diag}\big(s(t)\big) A x(t) - \gamma x(t)\Big)$$
$$\leq v_{\max}(\tau)^\top \Big(\beta \operatorname{diag}\big(s(\tau)\big) A x(t) - \gamma x(t)\Big) = (\beta\lambda_{\max}(\tau) - \gamma) v_{\max}(\tau)^\top x(t).$$

Therefore, we obtain

$$v_{\max}(\tau)^\top x(t) \leq (v_{\max}(\tau)^\top x(0)) \mathrm{e}^{(\beta\lambda_{\max}(\tau)-\gamma)t}.$$

The right-hand side exponentially decays to zero when $\beta\lambda_{\max}(\tau) < \gamma$. Therefore, $v_{\max}(\tau)^\top x(t)$ also decreases monotonically and exponentially to zero for all $t > \tau$.

Regarding statement (iva, note that based on the argument in (ia, we only need to consider the case when $x(0) \succ \mathbb{0}_n$. Left-multiplying $v_{\max}(0)^\top$ on both sides of equation (1.14b), we obtain:

$$\frac{d}{dt}\big(v_{\max}(0)^\top x(t)\big)\Big|_{t=0} = v_{\max}(0)^\top \Big(\beta \operatorname{diag}\big(s(t)\big) A x(t) - \gamma x(t)\Big)\Big|_{t=0} = (\beta\lambda_{\max}(0) - \gamma) v_{\max}(0)^\top x(0).$$

Since $\beta\lambda_{\max}(0) - \gamma > 0$, the initial time derivative of $v_{\max}(0)^\top x(t)$ is positive. Since $t \mapsto v_{\max}(0)^\top x(t)$ is a continuously differentiable function, there exists $\tau' > 0$ such that $\frac{d}{dt}\big(v_{\max}(0)^\top x(t)\big) > 0$ for any $t \in [0, \tau']$.

Regarding statement (ivb, since $\dot{s}(t) \preceq \mathbb{0}_n$ and is lower bounded by $\mathbb{0}_n$, we conclude that the limit $\lim_{t\to+\infty} s(t)$ exists. Moreover, since $s(t)$ is monotonically non-increasing, we have $\lim_{t\to+\infty} \dot{s}(t) = 0$, which implies either $\lim_{t\to+\infty} s(t) = \mathbb{0}_n$ or $\lim_{t\to+\infty} x(t) = \mathbb{0}_n$. If $s(t)$ converges to $\mathbb{0}_n$, then $\dot{x}(t)$ converges to $-\gamma x(t)$. Therefore, there exists $T > 0$ such that $\beta\lambda_{\max}(T) < \gamma$, which leads to $x(t) \to \mathbb{0}_n$ as $t \to +\infty$; If $s(t)$ converges to some $s^* > \mathbb{0}_n$, then $x(t)$ still converges to $\mathbb{0}_n$. Therefore, for any $\big(s(0), x(0)\big)$, the trajectory $\big(s(t), x(t)\big)$

converges to some equilibria with the form $(s^*, \mathbb{0}_n)$, where $s^* \succeq \mathbb{0}_n$. Let

$$s(t) = s^* + \delta_s(t), \quad \text{and } x(t) = \mathbb{0}_n + \delta_x(t).$$

We know that $\delta_s(t) \geq 0$ and $\delta_x(t) \geq 0$ for all $t \geq 0$. Moreover, $\delta_s(t)$ is monotonically non-increasing and converges to $\mathbb{0}_n$, and there exists $\tilde{T} > 0$ such that, for any $t \geq T$, $\delta_x(t)$ is monotonically non-increasing and converges to $\mathbb{0}_n$.

Let λ^* and v^* denote the dominant eigenvalue and the corresponding normalized left eigenvector of matrix $\operatorname{diag}(s^*)A$, respectively, that is, $v^{*\top} \operatorname{diag}(s^*)A = \lambda^* v^{*\top}$. First let us suppose $\beta\lambda^* - \gamma > 0$, then the linearized system of (1.12) around $(s^*, \mathbb{0}_n)$ is written as

$$\begin{aligned} \dot{\delta}_s &= -\beta \operatorname{diag}(s^*)A\delta_x, \\ \dot{\delta}_x &= \beta \operatorname{diag}(s^*)A\delta_x - \gamma\delta_x. \end{aligned}$$

Since $\beta\lambda^* - \gamma > 0$, the linearized system is exponentially unstable, which contradicts the fact that $\big(\delta_s(t), \delta_x(t)\big) \to (\mathbb{0}_n, \mathbb{0}_n)$ as $t \to +\infty$. Alternatively, suppose $\beta\lambda^* - \gamma = 0$. By left multiplying $v^{*\top}$ on both sides of the equation for $\dot{x}(t)$ in (1.12), we obtain

$$v^{*\top}\dot{\delta}_x = (\beta\lambda^* - \gamma)(v^{*\top}\delta_x) + \beta v^{*\top} \operatorname{diag}(\delta_s)A\delta_x = \beta v^{*\top} \operatorname{diag}(\delta_s)A\delta_x \succeq \mathbb{0}_n,$$

which contradicts $\delta_x(t) \to \mathbb{0}_n$ as $t \to +\infty$. Therefore, we conclude that $\beta\lambda^* - \gamma < 0$. Since $\lambda_{\max}(t)$ is continuous on t, we conclude that there exists $\tau < +\infty$ such that $\beta\lambda_{\max}(t) - \gamma < 0$.

■

Remark 1.5.3 *Consider the network SIR model as a parameterized dynamical system, with the curing rate γ as the parameter. The network SI model can be regarded the network*

SIR dynamics with $\gamma = 0$ and zero initial fraction of recovered individuals. However, due to the specific bifurcation behavior of the network SIR model at $\gamma = 0$, the dynamical properties of the network SIR model with $\gamma = 0$ are qualitatively different from the case when $\gamma > 0$. When $\gamma = 0$, the set given by statement (ii of Theorem 1.5.2 is only a subset of the equilibirum set. Points in the set of pairs $(\mathbb{0}_n, x^*)$ are also the equilibria of the network SIR with $\gamma = 0$. In addition, while statement (iva of Theorem 1.5.2 on the initial epidemic outbreak is still true, statements (ivb and (v on the eventual decay no longer hold for $\gamma = 0$.

In what follows, we present a novel result on an iterative algorithm that computes the limit state $\lim_{t\to\infty} \big(s(t), 0, r(t)\big)$ of the network SIR model (1.14) as a function of an arbitrary initial condition $\big(s(0), x(0), r(0)\big)$.

Note that, for the scalar SIR model (1.12), if we define

$$V\big(s(t), x(t)\big) := s(t)\mathrm{e}^{\frac{\beta}{\gamma}\big(1-x(t)-s(t)\big)}.$$

Simple calculations result in $dV\big(s(t), x(t)\big)/dt = 0$, which implies that the trajectories are on the level sets of V and in the set $\{(s, x) \in \mathbb{R}^2 \mid s \geq 0, x \geq 0, s + x \leq 1\}$. Here, we apply a similar approach to the network SIR system (1.14). Let

$$V_i(s, r) := s_i \mathrm{e}^{\frac{\beta}{\gamma}\sum_{j=1}^{n} a_{ij} r_j}, \quad \text{for any } i \in \{1, \dots, n\}.$$

One can check that, along any trajectory of dynamics (1.14), $dV_i/dt = 0$ for any $i \in \{1, \dots, n\}$. Therefore, the trajectories $(s(t), r(t))$ lie on the level curves of the functions $V_i(s, r)$ for $i \in \{1, \dots, n\}$.

Let $s(\infty) := \lim_{t\to+\infty} s(t)$, $x(\infty) := \lim_{t\to+\infty} x(t)$, and $r(\infty) := \lim_{t\to+\infty} r(t)$. Notice that $x(\infty) = \mathbb{0}_n$ and so $r(\infty) = \mathbb{1}_n - s(\infty)$. Since $dV_i/dt = 0$ for any $i \in \{1, \dots, n\}$, we have

$$s_i(\infty) = s_i(0)\mathrm{e}^{-\frac{\beta}{\gamma}\sum_{j=1}^{n} a_{ij}\left(1-r_j(0)\right)}\mathrm{e}^{\frac{\beta}{\gamma}\sum_{j=1}^{n} a_{ij}s_j(\infty)}. \tag{1.16}$$

Given any initial condition $\big(s(0), r(0)\big)$, the right-hand side of equation (1.16) defines a map

$$H(s) := e^{\frac{\beta}{\gamma}\operatorname{diag}\left(A(s-\mathbb{1}_n+r(0))\right)}s(0), \tag{1.17}$$

and $s(\infty)$ is a fixed point of H, that is, $s(\infty) = H\big(s(\infty)\big)$. The following theorem is novel.

Theorem 1.5.4 (Existence, uniqueness, and algorithm for the asymptotic point) *Consider the network SIR model* (1.14)*, with positive rates β and γ and with initial condition $\big(s(0), x(0), r(0)\big)$ satisfying $s(0) \succ \mathbb{0}_n$, $x(0) > \mathbb{0}_n$, $r(0) \succeq \mathbb{0}_n$ and $s(0)+x(0)+r(0) = \mathbb{1}_n$. Let $\big(s(\infty), \mathbb{0}_n, r(\infty)\big)$ be the asymptotic state of system* (1.14)*. The map $H : \mathbb{R}^n \to \mathbb{R}^n$ defined by equation* (1.17) *has the following properties:*

(i. there exists a unique fixed point s^* of the map H in the set $\{s \in \mathbb{R}^n \mid \mathbb{0}_n \preceq s \preceq \mathbb{1}_n - r(0)\}$. Moreover, $s^* = s(\infty)$ and $r(\infty) = \mathbb{1}_n - s^*$; and

(ii. any sequence $\{y(k)\}_{k\in\mathbb{N}}$ defined by $y(k+1) = H(y(k))$ and initial condition $\mathbb{0}_n \preceq y(0) \preceq \mathbb{1}_n - r(0)$ converges to the unique fixed point s^*.

Proof: Since A is a non-negative matrix, and $s(0) \preceq \mathbb{1} - r(0)$, one can easily observe that, if $\mathbb{0}_n \preceq p \preceq q \preceq \mathbb{1}_n - r(0)$, then $\mathbb{0}_n \preceq H(\mathbb{0}_n) \preceq H(p) \preceq H(q) \preceq H(\mathbb{1}_n - r(0)) \preceq \mathbb{1}_n - r(0)$. According to the Brower Fixed Point Theorem, the map H has at least one fixed point.

Define the sequence $\{p(k)\}_{k\in\mathbb{N}}$ by $p(k+1) = H(p(k))$ and $p(0) = \mathbb{0}_n$. Since

$$\mathbb{1}_n - r(0) \succeq p(1) = H(\mathbb{0}_n) = e^{\frac{\beta}{\gamma}\operatorname{diag}\left(-A\mathbb{1}_n+Ar(0)\right)}s(0) \succeq p(0),$$

we have $\mathbb{1}_n - r(0) \succeq p(2) = H(p(1)) \succeq H(p(0)) = p(1)$ and, by induction, $\mathbb{1}_n - r(0) \succeq p(k+1) \succeq p(k)$ for any $k \in \mathbb{N}$. Since $p(k)$ is non-decreasing and upper bounded by $\mathbb{1}_n - r(0)$, we conclude that the limit $p^* = \lim_{k\to\infty} p(k)$ exists, and p^* is a fixed point of the map H.

Similarly, define a sequence $\{q(k)\}_{k\in\mathbb{N}}$ by $q(k+1) = H(q(k))$ and $q(0) = \mathbb{1}_n - r(0)$. One can check that $q(k)$ is non-increasing and that $q^* = \lim_{k\to\infty} q(k)$ is a fixed point of map H. Moreover, since $p(0) \preceq q(0)$, we have $p(k) \preceq q(k)$ for any $k \in \mathbb{N}$ and thereby $p^* \preceq q^*$.

If $p^* = q^*$, then, for any $\mathbb{0}_n \preceq y(0) \preceq \mathbb{1}_n - r(0)$, the sequence $\{y(k)\}_{k\in\mathbb{N}}$ defined by $y(k+1) = H(y(k))$ satisfies $p(k) \preceq y(k) \preceq q(k)$ for any $k \in \mathbb{N}$. Therefore, $y^* = \lim_{k\to\infty} y(k)$ exists and $y^* = p^* = q^*$, which implies that the fixed point of map H is unique. According to equation (1.16), $s(\infty)$ is the unique fixed point. This concludes the proof for statement (i) and (ii).

Now we eliminate the case $p^* < q^*$ by contradiction. First of all we prove that $q^* \prec \mathbb{1}_n - r(0)$. Let $N_i = \{j \,|\, a_{ij} > 0\}$ and $\mathcal{I}(k) = \big\{i \,\big|\, q_i(\tau) < 1 - r_i(0) \text{ for any } \tau \geq k\big\}$. We have $\mathcal{I}(0) = \phi$. Since $x(0) > \mathbb{0}_n$, we have $q(1) = s(0) < 1 - r(0)$, that is, there exists i such that $q_i(1) < 1 - r_i(0)$. Moreover, since $q(k)$ is non-increasing, we have $q(k) \preceq q(1)$ for any $k \geq 1$. Therefore, for any i such that $q_i(1) < 1 - r_i(0)$, it satisfies $q_i(k) \leq q_i(1) < 1 - r_i(0)$ for any $k \geq 1$. Since $j \notin \mathcal{I}(1)$ if $q_j(1) = s_j(0) = 1 - r_j(0)$, we conclude that $\mathcal{I}(1) = \{i \mid s_i(0) < 1 - r_i(0)\}$. Moreover, for any given $k \geq 1$, since, for any i such that $N_i \cap \mathcal{I}(k) \neq \phi$,

$$q_i(k+1) = H(q(k))_i = e^{\frac{\beta}{\gamma}\sum_{j=1}^n a_{ij}\left(q_j(k) - 1 + r_j(0)\right)} s_i(0) < s_i(0) \leq 1 - r_i(0);$$

and for any i such that $N_i \cap \mathcal{I}(k) = \phi$ and $i \notin \mathcal{I}(k)$,

$$q_i(k+1) = H(q(k))_i = e^{\frac{\beta}{\gamma}\sum_{j=1}^n a_{ij}\left(q_j(k)-1+r_j(0)\right)} s_i(0) = s_i(0) = 1 - r_i(0),$$

we have $\mathcal{I}(k+1) = \{i \mid N_i \cap \mathcal{I}(k) \neq \phi\} \cup \mathcal{I}(k)$ for any $k \geq 1$. Because the graph associated with A is strongly connected, we can argue that $\mathcal{I}(k)$ contains all the indices when k is large enough. Therefore, $q^* \prec \mathbb{1}_n - r(0)$.

Now suppose $p^* < q^*$. Let

$$\alpha = \min_j \frac{1 - r_j(0) - p_j^*}{q_j^* - p_j^*}, \quad \text{and} \quad w = (1-\alpha)p^* + \alpha q^*.$$

We have $\alpha > 1$, $\mathbb{0}_n \preceq w < \mathbb{1}_n - r(0)$, and $w_i = 1 - r_i(0)$ for any i such that $\alpha_i = \big(1 - r_i(0) - p_i^*\big)/(q_j^* - p_j^*)$. Let $\mu = 1/\alpha$. Thereby $q^* = \mu w + (1-\mu)p^*$, where $0 < \mu < 1$. This means that q^* is a convex combination of p^* and w. Since $H(s)_i$ is a strictly convex function of s, we obtain that

$$q_i^* = H\big(\mu w + (1-\mu)p^*\big)_i < \mu H(w)_i + (1-\mu)p_i^* \leq \mu\big(1 - r_i(0)\big) + (1-\mu)p_i^* = q_i^*.$$

In the last inequality, we used the fact that $H(w)_i \leq 1 - r_i(0)$ for any $\mathbb{0}_n \preceq w \preceq \mathbb{1}_n - r(0)$. The previous inequality yields a contradiction. ■

In the rest of this section, we present some numerical results for the network SIR model for the famous Krackhardt?s advice network illustrated in Figure 1.5. This network reflects the data collected by [42] on the cognitive social structure of the management personnel in a high-tech machine manufacturing firm. In the network, each node represents an individual, and each directed link (i, j) means that individual i seeks advice from individual j. We refer the interested readers to [42] for more details.

Consider the epidemic spreading process on the Krackhardt?s advice network. The associated adjacency matrix A is binary. Unless otherwise stated, the system parameters are set as $\beta = 0.5$ and $\gamma = 0.4$. As for initial condition, we select one node fully infected (the dark-gray node in Figure 1.5, say, with index 1), 16 fully healthy individuals, and zero recovered fraction — corresponding to $x(0) = \mathbb{e}_1$, $r(0) = \mathbb{0}_n$, and $s(0) = \mathbb{1}_n - x(0)$. These parameters lead to an initial effective reproduction number $R(0) = 3.57$.

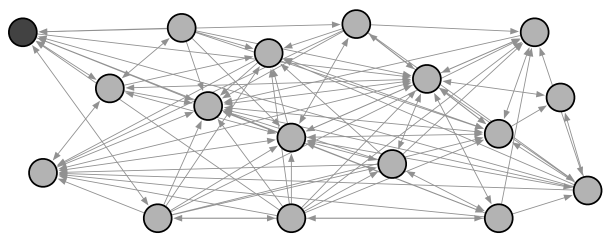


Figure 1.5: Main strongly-connected component of the Krackhardt digraph with 17 nodes

Figure 1.6 illustrates the time evolution of $(\beta/\gamma)\lambda_{\max}(t)$ with varying network parameters. Note that each evolution starts above the threshold, reaches the threshold value 1 in finite time, and converges to a final value below 1.

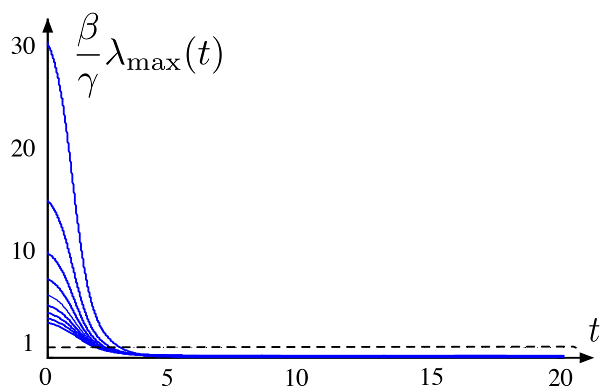


Figure 1.6: Evolution of the spectral radius of $(\beta/\gamma)\,\mathrm{diag}(s(t))A)$ over the strongly connected digraph in Figure 1.5. The parameter γ takes value in $.1, .2, \ldots, .9$, corresponding respectively to the curves from up to down in the time interval $[0, 5]$.

Figure 1.7 illustrates the behavior of the average susceptible, average infected and average recovered quantities in populations starting from a small initial infection fraction and with an effective reproduction number above 1 at time 0. Note that the evolution of the infected fraction of the population displays a unimodal dependence on time, like in the scalar model.

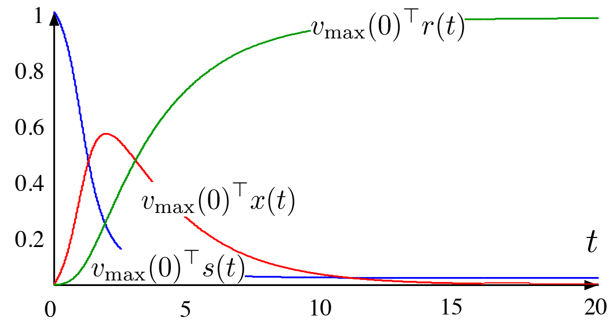


Figure 1.7: Evolution of the network SIR model from initial condition consisting of one node fully infected individual (the dark-gray node in Figure 1.5), 16 fully healthy individuals, and zero recovered fraction. The effective reproduction number satisfies $R(0) = 3.57$.

1.6 Conclusion

This chapter provides a comprehensive and consistent treatment of deterministic nonlinear continuous-time SI, SIS, and SIR propagation models over contact networks. We investigated the asymptotic behaviors (vanishing infection, steady-state epidemic, and full contagion). We studied the transient propagation of an epidemic starting from small initial fractions of infected nodes. We presented conditions under which a possible epidemic outbreak occurs or the infection monotonically vanishes for arbitrary fixed topology graphs. We introduced a network SI model and analyzed its behavior. Network SIS model sections includes improved properties over previously proposed works. New transient behavior, threshold condition, and system properties for the network SIR model were proposed. In addition, for the network SIR model, we provide a novel iterative algorithm to compute the asymptotic state of the system. In all cases, we show the results for network models are appropriate generalizations of those for the respective scalar models.

There are numerous potential future research directions regarding the deterministic network epidemic processes and the literature is still growing rapidly. Recent progress in this area includes but is not limited to the modeling and analysis of epidemic spreading on time-varying networks, e.g., see [43, 44], the optimal immunization strategies, e.g., see [45, 46], and the competitive propagation of multiple virus/memes, e.g., see [47, 48, 49].

Finally, we point out that, although the network SI, SIS, and SIR models have attracted enormous attention by researchers working on network epidemics, they are not the only deterministic models of epidemic spreading processes on networks. For example, there is another class of deterministic network models, referred to as the *multi-city model* or the *epidemic model in a patchy environment*. This class of models considers each node in the network as a city obeying the scalar SIS or SIR dynamics. The disease is spread via the traffic flows between those cities. We refer the interested reader to [50, 51, 52] for detailed treatments.

Chapter 2

Competitive Propagation and Quality-Seeding Games

2.1 Introduction

a) Motivation and problem description:

It is of great scientific interest to model some sociological phenomenon as dynamics on networks, such as consensus, polarization, synchronization and propagation. Indeed, the past fifteen years have witnessed a flourishing of research on propagation of diseases, opinions, commercial products etc, collectively referred to as memes, on social networks. Much progress has been made both on obtaining and analyzing empirical data [53, 54, 55, 56], and mathematical modeling [17, 57, 58, 59]. In a more recent set of extensions, scientists have begun studying the simultaneous propagation of multiple memes, in which not only the interaction between nodes (or equivalently referred to as individuals) in the network, but also the interplay of multiple memes, plays an important role in determining the system's dynamical behaviors. These two forms of interactions together add complexity and research value to the multi-meme propagation models.

This chapter proposed a series of mathematical models on the propagation of competing products. Three key elements: the interpersonal network, the individuals and the competing products, are modeled respectively as a graph with fixed topology, the nodes on the graph, and the states of nodes. Our models are based on the characterization of individuals' decision making behavior under social pressure. Two factors determine individuals' choices on which product to adopt: the endogenous factor and the exogenous factor. The endogenous factor is the social contact between nodes via social links, which forms a tendency of imitation, referred to as social pressure in this paper. The exogenous factor is what is unrelated to the network, e.g., the products' quality.

In the microscopic level, we model the endogenous and exogenous factors respectively as two types of product-adoption processes: the social conversion and the self conversion. In social conversion, any node randomly picks one of its neighbors and follows that neighbor's state with some given probability characterizing how open-minded the node is. In the self conversion, each node independently converts from one product to another with some given probability depending on the two products involved. Although individuals exhibit subjective preferences when they are choosing the products, statistics on a large scale of different individuals' actions often reveal that the relative qualities of the competing products are objective. For example, although some people may have special affections on feature phones, the fact that more people have converted from feature phones to smart phones, rather than the other way around, indicates that the latter is relatively better. We assume that the transit probabilities between the competing products are determined by their relative qualities and thus homogeneous among the individuals.

b) Literature review: Various models have been proposed to describe the propagation phenomena on networks, such as the percolation model on random graphs [10, 11], the independent cascade model [60, 61, 62], the linear threshold model [15, 63, 16] and the epidemic-like mean-field model [23, 25, 33].

As extensions to the propagation of a single meme, some recent papers have discussed the propagation of multiple memes, e.g., see [64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 47, 74]. Some of these papers adopt a Susceptible-Infected-Susceptible (SIS) epidemic-like model and discuss the long-term coexistence of multiple memes in single/multiple-layer networks, e.g., see [68, 69, 70]. Some papers focus instead on the strategy of initial seeding to maximize or prevent the propagation of one specific meme in the presence of adversaries [72, 73, 47, 74]. Among all these papers mentioned in this paragraph, our model is most closely related to the work by Stanoev et. al. [71] but the social contagion process in [71] is different from our model and theoretical analysis on the general model is not included.

c) Contribution: Firstly we propose a generalized and novel model for the competitive propagation on social networks. By taking into account both the endogenous and exogenous factors and by considering the individual variance as well as the interplay of the competing products, our model is general enough to describe a large class of multi-meme propagation processes. Moreover, many existing models have difficulty in dealing with the simultaneous contagions of multiple memes, and have to avoid this problem by adding an additional assumption of the infinitesimal step length that only allows the occurrence of a single contagion at every step. Different from these models, the problem of multiple contagions does not occur in our model since we model the contagion process as the individual's initiative choice under the social pressure, which is more suitable for the product-adoption process. In addition, compared with the independent cascade model, in which individuals' choices are irreversible, our models adopt a more realistic assumption that conversions from one product to another are reversible and occur persistently.

Secondly, we propose a new concept, the product-conversion graph, to characterize the interplay between the products. There are two graphs in our model: the social network describing the interpersonal connections, and the product-conversion graph defining the transitions between the products in self conversion, which in turn reflect the products' relative quality.

Thirdly, starting from the description of individuals' behavior, we develop two Markov-chain competitive propagation models different in the chronological order of the social conversion and the self conversion processes. Applying the independence approximation, we propose two corresponding network competitive propagation models, which are difference equations systems, such that the dimension of our problem is reduced and some theorems in the area of dynamical systems can be applied to the analysis of the approximation models.

Fourthly, both theoretical analysis and simulation results are presented on the dynamical properties of the network competitive propagation models. We discuss the existence, uniqueness and stability of the fixed point, as well as how the systems' asymptotic state probability distribution is determined by the social network structure, the individuals' open-mindedness, the initial condition and, most importantly, the structure of the product-conversion graph. We find that, if the product-conversion graph contains only one absorbing strongly connected component, then the self conversion dominates the system's asymptotic behavior; With multiple absorbing strongly connected components in the product-conversion graph, the system's asymptotic state probability distribution also depends on the initial condition, the network topology and the individual open-mindedness. In addition, simulation results are presented to show the high accuracy of the independence approximation and reveal that the original Markov-chain model also exhibits the same asymptotic behavior.

At last, based on the network competitive propagation model, we propose two classes of non-cooperative games. In both games the players are the competing companies with bounded investment budgets on seeding, e.g., advertisement and promotion, and improving their products' quality. The first model is a one-shot game, in which at each step the players myopically maximize their next-step pay-off. We investigate the unique Nash equilibrium at each stage. Theoretical analysis also reveals some strategic and realistic insights on the seeding-quality trade-off and the allocation of seeding resources among the individuals. The second model is a dynamic game with infinite horizon, in which the players aim to maximize their discounted accumulated pay-offs. The existence of Nash equilibrium for the two-player case is proved and numerical analysis is given on the comparison with the one-shot game.

d) Organization: The rest of this chapter is organized as follows. Section II give the assumptions for two Markov-chain propagation models. Section III and IV discuss the approximation of these two models respectively. In Section V, we discuss the two classes of games. Section VI is the conclusion.

2.2 Model Description and Notations

a) Social network as a graph: In this model, a social network is considered as an undirected, unweighted, fixed-topology graph $G = (V, E)$ with n nodes. The nodes are indexed by $i \in V = \{1, 2, \dots, n\}$. The adjacency matrix is denoted by $A = (a_{ij})_{n\times n}$ with $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ if $(i, j) \notin E$.

The row-normalized adjacency matrix is denoted by $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, where $\tilde{a}_{ij} = \frac{1}{N_i} a_{ij}$ with $N_i = \sum_{j=1}^{n} a_{ij}$. The graph $G = (V, E)$ is always assumed connected and there is no self loop, i.e., $\tilde{a}_{ii} = 0$ for any $i \in V$.

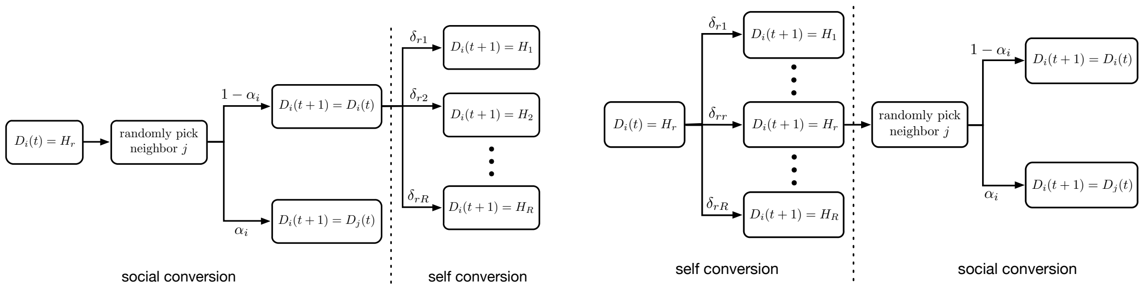
b) Competing products and the states of nodes: Suppose there are R competing products, denoted by $H_1, H_2, \dots, H_R$, propagating in the network. We consider a discrete-time model, i.e., $t \in \mathbb{N}$, and assume the products are mutually exclusive. We do not specify the state of adopting no product and collectively refer to all the states as "products". Denote by $D_i(t)$ the state of node i after time step t. For any $t \in \mathbb{N}$, $D_i(t) \in \{H_1, H_2, \dots, H_R\}$. For simplicity let $\Theta = \{1, 2, \dots, R\}$, i.e., the set of the product indexes.

c) Nodes' production adoption behavior: Two mechanisms define the individuals' behavior: the social conversion and the self conversion. The following two assumptions propose respectively two models different in the chronological order of the social and self conversions.

Assumption 2.1 (Social-self conversion model) *Consider the competitive propagation of R products in the network $G = (V, E)$. At time step $t+1$ for any $t \in \mathbb{N}$, suppose the previous state of any node i is $D_i(t) = H_r$. Node i first randomly picks one of its neighbor j and follows j's previous state, i.e., $D_i(t+1) = D_j(t)$, with probability α_i. If node i does not follow j's state in the social conversion, with probability $1-\alpha_i$, then node i converts to product H_s with probability δ_{rs} for any $s \neq r$, or stay in H_r with probability δ_{rr}.*

Assumption 2.2 (Self-social conversion model) *At any time step $t+1$, any node i with $D_i(t) = H_r$ converts to H_s with probability δ_{rs} for any $s \neq r$, or stay in the state H_r with probability δ_{rr}. If node i stays in H_r in the process above, then node i randomly picks a neighbor j and follows $D_j(t)$ with probability α_i, or still stay in H_r with probability $1 - \alpha_i$.*

Assumptions 2.1 and 2.2 are illustrated by Figure 2.1(a) and Figure 2.1(b) respectively. By introducing the parameters δ_{rs} we define a directed and weighted graph with the adjacency matrix $\mathscr{D} = (\delta_{rs})_{R\times R}$, referred to as the *product-conversion graph*. Figure 2.2 gives an example of the product-conversion graph for different smart phone operation systems. Based on either of the two assumptions, $\mathscr{D}$ is row-stochastic. In this chapter we discuss several types of structures of the product-conversion graph, e.g., the case when



(a) social-self conversion (b) self-social conversion

Figure 2.1: Diagram illustrations for the social-self conversion model and the self-social conversion model.

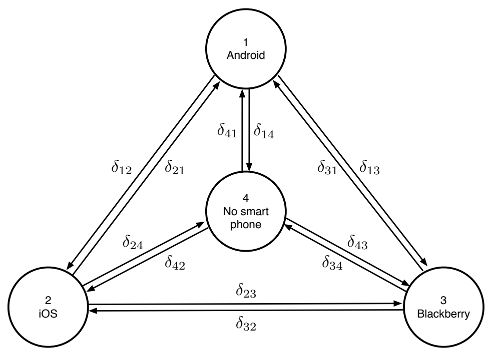


Figure 2.2: An example of the product-conversion graph for different smart phone operation systems. The self loops exist but are not displayed in this graph.

it is strongly connected, or consists of a transient subgraph and some isolated absorbing subgraphs. The parameter α_i characterizes node i's inclination to be influenced by social pressure. Define $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)^\top$ as the individual open-mindedness vector. Assume $0 < \alpha_i < 1$ for any $i \in V$.

d) Problem description: According to either Assumption 2.1 or Assumption 2.2, at any time step $t+1$, the probability distribution of any node's states depends on its own state as well as the states of all its neighbors at time t. Therefore, the collective evolution of nodes' states is a R^n-state discrete-time Markov chain. Define $p_{ir}(t)$ as the probability that node i is in state H_r after time step t, i.e., $p_{ir}(t) = \mathbb{P}[D_i(t) = H_r]$. We aim to understand the dynamics of $p_{ir}(t)$. Since the Markov chain models have exponential dimensions, we approximate it with lower-dimension difference equations systems and analyze instead the dynamical properties of the approximation systems.

e) Notations: Before proceeding to the next section, we introduce some frequently used notations in Table 2.1. In order to distinguish vectors from matrices, in this chapter, we use symbols in bold to denote vectors.

Table 2.1: Notations frequently used in this chapter

Θ	set of products. $\Theta = \{1, \dots, R\}$
$S_{nm}(\boldsymbol{a})$	the set $\{\boldsymbol{X} \in \mathbb{R}^{n\times m} \,\vert\, X \succeq \mathbb{0}_{n\times m}, X\mathbb{1}_m = \boldsymbol{a}\}$ for any $\boldsymbol{a} \in \mathbb{R}^n$
$\tilde{S}_{nm}(\boldsymbol{a})$	the set $\{\boldsymbol{X} \in \mathbb{R}^{n\times m} \,\vert\, X \succeq \mathbb{0}_{n\times m}, X\mathbb{1}_m \preceq \boldsymbol{a}\}$ for any $\boldsymbol{a} \in \mathbb{R}^n$
$\boldsymbol{x}^r$	the r-th column vector of the matrix $X \in \mathbb{R}^{n\times m}$
$\boldsymbol{x}^{(i)}$	the i-th row vector of the matrix $X \in \mathbb{R}^{n\times m}$
$\boldsymbol{x}^{(-i)}$	the i-th row vector of the matrix $\tilde{A}X \in \mathbb{R}^{n\times m}$, i.e., $\boldsymbol{x}^{(-i)} = (x_{-i1}, x_{-i2}, \dots, x_{-im})$ where $x_{-ir} = \sum_{j=1}^{n} \tilde{a}_{ij} x_{jr}$

2.3 Network Competitive Propagation Model with Social-self conversion

This section is based on Assumption 2.1. We first derive an approximation model for the time evolution of $p_{ir}(t)$, referred to as the *social-self conversion network competitive propagation model* (social-self NCPM), and then analyze the asymptotic behavior of the approximation model and its relation to the social network topology, the product-conversion graph, the initial condition and the individuals open-mindedness. Further simulation work is presented in the end of this section.

2.3.1 Derivation of the social-self NCPM

Some notations are used in this section.

Notation 2.3.1 *For the competitive propagation of products $\{H_1, H_2, \dots, H_R\}$ on the network $G = (V, E)$,*

(1) define the random variable $X_i^r(t)$ by $X_i^r(t) = 1$ if $D_i(t) = H_r$; $X_i^r(t) = 0$ if $D_i(t) \neq H_r$. Due to the mutual exclusiveness of the products, for any $i \in V$, if $X_i^r(t) = 1$, then $X_i^s(t) = 0$ for any $s \neq r$;

(2) Define the $n-1$ tuple $\boldsymbol{D}_{-i}(t) = (D_1(t), \dots, D_{i-1}(t), D_{i+1}(t), \dots, D_n(t))$, i.e., the states of all the nodes except node i after time step t;

(3) Define the following notations for simplicity:

$$
\begin{aligned}
P_{ij}^{rs}(t) &= \mathbb{P}[X_i^r(t) = 1 \,|\, X_j^s(t) = 1], \\
P_i^r(t; -i) &= \mathbb{P}[X_i^r(t) = 1 \,|\, \boldsymbol{D}_{-i}(t)], \\
\Gamma_i^r(t; s, -i) &= \mathbb{P}[X_i^r(t+1) = 1 \,|\, X_i^s(t) = 1, \boldsymbol{D}_{-i}(t)].
\end{aligned}
$$

In the derivation of the network competitive propagation model, the following approximation is adopted:

Approximation 2.3.2 (Independence Approximation) *For the competitive propagation of R products on the network $G = (V, E)$, approximate the conditional probability $P_{ij}^{ms}(t)$ by its corresponding total probability $p_{im}(t)$ for any $m, s \in \Theta$ and any $i, j \in V$.*

With the *independence approximation*, the social-self NCPM is presented in the theorem below.

Theorem 2.3.3 (Social-self NCPM) *Consider the competitive propagation based on Assumption 2.1, with the social network and the product-conversion graph represented by* *their adjacency matrices* $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ *and* $\mathscr{D} = (\delta_{rs})_{R\times R}$ *respectively. The probability* $p_{ir}(t)$ *satisfies*

$$\begin{aligned} p_{ir}(t+1) - p_{ir}(t) = &\sum_{s\neq r} \alpha_i \sum_{j=1}^{n} \tilde{a}_{ij} \big(P_{ij}^{sr}(t) p_{jr}(t) - P_{ij}^{rs}(t) p_{js}(t) \big) \\ &+ \sum_{s\neq r} (1-\alpha_i)(\delta_{sr} p_{is}(t) - \delta_{rs} p_{ir}(t)), \end{aligned} \tag{2.1}$$

for any $i \in V$ *and* $r \in \Theta$*. Applying the independence approximation, the approximation model for equation* (2.1)*, i.e., the social-self NCPM, is*

$$p_{ir}(t+1) = \alpha_i \sum_{j=1}^{n} \tilde{a}_{ij} p_{jr}(t) + (1-\alpha_i) \sum_{s=1}^{R} \delta_{sr} p_{is}(t). \tag{2.2}$$

Proof: By definition,

$$p_{ir}(t+1) - p_{ir}(t) = \mathbb{E}\big[\mathbb{E}[X_i^r(t+1) - X_i^r(t) \,|\, \boldsymbol{D}_{-i}(t)]\big],$$

where the conditional expectation is given by

$$\mathbb{E}[X_i^r(t+1) - X_i^r(t) \,|\, \boldsymbol{D}_{-i}(t)] = \sum_{s\neq r} \big(\Gamma_i^r(t; s, -i) P_i^s(t; -i) - \Gamma_i^s(t; r, -i) P_i^r(t; -i) \big).$$

According to Assumption 2.1,

$$\Gamma_i^r(t; s, -i) P_i^s(t; -i) \; = \alpha_i \sum_j \tilde{a}_{ij} X_j^r(t) P_i^s(t; -i) + (1-\alpha_i)\delta_{sr} P_i^s(t; -i).$$

Therefore,

$$\mathbb{E}[\Gamma_i^r(t; s, -i) P_i^s(t; -i)] = \alpha_i \sum_j \tilde{a}_{ij} \mathbb{E}[X_j^r(t) P_i^s(t; -i)] + (1-\alpha_i)\delta_{sr} \mathbb{E}[P_i^s(t; -i)].$$

One the right-hand side of the equation above, $\mathbb{E}[P_i^s(t;-i)] = p_{is}(t)$. Moreover,

$$\mathbb{E}[X_j^r(t)P_i^s(t;-i)] = \sum_{\boldsymbol{d}_{-i-j}} \mathbb{P}[X_i^s(t) = 1, X_j^r(t) = 1, \boldsymbol{D}_{-i-j}(t) = \boldsymbol{d}_{-i-j}] = P_{ij}^{sr}(t)p_{jr}(t).$$

Apply the same computation to $\mathbb{E}[\Gamma_i^s(t;r,-i)P_i^r(t;-i)]$ and then we obtain equation (2.1). Replace $P_{ij}^{sr}(t)$ and $P_{ij}^{rs}(t)$ by $p_{is}(t)$ and $p_{ir}(t)$ respectively and according to the equations $\sum_{s\neq r} p_{is}(t) = 1 - p_{ir}(t)$ and $\sum_{s\neq r} \delta_{rs} = 1 - \delta_{rr}$, we obtain equation (2.2). ■

The derivation of Theorem 2.3.3 is equivalent to the widely adopted mean-field approximation in the modeling of the network epidemic spreading [25, 75, 19]. Notice that the independence approximation neither neglects the correlation between any two nodes' states, nor destroys the network topology, since $p_{jr}(t)$, $p_{js}(t)$ and $\tilde{a}_{ij}$ all appear in the dynamics of $p_{ir}(t)$.

2.3.2 Asymptotic behavior of the social-self NCPM

Define the map $f : \mathbb{R}^{n\times R} \to \mathbb{R}^{n\times R}$ by

$$f(X) = \mathrm{diag}(\boldsymbol{\alpha})\tilde{A}X + (I - \mathrm{diag}(\boldsymbol{\alpha}))X\mathscr{D}. \tag{2.3}$$

According to equation (2.2), the matrix form of the social-self NCPM is written as

$$P(t+1) = f\big(P(t)\big), \tag{2.4}$$

where $P(t) = (p_{ir}(t))_{n\times R}$. We analyze how the asymptotic behavior of system (2.4), i.e., the existence, uniqueness and stability of the fixed point of the map f, is determined by the two graphs introduced in our model: the social network with the adjacency matrix $\tilde{A}$, and the product-conversion graph with the adjacency matrix $\mathscr{D}$.

Structures of the social network and the product-conversion graph

Assume that the social network $G(\tilde{A})$ has a globally reachable node. As for the product-conversion graph, we consider the more general case. Suppose that the product-conversion graph $G(\mathscr{D})$ has m absorbing strongly connected components (absorbing SCCs) and a transient subgraph. Re-index the products such that the product index set for any l-th absorbing SCCs is given by $\Theta_1 = \{1, 2, \dots, k_1\}$, and

$$\Theta_l = \Big\{ \sum_{u=1}^{l-1} k_u + 1, \sum_{u=1}^{l-1} k_u + 2, \dots, \sum_{u=1}^{l} k_u \Big\},$$

for any $l \in \{2, 3, \dots, m\}$, and the index set for the transient subgraph is $\Lambda = \{\sum_{l=1}^{m} k_l + 1, \dots, \sum_{l=1}^{m} k_l + 2, \dots, R\}$. then the adjacency matrix $\mathscr{D}$ of the product-conversion graph takes the following form:

$$\mathscr{D} = \begin{bmatrix} \bar{\mathscr{D}} & \mathbb{0}_{(R-k_0)\times k_0} \\ B_{k_0\times(R-k_0)} & \mathscr{D}_0 \end{bmatrix}, \tag{2.5}$$

where $\bar{\mathscr{D}} = \operatorname{diag}[\mathscr{D}_1, \mathscr{D}_2, \dots, \mathscr{D}_m]$ and $B = [B_1, B_2, \dots, B_m]$, with $B_l \in \mathbb{R}^{k_0\times k_l}$ for any $l \in \{1, 2, \dots, m\}$, is non-zero and entry-wise non-negative. Matrix $\mathscr{D}_l = (\delta_{rs}^{\Theta_l})_{k_l\times k_l}$, with $\delta_{rs}^{\Theta_1} = \delta_{rs}$ and $\delta_{rs}^{\Theta_l} = \delta_{\sum_{u=1}^{l-1} k_u + r, \sum_{u=1}^{l-1} k_u + s}$ for any $l \in \{2, 3, \dots, m\}$, is the adjacency matrix of the l-th absorbing SCC, and is thus irreducible and row-stochastic. The following definition classifies four types of structures of $G(\mathscr{D})$.

Definition 2.3.4 (Four sets of product-conversion graphs) *Based on whether the product-conversion graph $G(\mathscr{D})$ has a transient subgraph and a single or multiple absorbing SCCs, we classify the adjacency matrix $\mathscr{D}$ into the following four cases:*

(i. Case 1 (single SCC): The graph $G(\mathscr{D})$ is strongly connected, i.e., $\mathscr{D} = \mathscr{D}_1$, with $k_1 = R$;

(ii. *Case 2 (single SCC + transient subgraph): The graph $G(\mathscr{D})$ contains one absorbing SCC and a transient subgraph, i.e.,* $\bar{\mathscr{D}} = \mathscr{D}_1$ *and* $k_0 \geq 1$*;*

(iii. *Case 3 (multi-SCC): The graph $G(\mathscr{D})$ contains m absorbing SCCs, i.e.,* $\mathscr{D} = \mathrm{diag}[\mathscr{D}_1, \mathscr{D}_2, \dots, \mathscr{D}_m]$*, with* $\sum_{l=1}^{m} k_l = R$*;*

(iv. *Case 4 (multi-SCC + transient subgraph): The graph $G(\mathscr{D})$ contains m absorbing SCCs and a transient subgraph, with $\mathscr{D}$ given by equation* (2.5)*.*

Stability analysis of the social-self NCPM

The following theorem states the distinct asymptotic behaviors of the social-self NCPM, with different structures of the product-conversion graph.

Theorem 2.3.5 (Asymptotic behavior for social-self NCPM) *Consider the social-self NCPM on a strongly connected social network $G(\tilde{A})$, with the product-conversion graph $G(\mathscr{D})$. Assume that*

(i. *Each absorbing SCC $G(\mathscr{D}_l)$ of $G(\mathscr{D})$ is aperiodic;*

(ii. *For any $\mathscr{D}_l$, $l \in \{1, 2, \dots, m\}$, as least one column of $\mathscr{D}_l$ is entry-wise strictly positive;*

(iii. *For any $r \in \Lambda$, $\sum_{s\in\Lambda} \delta_{rs} < 1$, i.e., $\mathscr{D}_0 \mathbb{1}_{k_0} \prec \mathbb{1}_{k_0}$.*

Then, for any $P(0) \in S_{nR}(\mathbb{1}_n)$, the solution $P(t)$ to equation (2.4) *has the following properties, depending upon the structure of $\mathscr{D}$:*

(i. *in Case 1, $P(t)$ converges to $P^* = \mathbb{1}_n \boldsymbol{v}_{\mathrm{left}}(\mathscr{D})^\top$ exponentially fast, where P^* is the unique fixed point in $S_{nR}(\mathbb{1}_n)$ for the map f defined by equation* (2.3)*. Moreover, the convergence rate is $\epsilon(\mathscr{D}) = \alpha_{\max} + (1-\alpha_{\max})\zeta(\mathscr{D})$, where $\alpha_{\max} = \max_i \alpha_i$ and $\zeta(\mathscr{D}) = 1 - \sum_{r=1}^{R} \min_s \delta_{sr}$;*

(ii. in Case 2, for any $i \in V$,

$$\lim_{t\to\infty} p_{ir}(t) = \begin{cases} 0, & \text{for any } r \in \Lambda, \\ w_r(\mathscr{D}_1), & \text{for any } r \in \Theta_1; \end{cases}$$

(iii. in Case 3, for any $l \in \{1, 2, \dots, m\}$ and $i \in V$,

$$\lim_{t\to\infty} \boldsymbol{p}^{\Theta_l(i)}(t) = \big(\boldsymbol{v}_{\text{left}}^\top(M) P^{\Theta_l}(0) \mathbb{1}_{k_l}\big) \boldsymbol{v}_{\text{left}}^\top(\mathscr{D}_l),$$

where $M = \operatorname{diag}(\boldsymbol{\alpha})\tilde{A} + I - \operatorname{diag}\boldsymbol{\alpha}$ and $P^{\Theta_l}(t) = \big(p_{ir}^{\Theta_l}(t)\big)_{n\times k_l}$, with $p_{ir}^{\Theta_l}(t) = p_{i,\sum_{u=1}^{l-1} k_u + r}(t)$ and $\boldsymbol{p}^{\Theta_l(i)}(t)$ being the i-th row of $P^{\Theta_l}(t)$;

(iv. in Case 4, for any $l \in \{1, 2, \dots, m\}$ and $i \in V$,

$$\lim_{t\to\infty} p_{ir}(t) = \begin{cases} 0, & \text{for any } r \in \Lambda, \\ \gamma_l w_r(\mathscr{D}_l), & \text{for any } r \in \Theta_l, \end{cases}$$

where γ_l depends on $\tilde{A}$, B_l, $P^{\Theta_l}(0)$, $P^{\Lambda}(0)$ and satisfies $\sum_{l=1}^{m} \gamma_l = 1$.

Before proving the theorem above, a useful and well-known lemma is stated without the proof.

Lemma 2.3.6 (Row-stochastic matrices after pairwise-difference similarity transform) *Let $M \in \mathbb{R}^{n\times n}$ be row-stochastic. Suppose the graph $G(M)$ is aperiodic and has a globally* *reachable node. Then the nonsingular matrix*

$$Q = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ 1/n & \dots & 1/n & 1/n \end{bmatrix}$$

satisfies

$$QMQ^{-1} = \begin{bmatrix} M_{\text{red}} & \mathbb{0}_{n-1} \\ \boldsymbol{c}^\top & 1 \end{bmatrix}$$

for some $\boldsymbol{c} \in \mathbb{R}^{n-1}$ *and* $M_{\text{red}} \in \mathbb{R}^{(n-1)\times(n-1)}$*. Moreover,* M_{red} *is discrete-time exponentially stable.*

Proof of Theorem 2.3.5: (1) Case 1:

Since matrix $\mathscr{D}$ is row-stochastic, irreducible and aperiodic, according to the Perron-Frobenius theorem, $\boldsymbol{v}_{\text{left}}(\mathscr{D}) \in \mathbb{R}^R$ is well-defined. By substituting P^*, defined by $\boldsymbol{p}^{*(i)} = \boldsymbol{v}_{\text{left}}(\mathscr{D})^\top$ for any $i \in V$, into equation (2.3), we verify that P^* is a fixed point of f.

For any X and $Y \in \mathbb{R}^{n\times R}$, define the distance $d(\cdot,\cdot)$ by $d(X,Y) = \|X - Y\|_\infty$. Then $(S_{nR}(\mathbb{1}_n), d)$ is a complete metric space. For any $X \in S_{nR}(\mathbb{1}_n)$, it is easy to check that $f(X) \succeq \mathbb{0}_{n\times R}$ and

$$f(X)\mathbb{1}_R = \text{diag}(\boldsymbol{\alpha})\tilde{A}X\mathbb{1}_R + (I - \text{diag}(\boldsymbol{\alpha}))X\mathbb{1}_R = \mathbb{1}_n.$$

Therefore, f maps $S_{nR}(\mathbb{1}_n)$ to $S_{nR}(\mathbb{1}_n)$.

For any $X \in S_{nR}(\mathbb{1}_n)$, according to equation (2.3),

$$\|f(X)^{(i)} - f(P^*)^{(i)}\|_1 \leq \alpha_i\|\boldsymbol{x}^{(-i)} - \boldsymbol{p}^{*(-i)}\|_1 + (1-\alpha_i)\|(\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)})\mathscr{D}\|_1. \qquad (2.6)$$

The first term of the right-hand side of (2.6) satisfies

$$\|\boldsymbol{x}^{(-i)} - \boldsymbol{p}^{*(-i)}\|_1 \leq \sum_{r=1}^{R}\sum_{j=1}^{n} \tilde{a}_{ij}|x_{jr} - w_r(\mathscr{D})| \leq \|X - P^*\|_\infty.$$

The second term of the right-hand side of (2.6) satisfies

$$\|(\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)})\mathscr{D}\|_1 = \sum_{r=1}^{R}|\sum_{s=1}^{R}\big(x_{is} - w_s(\mathscr{D})\big)\delta_{sr}|.$$

If $\boldsymbol{x}^{(i)} = \boldsymbol{p}^{*(i)}$, then $\|f(X)^{(i)} - f(P^*)^{(i)}\|_1 \leq \alpha_i\|X - P^*\|_\infty$. If $\boldsymbol{x}^{(i)} \neq \boldsymbol{p}^{*(i)}$, since $\boldsymbol{x}^{(i)}\mathbb{1}_R = \boldsymbol{p}^{*(i)}\mathbb{1}_R = 1$, both the set $\theta_1 = \{s \,|\, x_{is} \geq w_s(\mathscr{D})\}$ and the set $\theta_2 = \{s \,|\, x_{is} < w_s(\mathscr{D})\}$ are nonempty and

$$\sum_{s\in\theta_1}\big(x_{is} - w_s(\mathscr{D})\big) = \sum_{s\in\theta_2}\big(w_s(\mathscr{D}) - x_{is}\big) = \frac{1}{2}\sum_{s=1}^{R}|x_{is} - w_s(\mathscr{D})|.$$

Therefore,

$$\begin{aligned}\|(\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)})\mathscr{D}\|_1 = &\sum_{r=1}^{R}\sum_{s=1}^{R}|x_{is} - w_s(\mathscr{D})|\delta_{sr}\\ &- 2\sum_{r=1}^{R}\min\{\sum_{s\in\theta_1}(x_{is} - w_s(\mathscr{D}))\delta_{sr}, \sum_{s\in\theta_2}(w_s(\mathscr{D}) - x_{is})\delta_{sr}\},\end{aligned} \tag{2.7}$$

where

$$\min\{\sum_{s\in\theta_1}(x_{is} - w_s(\mathscr{D}))\delta_{sr}, \sum_{s\in\theta_2}(w_s(\mathscr{D}) - x_{is})\delta_{sr}\} \geq \frac{1}{2}\min_s \delta_{sr}\|\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)}\|_1.$$

Substituting the inequality above into (2.7), we obtain

$$\|(\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)})\mathscr{D}\|_1 \leq \Big(1 - \sum_{r=1}^{R}\min_s \delta_{sr}\Big)\|\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)}\|_1.$$

Since $\sum_{r=1}^{R}\delta_{sr}=1$ for any s, $\sum_{r=1}^{R}\min_s\delta_{sr}$ is no larger than 1. In addition, since at least one column of $\mathscr{D}$ is strictly positive, $\sum_{r=1}^{R}\min_s\delta_{sr}>0$. Therefore, $0\le\zeta(\mathscr{D})=1-\sum_{r=1}^{R}\min_s\delta_{sr}<1$, and

$$\|f(X)^{(i)}-\boldsymbol{p}^{*(i)}\|_1\le\big(\alpha_i+(1-\alpha_i)\zeta(\mathscr{D})\big)|X-P^*\|_\infty.$$

This leads to

$$\|f(X)-f(P^*)\|_\infty\le\epsilon(\mathscr{D})\|X-P^*\|_\infty,$$

for any $X\in S_{nR}(\mathbb{1}_n)$ and $0<\epsilon(\mathscr{D})<1$. This concludes the proof for Case 1.

(2) Case 2:

For the transient subset Λ, define $P^\Lambda(t)=\big(p^\Lambda_{ir}(t)\big)_{n\times k_0}$, with $p^\Lambda_{ir}(t)=p_{i,r+k_1}(t)$, for any $i\in V$ and $r\in\{1,2,\dots,k_0\}$. Then,

$$P^\Lambda(t+1)=\mathrm{diag}(\boldsymbol{\alpha})\tilde{A}P^\Lambda(t)+(I-\mathrm{diag}(\boldsymbol{\alpha}))P^\Lambda(t)\mathscr{D}_0.$$

According to Assumption (iii) of Theorem 2.3.5,

$$c=\max_{r\in\{1,2,\dots,k_0\}}\sum_{s=1}^{k_0}\delta^\Lambda_{rs}<1,\quad\text{and}\quad\mathscr{D}_0\mathbb{1}_{k_0}\le c\mathbb{1}_{k_0}.$$

Therefore,

$$P^\Lambda(t+1)\mathbb{1}_{k_0}\preceq\Big(\mathrm{diag}(\boldsymbol{\alpha})\tilde{A}+c\big(I-\mathrm{diag}(\boldsymbol{\alpha})\big)\Big)P^\Lambda(t)\mathbb{1}_{k_0}.$$

Since $\rho\Big(\mathrm{diag}(\boldsymbol{\alpha})\tilde{A}+c\big(I-\mathrm{diag}(\boldsymbol{\alpha})\big)\Big)<1$, for any $P^\Lambda(0)\in\tilde{S}_{nk_0}(\mathbb{1}_n)$, $P^\Lambda(t)\to\mathbb{0}_{n\times k_0}$ exponentially fast.

Define $P^{\Theta_1}(t) = (p_{ir}(t))_{n\times k_1}$. then we have

$$P^{\Theta_1}(t+1) = \text{diag}(\boldsymbol{\alpha})\tilde{A}P^{\Theta_1}(t) + \big(I - \text{diag}(\boldsymbol{\alpha})\big)P^{\Theta_1}(t)\mathscr{D}_1 + \big(I - \text{diag}(\boldsymbol{\alpha})\big)P^{\Lambda}(t)B.$$

Since $P^{\Lambda}(t)$ converges to $\mathbb{0}_{n\times k_0}$ exponentially fast, we have: 1) there exists $C > 0$ and $0 < \xi < 1$ such that

$$\|\big(I - \text{diag}(\boldsymbol{\alpha})P^{\Lambda}(t)B\big)\|_\infty \leq C\xi^t;$$

2) $\|P^{\Theta_1}(t)\mathbb{1}_{k_1} - \mathbb{1}_{k_1}\|_\infty \to 0$ exponentially fast, which implies $d\big(P^{\Theta_1}(t), S_{nk_1}(\mathbb{1}_n)\big) \to 0$ exponentially fast.

For any $X \in \tilde{S}_{nk_1}(\mathbb{1}_n)$, define map $\tilde{f}$ by

$$\tilde{f}(X) = \text{diag}(\boldsymbol{\alpha})\tilde{A}X + \big(I - \text{diag}(\boldsymbol{\alpha})\big)X\mathscr{D}_1.$$

According to the proof for Case 1, there exists a unique fixed point $\tilde{P}^*$ for the map $\tilde{f}$ in $S_{nk_1}(\mathbb{1}_n)$, given by $\tilde{p}^*_{ir} = w_r(\mathscr{D}_1)$. Moreover, there exists $0 < \epsilon < 1$ such that, for any $X \in S_{nk_1}(\mathbb{1}_n)$,

$$\|\tilde{f}(X) - \tilde{P}^*\|_\infty \leq \epsilon\|X - \tilde{P}^*\|_\infty.$$

Since the function $\frac{\|\tilde{f}(X)-\tilde{P}^*\|_\infty}{\|X-\tilde{P}^*\|_\infty}$ is continuous in $\tilde{S}_{nk_1}(\mathbb{1}_n)\setminus\tilde{P}^*$ and $d\big(P^{\Theta_1}(t), S_{nk_1}(\mathbb{1}_n)\big) \to 0$, there exists $T > 0$ and $0 < \eta < 1$ such that, for any $t > T$,

$$\|\tilde{f}\big(P^{\Theta_1}(t)\big) - \tilde{P}^*\|_\infty \leq \eta\|P^{\Theta_1}(t) - \tilde{P}^*\|_\infty.$$

For $t \in \mathbb{N}$ much larger than T,

$$\|P^{\Theta_1}(t) - \tilde{P}^*\|_\infty \leq \eta^{t-T}\|P^{\Theta_1}(T) - \tilde{P}^*\|_\infty + C\frac{\xi^t - \eta^{t-T}\xi^T}{\eta/\xi}.$$

Since $0 < \eta < 1$, $0 < \xi < 1$, as $t \to \infty$, $\|P^{\Theta_1}(t) - \tilde{P}^*\|_\infty \to 0$. This concludes the proof for Case 2.

(3) Case 3:

For any $l \in \{1, 2, \dots, m\}$,

$$P^{\Theta_l}(t+1) = \hat{f}\big(P^{\Theta_l}(t)\big) = \big(I - \mathrm{diag}(\boldsymbol{\alpha})\big)P^{\Theta_l}(t)\mathscr{D}_l + \mathrm{diag}(\boldsymbol{\alpha})\tilde{A}P^{\Theta_l}(t),$$

where $\mathscr{D}_l \mathbb{1}_{k_l} = \mathbb{1}_{k_l}$ since Θ_l is absorbing and strongly connected. Therefore,

$$P^{\Theta_l}(t+1)\mathbb{1}_{k_l} = MP^{\Theta_l}(t)\mathbb{1}_{k_l},$$

where $M = I - \mathrm{diag}(\boldsymbol{\alpha}) + \mathrm{diag}(\boldsymbol{\alpha})\tilde{A}$ is row-stochastic and aperiodic. Moreover, the graph $G(M)$ has a globally reachable node and therefore the matrix M has a normalized dominant left eigenvector $\boldsymbol{v}_{\mathrm{left}}(M)$. Applying the Perron-Frobenius theorem,

$$\lim_{t\to\infty} P^{\Theta_l}(t)\mathbb{1}_{k_l} = \big(\boldsymbol{v}_{\mathrm{left}}^\top(M)P^{\Theta_l}(0)\mathbb{1}_{k_l}\big)\mathbb{1}_n.$$

Let $c_l = \boldsymbol{v}_{\mathrm{left}}^\top(M)P^{\Theta_l}(0)\mathbb{1}_{k_l}$. Following the same line of argument in the proof for Case 2, $\hat{f}$ maps $S_{nk_l}(c_l\mathbb{1}_n)$ to $S_{nk_l}(c_l\mathbb{1}_n)$, and maps $\tilde{S}_{nk_l}(c_l\mathbb{1}_n)$ to $\tilde{S}_{nk_l}(c_l\mathbb{1}_n)$. Moreover, $\hat{P}^* \in \mathbb{R}^{n\times k_l}$ with $\hat{\boldsymbol{p}}^{*(i)} = c_l\boldsymbol{v}_{\mathrm{left}}(\mathscr{D}_l)^\top$, for any $i \in V$, is the unique fixed point of the map $\hat{f}$ in $S_{nk_l}(c_l\mathbb{1}_n)$. In addition, there exists $0 < \epsilon < 1$ such that for any $X \in S_{nk_l}(c_l\mathbb{1}_n)$,

$$\|\hat{f}(X) - \hat{P}^*\|_\infty \leq \epsilon\|X - \hat{P}^*\|_\infty.$$

The function $\hat{h}(X) = \frac{\|\hat{f}(X)-\hat{P}^*\|_\infty}{\|X-\hat{P}^*\|_\infty}$ is continuous in $\tilde{S}_{nk_l}(c_l\mathbb{1}_n) \setminus \hat{P}^*$. Since for any $P^{\Theta_l}(0) \in \tilde{S}_{nk_l}(c_l\mathbb{1}_n)\setminus\hat{P}^*$, we have $P^{\Theta_l}(t)\mathbb{1}_{k_l} \to c_l\mathbb{1}_{k_l}$, which implies $d\big(P^{\Theta_l}(t), S_{nk_l}(c_l\mathbb{1}_{k_l})\big) \to$

0 as $t \to 0$. Therefore, there exists $0 < \eta < 1$ and $T > 0$ such that for any $t > T$,

$$\|\hat{f}\big(P^{\Theta_l}(t)\big) - \hat{P}^*\|_\infty \leq \eta \|P^{\Theta_l}(t) - \hat{P}^*\|_\infty.$$

Therefore, $P^{\Theta_l}(t) \to \hat{P}^*$ as $t \to \infty$.

(4) Case 4:

$$P^{\Theta_l}(t+1) = \operatorname{diag}(\boldsymbol{\alpha})\tilde{A}P^{\Theta_l}(t) + \big(I - \operatorname{diag}(\boldsymbol{\alpha})\big)P^{\Theta_l}(t)\mathscr{D}_l + \big(I - \operatorname{diag}(\boldsymbol{\alpha})\big)P^{\Lambda}(t)B_l.$$

for any $l \in \{1, 2, \dots, m\}$. Therefore,

$$P^{\Theta_l}(t+1)\mathbb{1}_{k_l} = MP^{\Theta_l}(t)\mathbb{1}_{k_l} + \boldsymbol{\phi}(t), \tag{2.8}$$

where $M = \operatorname{diag}(\boldsymbol{\alpha})\tilde{A} + I - \operatorname{diag}(\boldsymbol{\alpha})$ is row-stochastic and primitive. The vector $\boldsymbol{\phi}(t)$ is a vanishing perturbation according to the proof for Case 2.

Let $\boldsymbol{x}(t) = P^{\Theta_l}(t)\mathbb{1}_{k_l}$ and $\boldsymbol{y}(t) = Q\boldsymbol{x}(t)$ with Q defined in Lemma 2.3.6. Let $\boldsymbol{y}_{\mathrm{err}}(t) = (y_1(t), y_2(t), \dots, y_{n-1}(t))^\top$, where $y_i(t) = x_{i+1}(t) - x_i(t)$ for any $i = 1, 2, \dots, n-1$. Then we have

$$\boldsymbol{y}(t+1) = QMQ^{-1}\boldsymbol{y}(t) + Q\boldsymbol{\phi}(t).$$

Let $\boldsymbol{\varphi}(t) = \big(\varphi_1(t), \varphi_2(t), \dots, \varphi_{n-1}(t)\big)^\top$ with $\varphi_i(t) = \sum_j Q_{ij}\phi_j(t)$. $\boldsymbol{\varphi}(t)$ is also a vanishing perturbation and

$$\boldsymbol{y}_{\mathrm{err}}(t+1) = M_{\mathrm{red}}\boldsymbol{y}_{\mathrm{err}}(t) + \boldsymbol{\varphi}(t).$$

The equation above is an exponentially stable linear system with a vanishing perturbation. Since $\rho(M_{\mathrm{red}}) < 1$, $\boldsymbol{y}_{\mathrm{err}} \to \mathbb{0}_{n-1}$ as $t \to \infty$, which implies that $P^{\Theta_l}(t)\mathbb{1}_{k_l} \to \gamma\mathbb{1}_n$ and γ_l depends on M, B_l, $P^{\Theta_l}(0)$ and $P^{\Lambda}(0)$. Moreover, $\sum_l \gamma_l = 1$ since $P(t)\mathbb{1}_R = \mathbb{1}_n$.

Following the same argument in the proof for Case 3, we obtain

$$\lim_{t\to\infty} \boldsymbol{p}^{\Theta_l(i)}(t) = \gamma_l \boldsymbol{v}_{\mathrm{left}}^{\top}(\mathscr{D}_l).$$

■

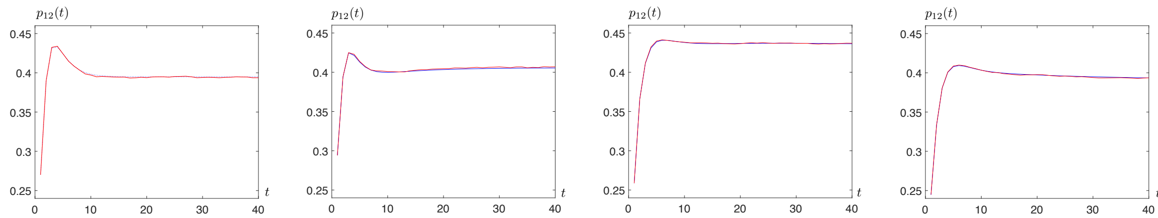
Interpretations of Theorem 2.3.5

Analysis on Case 1 to 4 leads to the following conclusions: 1) The probability of adopting any product in the transient subgraph eventually decays to zero; 2) For the product-conversion graph with only one absorbing SCC $G(\mathscr{D}_1)$, the system's asymptotic product-adoption probability distribution only depends on $\boldsymbol{v}_{\mathrm{left}}(\mathscr{D}_1)$. In this case, the self conversion dominates the competitive propagation process; 3) With multiple absorbing SCCs in the product-conversion graph, the initial condition $P(t)$ and the structure of the social network $G(\tilde{A})$ together determine the fraction each absorbing SCC eventually takes in the total probability 1; 4) In each absorbing SCC $G(\mathscr{D}_l)$, the asymptotic adoption probability for each product is proportional to its corresponding entry of $\mathscr{D}_l$.

2.3.3 Further simulation work

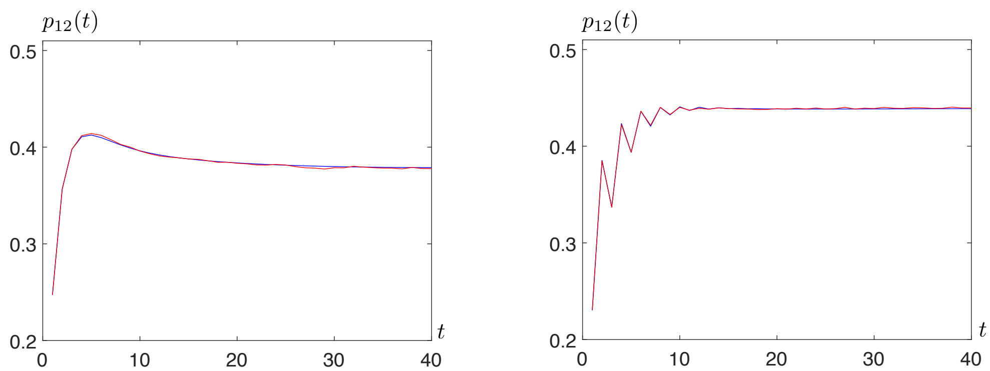
a) Accuracy of the social-self NCPM solution: Simulation results have been presented to compare the solution to the social-self NCPM with the solution to the original Markov chain model defined by Assumption 2.1. Let the matrix $\mathscr{D}$ take the following form

$$\mathscr{D} = \begin{bmatrix} \mathscr{D}_1 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathscr{D}_2 & \mathbb{0} \\ B_1 & B_2 & \mathscr{D}_0 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.8 & 0 & 0.2 \end{bmatrix}. \tag{2.9}$$



(a) $n = 5$, $p = 1$ (b) $n = 10$, $p = 0.5$ (c) $n = 50$, $p = 0.1$ (d) $n = 50$, $p = 1$

Figure 2.3: Difference between the solutions to the social-self NCPM (blue dash) and the original Markov-chain model (red) in complete graphs or Erdős-Rényi graphs.



(a) power-law graph (b) star graph

Figure 2.4: Difference between the solutions to the social-self NCPM (blue dash) and the original Markov-chain model (red) in the power-law graph and the star graph. The power-law graph has 100 nodes, with the degree distribution $p(k) = 1010k^{-2.87}$. The star graph consists of 10 nodes with node 1 as the center.

The Markov-chin solution is computed by the Monte Carlo method. In each sampling, A, $\boldsymbol{\alpha}$ and $P(0)$ are randomly generated and set identical for the Markov chain and the NCPM. The probability $p_{12}(t)$ is plotted for both models on different types of social networks, such as the complete graph, the Erdős-Rényi graph, the power-law graph and the star graph. As shown in Figure 2.3 and Figure 2.4, the solution to the social-self NCPM nearly overlaps with the Markov-chain solution in every plot, due to the i.i.d self conversion process.

b) Asymptotic behavior of the Markov chain model In Figure 2.5 and Figure 2.6, all the trajectories $p_{ir}(t)$, for the Markov-chain model on an Erdős-Rényi graph with $n = 5$, $p = 0.4$ and randomly generated $\boldsymbol{\alpha}$, are computed by the Monte Carlo method. Figure 2.5(a) corresponds to the structure of the product-conversion graph defined by Case 4 in Definition 2.3.4 with

$$\mathscr{D}_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \mathscr{D}_2 = 1, \mathscr{D}_0 = 0.2, B = [0 \ 0.8 \ 0].$$

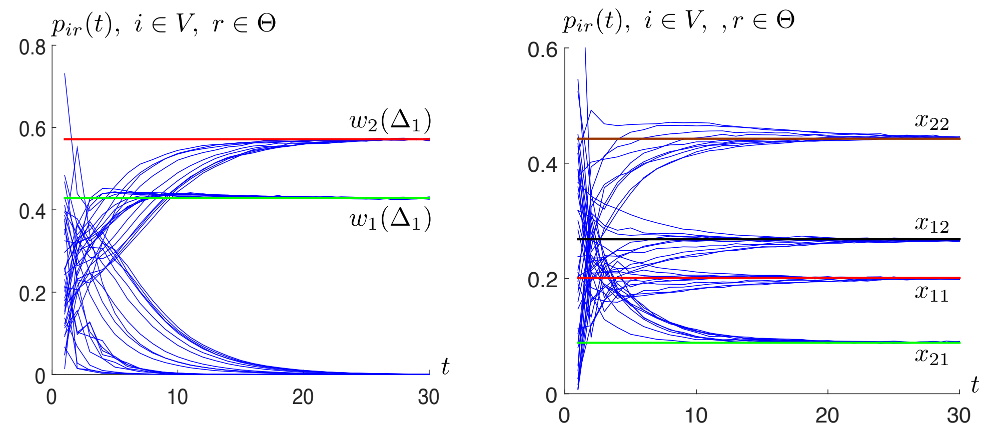
The transient subgraph is only connected to SCC Θ_1 and the intial adoption probability for H_3 is 0. Figure 2.5(b) corresponds to the structure of the product-conversion graph defined by Case 3 in Defintion 2.3.4 with

$$\mathscr{D} = \begin{bmatrix} \mathscr{D}_1 & \mathbb{0} \\ \mathbb{0} & \mathscr{D}_2 \end{bmatrix}, \mathscr{D}_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \mathscr{D}_2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix}.$$

The simulation results shows that, in these two cases the Markov-chain solutions converge exactly to the values indicated by the social-self NCPM, regardless of the initial condition. The matrix $\mathscr{D}$ used in Figure 2.6 is given by equation (2.9). As illustrated by Figure 2.6, the asymptotic adoption probabilities vary with the initial condition in the Markov-chain model, in consistence with the results of Theorem 2.3.5.

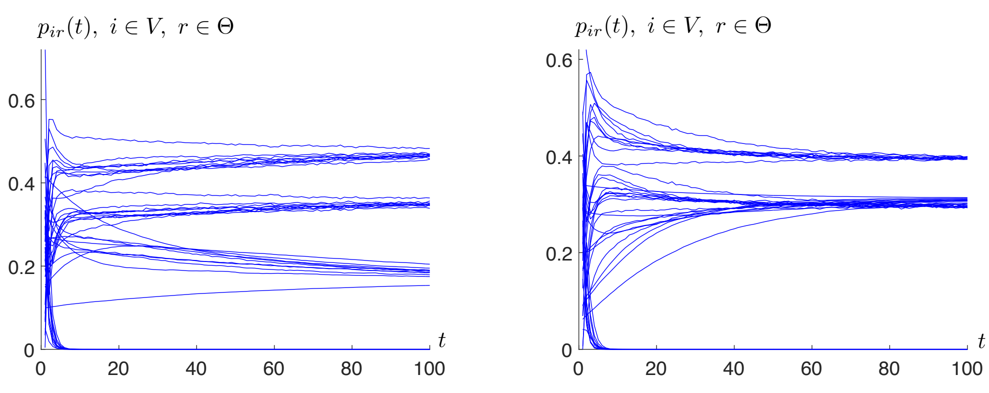
2.4 Analysis of the Self-social Network Competitive Propagation Model

In this section we discuss the network competitive propagation model based on Assumption 2.2, i.e, the case in which self conversion occurs before social conversion at each time step. Firstly we propose an approximation model, referred to as the *self-social net-*



(a) multi-SCC + transient subgraph (b) multi-SCC

Figure 2.5: Asymptotic behavior of the Markov chain model with the production-conversion graphs defined by Case 3 or Case 4 in Defintion 2.3.4. Every curve in this plot is a trajectory $p_{ir}(t)$ for $i \in V$ and $r \in \Theta$. Here $x_{lr} = \boldsymbol{v}_{\text{left}}^{\top}(M) P^{\Theta_l}(0) \mathbb{1}_{k_l} w_r(\mathscr{D}_l)$.



(a) initial condition 1 (b) initial condition 2

Figure 2.6: Asymptotic behavior of the Markov chain model with the production-conversion graph consisting of multiple SCCs and a transient subgraph. Every curve in this plot is a trajectory $p_{ir}(t)$ for $i \in V$ and $r \in \Theta$.

work competitive propagation model (self-social NCPM), and then analyze the dynamical properties of this approximation model.

Theorem 2.4.1 (Self-social NCPM) *Consider the competitive propagation model based on Assumption 2.2, with the social network and the product-conversion graph represented by their adjacency matrices $\tilde{A}$ and $\mathscr{D}$ respectively. The probability $p_{ir}(t)$ satisfies*

$$p_{ir}(t+1) - p_{ir}(t) = \sum_{s\neq r} \big(\delta_{sr} p_{is}(t) - \delta_{rs} p_{ir}(t)\big) + \sum_{s\neq r} \delta_{ss}\alpha_i \sum_{j=1}^{n} \tilde{a}_{ij} p_{is}(t) P_{ji}^{rs}(t)$$
$$- \sum_{s\neq r} \delta_{rr}\alpha_i \sum_{j=1}^{n} \tilde{a}_{ij} p_{ir}(t) P_{ji}^{sr}(t),$$

for any $i \in V$ and $r \in \Theta$. Applying the independence assumption, the matrix form of the self-social NCPM is

$$P(t+1) = P(t)\mathscr{D} + \operatorname{diag}(\boldsymbol{\alpha}) \operatorname{diag}\big(P(t)\boldsymbol{\delta}\big)\tilde{A}P(t) - \operatorname{diag}(\boldsymbol{\alpha})P(t)\operatorname{diag}(\boldsymbol{\delta}), \tag{2.10}$$

with $P(t) = (p_{ir}(t))_{n\times R}$ and $\boldsymbol{\delta} = (\delta_{11}, \delta_{22}, \dots, \delta_{RR})^\top$.

It is straightforward to check that, for any $P(t) \in S_{nR}(\mathbb{1}_n)$, $P(t+1)$ is still in $S_{nR}(\mathbb{1}_n)$. According to the Brower fixed point theorem, there exists at least one fixed point for the system (2.10) in $S_{nR}(\mathbb{1}_n)$. Since the nonlinearity of equation (2.10) add much difficulty to the analysis of it, in the remaining part of this section we discuss the special case when $R = 2$.

For simplicity, in this section, let $\boldsymbol{p}(t) = \boldsymbol{p}_2(t) = \big(p_{12}(t), p_{22}(t), \dots, p_{n2}(t)\big)^\top$. Without loss of generality, assume $\delta_{22} \geq \delta_{11}$. Define the map $h : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{aligned} h(\boldsymbol{x}) = \delta_{12}\mathbb{1}_n + (1 - \delta_{12} - \delta_{21})\boldsymbol{x} + \delta_{11} \operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{x} - \delta_{22} \operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{x} \\ + (\delta_{22} - \delta_{11}) \operatorname{diag}(\boldsymbol{\alpha}) \operatorname{diag}(\boldsymbol{x})\tilde{A}\boldsymbol{x}. \end{aligned} \tag{2.11}$$

Then the self-social NCPM for $R = 2$ is written as

$$\boldsymbol{p}(t+1) = h(\boldsymbol{p}(t)), \tag{2.12}$$

and $\boldsymbol{p}_1(t)$ is computed by $\boldsymbol{p}_1(t) = \mathbb{1}_n - \boldsymbol{p}(t)$.

We present below the main theorem of this section.

Theorem 2.4.2 (Dynamical behavior of self-social NCPM with $R = 2$) *Consider the two-product self-social NCPM, given by equations* (2.11) *and* (2.12)*, with the parameters* $\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \alpha_1, \dots, \alpha_n$ *all in the interval* $(0, 1)$*, and* $\delta_{22} \geq \delta_{11}$*. We conclude that,*

(i. system (2.12) *has a unique fixed point* $\mathbf{p}^* \in [0,1]^n$*;*

(ii. the unique fixed point $\mathbf{p}^*$ *satisfies*

$$\frac{1}{2}\mathbb{1}_n \preceq \boldsymbol{p}^* \preceq \frac{\delta_{12}}{\delta_{12} + \delta_{21}}\mathbb{1}_n, \quad \textit{and} \tag{2.13}$$

$$p_i^* - p_{-i}^* \leq \frac{1 - \frac{1}{2}\alpha_i}{\alpha_i}\frac{\delta_{22} - \delta_{11}}{\delta_{22} + \delta_{11}}; \tag{2.14}$$

(iii. if $\delta_{22} = \delta_{11}$*, the unique fixed point* $\mathbf{p}^*$ *for system* (2.12) *is globally exponentially stable; (By "globally" we mean "for any* $\boldsymbol{p}(0) \in [0,1]^n$*.")*

(iv. if $\delta_{22} > \delta_{11}$*, and*

$$\alpha_i < \frac{8\delta_{11}\delta_{22}}{(\delta_{22} - \delta_{11})^2 + 8\delta_{11}\delta_{22}} \textit{ for any } i \in V, \tag{2.15}$$

then $\mathbf{p}^*$ *is locally asymptotically stable;*

(v. if $\delta_{22} > \delta_{11}$, and

$$\alpha_i < \frac{\delta_{22} + \delta_{11}}{3\delta_{22} - \delta_{11}} \text{ for any } i \in V, \tag{2.16}$$

then $\boldsymbol{p}^*$ is globally exponentially stable. Moreover, the convergence rate is upper bounded by $\max_i \big(\max\{\epsilon_i, K_i\epsilon_i + K_i - 1\}\big)$, where ϵ_i and K_i are defined as $\epsilon_i = (2\delta_{22} - \delta_{11})\alpha_i/K_i$ and $K_i = \delta_{12} + \delta_{21} + \delta_{22}\alpha_i$, respectively.

Proof: We start the proof by establishing that h is a continuous map from $[0,1]^n$ to $[0,1]^n$ itself. Firstly, since

$$\begin{aligned} h(\boldsymbol{x}) = &\ \delta_{12}(\mathbb{1}_n - \boldsymbol{x}) + \delta_{11}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{x} + (1-\delta_{21})\boldsymbol{x} - \delta_{22}\operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{x} \\ &+ (\delta_{22} - \delta_{11})\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{x})\tilde{A}\boldsymbol{x}, \end{aligned}$$

and

$$(1-\delta_{21})\boldsymbol{x} - \delta_{22}\operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{x} \succeq (1 - \delta_{21} - \delta_{22})\boldsymbol{x} = \mathbb{0}_n,$$

the right-hand side of the expression of h is non-negative. Therefore, for any $\boldsymbol{x} \in [0,1]^n$, $h(\boldsymbol{x}) \succeq \mathbb{0}_n$. Secondly, recall that $x_{-i} = (\tilde{A}\boldsymbol{x})_i = \sum_j \tilde{a}_{ij}x_j$. That is, x_{-i} is the weighted average of all the x_j's except x_i and the value of x_{-i} does not depend on x_i since $\tilde{a}_{ii} = 0$. Moreover, since $\sum_j \tilde{a}_{ij} = 1$ for any $i \in V$, x_{-i} is also in the interval $[0,1]$. According to equation (2.11), rewrite the i-th entry of $h(\boldsymbol{x})$ as

$$h(\boldsymbol{x})_i = \delta_{12} + \delta_{11}\alpha_i x_{-i} + \eta_i x_i,$$

where $\eta_i = 1 - \delta_{12} - \delta_{21} - \delta_{22}\alpha_i + (\delta_{22} - \delta_{11})\alpha_i x_{-i}$. The maximum value of η_i is $1 - \delta_{12} - \delta_{21} - \delta_{11}\alpha_i$, obtained when $x_{-i} = 1$. Therefore,

$$\eta_i x_i \leq \max(1 - \delta_{12} - \delta_{21} - \delta_{11}\alpha_i,\ 0).$$

Then we have

$$h(\boldsymbol{x})_i \leq \delta_{12} + \delta_{11}\alpha_i + \max(1 - \delta_{12} - \delta_{21} - \delta_{11}\alpha_i,\ 0) = \max(\delta_{22},\ \delta_{12} + \delta_{11}\alpha_i) < 1.$$

The inequality above leads to $h(\boldsymbol{x}) \preceq \mathbb{1}_n$ for any $\boldsymbol{x} \in [0,1]^n$. Since h maps $[0,1]^n$ to $[0,1]^n$ itself, according to the Brower fixed point theorem, there exists $\boldsymbol{p}^*$ such that $h(\boldsymbol{p}^*) = \boldsymbol{p}^*$. This concludes the proof of the existence of a fixed point.

Any fixed point of h should satisfy $h(\boldsymbol{p}^*) = \boldsymbol{p}^*$, i.e.,

$$\begin{aligned} \mathbb{0}_n = {} & \delta_{12}\mathbb{1}_n + \delta_{11}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{p}^* + (\delta_{22} - \delta_{11})\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{p}^*)\tilde{A}\boldsymbol{p}^* \\ & - (\delta_{12} + \delta_{21})\boldsymbol{p}^* - \delta_{22}\operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{p}^*. \end{aligned} \tag{2.17}$$

Therefore,

$$\boldsymbol{p}^* = \delta_{12}K^{-1}\mathbb{1}_n + \delta_{11}K^{-1}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{p}^* + (\delta_{22} - \delta_{11})K^{-1}\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{p}^*)\tilde{A}\boldsymbol{p}^*,$$

where $K = (\delta_{12} + \delta_{21})I + \delta_{22}\operatorname{diag}(\boldsymbol{\alpha})$ is a positive diagonal matrix. Define a map $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(\boldsymbol{x}) = \delta_{12}K^{-1}\mathbb{1}_n + \delta_{11}K^{-1}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{x} + (\delta_{22} - \delta_{11})K^{-1}\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{x})\tilde{A}\boldsymbol{x}. \tag{2.18}$$

We have that map h has a unique fixed point if and only if map T has a unique fixed point. For any $\boldsymbol{x}$ and $\boldsymbol{y} \in [0,1]^n$, define the distance $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|_\infty$. Then $([0,1]^n, d)$ is a complete metric space. According to equation (2.18), since K^{-1}, $\operatorname{diag}(\boldsymbol{\alpha})$, $\tilde{A}$, $\delta_{22}-\delta_{11}$ and $\operatorname{diag}(\boldsymbol{x})$ are all nonnegative, for any $\boldsymbol{x}$, $\boldsymbol{y} \in [0,1]^n$ and $\boldsymbol{x} \preceq \boldsymbol{y}$, we have $T(\boldsymbol{x}) \preceq T(\boldsymbol{y})$. Moreover,

$$T(\mathbb{0}_n) = \delta_{12}K^{-1}\mathbb{1}_n \succ \mathbb{0}_n, \quad \text{and}$$

$$T(\mathbb{1}_n) = \delta_{12}K^{-1}\mathbb{1}_n + \delta_{11}K^{-1}\boldsymbol{\alpha} + (\delta_{22} - \delta_{11})K^{-1}\boldsymbol{\alpha} = \delta_{12}K^{-1}\mathbb{1}_n + \delta_{22}K^{-1}\boldsymbol{\alpha}.$$

Since

$$T(\mathbb{1}_n)_i = \frac{\delta_{12} + \delta_{22}\alpha_i}{\delta_{12} + \delta_{21} + \delta_{22}\alpha_i} < 1,$$

we have $T(\mathbb{1}_n) \prec \mathbb{1}_n$. Therefore, T maps $[0,1]^n$ to $[0,1]^n$. For any $\boldsymbol{x}$, $\boldsymbol{y} \in [0,1]^n$,

$$T(\boldsymbol{x})_i - T(\boldsymbol{y})_i = \frac{\delta_{11}\alpha_i}{K_i}(x_{-i} - y_{-i}) + \frac{(\delta_{22} - \delta_{11})\alpha_i}{K_i}(x_i x_{-i} - y_i y_{-i}).$$

Moreover,

$$|x_{-i} - y_{-i}| \leq (\sum_{j=1}^{n} \tilde{a}_{ij}) \max_j |x_j - y_j| = \|\boldsymbol{x} - \boldsymbol{y}\|_\infty,$$

and

$$|x_i x_{-i} - y_i y_{-i}| \leq \max\big(\max_i y_i^2 - \min_i x_i^2,\ \ \max_i x_i^2 - \min_i y_i^2\big) \leq 2\|\boldsymbol{x} - \boldsymbol{y}\|_\infty.$$

Therefore,

$$|T(\boldsymbol{x})_i - T(\boldsymbol{y})_i| \leq \epsilon_i \|\boldsymbol{x} - \boldsymbol{y}\|_\infty,$$

where $\epsilon_i = \frac{(2\delta_{22} - \delta_{11})\alpha_i}{\delta_{12} + \delta_{21} + \delta_{22}\alpha_i}$. One can check that $\epsilon_i < 1$ for any $i \in V$ and ϵ_i does not depend on the $\boldsymbol{x}$ and $\boldsymbol{y}$. Let $\epsilon = \max_i \epsilon_i$. Then for any $\boldsymbol{x}$, $\boldsymbol{y} \in [0,1]^n$,

$$\|T(\boldsymbol{x}) - T(\boldsymbol{y})\|_\infty \leq \epsilon \|\boldsymbol{x} - \boldsymbol{y}\|_\infty \quad \text{with } \epsilon < 1.$$

Applying the Banach fixed point theorem, we know that the map T possesses a unique fixed point $\boldsymbol{p}^*$ in $[0,1]^n$. In addition, for any $\boldsymbol{p}(0)$, the sequence $\{\boldsymbol{p}(t)\}_{t\in\mathbb{N}}$ defined by $\boldsymbol{p}(t+1) = T\big(\boldsymbol{p}(t)\big)$ satisfies $\lim_{t\to\infty} \boldsymbol{p}(t) = \boldsymbol{p}^*$. This concludes the proof of statement (i).

For statement (ii), one can check that T maps $S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \frac{1}{2}\mathbb{1}_n \preceq \boldsymbol{x} \preceq \frac{\delta_{12}}{\delta_{12}+\delta_{21}}\mathbb{1}_n\}$ to S itself. Since T is a contraction map, the unique fixed point $\boldsymbol{p}^*$ is in S. The concludes the proof for equation (2.13). According to equation (2.17), we have $C_i p_i^* - C_{-i} p_{-i}^* = \delta_{12} - \delta_{12} p_i^*$, where $C_i = \delta_{21} + \delta_{22}\alpha_i$ and $C_{-i} = \delta_{11}\alpha_i + (\delta_{22} - \delta_{11})\alpha_i p_i^*$. Firstly we point out that $C_i > C_{-i}$, since $C_i - C_{-i} = \delta_{21} + \alpha_i(\delta_{22} - \delta_{11})(1 - p_i^*) > 0$. Moreover,

$$p_i^* - p_{-i}^* = \frac{\delta_{12} - \big(\delta_{12} + \delta_{21} + \alpha_i(\delta_{22} - \delta_{11})(1 - p_i^*)\big)p_i^*}{\delta_{11}\alpha_i + (\delta_{22} - \delta_{11})\alpha_i p_i^*}.$$

The right-hand side of the equation above with $\frac{1}{2} \leq p_i^* \leq \frac{\delta_{12}}{\delta_{12}+\delta_{21}}$ achieves its maximum value $\frac{1-\frac{1}{2}\alpha_i}{\alpha_i}\frac{\delta_{22}-\delta_{11}}{\delta_{22}+\delta_{11}}$ at $p_i^* = \frac{1}{2}$. This concludes the proof for equation (2.14).

Now we prove statement (iii). With $\delta_{11} = \delta_{22}$,

$$h\big(\boldsymbol{x}\big) = \boldsymbol{x} + \delta_{12}\mathbb{1}_n - 2\delta_{12}\boldsymbol{x} + \delta_{11}\,\mathrm{diag}(\boldsymbol{\alpha})\big(\tilde{A}\boldsymbol{x} - \boldsymbol{x}\big).$$

One can check that $\boldsymbol{p}^* = \frac{1}{2}\mathbb{1}_n$ is a fixed point. According to statement (i), the fixed point is unique. Let $\boldsymbol{p}(t) = \boldsymbol{y}(t) + \frac{1}{2}\mathbb{1}_n$. Then the two-product self-social NCPM becomes $\boldsymbol{y}(t+1) = M\boldsymbol{y}(t)$, where $M = (1 - 2\delta_{12})I + \delta_{11}\,\mathrm{diag}(\boldsymbol{\alpha})\tilde{A} - \delta_{11}\,\mathrm{diag}(\boldsymbol{\alpha})$. For any $i \in V$, if $1 - 2\delta_{12} - \delta_{11}\alpha_i \geq 0$, then

$$\sum_{j=1}^{n}|M_{ij}| = 1 - 2\delta_{12} - \delta_{11}\alpha_i + \delta_{11}\alpha_i = 1 - 2\delta_{12} < 1;$$

and, if $1 - 2\delta_{12} - \delta_{11}\alpha_i < 0$, then

$$\sum_{j=1}^{n}|M_{ij}| = 2\delta_{12} + \delta_{11}\alpha_i + \delta_{11}\alpha_i - 1 < 1.$$

Since $\rho(M) \leq \|M\|_\infty = \max_i \sum_{j=1}^{n}|M_{ij}|$, the spectral radius of M is strictly less than 1. Therefore, the fixed point $\boldsymbol{p}^* = \frac{1}{2}\mathbb{1}_n$ is globally exponentially stable.

Now consider the case when $\delta_{22} > \delta_{11}$. Let $\boldsymbol{p}(t) = \boldsymbol{y}(t) + \boldsymbol{p}^*$. Then system (2.12) becomes

$$\boldsymbol{y}(t+1) = M\boldsymbol{y} + (\delta_{22} - \delta_{11})\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{y}(t))\tilde{A}\boldsymbol{y}(t).$$

The right-hand side of the equation above is a linear term $M\boldsymbol{y}(t)$ with a constant matrix M, plus a quadratic term. The matrix M can be decomposed as $M = \tilde{M} - \delta_{12}I$ and $\tilde{M} = \tilde{M}^{(1)} + \tilde{M}^{(2)}$ is further decomposed as a diagonal matrix $\tilde{M}^{(1)}$ plus a matrix $\tilde{M}^{(2)}$ in which all the diagonal entries are 0. Since

$$\tilde{M}^{(1)} = (1 - \delta_{12})I - \delta_{22}\operatorname{diag}(\boldsymbol{\alpha}) + (\delta_{22} - \delta_{11})\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\tilde{A}\boldsymbol{p}^*)$$

is a positive diagonal matrix, and

$$\tilde{M}^{(2)} = \delta_{11}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A} + (\delta_{22} - \delta_{11})\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{p}^*)\tilde{A}$$

is a matrix with all the diagonal entries being zero and all the off-diagonal entries being nonnegative. The matrix $\tilde{M} = \tilde{M}^{(1)} + \tilde{M}^{(2)}$ is nonnegative.

Since $\tilde{A} = \operatorname{diag}(\frac{1}{N_1}, \frac{1}{N_2}, \dots, \frac{1}{N_n})A$, the matrix $\tilde{M}$ can be written in the form $DA + E$, where A is symmetric and D, E are positive diagonal matrix. One can easily prove that all the eigenvalues of any matrix in the form $\tilde{M} = DA + E$ are real since $\tilde{M}$ is similar to the symmetric matrix $D^{\frac{1}{2}}(A + D^{-1}E)D^{\frac{1}{2}}$.

The local stability of $\boldsymbol{p}^*$ is equivalent to the inequality $\rho(M) < 1$, which is in turn equivalent to the intersection of the following two conditions: $\lambda_{\max}(\tilde{M}) < 1 + \delta_{12}$ and $\lambda_{\min}(\tilde{M}) > -1 + \delta_{12}$. First we prove $\lambda_{\max}(\tilde{M}) < 1 + \delta_{12}$. Since A is irreducible and $\boldsymbol{\alpha} \succ \mathbb{0}_n$, $\boldsymbol{p}^* \succ \mathbb{0}_n$, we have $\tilde{M}_{ij} > 0$ if and only if $a_{ij} > 0$ for any $i \neq j$. In addition, $\tilde{M}_{ii} > 0$ for any $i \in V$. Therefore, $\tilde{M}$ is irreducible, aperiodic and thus primitive. According to the Perron-Frobenius theorem, $\lambda_{\max}(\tilde{M}) = \rho(\tilde{M})$. We have $\rho(\tilde{M}) \leq \|\tilde{M}\|_\infty$ and for any

$i \in V$,

$$\sum_j |\tilde{M}_{ij}| = 1 - \delta_{21} + (\delta_{22} - \delta_{11})\big(\alpha_i(p^*_{-i} + p^*_i) - \alpha_i\big).$$

According to equation (2.13), for any $i \in V$,

$$1 - \delta_{21} \leq \sum_j |\tilde{M}_{ij}| \leq 1 - \delta_{21} + \frac{(\delta_{12} - \delta_{21})^2}{\delta_{12} + \delta_{21}}\alpha_i < 1 + \delta_{12}.$$

Therefore,

$$\lambda_{\max}(\tilde{M}) \leq 1 - \delta_{21} + \frac{(\delta_{12} - \delta_{21})^2}{\delta_{12} + \delta_{21}}\alpha_i < 1 + \delta_{12}.$$

Now we prove $\lambda_{\min}(\tilde{M}) > -1 + \delta_{12}$. According to the Gershgorin circle theorem,

$$\lambda_{\min}(\tilde{M}) \geq \min_i (\tilde{M}_{ii} - \sum_{j \neq i} |\tilde{M}_{ij}|).$$

For any $i \in V$,

$$\tilde{M}_{ii} - \sum_{j \neq i} |\tilde{M}_{ij}| = 1 - \delta_{21} - \alpha_i(\delta_{22} + \delta_{11}) - \alpha_i(\delta_{22} - \delta_{11})(p^*_i - p^*_{-i}).$$

According to equation (2.14),

$$p^*_i - p^*_{-i} \leq \frac{1 - \frac{1}{2}\alpha_i}{\alpha_i}\frac{\delta_{22} - \delta_{11}}{\delta_{22} + \delta_{11}}.$$

Moreover, inequality (2.15) is necessary and sufficient to

$$\frac{1 - \frac{1}{2}\alpha_i}{\alpha_i}\frac{\delta_{22} - \delta_{11}}{\delta_{22} + \delta_{11}} < \frac{1 - \alpha_i}{\alpha_i}\frac{\delta_{22} + \delta_{11}}{\delta_{22} - \delta_{11}}.$$

Therefore,

$$\tilde{M}_{ii} - \sum_{j\neq i} |\tilde{M}_{ij}| > 1 - \delta_{21} - \alpha_i(\delta_{22} + \delta_{11}) - (1 - \alpha_i)(\delta_{22} + \delta_{11}) = -1 + \delta_{12},$$

for any $i \in V$. That is to say, the inequality (2.15) is sufficient for $\rho(M) < 1$, i.e., the local stability of $\boldsymbol{p}^*$. This concludes the proof for statement (iv).

For statement (v), observe that the maps h and T satisfy the following relation:

$$h(\boldsymbol{x}) = KT(\boldsymbol{x}) + (I - K)\boldsymbol{x},$$

for any $\boldsymbol{x} \in [0, 1]^n$, where $K = (\delta_{12} + \delta_{21})I + \delta_{22}\,\mathrm{diag}(\boldsymbol{\alpha})$. For any $\boldsymbol{x}$, $\boldsymbol{y} \in [0, 1]^n$,

$$|h(\boldsymbol{x})_i - h(\boldsymbol{y})_i| = |K_i\big(T(\boldsymbol{x})_i - T(\boldsymbol{y})_i\big) + (1 - K_i)(x_i - y_i)|.$$

We estimate the upper bound of $|h(\boldsymbol{x})_i - h(\boldsymbol{y})_i|$ in terms of $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty$ in two cases.

Case 1: $\delta_{12} + \delta_{21} + \delta_{22}\alpha_i < 1$ for any i. Firstly,

$$\frac{\delta_{11}}{\delta_{22}} + 1 - \frac{1}{\delta_{22}} < \frac{\delta_{11} + \delta_{22}}{3\delta_{22} - \delta_{11}}$$

always holds as long as $\delta_{11} < \delta_{22}$. Then recall that, for any $\boldsymbol{x}$, $\boldsymbol{y} \in [0, 1]^n$,

$$|T(\boldsymbol{x})_i - T(\boldsymbol{y})_i| \le \epsilon_i \|\boldsymbol{x} - \boldsymbol{y}\|_\infty,$$

where $\epsilon_i = \frac{(2\delta_{22} - \delta_{11})\alpha_i}{K_i} < 1$. Therefore,

$$|h(\boldsymbol{x})_i - h(\boldsymbol{y})_i| \le (K_i\epsilon_i + 1 - K_i)\|\boldsymbol{x} - \boldsymbol{y}\|_\infty,$$

for any $i \in V$. The coefficient $K_i\epsilon_i + 1 - K_i$ is always strictly less than 1 because it is a convex combination of $\epsilon_i < 1$ and 1. Therefore, h is a contraction map.

Case 2: There exists some i such that $\delta_{12} + \delta_{21} + \delta_{22}\alpha_i \geq 1$. In this case, for any such i,

$$|h(\boldsymbol{x})_i - h(\boldsymbol{y})_i| \leq (K_i\epsilon_i + K_i - 1)\|\boldsymbol{x} - \boldsymbol{y}\|_\infty.$$

If $\alpha_i < \frac{\delta_{11}+\delta_{22}}{3\delta_{22}-\delta_{11}}$, then we have

$$K_i\epsilon_i + K_i - 1 = (3\delta_{22} - \delta_{11})\alpha_i + \delta_{12} + \delta_{21} - 1 < \delta_{11} + \delta_{22} + \delta_{12} + \delta_{21} - 1 = 1.$$

Therefore, h is also a contraction map.

Combining Case 1 and Case 2 we conclude that if $\alpha_i < \frac{\delta_{11}+\delta_{22}}{3\delta_{22}-\delta_{11}}$ for any $i \in V$, then h is a contraction map. According to the proof for statement (i), h maps $[0,1]^n$ to $[0,1]^n$. Therefore, according to the Banach fixed point theorem, for any initial condition $\boldsymbol{p}(0) \in [0,1]^n$, the solution $\boldsymbol{p}(t)$ converges to $\boldsymbol{p}^*$ exponentially fast and the convergence rate is upper bounded by $\max_i \big(\max(\epsilon_i, K_i\epsilon_i + K_i - 1)\big)$. ■

The rest of this section are some remarks of Theorem 2.4.2. Firstly, equation (2.13) has a meaningful interpretation: The condition $\delta_{22} \geq \delta_{11}$ implies that product H_2 is advantageous to H_1, in the sense that the nodes in state H_1 have a higher or equal tendency of converting to H_2 than the other way around. As the result, the fixed point is in favor of H_2, i.e., $\boldsymbol{p}^* \geq \frac{1}{2}\mathbb{1}_n$.

From the proof of statement (iv), we know that, around the unique fixed point, the linearized system is $\boldsymbol{y}(t+1) = M\boldsymbol{y}(t)$, where M is a Metzler matrix and is Hurwitz stable. Usually the Metzler matrices are presented in continuous-time network dynamics models, e.g., the epidemic spreading model [26, 27]. In the proof of Theorem 2.4.2 (iv), we provide an example of the Metzler matrix in a stable discrete-time system.

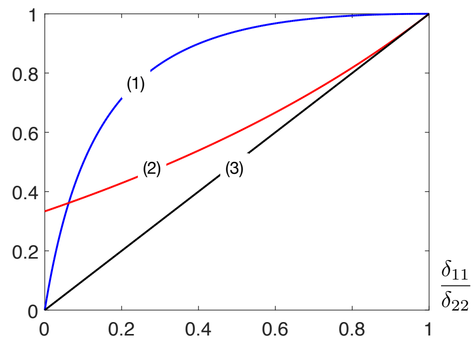


Figure 2.7: This figure illustrates how the conditions for the local stability and global stability change with the ratio δ_{11}/δ_{22}. Curve (1) is $8\delta_{11}\delta_{22}/((\delta_{22})-\delta_{11})^2+8\delta_{11}\delta_{22})$, i.e, corresponding to the condition for local stability. Curve (2) is $(\delta_{22}+\delta_{11})/(3\delta_{22}-\delta_{11})$, corresponding to the condition for global stability. Curve (3) is δ_{11}/δ_{22}.

Figure 2.7 plots the right-hand sides of inequalities (2.15) and (2.16), respectively, as functions of the ratio $\frac{\delta_{11}}{\delta_{22}}$, for the case when $0 < \frac{\delta_{11}}{\delta_{22}} < 1$. One can observe that, for a large range of $\frac{\delta_{11}}{\delta_{22}}$, the sufficient condition we propose for the global stability is more conservative than the sufficient condition for the local stability.

One major difference between the self-social and the social-self NCPM in the asymptotic property is that, in the self-social NCPM, every individual's state probability distribution is not necessarily identical. Moreover, distinct from the social-self NCPM, for any of the four cases of $G(\mathscr{D})$ defined in Definition 2.3.4, the asymptotic behavior of the self-social NCPM depends on not only the structure of $G(\mathscr{D})$, but also the structure of the social network $G(\tilde{A})$ and the individual open-mindedness $\boldsymbol{\alpha}$.

2.5 Non-cooperative Quality-Seeding Games

Based on the social-self NCPM given by equation (2.4), we propose two non-cooperative multi-player games distinct in the pay-off functions, and analyze their Nash equilibria. These two games share the common idea that, companies benefit from the adoption of their products, and thereby invest on both improving their products' quality, and seeding, e.g., advertisement and promotion, to maximize their products' adoption probabilities. All the notations in Table 2.1 and the previous sections still apply, and, in Table 2.2, we introduce some additional notations and functions exclusively for this section.

Table 2.2: Notations and functions used in Section V

$X(t)$	seeding matrix at time t. $X(t) = \big(x_{ir}(t)\big)_{n\times R}$, where $x_{ir}(t) \geq 0$ is company r's investment on seeding for individual i. $\boldsymbol{x}_r(t)$ is the r-th column of $X(t)$ and $\boldsymbol{x}^{(i)}(t)$ is the i-th column of $X(t)$
$\boldsymbol{w}(t)$	the quality investment vector at time t. $\boldsymbol{w}(t) \in \mathbb{R}^{R\times 1}$, and each entry $w_r(t) \geq 0$ is company r's investment at time t on product H_r's quality
$Y(t)$	action matrix at time t. $Y(t) = \big(X(t)^\top, \boldsymbol{w}(t)\big)^\top$, in which any $\boldsymbol{y}_r(t) = (\boldsymbol{x}_r(t)^\top, w_r(t))^\top$ is Player r's action at t.
$\boldsymbol{c}$	the budget vector. $\boldsymbol{c} \in \mathbb{R}^{R\times 1}$ and $\boldsymbol{c} \succ \mathbb{0}_R$. entry c_r is the budget limit for company r
Ω_r	player r's action set. $\Omega_r = \{\boldsymbol{y} \in \mathbb{R}^{n+1}_{\geq 0} \,\vert\, \mathbb{1}^\top_{n+1}\boldsymbol{y} \leq c_r\}$
$\psi_r(\boldsymbol{x}^{(i)};\gamma)$	$\psi_r : \mathbb{R}^{1\times R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $\psi_r(\boldsymbol{x}^{(i)};\gamma) = x_{ir}/(\boldsymbol{x}^{(i)}\mathbb{1}_R + \gamma)$, with model parameter $\gamma > 0$
$g_r(\boldsymbol{w};\boldsymbol{\varsigma})$	$g_r : \mathbb{R}^{R\times 1}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $g_r(\boldsymbol{w};\boldsymbol{\varsigma}) = (w_r + \varsigma_r)/\mathbb{1}^\top_R(\boldsymbol{w} + \boldsymbol{\varsigma})$, where $\boldsymbol{\varsigma} \in \mathbb{R}^R_{>0}$
$\boldsymbol{\beta}_r(t)$	$\boldsymbol{\beta}_r(t) = \big(\beta_{1r}(t), \dots, \beta_{nr}(t)\big)^\top = \tilde{A}\boldsymbol{p}_r(t)$
$u_r(P)$	single-stage reward for player r with system state P. $u_r(P) = \mathbb{1}^\top_n \boldsymbol{p}_r$

2.5.1 One-shot quality-seeding game

Game set-up and analysis

In this subsection we consider the scenario in which the companies allocate their investments aiming to maximize their instant pay-offs. The set-up is grounded in the natural assumption that the managers of the competing companies' are motivated to make investment decisions aimed at maximizing their companies' profits during their terms of service.

The game is referred to as the *one-shot quality-seeding game*, and is formalized as follows.

(a) Players: The players are the R companies. Each company r has a product H_r competing on the network.

(b) Players' actions: At each stage (or time step equivalently) t, each company r has two types of investments. The investment on seeding, i.e., $\boldsymbol{x}_r(t)$, and the investment on quality, i.e., $w_r(t)$. The total investment is bounded by a fixed budget c_r, i.e., $\mathbb{1}_n^\top x_r(t) + w_r(t) \leq c_r$.

(c) Rules: The investment on seeding changes the individuals' product-adoption probability in the social conversion process. For any individual $i \in V$, each company r's investment $x_{ir}(t)$ creates a "virtual node" in the network, who is always adopting the product H_r. In the social conversion process, the probability that individual i picks company r's virtual node is $\psi_r\big(\boldsymbol{x}^{(i)}(t);\gamma\big)$ for any $i \in V$ and $r \in \Theta$. The probability that individual i picks individual j in the social conversion process is then given by $\Big(1 - \sum_{s=1}^R \psi_s\big(\boldsymbol{x}^{(i)};\gamma\big)\Big)\tilde{a}_{ij}$. The investment on quality, i.e., $w_r(t)$, influences the product-conversion graph. We assume that the product-conversion graph is associated with a rank-one adjacency matrix $[\delta_1\mathbb{1}_n, \delta_2\mathbb{1}_n, \dots, \delta_R\mathbb{1}_n]$ and $\delta_r = g_r(\boldsymbol{w}(t);\boldsymbol{\varsigma})$ is determined by all the companies' investments on product quality and the products' preset qualities $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_R)^\top \succ \mathbb{0}_R$. With each company r's action $\boldsymbol{y}_r(t) = \big(\boldsymbol{x}_r(t)^\top, w_r(t)\big)^\top$ at time t, the dynamics of the product-adoption probabilities $P(t) \in \mathbb{R}_{\geq 0}^{n\times R}$ is given by

$$P(t+1) = H\big(P(t), \boldsymbol{y}_1(t), \dots, \boldsymbol{y}_R(t)\big), \tag{2.19}$$

where the map H is defined by

$$H\big(P, \boldsymbol{y}_1(t), \dots, \boldsymbol{y}_R(t)\big)_{ir} = \alpha_i \frac{\gamma}{\boldsymbol{x}^{(i)}(t)\mathbb{1}_R + \gamma} \sum_{k=1}^{n} \tilde{a}_{ik} p_{kr} + \alpha_i \psi_r\big(\boldsymbol{x}^{(i)}(t); \gamma\big) + (1-\alpha_i) g_r\big(\boldsymbol{w}(t); \boldsymbol{\varsigma}\big),$$

for any $P \in S_{nR}(\mathbb{1}_n)$, $i \in V$, and $r \in \Theta$.

(d) Pay-offs and goals: At each stage t, each player r chooses its action $\boldsymbol{y}_r(t)$, in order to maximize the pay-off $u_r(P(t+1)) = \mathbb{1}_n^\top \boldsymbol{p}_r(t+1)$, i.e., the total adoption probability of product H_r at the next stage.

The following theorem gives a closed-form expression of the Nash equilibrium at each stage and the system's asymptotic behavior when every player is adopting the policy at the Nash equilibrium.

Theorem 2.5.1 (One-shot quality-seeding game) *Consider the R-player quality-seeding game described in this subsection. Further assume that the budget limit c_r for any company r satisfies*

$$c_r \geq \max\Big\{(\frac{n}{\min_i \alpha_i} - 1)\gamma - \varsigma_r, \frac{\mathbb{1}_n^\top \boldsymbol{\alpha}}{n - \mathbb{1}_n^\top \boldsymbol{\alpha}} \varsigma_r\Big\}. \tag{2.20}$$

Then we have the following conclusions:

i) for each t, there exists a unique pure-strategy Nash equilibrium $Y^*(t) = \big(X^*(t)^\top, \boldsymbol{w}^*(t)\big)^\top$, given by

$$x_{ir}^*(t) = \frac{\alpha_i}{n} c_r + \frac{\alpha_i \gamma}{n} \mathbb{1}_n^\top \boldsymbol{\beta}_r(t) + \frac{\alpha_i}{n} \varsigma_r - \beta_{ir}(t)\gamma, \tag{2.21}$$

$$w_r^*(t) = \big(1 - \frac{\mathbb{1}_n^\top \boldsymbol{\alpha}}{n}\big)\big(c_r + \mathbb{1}_n^\top \boldsymbol{\beta}_r(t)\gamma\big) - \frac{\mathbb{1}_n^\top \boldsymbol{\alpha}}{n} \varsigma_r, \tag{2.22}$$

and $x_{ir}^*(t) \geq 0$, $w_r^*(t) \geq 0$ for any $i \in V, r \in \Theta$;

ii) if $\big(X(t), \boldsymbol{w}(t)\big) = \big(X^*(t), \boldsymbol{w}^*(t)\big)$ for any $t \in \mathbb{N}$ and $P(0) \in S_{nR}(\mathbb{1}_n)$, then $P(t)$

obeys the following iteration equations:

$$\boldsymbol{p}_r(t+1) = \frac{c_r + \varsigma_r + \mathbb{1}_n^\top \tilde{A} \boldsymbol{p}_r(t)\gamma}{\mathbb{1}_R^\top \boldsymbol{c} + \mathbb{1}_R^\top \boldsymbol{\varsigma} + n\gamma} \mathbb{1}_n, \tag{2.23}$$

for any $r \in \Theta$, $t \in \mathbb{N}$. *As the result,* $\boldsymbol{p}_r(t)$ *converges to* $(c_r+\varsigma_r)\big/\big(\mathbb{1}_R^\top(\boldsymbol{c}+\boldsymbol{\varsigma})\big)$ *exponentially fast with the rate* $n\gamma\big/\big(\mathbb{1}_R^\top(\boldsymbol{c}+\boldsymbol{\varsigma})+n\gamma\big)$.

Proof: Since we only discuss the actions at stage t in this proof, for simplicity of notations and without causing any confusion, we use x_{ir} (w_r, x_{ir}^*, w_r^* resp.) for $x_{ir}(t)$ ($w_r(t)$, $x_{ir}^*(t)$, $w_r^*(t)$ resp.).

If company r knows the actions of all the other companies at time step t, i.e., $\boldsymbol{y}_s$, for any $s \neq r$, the optimal response for company r is the solution to the following optimization problem:

$$\begin{aligned} \underset{(\boldsymbol{x},w)\in\Omega_r}{\text{minimize}} \quad & -\mathbb{1}_n^\top \boldsymbol{p}_r(t+1) \\ \text{subject to} \quad & \mathbb{1}_n^\top \boldsymbol{x} + w - c_r \leq 0. \end{aligned} \tag{2.24}$$

Let $\tilde{x}_{ir} = x_{ir} + \beta_{ir}(t)\gamma$, $\tilde{w}_r = w_r + \varsigma_r$, and $L_r(\boldsymbol{x}_r, w_r, \mu_r) = -\mathbb{1}_n^\top \boldsymbol{p}_r(t+1) + \mu_r \mathbb{1}_n^\top \boldsymbol{x}_r + \mu_r w_r - \mu_r c_r$, for any $i \in V$ and $r \in \Theta$. The solution to the optimization problem (2.24) satisfies

$$\frac{\partial L_r}{\partial x_{ir}} = -\alpha_i \frac{\sum_{s\neq r} \tilde{x}_{is}}{(\sum_{s=1}^R \tilde{x}_{is})^2} + \mu_r = 0, \tag{2.25}$$

$$\frac{\partial L_r}{\partial w_r} = -\mathbb{1}_n^\top (\mathbb{1}_n - \boldsymbol{\alpha}) \frac{\sum_{s\neq r} \tilde{w}_s}{(\mathbb{1}_R^\top \tilde{\boldsymbol{w}})^2} + \mu_r = 0, \tag{2.26}$$

$$\frac{\partial L_r}{\partial \mu_r} = \mathbb{1}_n^\top \boldsymbol{x}_r + w_r - c_r = 0. \tag{2.27}$$

According to the definition of Nash equilibrium, $(\boldsymbol{x}_r^*, w_r^*)$ solves the optimization problem (2.24) with $(\boldsymbol{x}_s, w_s) = (\boldsymbol{x}_s^*, w_s^*)$ for any $s \neq r$. One immediate result is that

$\mathbb{1}_n^\top \boldsymbol{x}_r^* + w_r^* - c_r = 0$ for any $r \in \Theta$. Moreover, equation (2.25) leads to:

$$\frac{1}{\sqrt{\mu_r}} = \frac{1}{\sum_{k=1}^n \sqrt{\alpha_k \sum\limits_{s \neq r} \tilde{x}_{ks}^*}} \sum_{s=1}^R \big(c_s - w_s^* + \mathbb{1}_n^\top \boldsymbol{\beta}_s(t) \gamma \big),$$

and therefore,

$$\frac{\sqrt{\alpha_i \sum_{s \neq r} \tilde{x}_{is}^*}}{\sum\limits_{k=1}^n \sqrt{\alpha_k \sum\limits_{s \neq r} \tilde{x}_{ks}^*}} = \frac{\sum\limits_{s=1}^R \tilde{x}_{is}^*}{\sum\limits_{s=1}^R \big(c_s - w_s^* + \mathbb{1}_n^\top \boldsymbol{\beta}_s(t) \gamma \big)}. \tag{2.28}$$

The right-hand side of the equation above does not depend on the product index r. Therefore,

$$\frac{\sum_{s \neq r} \tilde{x}_{is}^*}{\sum_{s \neq \tau} \tilde{x}_{is}^*} = \left(\frac{\sum_{k=1}^n \sqrt{\alpha_k \sum_{s \neq r} \tilde{x}_{ks}^*}}{\sum_{k=1}^n \sqrt{\alpha_k \sum_{s \neq \tau} \tilde{x}_{ks}^*}} \right)^2,$$

for any $r, \tau \in \Theta$. Since the right-hand side of the equation above does not depend on i, we have

$$\frac{\sum_{s \neq r} \tilde{x}_{is}^*}{\sum_{s \neq r} \tilde{x}_{js}^*} = \frac{\sum_{s \neq \tau} \tilde{x}_{is}^*}{\sum_{s \neq \tau} \tilde{x}_{js}^*} = \frac{\sum_{s=1}^R \tilde{x}_{is}^*}{\sum_{s=1}^R \tilde{x}_{js}^*} = \frac{\tilde{x}_{ir}^*}{\tilde{x}_{jr}^*},$$

for any $r, \tau \in \Theta$. Combine the equation above with equation (2.28) and then we obtain

$$\frac{\sum_{s=1}^R \tilde{x}_{is}^*}{\sum_{s=1}^R \tilde{x}_{js}^*} = \sqrt{\frac{\alpha_i}{\alpha_j}} \sqrt{\frac{\sum_{s \neq r} \tilde{x}_{is}^*}{\sum_{s \neq r} \tilde{x}_{js}^*}} \quad \Rightarrow \quad \frac{\tilde{x}_{ir}^*}{\tilde{x}_{jr}^*} = \frac{\alpha_i}{\alpha_j},$$

for any $r \in \Theta$. Therefore,

$$\tilde{x}_{ir}^* = \frac{\alpha_i}{\mathbb{1}_n^\top \boldsymbol{\alpha}} \big(c_r - w_r^* + \mathbb{1}_n^\top \boldsymbol{\beta}_r(t) \gamma \big). \tag{2.29}$$

Combining equation (2.29) and (2.26), we obtain

$$\frac{c_r - w_r^* + \mathbb{1}_n^\top \boldsymbol{\beta}_r(t)\gamma}{\tilde{w}_r^*} = \frac{c_\tau - w_\tau^* + \mathbb{1}_n^\top \boldsymbol{\beta}_\tau(t)\gamma}{\tilde{w}_\tau^*} = \eta,$$

for any $r, \tau \in \Theta$ and some constant η. Substitute the equation above into equation (2.26), we solve that $\eta = \mathbb{1}_n^\top \boldsymbol{\alpha} / \mathbb{1}_n^\top (\mathbb{1}_n - \boldsymbol{\alpha})$. Therefore, we obtain equation (2.22), and by substituting equation (2.22) into equation (2.29) we obtain equation (2.21). The uniqueness of the pure-strategy Nash equilibrium $(X^{*\top}, \boldsymbol{w})^\top$ is implied from the computation. Moreover, equation (2.20) guarantees $\tilde{x}_{ir}^* \geq 0$ and $w_r^* \geq 0$ for any $i \in V$ and $r \in \Theta$.

Substituting equation (2.21) and (2.22) into the dynamical system (2.19), after simplification, we obtain equation (2.23) and thereby all the results in Conclusion ii). ■

Interpretations and Remarks:

The basic idea of seeding-quality trade-off in the competitive seeding-quality game is similar to the work by Fazeli et. al. [74], but, in our model, players take actions at every step, instead of only at the beginning of the game. Moreover, our model is based on a different propagation model.

Theorem 2.5.1 reveals the behavior of the competitive propagation dynamics under the players' rational but myopic actions, and provides some strategic insights on the investment decisions and the seeding-quality trade-off for short-term reward maximization.

(a) Interpretation of $\beta_{ir}(t)$: By definition, $\beta_{ir}(t)$ is the average probability, among all the neighbors of individual i, of adopting product H_r at time step t. The larger $\beta_{ir}(t)$, the more individual i is inclined to adopt H_r via social conversion. Therefore, $\beta_{ir}(t)$ characterizes the current "social attraction" of H_r for individual i, and $\mathbb{1}_n^\top \boldsymbol{\beta}_r(t)/n$ characterizes the current overall social attraction of product H_r in the network.

(b) Seeding-quality trade-off: According to equation (2.22), at the Nash equilibrium,

the investment on H_r's product quality monotonically decreases with $\mathbb{1}_n^\top \boldsymbol{\alpha}/n$, and increases with $\mathbb{1}_n^\top \boldsymbol{\beta}_r$. This observation implies that: 1) in a society with relatively low open-mindedness, the competing companies should relatively emphasize more on improving their products' quality, rather than seeding, and vice versa; 2) for products which do not have much social attraction, seeding is more efficient than improving the product's quality.

(c) Allocation of seeding resources among the individuals: According to equation (2.21), for any company r, at the Nash equilibrium at each time step t, the investment on seeding for any individual i, i.e., $x_{ir}(t)$, increases with individual i's open-mindedness, since it is easier for a more open-minded individual to be influence by seeding. Moreover, by rewriting equation (2.21), one would observe that $x^*_{ir}(t)$ monotonically decreases with $\beta_{ir}(t)$. A possible interpretation is that, seeding is relatively not efficient for products with strong social attraction. Moreover, one can also observe that $x^*_{ir}(t)$ increases with $\sum_{l=1}^n \tilde{a}_{li} p_{ir}(t)$, in which $\sum_{l=1}^n \tilde{a}_{li}$ is individual i's in-degree, reflecting i's potential of influencing the others, and $\sum_{l=1}^n \tilde{a}_{li} p_{ir}(t)$ characterizes individual i's potential of converting other individuals to product H_r.

(d) Nash equilibrium on the boundary: Without equation (2.20), the right-hand sides of equation (2.21) and (2.22) could be non-positive. In this case, the Nash equilibrium would be on the boundary of the feasible action set, i.e., some of the $x^*_{ir}(t)$ or $w^*_r(t)$ might be 0.

2.5.2 Dynamic quality-seeding game with infinite-horizon

In this subsection we introduce a game among more farsighted players than in the previous subsection. The players aim to maximize the accumulated pay-offs of all the stages. We refer to this game as the *dynamic quality-seeding game*. The model set-up is the same with the game defined in the previous subsection, except for the following two modifications:

(a) Players' policies: Denote by $\mathcal{Y}_r$ the set of functions mapping $S_{nR}(\mathbb{1}_n)$ to Ω_r. Each player r's policy is a sequence of maps, denoted by $\mathscr{Y}_r = \{\mathscr{Y}_{r,t}\}_{t\in\mathbb{N}}$, where $\mathscr{Y}_{r,t} \in \mathcal{Y}_r$ for any t. Player r's action at each stage t is thus given by $\boldsymbol{y}_t = \mathscr{Y}_{r,t}\big(P(t)\big)$. We refer to $\mathscr{Y}_r = \{\mathscr{Y}_{r,t}\}_{t\in\mathbb{N}}$ as stationary policy if $\mathscr{Y}_{r,t} = \mathscr{Y}_{r,\tau}$ for any $t \neq \tau$, and simply use $\mathscr{Y}_r$ for the map at each stage.

(b) Pay-offs and goals: Denote by $v_r(P; \mathscr{Y}_1, \dots, \mathscr{Y}_R)$ the pay-off of Player r, with initial condition $P(0) = P$ and each Player s adopting the policy $\mathscr{Y}_s$. The pay-off $v_r(P; \mathscr{Y}_1, \dots, \mathscr{Y}_R)$ is given by the accumulated step pay-offs with discount, that is,

$$v_r(P; \mathscr{Y}_1, \dots, \mathscr{Y}_R) = \sum_{t=0}^{\infty} \varepsilon^t u_r(P(t)),$$

where $P(0) = P$ and $P(t+1) = H\big(P(t); \mathscr{Y}_1(P(t)), \dots, \mathscr{Y}_R(P(t))\big)$ for any $t \in \mathbb{N}$.

This model set-up defines a non-cooperative dynamic game with infinite horizon. One interpretation of the discounted accumulated pay-off is that, people tend to value the immediate profit more than the future profit. An alternative explanation is that, the discount factor ε characterizes the bank interest rate $1/\varepsilon - 1$.

The R-tuple $(\mathscr{Y}_1^*, \dots, \mathscr{Y}_R^*)$ is a Nash equilibrium if, for any $P \in S_{nR}(\mathbb{1}_n)$ and $r \in \Theta$, $v_r(P; \mathscr{Y}_1^*, \dots, \mathscr{Y}_R^*) \geq v_r(P; \mathscr{Y}_1^*, \dots, \mathscr{Y}_{r-1}^*, \mathscr{Y}_r, \mathscr{Y}_{r+1}^*, \dots, \mathscr{Y}_R^*)$, for any $\mathscr{Y}_r \in \mathcal{Y}_r^\infty = \mathcal{Y}_r \times \mathcal{Y}_r \times \dots$. In this subsection, we limit our discussion to the case of two players. The following theorem presents some results on the stationary Nash equilibrium and the equilibrium pay-off function for this dynamic quality-seeding game.

Theorem 2.5.2 (Two-player infinite-horizon game) *Consider the dynamic quality-seeding game defined in this subsection, with $R = 2$. Define the subset of continuously* *differentiable functions* $\mathcal{V} = \{v : [0,1]^n \to \mathbb{R} \,|\, v$ *satisfies properties* $\mathcal{P}_1$ *and* $\mathcal{P}_2\}$*, where*

$$\mathcal{P}_1 : \boldsymbol{p} \preceq \hat{\boldsymbol{p}} \Rightarrow v(\boldsymbol{p}) \leq v(\hat{\boldsymbol{p}}) \text{ for any } \boldsymbol{p}, \hat{\boldsymbol{p}} \in [0,1]^n,$$
$$\mathcal{P}_2 : v(\boldsymbol{p}) \text{ is convex in } \boldsymbol{p}.$$

We conclude that:

(i. *There exists a Nash equilibrium* $(\mathscr{Y}_1^*, \mathscr{Y}_2^*)$*, where* $\mathscr{Y}_1^*$ *and* $\mathscr{Y}_2^*$ *are both stationary policies;*

(ii. *The total pay-off for Player 2 at this Nash equilibrium is given by* $v_2(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*) = v^*(P\mathbb{e}_2)$*, where* $\mathbb{e}_2$ *is the second standard basis vector of* $\mathbb{R}^2$*, and* v^* *is the unique fixed point of the map* $\mathcal{T} : \mathcal{V} \to \mathcal{V}$*, defined by*

$$\mathcal{T} v(\boldsymbol{p}) = \mathbb{1}_n^\top \boldsymbol{p} + \varepsilon \sup_{\boldsymbol{y}_2 \in \Omega_2} \inf_{\boldsymbol{y}_1 \in \Omega_1} v\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\mathbb{e}_2\big),$$

where $P = [\mathbb{1}_n - \boldsymbol{p}, \boldsymbol{p}] \in \mathbb{R}^{n \times 2}$*. As a result,* $v_1(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*) = n/(1-\varepsilon) - v_2(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*)$*;*

(iii. *The stationary Nash policies* $\mathscr{Y}_1^*$*,* $\mathscr{Y}_2^*$ *are given by*

$$\mathscr{Y}_1^*(P) = \operatorname*{argmin}_{\boldsymbol{y}_1 \in \Omega_1} \sup_{\boldsymbol{y}_2 \in \Omega_2} v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\mathbb{e}_2\big),$$
$$\mathscr{Y}_2^*(P) = \operatorname*{argmax}_{\boldsymbol{y}_2 \in \Omega_2} \inf_{\boldsymbol{y}_1 \in \Omega_1} v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\mathbb{e}_2\big).$$

Before proving the theorem above, we summarize Theorem 4.4 and Property 4.1 in [76], on the two-player zero-sum continuous games, into the following lemma.

Lemma 2.5.3 (Pure-strategy Nash equilibrium) *Consider the two-player zero-sum continuous game with Player 1 as the minimizer and Player 2 as the maximizer. Sup-*

pose the action sets of Player 1 and 2, denoted by Ω_1 and Ω_2 respectively, are both compact and convex subsets of finite-dimension Euclidean spaces. If the cost function $v(\boldsymbol{y}_1, \boldsymbol{y}_2) : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is continuously differentiable, convex in $\boldsymbol{y}_1$, and concave in $\boldsymbol{y}_2$, then: (1) the game admits at least one saddle-point Nash equilibrium in pure strategies; (2) if there are multiple saddle points, the saddle points satisfy the ordered interchangeability property. That is, if $(\boldsymbol{y}_1^*, \boldsymbol{y}_2^*)$ and $(\tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2)$ are saddle points, so are $(\boldsymbol{y}_1^*, \tilde{\boldsymbol{y}}_2)$ and $(\tilde{\boldsymbol{y}}_1, \boldsymbol{y}_2^*)$.

Proof of Theorem 2.5.2: In this proof, for simplicity, denote by $\boldsymbol{p}$ the second column of the matrix P, i.e., $P = [\mathbb{1}_n - \boldsymbol{p}, \boldsymbol{p}]$, and correspondingly, $\hat{P} = [\mathbb{1}_n - \hat{\boldsymbol{p}}, \hat{\boldsymbol{p}}]$. Since Ω_1 and Ω_2 are compact subsets of $\mathbb{R}^{n+1}$, for any $v \in \mathcal{V}$, there exists $(\boldsymbol{y}_1, \boldsymbol{y}_2)$ such that $\mathcal{T}v(\boldsymbol{p}) = \mathbb{1}_n^\top \boldsymbol{p} + \varepsilon v\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\mathbb{e}_2\big)$. Moreover, from the expression of map H, one can deduce that $H(P, \boldsymbol{y}_1, \boldsymbol{y}_2)$ satisfies

$$\boldsymbol{p} \preceq \hat{\boldsymbol{p}} \Rightarrow H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\mathbb{e}_2 \preceq H(\hat{P}; \boldsymbol{y}_1, \boldsymbol{y}_2)\mathbb{e}_2,$$

for any $(\boldsymbol{y}_1, \boldsymbol{y}_2) \in \Omega_1 \times \Omega_2$ and $\boldsymbol{p}, \hat{\boldsymbol{p}} \in [0,1]^n$. This leads to the conclusion that $\mathcal{T}v$ also satisfies property $\mathcal{P}_1$. Moreover, by definition, $H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)$ is linear in P. Since $v(\boldsymbol{p})$ is convex in $\boldsymbol{p}$, one can check that $\mathcal{T}v(\boldsymbol{p})$ is also convex in $\boldsymbol{p}$. Therefore, $\mathcal{T}$ satisfies property $\mathcal{P}_2$ and maps $\mathcal{V}$ to $\mathcal{V}$ itself. Now we prove that $\mathcal{T}$ is a contraction map. Define the function norm $\|\cdot\|$ for any $v \in \mathcal{V}$ as $\|v\| = \sup_{\boldsymbol{p}\in[0,1]^n} |v(\boldsymbol{p})|$. For any $v, \hat{v} \in \mathcal{V}$, we have

$$\|\mathcal{T}v - \mathcal{T}\hat{v}\| = \varepsilon \sup_{\boldsymbol{p}\in[0,1]^n} |\mathcal{T}v(\boldsymbol{p}) - \mathcal{T}\hat{v}(\boldsymbol{p})| \le \varepsilon \sup_{\boldsymbol{p}\in[0,1]^n} \sup_{\boldsymbol{y}_2\in\Omega_2} \sup_{\boldsymbol{y}_1\in\Omega_1} |v(\boldsymbol{p}) - \hat{v}(\boldsymbol{p})| \le \varepsilon \|v - \hat{v}\|.$$

According to the Banach fixed-point theorem, there exists a unique $v^* \in \mathcal{V}$ satisfying

$$v^*(P\mathbb{e}_2) = \mathbb{1}_n^\top P\mathbb{e}_2 + \varepsilon \sup_{\boldsymbol{y}_2\in\Omega_2} \inf_{\boldsymbol{y}_1\in\Omega_1} v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\mathbb{e}_2\big).$$

According to the expression of the map $H(P;\boldsymbol{y}_1,\boldsymbol{y}_2)$, one can check that, for any $\eta\in[0,1]$, $P\in S_{nR}(\mathbb{1}_n)$, and $\boldsymbol{y}_1,\hat{\boldsymbol{y}}_1\in\Omega_1$,

$$H\big(P;\eta\boldsymbol{y}_1+(1-\eta)\hat{\boldsymbol{y}}_1,\boldsymbol{y}_2\big)\mathbb{e}_2\preceq\eta H(P;\boldsymbol{y}_1,\boldsymbol{y}_2)+(1-\eta)H(P;\hat{\boldsymbol{y}}_1,\boldsymbol{y}_2).$$

Since $v^*(\boldsymbol{p})$ satisfies properties $\mathcal{P}_1$ and $\mathcal{P}_2$,

$$v^*\big(H(P;\eta\boldsymbol{y}_1+(1-\eta)\hat{\boldsymbol{y}}_1,\boldsymbol{y}_2)\mathbb{e}_2\big)\leq\eta v^*\big(H(P;\boldsymbol{y}_1,\boldsymbol{y}_2)\mathbb{e}_2\big)+(1-\eta)v^*\big(H(P;\hat{\boldsymbol{y}}_1,\boldsymbol{y}_2)\mathbb{e}_2\big).$$

That is, $v^*\big(H(P;\boldsymbol{y}_1,\boldsymbol{y}_2)\mathbb{e}_2\big)$ is convex in $\boldsymbol{y}_1$. Similarly, we have $v^*\big(H(P;\boldsymbol{y}_1,\boldsymbol{y}_2)\mathbb{e}_2\big)$ is concave in $\boldsymbol{y}_2$.

According to Lemma 2.5.3, for any $P\in S_{nR}(\mathbb{1}_n)$ and the two-player zeros-sum game with cost function $v^*\big(H(P;\boldsymbol{y}_1,\boldsymbol{y}_2)\mathbb{e}_2\big)$, there exists a saddle-point Nash equilibrium $(\boldsymbol{y}_1^*,\boldsymbol{y}_2^*)\in\Omega_1\times\Omega_2$ such that

$$v^*\big(H(P;\boldsymbol{y}_1^*,\boldsymbol{y}_2^*)\mathbb{e}_2\big)=\sup_{\boldsymbol{y}_2\in\Omega_2}\inf_{\boldsymbol{y}_1\in\Omega_1}v^*\big(H(P;\boldsymbol{y}_1^*,\boldsymbol{y}_2^*)\mathbb{e}_2\big)=\inf_{\boldsymbol{y}_1\in\Omega_1}\sup_{\boldsymbol{y}_2\in\Omega_2}v^*\big(H(P;\boldsymbol{y}_1,\boldsymbol{y}_2)\mathbb{e}_2\big).$$

Therefore, there exists functions $\mathscr{Y}_1$, $\mathscr{Y}_2$ such that $\boldsymbol{y}_1^*=\mathscr{Y}_1^*(P)$ and $\boldsymbol{y}_2^*=\mathscr{Y}_2^*(P)$ satisfy the equation above, for any $P\in S_{nR}(\mathbb{1}_n)$. Moreover, since

$$\begin{aligned}v^*(P\mathbb{e}_2)-v_2(P;\mathscr{Y}_1^*,\mathscr{Y}_2^*)=\varepsilon\Big(&v^*\big(H(P;\mathscr{Y}_1^*(P),\mathscr{Y}_2^*(P))\mathbb{e}_2\big)\\&-v_2\big(H(P;\mathscr{Y}_1^*(P),\mathscr{Y}_2^*(P));\mathscr{Y}_1^*,\mathscr{Y}_2^*\big)\Big),\end{aligned}$$

for any $P\in S_{nR}(\mathbb{1}_n)$, and functions v and v_2 are bounded, we conclude that $v^*(P\mathbb{e}_2)=$

$v_2(P;\mathscr{Y}_1^*,\mathscr{Y}_2^*)$. Therefore, for any $\tau\in\mathbb{N}$, we have

$$v_2(P;\mathscr{Y}_1^*,\mathscr{Y}_2^*)\geq\sum_{t=0}^{\tau-1}\varepsilon^t u_2(P(t))+\varepsilon^\tau v_2\Big(H\big(P(\tau);\mathscr{Y}_1^*(P(\tau)),\boldsymbol{y}_2);\mathscr{Y}_1^*,\mathscr{Y}_2^*\big)\Big),$$

for any $\boldsymbol{y}_2\in\Omega_2$, and, due to the fact that $v_1(P;\mathscr{Y}_1,\mathscr{Y}_2)=n/(1-\varepsilon)-v_2(P;\mathscr{Y}_1,\mathscr{Y}_2)$ for any $(\mathscr{Y}_1,\mathscr{Y}_2)$, we have

$$v_1(P;\mathscr{Y}_1^*,\mathscr{Y}_2^*)\geq\sum_{t=0}^{\tau-1}\varepsilon^t u_1(P(t))+\varepsilon^\tau v_1\Big(H\big(P(\tau);\boldsymbol{y}_1,\mathscr{Y}_2^*(P(\tau))\big);\mathscr{Y}_1^*,\mathscr{Y}_2^*\big)\Big),$$

for any $\boldsymbol{y}_1\in\Omega_1$. Since both $v_1(P;\mathscr{Y}_1,\mathscr{Y}_2)$ and $v_2(P;\mathscr{Y}_1,\mathscr{Y}_2)$ satisfy the property of continuity at infinity, according to the one-stage deviation principle, $(\mathscr{Y}_1^*,\mathscr{Y}_2^*)$ is a Nash equilibrium of the dynamics game. This concludes the proof. ■

Theorem 2.5.2 provides an iteration algorithm to compute the stationary Nash policy $(\mathscr{Y}_1^*,\mathscr{Y}_2^*)$, and the players' respective pay-offs at the Nash equilibrium. A comparison by simulation is given in Figure 2.8, between the Nash policies for the dynamic game discussed in this subsection, and the one-shot game in the previous subsection. The model parameters are set as $n=3$, $\boldsymbol{\alpha}=(0.51,0.87,0.77)^\top$, $\gamma=5$, $\varsigma_1=\varsigma_2=1$, $c_1=30$, $c_2=60$, $\varepsilon=0.8$, and $\tilde{A}$ such that $\tilde{a}_{13}=\tilde{a}_{23}=1$, $\tilde{a}_{31}=\tilde{a}_{32}=0.5$, and $\tilde{a}_{ij}=0$ otherwise. Simulation results show that, with the same initial condition, for the two types of games, the players' total pay-offs at the corresponding Nash equilibria are very close to each other. Moreover, from Figure 2.8 we can observe that, for each of the two games, the players' pay-offs are almost linear to the initial average probability of adopting H_2.

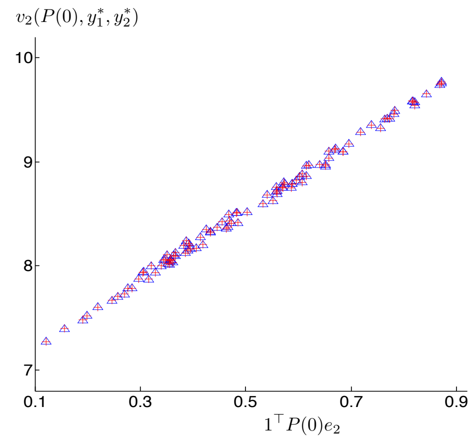


Figure 2.8: Comparison between the Nash policies for the dynamic game discussed in Subsection V.B (blue triangles), and the one-shot game in Subsection V.A (red crosses).

2.6 Conclusion

This chapter discusses a class of competitive propagation models based on two product-adoption mechanisms: the social conversion and the self conversion. Applying the independence approximation we propose two difference equations systems, referred to as the social-self NCPM and the self-social NCPM respectively. Theoretical analysis reveals that the structure of the product-conversion graph plays an important role in determining the nodes' asymptotic state probability distributions. Simulation results reveal the high accuracy of the independence approximation and the asymptotic behavior of the original social-self Markov chain model. Based on the social-self NCPM, we propose two-types of competitive propagation games and discuss their Nash equilibria, as well as the trade-off between seeding and quality for the one-shot game. One possible future work is the deliberative investigation on the Nash equilibrium on the boundary. It is also of research value to explore the extension of the analysis in Section V.B to the case of multiple-player dynamic games. Another open problem is the stability analysis of the self-social NCPM with $R > 2$. Simulation results support the claim that, for the case when $R > 2$, there also exists a unique fixed point P^* and, for any initial condition $P(0) \in S_{nR}(\mathbb{1}_n)$, the solution $P(t)$ to equation (2.10) converges to P^*. We leave this statement as a conjecture.

Chapter 3

Sequential Decision Aggregation with Social Pressure

3.1 Introduction

3.1.1 Motivation and problem set-up

Decision making has been a classic research topic in the areas of industrial engineering as well as social science. In a centralized decision making model, all the signals are available to one decision maker, based on which the decision maker makes a choice among some candidate hypotheses according to some prescribed decision making policy. Numerous centralized decision making policies have been proposed. However, an isolated decision maker is always limited in decision accuracy and reliability. Moreover, in the context of sociological psychology, if we consider the decision maker as an individual in a social network, the individual is not likely to have access to all the disseminated information and make decisions independently. Instead, individuals have their private information sets and their decision making behaviors are influenced by others in the network. Therefore, it is of great research interest to study the group decision making problem. Recent years have seen much research on this topic with a focus on two objectives. The first is to establish the optimal group decision making policy. The second aspect is to build models to describe and understand the observed sociological phenomena. This chapter aims to understand how grouping individual decision makers and their mutual interactions affect the accuracy and speed with which these individuals reach a collective decision.

In this chapter, we consider a system consisting of a group of *sequential decision makers* (SDMs) and a fusion center. The SDMs are doing the sequential hypothesis test between two candidate hypotheses. The fusion center collects individual decisions and makes the global final decision. In our model, the individual SDMs make individual decisions based on both their private observations and the decisions of other SDMs. The latter amounts to a form of *social pressure*. We aim to relate the fusion center's global accuracy and expected decision time to the individuals' accuracy and expected decision time.

3.1.2 Literature Review

Group decision making has been extensively studied by numerous literature in both the engineering community [77, 78, 79, 80, 81, 82, 83, 84, 85, 86], and the area of sociological psychology [87, 88, 89, 90, 91, 92]. In engineering areas, such as control system and signal processing, two problems on group decision making are emphasized: 1) the communication between the individual decision makers and the fusion center; 2) the optimal decision making policy either in the individual level, or in the global level, to maximize the system's performance. In sociological psychology, researchers aim to investigate individuals' cognitive behavior in presence of social pressure and interactions, and the factors which influence individual or group decision making performance. Our model is closest to the work by Dandach et al. [84], of which the key feature is that, different from models in [77, 78, 79, 80, 81, 82], the fusion center in [84] does not need to wait for all the SDMs' decisions. Our model generalizes [84] by allowing mutual interactions among SDMs.

The process, with which a decision maker updates its posterior belief, or likelihood function, according to the Bayesian formula and based on its private information set, is sometimes collectively referred to as *Bayesian learning*, e.g. [93, 94]. Bayesian learning has been used to model the individuals' rational behavior. As long as the signal-generation mechanism and the decision policy are given, the individual's decision making probabilities at any given time can be predicted. In this chapter, we do not specify the signal structure and decision policy for an individual SDM, but assume that, when isolated, the SDM is adopting some Bayesian learning policy and its decision probabilities at each time step are given. On the other hand, *non-Bayesian learning* is a wording usually adopted to denote irrational decisions due to influence of other individuals in the system, or any other rule of thumb [95]. In our model, the non-Bayesian learning is characterized by the influence of social pressure. Therefore, our model can be considered as the combination of Bayesian learning [96, 83, 8] and non-Bayesian learning processes. Examples of the combination of Bayesian and non-Bayesian learning, either discrete-time or continuous-time, can be seen in [97, 86, 98], whereby individuals do not make any final decision but just update their posterior belief based on accumulated private information set (Bayesian), and combine it with the belief of their neighbors in the network (non-Bayesian).

In our model, the way that the decisions on either hypothesis propagates in the group through social pressure is similar to the *independent cascade model* [60, 99, 62, 72, 73, 47] used in the computer science community to model the network contagion process. However, in the independent cascade model, the individuals are infected passively via the activated edges while in our model the decision makers proactively pick the other decision makers and follow the picked individuals' decisions with some probability.

3.1.3 Contribution

As the first contribution of this chapter, we propose an algorithm to compute the fusion center's decision probabilities at each time step, based on the individual SDMs' decision probabilities. By introducing the concept of system state, we simplify our model, which is an exponential-dimension Markov chain, to a lumped polynomial-dimension Markov chain. The computation complexity of the iterative algorithm to compute the fusion center's decision probabilities is also polynomial. In addition, the algorithm does not rely on the specific decision making policies of the individual SDMs.

As the second contribution, we analyze the asymptotic accuracy and expected decision time of the fusion center as the system size n tends to infinity. We focus on two specific group decision making rules: the fastest rule and the majority rule. We give the exact expressions for the asymptotic accuracy and expected decision time in these two cases. Our model under the fastest rule has the same asymptotic performance as the model under the fastest rule in [84]. The analysis of the majority rule is based on the result on the mean-field convergence analysis proposed by Le Boudec et al. [100]. The asymptotic performance of the majority rule in our model is distinct from the model in [84] in that our model achieves faster decision speed, while at the cost of less accuracy, with the same individual SDMs. In addition, in our model under the majority rule, the decision speed and the global accuracy can simultaneously be better than the isolated SDM, which is not achieved by the model [84] without social pressure. Besides, leading order of a model parameter, which characterizes the individual SDMs' tendency of being influenced by the social pressure, is analyzed for the mean-field approximation of our system.

In addition, we present simulation work to validate the theoretical results and show how the accuracy and decision speed of our system vary with the system size, the group decision policies and the inclination of the decision makers to be influenced by the social pressure. We discuss how to adjust the model parameters to trade off between the system's accuracy and expected decision time.

3.1.4 Organization

The rest of this chapter is organized as follows. Section 2 is the model description and problem statement. Section 3 provides the algorithm of computing the fusion center's decision probabilities for finite system sizes. Section 4 is the discussion of the asymptotic behavior as the system size tends to infinity. Some further simulation is provided in Section 5. Section 6 is the conclusion and discussion.

3.2 Notations, Model Description, and Problem Statement

The group decision making system discussed in this chapter consists of a fusion center and n identical individual decision makers indexed by $i \in V = \{1, 2, \dots, n\}$. The individual decision makers are taking sequential hypothesis test between two hypotheses, H_1 and H_0, and are thus referred to as the *sequential decision makers* (SDMs). The SDMs make individual decisions based on both their private signals and communication with other SDMs in the system. The fusion center collects individual decisions and reach a global decision according to the *q-out-of-n* aggregation rule. Before the model description, we present in Table 3.2 all the notations frequently used in this chapter.

Table 3.1: Notations frequently used in this chapter

$D_i(t)$	decision of SDM i after time step t. $D_i(t) \in \{H_1, H_0.H_{\mathrm{nd}}\}$
$p_r(t)$	isolated SDM's probability of deciding H_r, for $r \in \{1, 0\}$, at time step t, on condition that it has not decided H_1 or H_0 before time t
$p_{\mathrm{nd}}(t)$	isolated SDM's probability of not deciding H_1 or H_0 at time step t, on condition that it has not decided H_1 or H_0 before time t
$f_r(t \mid N_1, N_0)$	SDM's probability of deciding H_r, $r \in \{1, 0\}$, after time step t, on condition that it has not decide H_1 or H_0, and N_1 (N_0 resp.) SDMs have already decided H_1 (H_0 resp.) before time t
$f_{\mathrm{nd}}(t \mid N_1, N_0)$	SDM's probability of not deciding H_1 or H_0 after time step t, on condition that it has not decide H_1 or H_0, and N_1 (N_0 resp.) SDMs have already decided H_1 (H_0 resp.) before time t
$N_1(t)$ ($N_0(t)$ resp.)	the number of SDMs who have decided H_1 (H_0 resp.) up to time step t
$p_r(t; n, q)$	the probability that the fusion center, running the q-out-of-n rule, decides H_r, $r \in \{1, 0\}$, right at time step t
T_{fc}	decision time of the fusion center, which is a random variable
$p_c(n, q)$	the probability that the fusion center, running the q-out-of-n rule, makes the correct global decision, i.e., the accuracy of the fusion center
$\mathbb{E}[T_{\mathrm{fc}} \mid n, q]$	the expected decision time for the fusion center running the q-out-of-n rule

3.2.1 Behavior of an isolated SDM

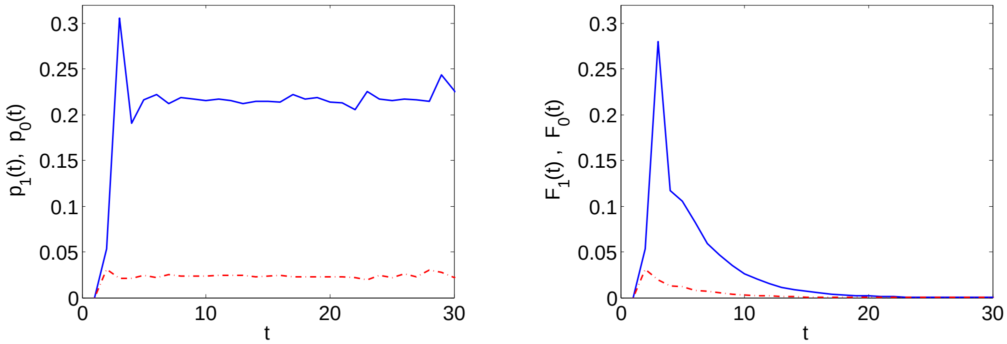
Our model of the isolated SDM is the same as that studied by Dandach et al. [84]. Suppose H_1 and H_0 are the candidate hypotheses and $H_{\mathrm{nd}}(t)$ corresponds to the state of "not deciding either H_1 or H_0". Without loss of generality, we always assume H_1 to be the correct hypothesis. Denote by $D_i(t)$ the decision state of SDM i at any time t, thereby $D_i(t) \in \{H_1, H_0, H_{\mathrm{nd}}\}$, and assume that the decision on H_1 or H_0 is irreversible. We assume that, when isolated from other SDMs, an SDM adopts some prescribed Bayesian learning and decision policy. We do not specify what the policy is, but just assume that the decision probabilities at each time, which can be predicted by the signal structure and the learning and decision policy, are given as the *individual decision probability sequence* (IDPS) $\{p_1(t), p_0(t), p_{\mathrm{nd}}(t)\}_{t\in\mathbb{N}}$, where

$$\begin{aligned} p_r(t) &= \mathbb{P}\big[D_i(t) = H_r \,|\, D_i(t-1) = H_{\mathrm{nd}}\big] \quad \text{for any } r \in \{1, 0\}, \text{ and} \\ p_{\mathrm{nd}}(t) &= \mathbb{P}\big[D_i(t) = H_{\mathrm{nd}} | D_i(t-1) = H_{\mathrm{nd}}\big]. \end{aligned} \tag{3.1}$$

Example: The *Sequential Probability Ratio Test* (SPRT) is a type of discrete-time Bayesian learning and decision policy, which achieves the minimum expected decision time for any prescribed error rate [93]. For an SDM running the SPRT, a signal S_t is received at each time step t, and, based on the accumulated information set $I_t = \{s_1, s_2, \dots, s_t\}$, the SDM calculate the *log-likelihood function*

$$\Lambda(t) = \log\left(\frac{\mathbb{P}[S_1 = s_1, S_2 = s_2, \dots, S_t = s_t \,|\, \theta = H_1]}{\mathbb{P}[S_1 = s_1, S_2 = s_2, \dots, S_t = s_t \,|\, \theta = H_0]}\right),$$

according to the Bayesian formula, where θ denotes the underlying hypothesis. Prescribed thresholds $\eta_1 > 0$ and $\eta_0 < 0$ are used to manipulate the trade-off between decision accuracy and speed. Whenever $\Lambda(t) > \eta_1$ ($\Lambda(t) < \eta_0$ resp.), the SDM decides H_1 (H_0 resp.) at time step t. Given the signal structure, i.e., $f_{S|\theta=H_1}(s)$ and $f_{S|\theta=H_0}(s)$, and



(a) $p_1(t)$ and $p_0(t)$ (b) $F_1(t)$ and $F_0(t)$

Figure 3.1: IDPS for an SDM implementing an SPRT. In Figure 3.1(a), the blue solid curve represents $p_1(t)$ while the red dash curve represent $p_0(t)$. In Figure 3.1(b), the blue solid curve represent $F_1(t) = \mathbb{P}[D_i(t) = H_1,\ D_i(t-1) = H_{\mathrm{nd}}]$, i.e., the probability of deciding H_1 right at time step t. The red dash curve represents $F_0(t) = \mathbb{P}[D_i(t) = H_0,\ D_i(t-1) = H_{\mathrm{nd}}]$.

the thresholds η_1 and η_0, the IDPS, i.e., the probabilities of deciding H_1 or H_0 at each time step, can be predicted before the SPRT process occurs. We refer the computation algorithm to Appendix B in [84]. Figure 3.1 is an example of the IDPS for an SDM running the SPRT with $\eta_1 = 2.94$ and $\eta_0 = -2.94$. In this case the false-alarm and mis-detection probabilities are both 0.05.

In our model the IDPS of an isolated SDM are assumed to have the following property.

Assumption 3.1 (Isolated SDMs' almost-sure decision and decision speed) *The isolated SDM, with the IDPS $\{p_1(t), p_0(t), p_{\mathrm{nd}}(t)\}_{t\in\mathbb{N}}$, makes the final individual decision almost surely, that is, $\prod_{t=1}^{\infty} p_{\mathrm{nd}}(t) = 0$. Moreover, the isolated SDM has finite expected decision time, i.e.,*

$$p_1(1) + p_0(1) + \sum_{t=2}^{\infty} t\Big(\big(p_1(t) + p_0(t)\big)\prod_{\tau=1}^{t-1} p_{\mathrm{nd}}(\tau)\Big) < \infty.$$

3.2.2 The n-SDM system

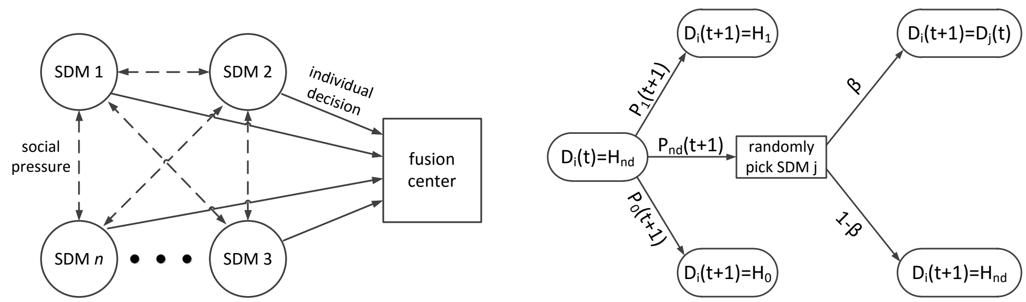
By n-SDM system we mean the system consisting of one fusion center and n identical and interacting SDMs. The behavior of the individual SDMs is described by the following assumption.

Assumption 3.2 (Individual decision making behavior in a n-SDM system) *In the n-SDM system, at each time step t, the following process occurs independently for any SDM $i \in V$ who has not made the final decision between H_1 and H_0:*

(i. SDM i first runs the sequential hypothesis test as an isolated SDM, i.e., SDM i decides H_1 (resp. H_0) with the probability $p_1(t)$ (resp. $p_0(t)$);

(ii. If no final decision is made in Step (i, SDM i will randomly pick one SDM j (can be SDM i itself) in the system and follow SDM j's previous decision state, i.e., $D_j(t-1)$, with some probability β.

In our model the more SDMs who have already made the decision H_1 (resp. H_0), the higher probability that the remaining SDMs decide H_1 (resp. H_0) at the current time step, that is, those SDMs who have made the final decision form the *social pressure*, which pushes other SDMs towards the final decisions. The probability β characterizes the inclination of the SDMs to be influenced by the social pressure. The model proposed by Dandach et. al. [84] is a special case when $\beta = 0$. Denote by $f_r(t \mid N_1, N_0)$, $r \in \{1, 0\}$, the probability that an SDM in the n-SDM system decides H_r at time step t, on condition that it has not made the final decision up to time $t-1$ and N_1 (resp. N_0) numbers of SDMs have decided H_1 (resp. H_0) before time t. Denote by $f_{\mathrm{nd}}(t \mid N_1, N_0)$ the probability that an SDM does not make the final decision at time step t, on condition that it has not made the final decision up to time $t-1$ and N_1 (resp. N_0) numbers of SDMs have



(a) The n-SDM system (b) Individual SDM's behavior at step $t+1$

Figure 3.2: The first diagram shows the structure of the n-SDM system. The connections between the SDMs are bilateral with self loops. Therefore any SDM can be picked by any other SDM or itself. Once an individual final decision is made, the decision is sent to the fusion center. The second diagram describes how an SDM in the n-SDM system makes the individual decision at time step $t+1$.

decided H_1 (resp. H_0) before time t. According to Assumption 3.2,

$$\begin{aligned} f_r(t|N_1,N_0) &= p_r(t) + \beta p_{\mathrm{nd}}(t)\frac{N_r}{n} \quad \text{for } r\in\{1,0\}, \text{ and} \\ f_{\mathrm{nd}}(t|N_1,N_0) &= p_{\mathrm{nd}}(t)\left(\frac{n-N_1-N_0}{n} + (1-\beta)\frac{N_1+N_0}{n}\right). \end{aligned} \tag{3.2}$$

One can easily check that $f_1(t|N_1,N_0) + f_0(t|N_1,N_0) + f_{\mathrm{nd}}(t|N_1,N_0) = 1$ for any t, N_1 and N_0.

Denote by $N_1(t)$ (resp. $N_0(t)$) the numbers of SDMs who have decided H_1 (resp. H_0) up to time step t. The fusion center receives each final individual decision from the SDMs and records $N_1(t)$ and $N_0(t)$. The global decision is made based on $N_1(t)$ and $N_0(t)$, according to the q-out-of-n rule defined below.

Definition 3.2.1 (The q-out-of-n rule) *In an n-SDM sequential decision aggregation system, the fusion center running a q-out-of-n rule decides H_1 at time step t whenever $N_1(t) > N_0(t)$ and $N_1(t) \geq q$, where q is a prescribed threshold. The global decision H_0 is made if $N_0(t) > N_1(t)$ and $N_0(t) \geq q$.*

Figure 3.2 gives a visual depiction of the n-SDM system structure and the individual SDMs' behavior.

3.2.3 Problem Statement

With the n-SDM system described in Section 2.1 and 2.2, we aim to solve the following problems.

Problem 1 (Finite-system behavior) *For the fusion center running the q-out-of-n rule in a system with finite SDMs, given the IDPS* $\{p_1(t), p_0(t), p_{\mathrm{nd}}(t)\}_{t\in\mathbb{N}}$*, compute the probabilities* $p_1(t;n,q)$*,* $p_0(t;n,q)$*,* $p_c(n,q)$*, and the expected decision time* $\mathbb{E}[T_{\mathrm{fc}}|n,q]$*, as defined in Table 1.*

Problem 2 (Asymptotic behavior) *For the fusion center running the q-out-of-n rule in a n-SDM system, given the IDPS, compute the limit of the fusion center's accuracy and expected decision time as n tends to infinity, especially in the cases when* $q = 1$ *or* $q = \lceil n/2 \rceil$*.*

3.3 The Behavior of the Fusion Center in a Finite n-SDM System

In this section we solve Problem 1, i.e., the fusion center's behavior in a system with finite SDMs. Firstly, we state a proposition on the almost-sure decision and finite expected decision time for the fusion center.

Proposition 3.3.1 (Almost-sure decision and finite expected decision time) *Consider an n-SDM system, assume that for the isolated SDM, there exists some* $\tilde{t} \in \mathbb{N}$ *such that*

$p_1(\tilde{t}) \neq 0$, $p_0(\tilde{t}) \neq 0$ and $\prod_{\tau=1}^{\tilde{t}-1} p_{\mathrm{nd}}(\tau) \neq 0$, then the fusion center has the almost-sure decision property if and only if

(i. the isolated SDMs have almost-sure decision property;

(ii. the system size n is an odd number;

(iii. the threshold q satisfies $1 \leq q \leq \lceil n/2 \rceil$.

Moreover, in addition to the conditions (ii and (iii, if the isolated SDMs have finite expected decision time, then the fusion center also has finite expected decision time.

Proof: We first prove the contrapositive of the statement that the almost-sure decision of the fusion center leads to the conditions (i, (ii, and (iii.

(1) If the individual SDMs do not have the almost-sure decision property, i.e., $p_{\mathrm{nd}} = \prod_{t=1} p_{\mathrm{nd}}(t) \neq 0$, then the probability that none of the SDMs makes any final decision in the n-SDM system is equal to p_{nd}^n. Therefore, the probability that the fusion center does not make any global decision at all is no less than $p_{\mathrm{nd}}^n > 0$.

(2) If n is even, the event "no SDM has made any final decision after time $\tilde{t}-1$, at time $\tilde{t}$, $n/2$ SDMs decide H_1 while $n/2$ SDMs decide H_0" has probability

$$\left(\prod_{\tau=1}^{\tilde{t}-1} p_{\mathrm{nd}}(\tau)\right)^n \binom{n}{n/2} p_1(\tilde{t})^{n/2} p_0(\tilde{t})^{n/2} > 0.$$

If this event occurs, then the fusion center will never make a global decision.

(3) If $q > \lceil n/2 \rceil$, then consider the following event: "No SDM has decided up to $\tilde{t}-1$. At $\tilde{t}$, $\lceil n/2 \rceil$ SDMs decide H_1 while $\lfloor n/2 \rfloor$ SDMs decide H_0." This event has the probability

$$\left(\prod_{\tau=1}^{\tilde{t}-1} p_{\mathrm{nd}}(\tau)\right)^n \binom{n}{\lceil n/2 \rceil} p_1(\tilde{t})^{\lceil n/2 \rceil} p_0(\tilde{t})^{\lfloor n/2 \rfloor} > 0.$$

In this case, neither N_1 nor N_0 has a chance to exceed the threshold, therefore the fusion center has a non-zero probability of making no global decision. Combining (1), (2) and (3) we conclude that the fusion center having the almost-sure decision property implies conditions (i, (ii, and (iii.

Next, we prove that conditions (i, (ii, and (iii lead to the almost-sure decision of the fusion center. Before the argument, we introduce some notations used in this proof. Define the random variable T_i as the decision time of SDM i when it is isolated, and define $T_i^{(n)}$ as the decision time of SDM i in an n-SDM system. Define $T_{\max}^{(n)}$ as $\max_i T_i^{(n)}$, i.e., the time instant when the last SDM makes the final individual decision. By definition, the fusion center's decision time must be prior or equal to $T_{\max}^{(n)}$. Let $\boldsymbol{T}_{-i}^{(n)} = (T_1^{(n)}, \dots, T_{i-1}^{(n)}, T_{i+1}^{(n)}, \dots, T_n^{(n)})$, i.e., the $(n-1)$-tuple of the decision time instants of all the SDMs except SDM i. Denote by ω one possible "trajectory" of the n-SDM system, i.e., a sequence of 2-tuples $\{(n_1(t), n_0(t))\}_{t\in\mathbb{N}}$, where $n_1(t)$, $n_0(t) \in \mathbb{N}$ and $n_1(t) + n_0(t) \le n$ for any $t \in \mathbb{N}$. For simplicity, let $f_\alpha(t\,|\,\omega) = f_\alpha\big(t\,|\,n_1(t-1), n_0(t-1)\big)$ with the right-hand side of the equation defined by equations (3.2) for $\alpha = 1$, or 0, or "nd". Denote by Ω the set of all the possible trajectories, i.e., $\omega \in \Omega$.

Due to equations (3.2), $f_1(t\,|\,\omega) \ge p_1(t)$, $f_0(t\,|\,\omega) \ge p_0(t)$ and $f_{\mathrm{nd}}(t\,|\,\omega) \le p_{\mathrm{nd}}(t)$ for any $\omega \in \Omega$. Since

$$\begin{aligned}
\mathbb{P}[T_i^{(n)} < \infty\,|\,\boldsymbol{T}_{-i}^{(n)} < \infty] &= \sum_{\omega\in\Omega} \mathbb{P}[T_i^{(n)} < \infty\,|\,\omega, \boldsymbol{T}_{-i}^{(n)} < \infty]\mathbb{P}[\omega\,|\,\boldsymbol{T}_{-i}^{(n)} < \infty] \\
&= \sum_{\omega\in\Omega} \big(1 - \prod_{t=1}^{\infty} f_{\mathrm{nd}}(t\,|\,\omega)\big)\mathbb{P}[\omega\,|\,\boldsymbol{T}_{-i}^{(n)} < \infty] \\
&\ge \sum_{\omega\in\Omega} \big(1 - \prod_{t=1}^{\infty} p_{\mathrm{nd}}(t)\big)\mathbb{P}[\omega\,|\,\boldsymbol{T}_{-i}^{(n)} < \infty] = \mathbb{P}[T_i < \infty] = 1,
\end{aligned}$$

we have

$$\mathbb{P}[T_{\max}^{(n)} < \infty] = \mathbb{P}[T_1^{(n)} < \infty, T_2^{(n)} < \infty, \dots, T_n^{(n)} < \infty] \geq \prod_{i=1}^{n} \mathbb{P}[T_i < \infty] = 1.$$

Therefore, $\mathbb{P}[T_{\max}^{(n)} < \infty] = 1$. Due to conditions (ii and (iii, the q-out-of-n rule must have been triggered no later than $T_{\max}^{(n)}$. Therefore, the fusion center makes the global decision almost surely.

We now prove the finite expected decision time for the fusion center. Conditions (ii and (iii lead to the inequality $T_{\text{fc}} \leq T_{\max}^{(n)} \leq T_1^{(n)} + T_2^{(n)} + \dots + T_n^{(n)}$ for any $\omega \in \Omega$. Moreover,

$$\begin{aligned}\mathbb{E}[T_i^{(n)}] &= \sum_{t=1}^{\infty} \mathbb{P}[T_i^{(n)} \geq t] = \sum_{t=1}^{\infty} \sum_{\omega \in \Omega} \mathbb{P}[T_i^{(n)} \geq t \,|\, \omega] \mathbb{P}[\omega] = 1 + \sum_{t=2}^{\infty} \sum_{\omega \in \Omega} \prod_{\tau=1}^{t-1} f_{\text{nd}}(t \,|\, \omega) \mathbb{P}[\omega] \\ &\leq 1 + \sum_{t=2}^{\infty} \sum_{\omega \in \Omega} \prod_{\tau=1}^{t-1} p_{\text{nd}}(t) \mathbb{P}[\omega] = 1 + \sum_{t=2}^{\infty} \mathbb{P}[T_i \geq t] = \mathbb{E}[T_i].\end{aligned}$$

Therefore, $\mathbb{E}[T_{\text{fc}} \,|\, n, q] \leq n\mathbb{E}[T_i] < \infty$ for any $1 \leq q \leq \lceil n/2 \rceil$. This concludes the proof. ■

In the rest of this section, we quantitatively analyze the behavior of the fusion center in an n-SDM system, given the IDPS of the isolated SDM. We compute the probabilities of deciding either H_1 or H_0 at each time step, the accuracy, and the expected decision time of the fusion center.

1) The n-SDM system as a lumped Markov chain: The n-SDM sequential decision aggregation system is a 3^n-state Markov chain, since $D_i(t) \in \{H_1, H_0, H_{\text{nd}}\}$ for any $i \in V$ and at any time step the decision of any SDM only depends on the states of all the SDMs after the previous time step as well as the IDPS. Instead of focusing on any individual SDM's decision state, we discuss the time evolution of $N_1(t)$ and $N_0(t)$. Then the system is reduced to to $\frac{(n+1)(n+2)}{2}$-state Markov chain.

Definition 3.3.2 *Consider the n-SDM sequential aggregation system. Define the system state after time step t by $\boldsymbol{N}(t) = (N_1(t), N_0(t))^T$ and define $p(t, N_1, N_0)$ as the probability distribution of the system state after time t. Define $\Gamma(t, \Delta N_1, \Delta N_0 \,|\, N_1, N_0)$ as the* state transition function*, which correspond to the probability of the following event: "on condition that N_1 SDMs have decided H_1 and N_0 SDMs have decided H_0 after time step $t-1$, ΔN_1 SDMs decide H_1 and ΔN_0 SDMs decide H_0 at time t."*

The computation algorithm of the system's state probability distribution at any time t is given by the following proposition. The proof is a straightforward application of probability theory and thus omitted.

Proposition 3.3.3 (System state probability distribution) *The probability distribution of the n-SDM system state is given by the formulas below:*

(i. For $t = 1$, $p(1, N_1, N_0) = \Gamma(1, N_1, N_0 \,|\, 0, 0)$;

(ii. For $t \geq 2$, the probability distribution of the system state is computed from the distribution at last time step as

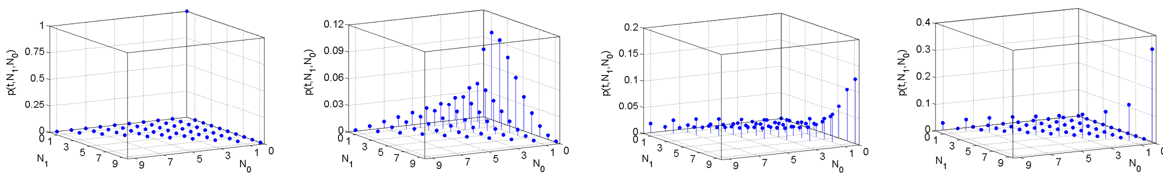
$$p(t, N_1, N_0) = \sum_{l=0}^{N_1} \sum_{k=0}^{N_0} p(t-1, l, k)\Gamma(t, N_1 - l, N_0 - k \,|\, l, k).$$

Here, the state transition function $\Gamma(t, \Delta N_1, \Delta N_0 \,|\, N_1, N_0)$ is computed by

$$\begin{aligned}\Gamma(t, \Delta N_1, \Delta N_0 \,|\, N_1, N_0) = &\binom{n - N_1 - N_0}{\Delta N_1} \binom{n - N_1 - N_0 - \Delta N_1}{\Delta N_0} \\ &\times f_1^{\Delta N_1}(t|N_1, N_0) f_0^{\Delta N_0}(t|N_1, N_0) f_{\mathrm{nd}}^{n - N_1 - N_0 - \Delta N_1 - \Delta N_0}(t|N_1, N_0),\end{aligned}$$

where $t \in \mathbb{N}$ and $0 \leq \Delta N_1 + \Delta N_0 + N_1 + N_0 \leq n$.

Figure 3.3 illustrates the evolution of the probability distribution of the system state for a group of 9 SDMs in which all the SDMs are running the SPRT as shown in Fig-



(a) $t = 1$ (b) $t = 21$ (c) $t = 41$ (d) $t = 101$

Figure 3.3: The time-evolution of the state probability distribution of a 9-SDM system with social pressure. The IDPS for individual SDMs are as shown in Figure 3.1.

ure 3.1. Initially, $(N_1, N_0) = (0, 0)$ is the only state with non-zero probability and then the states with non-zero probability spread out and finally aggregate on the diagonal line $N_1 + N_0 = 9$.

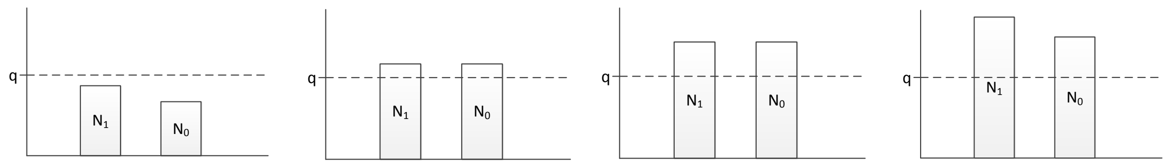
2) Computation of $p_1(t; n, q)$ and $p_0(t; , n, q)$: With the n-SDM system's state probability distribution at any time t, i..e, $p(t, N_1, N_0)$, we can compute $P_1(t; n, q)$ and $p_0(t; n, q)$ defined in Problem 1, that is, the probabilities that the fusion center running the q-out-of-n rule makes the global decision H_1 and H_0 respectively right at time step t. Notice that, in the sequential decision aggregation process for $1 \leq q \leq \lfloor n/2 \rfloor$, the *cancel-out* case may occur. The cancel-out case in which the fusion center finally decides H_1 corresponds to the intersection of the following three events:

(i. $N_1(\tau^* - 1) < q$ and $N_1(\tau^*) \geq q$ for some $\tau^* < t$;

(ii. For $\tau \in \{\tau^*, \tau^* + 1, \ldots, t - 1\}$, $N_1(\tau) = N_0(\tau) \geq q$;

(iii. After time step t, $N_1(t) > N_0(t) \geq q$.

If the notations $N_1(t)$ and $N_0(t)$ are exchanged, the intersection of events (i), (ii) and (iii) corresponds to the cancel-out case in which the fusion center decides H_0. An example of the cancel-out case is illustrated by Figure 3.4. Based on whether the cancel-out case



(a) $\tau = \tau^* - 1$ (b) $\tau = \tau^*$ (c) $\tau = t - 1$ (d) $\tau = t$

Figure 3.4: The cancel-out case in which the number of votes for H_1 and H_0 both exceed the threshold q at time τ^* and remain equal till $t-1$. At time t, the vote for H_1 outnumbers H_0 and the fusion center decides H_1 at time t.

may occur, we discuss the computation of $p_1(t; n, q)$ and $p_0(t; n, q)$ in two cases, Case 1: $1 \le q \le \lfloor n/2 \rfloor$ and Case 2: $\lceil n/2 \rceil \le q \le n$.

Proposition 3.3.4 (Computation of $p_1(t; n, q)$ in Case 1) *Consider the n-SDM sequential decision aggregation system with the fusion center running the q-out-of-n rule and the individual SDMs with the IDPS $\{p_1(t), p_0(t), p_{\mathrm{nd}}(t)\}_{t\in\mathbb{N}}$. For $1 \le q \le \lfloor n/2 \rfloor$, the probability $p_1(t; n, q)$ defined in Problem 1 is computed by the following formulas:*

(i. For $t = 1$,

$$p_1(1; n, q) = \sum_{N_1=q}^{n} \sum_{N_0=0}^{\tilde{m}} p(1, N_1, N_0); \tag{3.3}$$

(ii. For $t \ge 2$,

$$\begin{aligned} p_1(t; n, q) = &\sum_{l=0}^{q-1} \sum_{k=0}^{q-1} p(t-1, l, k) \sum_{\Delta N_1 = q-l}^{n-l-k} \sum_{\Delta N_0 = 0}^{\bar{m}} \Gamma(t, \Delta N_1, \Delta N_0 \,|\, l, k) \\ &+ \sum_{s=q}^{\lfloor n/2 \rfloor} p_{\mathrm{even}}(t-1, s) \sum_{\Delta N_1 = 1}^{n-2s} \sum_{\Delta N_0 = 0}^{m^*} \Gamma(t, \Delta N_1, \Delta N_0 \,|\, s, s), \end{aligned} \tag{3.4}$$

where $\tilde{m} = \min\{N_1 - 1, n - N_1\}$, $\bar{m} = \min\{\Delta N_1 + l - k - 1, n - l - k - \Delta N_1\}$ and $m^* = \min\{\Delta N_1 - 1, n - 2s - \Delta N_1\}$. The probability $p(t-1, l, k)$ for any $t \in \mathbb{N}$ and $0 \le l + k \le n$ is computed by Proposition 3.3.3 and the function $p_{\mathrm{even}}(t, s)$ for any $t \in \mathbb{N}$

and $q \leq s \leq \lfloor n/2 \rfloor$ is given by the following iteration formulas:

(i. *For $t = 1$, $p_{\text{even}}(t, s) = p(1, s, s)$;*

(ii. *For $t \geq 2$,*

$$p_{\text{even}}(t, s) = \sum_{l=0}^{q-1} \sum_{k=0}^{q-1} p(t-1, l, k)\Gamma(t, s-l, s-k \,|\, l, k) + \sum_{h=q}^{s} p_{\text{even}}(t-1, h)\Gamma(t, s-h, s-h \,|\, h, h). \tag{3.5}$$

Proof: First we define $p_{\text{even}}(t, s)$ as the probability of the intersection of the following tree events:

(i. $N_1(\tilde{\tau}) < q$ and $N_0(\tilde{\tau}) < q$ for some $\tilde{\tau} < t$;

(ii. For $\tau \in \{\tilde{\tau}, \tilde{\tau} + 1, \ldots, t\}$, $N_1(\tau) = N_0(\tau)$;

(iii. After time step t, $N_1(t) = N_0(t) = s \geq q$.

Then equation (3.5) is a straightforward application of the total probability formula. For $t = 1$, $p_{\text{even}}(1, s)$ is equal to $p(1, s, s)$ by definition. For the case $t \geq 2$, the first term of the right-hand side of equation (3.5) corresponds to the probability that both $N_1(t-1)$ and $N_0(t-1)$ are under the threshold q and $N_1(t) = N_0(t) = s \geq q$. The second term is the probability that, for any $\tau \leq t-1$, $N_1(\tau)$ and $N_0(\tau)$ remain equal if either of them exceeds the threshold q, and $N_1(t) = N_0(t) = s \geq q$.

With the computation algorithm of $p_{\text{even}}(t, s)$, now we derive the formula for $p_1(t; , n, q)$. If the fusion center decides H_1 at $t = 1$, then $N_1(1) \geq q$ and $N_1(1) > N_0(1)$. Since all the system states $(N_1(1), N_0(1))$ are mutually exclusive, the probability that the fusion center decides H_1 at $t = 1$ is the sum of all the $p(1, N_1, N_0)$ satisfying $N_1 > N_0$ and $N_1 \geq q$. This concludes the proof of equation (3.3).

For $t \geq 2$, first we consider the case when the cancel-out case does not occur. The probability of the intersection of the following two events:

(i. At time $t-1$, both $N_1(t-1)$ and $N_0(t-1)$ are below the threshold. The probability of this event is $\sum_{l=0}^{q-1}\sum_{k=0}^{q-1} p(t-1,l,k)$;

(ii. On condition that after time $t-1$, the system is in some state (l,k) below the threshold, i.e., $l<q$ and $k<q$, the votes for H_1 outnumbers the votes for H_0 and exceeds the threshold at time step t,

is equal to

$$\sum_{\Delta N_1=q-l}^{n-l-k}\sum_{\Delta N_0=0}^{\bar{m}} \Gamma(t,\Delta N_1,\Delta N_0\,|\,l,k).$$

Applying the total probability formula we obtain the probability that the fusion center decides H_1 at t when the cancel-out case does not occur, which is the first term of the right-hand side of equation (3.4).

In the cancel-out case, the q-out-of-n condition is not triggered before t. After time step $t-1$, both $N_1(t-1)$ and $N_0(t-1)$ must have exceeded the threshold q and they are equal to s with probability $p_{\text{even}}(t-1,s)$ for any $s\in\{q,q+1,\dots,\lfloor n/2\rfloor\}$. On condition that $N_1(t-1)=N_0(t-1)=s\geq q$, the probability that $N_1(t)>N_0(t)\geq q$ is equal to $\sum_{\Delta N_1=1}^{n-2s}\sum_{\Delta N_0=0}^{m^*}\Gamma(t,\Delta N_1,\Delta N_0\,|\,s,s)$. According to the total probability formula, we obtain the second term of the right hand side of equation (3.4). This concludes the proof.■

The computation of $p_1(t;n,q)$ in the case $\lceil n/2\rceil$, in which there is no cancel-out case, is given by the proposition below. The proof is a straightforward application of the total probability formula.

Proposition 3.3.5 (Computation of $p_1(t;n,q)$ in Case 2) *Consider the n-SDM sequential decision aggregation process with the fusion center running the q-out-of-n rule. For $\lceil n/2\rceil\leq q\leq n$, the probability $p_1(t;n,q)$ is computed by the following formulas:*

(i. For $t = 1$,

$$p_1(t;n,q) = \sum_{N_1=q}^{n} \sum_{N_0=0}^{n-N_1} p(1, N_1, N_0); \tag{3.6}$$

(ii. For $t \geq 2$,

$$p_1(t;n,q) = \sum_{l=0}^{q-1} \sum_{k=0}^{n-q} p(t-1, l, k) \sum_{\Delta N_0 = q-l}^{n-l-k} \sum_{\Delta N_0=0}^{\bar{m}} \Gamma(t, \Delta N_1, \Delta N_0 \,|\, l, k), \tag{3.7}$$

where $\bar{m} = n - l - k - \Delta N_1$.

To compute $p_0(t;n,q)$ we just need to switch all the indexes corresponding to H_1 and H_0 in equations (3.3), (3.4), (3.6), and (3.7).

3) Accuracy and expected decision time of the fusion center and the overall computation complexity: With the algorithm of computing $p_1(t;n,q)$ and $p_0(t;n,q)$, the fusion center's accuracy and expected decision time is given by the following equations:

$$p_c(n,q) = \sum_{t=1}^{\infty} p_1(t;n,q), \tag{3.8}$$

and

$$\mathbb{E}[T_{\text{fc}} \,|\, n, q] = \sum_{t=1}^{\infty} t\big(p_1(t;n,q) + p_0(t;n,q)\big). \tag{3.9}$$

The state transition function $\Gamma(t, \Delta N_1, \Delta N_0 \,|\, N_1, N_0)$ is given by a closed form with the computation complexity $O(1)$. According to Proposition 3.3.3, the computation complxity for $p(t, N)1, N_0)$ is $O(1)$ for $t = 1$ and $O(n^2)$ for $t \geq 2$. Knowing $p(t-, N_1, N_0)$ for any $0 \leq N_1 \leq n$, $0 \leq N_0 \leq n$ and $0 \leq N_1 + N_0 \leq n$, the algorithm of computing $p_{\text{even}}(t, s)$ has the complexity $O(n^2)$. Therefore, according to Proposition 3.3.4 and Proposition 3.3.5 we know that the computation complexity for $p_1(t;n,q)$ is $O(n^5)$ when

$1 \leq q \leq \lfloor n/2 \rfloor$ and is $O(n^4)$ when $\lceil n/2 \rceil \leq q \leq n$.

3.4 Asymptotic Behaviors of the q-out-of-n Decision Aggregation System

By asymptotic behavior we mean the behavior of the fusion center in the n-SDM system as n tends to infinity. In this section, firstly we relate the accuracy and the expected decision time of the fusion center to the IDPS of the isolated SDMs, particularly for two special q-out-of-n rules: the fastest rule with $q = 1$ and the majority rule with $q = \lceil n/2 \rceil$. Then we discuss the influence of the parameter β on the sequential decision aggregation system as $n \to \infty$.

3.4.1 The fastest rule

According to Proposition IV.1 in the paper by Dandach et. al. [84], which is a n-SDM system with $\beta = 0$, the asymptotic accuracy and expected decision time of the fusion center running the fastest rule only depends on the first time instance when either $p_1(t) \neq 0$ or $p_0(t) \neq 0$. The following theorem states that the n-SDM system under the fastest rule leads to the same result for any $0 \leq \beta \leq 1$.

Theorem 3.4.1 (Asymptotic behavior for the fastest rule) *Consider the sequential decision aggregation system in which the fusion center is running the fastest rule. Define the earliest possible decision time $\bar{t}$ as*

$$\bar{t} = \min\{t \in \mathbb{N} \,|\, p_1(t) \neq 0 \text{ or } p_0(t) \neq 0\}.$$

Then the asymptotic accuracy of the fusion center satisfies

$$\lim_{n\to\infty} p_c(n,1) = \begin{cases} 1, & \text{if } p_1(\bar{t}) > p_0(\bar{t}), \\ 0, & \text{if } p_1(\bar{t}) < p_0(\bar{t}), \\ 1/2, & \text{if } p_1(\bar{t}) = p_0(\bar{t}), \end{cases} \tag{3.10}$$

and the asymptotic expected decision time satisfies

$$\lim_{n\to\infty} \mathbb{E}[T_{\text{fc}}|n,1] = \bar{t}. \tag{3.11}$$

Proof: In this proof it is convenient to modify our notation as follows: several systems with different IDPS are indexed by subscripts. Denote by $S_r^{(n)}$ the n-SDM system with index r and the IDPS $\{p_1^r(t), p_0^r(t), p_{\text{nd}}^r(t)\}_{t\in\mathbb{N}}$. Notice that here r is the system index rather than the power. The accuracy and expected decision time for the fusion center are denoted by $p_c(S_r^{(n)}, q)$ and $\mathbb{E}[T_{\text{fc}}|S_r^{(n)}, q]$ respectively.

We introduce three different n-SDM systems. Define

(i. $S_1^{(n)}$ as the n-SDM system with IDPS $\{p_1^1(t), p_0^1(t), p_{\text{nd}}^1(t)\}_{t\in\mathbb{N}}$, for which the earliest possible decision time $\bar{t}$ is defined by $\bar{t} = \min\{t \in \mathbb{N} \,|\, p_1^1(t) \neq 0 \text{ or } p_0^1(t) \neq 0\}$;

(ii. $S_2^{(n)}$ as the n-SDM system with $\beta = 0$, i.e., no social pressure, and the corresponding IDPS satisfying

$$\begin{cases} p_1^2(t) = p_1^1(t) \text{ and } p_0^2(t) = p_0^1(t), \text{ for } \forall t \leq \bar{t}, \\ p_1^2(\bar{t}+1) = 1 \text{ and } p_0^2(\bar{t}+1) = 0, \\ p_1^2(t) = p_0^2(t) = 0 \text{ for } \forall t > \bar{t}+1; \end{cases}$$

(iii. $S_3^{(n)}$ as the n-SDM system with $\beta = 0$ and the IDPS satisfying

$$\begin{cases} p_1^3(t) = p_1^1(t) \text{ and } p_0^3(t) = p_0^1(t), \text{ for } \forall t \leq \bar{t}, \\ p_1^3(\bar{t}+1) = 0 \text{ and } p_0^3(\bar{t}+1) = 1, \\ p_1^3(t) = p_0^3(t) = 0 \text{ for } \forall t > \bar{t}+1. \end{cases}$$

First we compare the accuracy of $S_1^{(n)}$ and the accuracy of $S_2^{(n)}$ when both are running the fastest rule. The systems $S_1^{(n)}$ and $S_2^{(n)}$ are identical for $t \leq \bar{t}$ since the social pressure terms $\beta p_{\text{nd}}(t)N_1(t)/n$ and $\beta p_{\text{nd}}(t)N_0(t)/n$ remain zero. For system $S_2^{(n)}$, at time step $\bar{t}+1$, all the SDMs who have not made final individual decisions will decide H_1. Therefore, $p_c(S_1^{(n)}, 1) \leq p_c(S_2^{(n)}, 1)$. Applying the same argument we have $p_c(S_3^{(n)}, 1) \leq p_c(S_1^{(n)}, 1)$. Moreover, according to Proposition IV.1 in [84], as n tends to infinity,

$$\begin{aligned} \lim_{n\to\infty} p_c(S_2^{(n)}, 1) &= \lim_{n\to\infty} p_c(S_3^{(n)}, 1) \\ &= \begin{cases} 1, & \text{if } p_1^1(\bar{t}) > p_0^1(\bar{t}), \\ 0, & \text{if } p_1^1(\bar{t}) < p_0^1(\bar{t}), \\ \frac{1}{2}, & \text{if } p_1^1(\bar{t}) = p_0^1(\bar{t}). \end{cases} \end{aligned} \tag{3.12}$$

This leads to equation (3.10).

Now we discuss the asymptotic expected decision time. If $p_1^1(\bar{t}) + p_0^1(\bar{t}) = 1$, obviously the fusion center's expected decision time would be $\bar{t}$ for any n. Suppose $0 < p_1^1(\bar{t}) + p_0^1(\bar{t}) < 1$. Define another system $S_4^{(n)}$ with the IDPS $\{p_1^4(t), p_0^4(t), p_{\text{nd}}^4(t)\}_{t\in\mathbb{N}}$ satisfies

$$\begin{aligned} &p_1^4(\bar{t}) = p_0^4(\bar{t}) = 0, \ p_{\text{nd}}^4(\bar{t}) = 1, \text{ and} \\ &p_1^4(t) = p_1^1(t), \ p_0^4(t) = p_0^1(t), \ p_{\text{nd}}^4(t) = p_{\text{nd}}^1(t) \text{ for any } t \neq \bar{t}, \end{aligned}$$

and the fusion center in system $S_4^{(n)}$ makes the global decision after the SDMs have decided H_1 or H_0. As long as $p_1^1(\bar{t}) + p_0^1(\bar{t}) < 1$, the isolated SDMs with the IDPS $\{p_1^4(t), p_0^4(t), p_{\mathrm{nd}}^4(t)\}_{t\in\mathbb{N}}$ still have almost-sure decision and finite expected decision time.

For system $S_1^{(n)}$,

$$\mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1] = \bar{t}\mathbb{P}[T_{\mathrm{fc}} = \bar{t} \,|\, S_1^{(n)}, q=1] + \mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1, T_{\mathrm{fc}} > \bar{t}]\mathbb{P}[T_{\mathrm{fc}} > \bar{t} \,|\, S_1^{(n)}, q=1].$$

By definition and according to the proof of Proposition 3.3.1,

$$\mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1, T_{\mathrm{fc}} > \bar{t}] \leq \mathbb{E}[T_{\max}^{(n)} \,|\, S_4^{(n)}] \leq n\mathbb{E}[T_i | S_4^{(n)}].$$

Moreover, according to the proof of Proposition IV.1 in [84], the term $\mathbb{P}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1]$ is in order $O(\epsilon)$ for some $0 < \epsilon < 1$ and $\lim_{n\to\infty} \mathbb{P}[T_{\mathrm{fc}} = \bar{t} \,|\, S_1^{(n)}, q=1] = 1$. Therefore,

$$\lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1, T_{\mathrm{fc}} > \bar{t}]\mathbb{P}[T_{\mathrm{fc}} > \bar{t} \,|\, S_1^{(n)}, q=1] = 0, \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1] = \bar{t}.$$

■

3.4.2 The majority rule

Before analyzing the accuracy and expected decision time of the fusion center under the majority rule, we introduce a main result in the paper [100] on the mean-field convergence for systems with interacting objects, which can be applied to our model.

Consider a discrete-time Markov chain with n individuals. Denote by $X_i(t)$ the state of individual i after time step t. The individual states set is identical for all the individuals and is denoted by $\Theta = \{1, 2, \ldots, S\}$, i.e., $X_i(t) \in \Theta$ for any $i \in \{1, 2, \ldots, n\}$ and $t \in \mathbb{N}$.

Define the *occupancy measure* $\boldsymbol{M}^{(n)}(t) \in \mathbb{R}^{1\times S}$ by $M_r^{(n)}(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i(t)=r\}}$ for any $r \in \Theta$. Define the *memory* $\boldsymbol{R}^{(n)}(t)$ as some d-dimension row vector, which is updated according to some continuous function $g : \mathbb{R}^{1\times d} \times \mathbb{R}^{1\times S} \to \mathbb{R}^{1\times d}$, that is, $\boldsymbol{R}^{(n)}(t+1) = g(\boldsymbol{R}^{(n)}(t), \boldsymbol{M}^{(n)}(t))$. Denote the individual state transition matrix by $K^{(n)}(t) = \big(K^{(n)}_{rm}(t)\big)_{S\times S}$, that is,

$$K^{(n)}_{rm}(t) = \mathbb{P}[X^{(n)}_i(t+1) = m \,|\, X^{(n)}_i(t) = r],$$

and $K^{(n)}_{rm}$ is an explicit function of $\boldsymbol{R}^{(n)}(t)$, i.e., $K^{(n)}(t) = \big(K^{(n)}_{rm}(\boldsymbol{R}^{(n)}(t))\big)_{S\times S}$. We rewrite [100, Theorem 4.1] as follows.

Lemma 3.4.2 (Mean-field convergence) *Consider the discrete-time Markov chain described above. Assume that,*

(i. For any $r, m \in \Theta$, as $n \to \infty$, $K^{(n)}_{rm}(\boldsymbol{r})$ converges uniformly in $\boldsymbol{r} \in \mathbb{R}^{1\times d}$ to some $K_{rm}(\boldsymbol{r})$, which is a continuous function of $\boldsymbol{r}$;

(ii. The vectors $\boldsymbol{M}^{(n)}(0)$ and $\boldsymbol{R}^{(n)}(0)$ converge almost surely to some deterministic limits $\boldsymbol{\mu}(t)$ and $\boldsymbol{\rho}(0)$.

Then for any fixed t, almost surely,

$$\lim_{n\to\infty} \boldsymbol{M}^{(n)}(t) = \boldsymbol{\mu}(t), \quad \textit{and} \quad \lim_{n\to\infty} \boldsymbol{R}^{(n)}(t) = \boldsymbol{\rho}(t),$$

where $\boldsymbol{\mu}(t)$ and $\boldsymbol{\rho}(t)$ are defined by the following iteration formulas:

$$\boldsymbol{\mu}(t+1) = \boldsymbol{\mu}(t) K\big(\boldsymbol{\rho}(t)\big), \quad \textit{and} \quad \boldsymbol{\rho}(t+1) = g\big(\boldsymbol{\rho}(t), \boldsymbol{\mu}(t+1)\big).$$

In the lemma above, the deterministic vector $\boldsymbol{\mu}(t)$ is referred to as the *mean-field limit* of $\boldsymbol{M}^{(n)}(t)$ as $n \to \infty$. Now we apply this lemma to our model. Define the occupancy measure $\boldsymbol{M}^{(n)}(t)$ by

$$\boldsymbol{M}^{(n)}(t) = \left(\frac{N_1(t)}{n}, \frac{N_0(t)}{n}, \frac{n - N_1(t) - N_0(t)}{n}\right), \tag{3.13}$$

and define the vector sequence $\{\boldsymbol{\mu}(t)\}_{t\in\mathbb{N}}$ by

$$\begin{aligned} \boldsymbol{\mu}(0) &= (0, 0, 1), \\ \mu_1(t+1) &= \mu_1(t) + \mu_3(t)\big(p_1(t+1) + \beta p_{\mathrm{nd}}(t+1)\mu_1(t)\big), \\ \mu_2(t+1) &= \mu_2(t) + \mu_3(t)\big(p_0(t+1) + \beta p_{\mathrm{nd}}(t+1)\mu_2(t)\big), \\ \mu_3(t+1) &= 1 - \mu_1(t+1) - \mu_2(t+1). \end{aligned} \tag{3.14}$$

The following proposition states that, as n tends to infinity, the occupancy measure $\boldsymbol{M}^{(n)}(t)$ in our model converges almost surely to the mean-field limit $\boldsymbol{\mu}(t)$.

Proposition 3.4.3 (Mean-field convergence in the n-SDM system) *Consider the n-SDM sequential decision aggregation system. For any $t \in \mathbb{N}$, as the system size n tends to infinity, the occupancy measure $\boldsymbol{M}^{(n)}(t)$, defined by equation (3.13), satisfies*

$$\lim_{n\to\infty} \boldsymbol{M}^{(n)}(t) = \boldsymbol{\mu}(t) \quad \textit{almost surely,} \tag{3.15}$$

where $\boldsymbol{\mu}(t)$ is defined by equation (3.14).

Proof: Define the memory vector by

$$\boldsymbol{R}^{(n)}(t) = \big(t, M_1^{(n)}(t), M_2^{(n)}(t)\big) = \left(t, \frac{N_1(t)}{n}, \frac{N_0(t)}{n}\right).$$

Therefore the function $g = (g_1, g_2, g_3)$ becomes:

$$g_1\big(\boldsymbol{R}^{(n)}(t), \boldsymbol{M}^{(n)}(t+1)\big) = R_1^{(n)}(t) + 1 = t + 1,$$
$$g_2\big(\boldsymbol{R}^{(n)}(t), \boldsymbol{M}^{(n)}(t+1)\big) = M_1^{(n)}(t+1) = \frac{N_1(t+1)}{n},$$
$$g_3\big(\boldsymbol{R}^{(n)}(t), \boldsymbol{M}^{(n)}(t+1)\big) = M_2^{(n)}(t+1) = \frac{N_0(t+1)}{n}.$$

Let the individual states set be $\Theta = \{1, 2, 3\}$, where the indexes 1, 2 and 3 correspond to H_1, H_0 and H_{nd} respectively. Define the matrix $K(\boldsymbol{r})$ by

$$K_{11}(\boldsymbol{r}) = 1, \quad K_{12}(\boldsymbol{r}) = 0, \quad K_{13}(\boldsymbol{r}) = 0; \quad K_{21}(\boldsymbol{r}) = 0, \quad K_{22}(\boldsymbol{r}) = 1, \quad K_{23}(\boldsymbol{r}) = 0;$$
$$K_{31}(\boldsymbol{r}) = p_1(r_1+1) + \beta p_{\mathrm{nd}}(r_1+1)r_2, \quad K_{32}(\boldsymbol{r}) = p_0(r_1+1) + \beta p_{\mathrm{nd}}(r_1+1)r_3,$$
$$K_{33}(\boldsymbol{r}) = 1 - K_{31}(\boldsymbol{r}) - K_{32}(\boldsymbol{r}).$$

Based on Assumption 3.2 and equations (3.1) and (3.2), in our model, the individual state transition matrix with any memory $\boldsymbol{r}$ satisfies $K^{(n)}(\boldsymbol{r}) = K(\boldsymbol{r})$, for any $n \in \mathbb{Z}_+$. Moreover, initially $\boldsymbol{M}^{(n)}(0) = \boldsymbol{\mu}(0)$ and $\boldsymbol{R}^{(n)}(0) = \boldsymbol{\rho}(0)$. According to Lemma 3.4.2, we obtain equation (3.15). ■

Having completed all preparations, we now present the theorem on the asymptotic accuracy and expected decision time of the fusion center running the majority rule.

Theorem 3.4.4 (Asymptotic behavior for the majority rule) *Consider the n-SDM sequential decision aggregation system with the IDPS $\{p_1(t), p_0(t), p_{\mathrm{nd}}(t)\}_{t\in\mathbb{N}}$ known. Define the vector sequence $\{\boldsymbol{\mu}(t)\}_{t\in\mathbb{N}}$ by equation* (3.14). *As the system size n tends to* *infinity, the accuracy of the fusion center satisfies:*

$$\lim_{n\to\infty} p_c(n, \lceil n/2\rceil) = \begin{cases} 1, & \textit{if } \lim_{t\to\infty}\mu_1(t) > 1/2, \\ 0, & \textit{if } \lim_{t\to\infty}\mu_2(t) > 1/2, \\ 1/2, & \textit{if } \exists T\in\mathbb{N},\ \textit{s.t. } \mu_1(T)=\mu_2(T)=1/2. \end{cases} \tag{3.16}$$

As for the asymptotic expected decision time,

(i. *if* $\lim_{t\to\infty}\mu_1(t) > 1/2$ *or* $\lim_{t\to\infty}\mu_2(t) > 1/2$*, then*

$$t_{<\frac{1}{2}} + 1 \le \lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, n, \lceil n/2\rceil] \le t_{>\frac{1}{2}},$$

where $t_{>\frac{1}{2}} = \min\{t\in\mathbb{N} \,|\, \max\big(\mu_1(t),\mu_2(t)\big) > 1/2\}$ *and* $t_{<\frac{1}{2}} = \max\{t\in\mathbb{N} \,|\, \max\big(\mu_1(t),\mu_2(t)\big) < 1/2\}$*. Particularly, if there does not exists any* $T\in\mathbb{N}$ *such that* $\mu_1(T)=1/2$ *or* $\mu_2(T)=1/2$*, then* $\lim_{n\to\infty}\mathbb{E}[T_{\mathrm{fc}} \,|\, n, \lceil n/2\rceil] = t_{>\frac{1}{2}}$*;*

(ii. *if there exists* $T\in\mathbb{N}$ *such that* $\mu_1(T)=\mu_2(T)=1/2$*, then*

$$\lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, n, \lceil n/2\rceil] = t_{\frac{1}{2}},$$

where $t_{\frac{1}{2}} = \min\{t\in\mathbb{N} \,|\, \mu_1(t)=\mu_2(t)=1/2\}$*;*

(iii. *if for any* $t\in\mathbb{N}$, $\mu_1(t)<1/2$ *and* $\mu_2(t)<1/2$*, while* $\lim_{t\to\infty}\mu_1(t) = \lim_{t\to\infty}\mu_2(t) = 1/2$*, then the fusion center's expected decision time tends to infinity as* $n\to\infty$ *almost surely.*

Proof: First we discuss the asymptotic accuracy. If $\lim_{t\to\infty}\mu_1(t) > 1/2$, there exists $\tilde{t}\in\mathbb{N}$ such that $\mu_1(\tilde{t}) > 1/2$. Since $\boldsymbol{M}^{(n)}(t)$ converges to $\boldsymbol{\mu}(t)$ almost surely, $M_1^{(n)}(\tilde{t}) =$

$\frac{N_1(\tilde{t})}{n} > 1/2$ almost surely as $n \to \infty$. According to the majority rule,

$$\lim_{n\to\infty} \mathbb{P}[\text{The fusion center decides } H_1 \text{ no later than } \tilde{t} \,|\, n, \lceil n/2 \rceil] = 1,$$

that is, $p_c(n, \lceil n/2 \rceil) \to 1$ as $n \to \infty$. Following the same argument we have $p_c(n, \lceil n/2 \rceil) \to 0$ when $\lim_{t\to\infty} \mu_2(t) > 1/2$.

Now consider the case when there exists $T \in \mathbb{N}$ such that $\mu_1(T) = \mu_2(T) = 1/2$. Define $\bar{t} = \min\{t \,|\, \mu_1(t) = \mu_2(t) = 1/2\}$. According to equation (3.14), for any $t < \bar{t}$, $\mu_1(t) < 1/2$ and $\mu_2(t) < 1/2$, which implies $N_1(t)/n < 1/2$ and $N_0(t)/n < 1/2$ almost surely as $n \to \infty$. Therefore, no global decision is made before $\bar{t}$ and after time step $\bar{t}$ the fusion center decides H_1 with probability $1/2$ due to the symmetry.

Now we prove the results on the asymptotic expected decision time. First, we discuss the case when $\lim_{t\to\infty} \mu_1(t) > \frac{1}{2}$. The case $\lim_{t\to\infty} \mu_2(t) > \frac{1}{2}$ follows the same line of argument. For any $t \leq t_{<\frac{1}{2}}$, $\mu_1(t) < \frac{1}{2}$, $\mu_2(t) < \frac{1}{2}$, and therefore

$$\mathbb{P}\Big[\lim_{n\to\infty} \frac{N_1(t)}{n} = \mu_1(t) < \frac{1}{2}\Big] = 1.$$

The fusion center makes no decision before $t_{<\frac{1}{2}} + 1$, almost surely. For $t = t_{>\frac{1}{2}}$, $\mu_1(t) > \frac{1}{2}$, $\mu_2(t) < \frac{1}{2}$. We have

$$\mathbb{P}\Big[\lim_{n\to\infty} \frac{N_1(t_{>\frac{1}{2}})}{n} = \mu_1(t_{>\frac{1}{2}}) > \frac{1}{2}\Big] = 1.$$

Therefore, almost surely, $t_{<\frac{1}{2}} + 1 \leq T_{\text{fc}} \leq t_{>\frac{1}{2}}$. Particularly, if there does not exist any T such that $\mu_1(T) = 1/2$, then $t_{<\frac{1}{2}} + 1 = t_{>\frac{1}{2}}$. This concludes the proof for Case (i.

In Case (ii, when $\mu_1(t_{\frac{1}{2}}) = \mu_2(t_{\frac{1}{2}}) = \frac{1}{2}$ for any $t < t_{\frac{1}{2}}$, we have $\mu_1(t) < \frac{1}{2}$ and $\mu_2(t) < \frac{1}{2}$. Therefore, as n tends to infinity, the fusion center makes the global decision at $t_{\frac{1}{2}}$ almost surely. The asymptotic expected decision time is $t_{\frac{1}{2}}$.

In Case (iii, since $\mathbb{P}[\lim_{n\to\infty} N_1(t)/n = \mu_1(t) < 1/2] = \mathbb{P}[\lim_{n\to\infty} N_0(t)/n = \mu_2(t) <$

$1/2] = 1$ for any $t \in \mathbb{N}$, the fusion center almost surely makes no global decision at any time. Therefore, $\lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, n, q] = \infty$. ■

3.4.3 Analysis of the influence of parameter β

According to Proposition 3.4.3, $\mu_1(t)$ ($\mu_2(t)$, $\mu_3(t)$ resp.) is a mean-field approximation of $N_1(t)/n$ ($N_0(t)/n$, $\big(n - N_1(t) - N_0(t)\big)/n$ resp.) for large n. The parameter β plays an important role in the iteration of $\boldsymbol{\mu}(t)$. In this subsection we discuss the dynamical behavior of $\boldsymbol{\mu}(t)$ as a function of the parameter β.

1) β=0: The case $\beta = 0$ corresponds to the system without social pressure. In this scenario the n-SDM system is degenerated to the model discussed in [84]. Denote by $\boldsymbol{\nu}(t) = \big(\nu_1(t), \nu_2(t), \nu_3(t)\big)$ the solution to equation (3.14) with $\beta = 0$. Then we have

$$\boldsymbol{\nu}(t+1) = \boldsymbol{\nu}(t)A(t+1), \quad \text{with } A(t+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_1(t+1) & p_0(t+1) & p_{\mathrm{nd}}(t+1) \end{bmatrix}, \tag{3.17}$$

and $\boldsymbol{\nu}(0) = (0, 0, 1)$. It is straightforward to check that the closed form of $\boldsymbol{\nu}(t)$ is given by

$$\begin{aligned} \nu_1(t) &= \begin{cases} p_1(1), & \text{for } t = 1, \\ p_1(1) + \sum_{s=1}^{t-1} p_1(s+1) \prod_{\tau=1}^{s} p_{\mathrm{nd}}(\tau), & \text{for } t \geq 2, \end{cases} \\ \nu_2(t) &= \begin{cases} p_0(1), & \text{for } t = 1, \\ p_0(1) + \sum_{s=1}^{t-1} p_0(s+1) \prod_{\tau=1}^{s} p_{\mathrm{nd}}(\tau), & \text{for } t \geq 2, \end{cases} \\ \nu_3(t) &= \prod_{\tau=1}^{t} p_{\mathrm{nd}}(\tau). \end{aligned} \tag{3.18}$$

According to Assumption 3.1, $\lim_{t\to\infty} \nu_3(t) = 0$. According to the iteration equations (3.17), $\nu_1(t)$ and $\nu_2(t)$ is non-decreasing with t and are both upper bounded by 1. Therefore, $\lim_{t\to\infty}\nu_1(t)$ and $\lim_{\to\infty}\nu_2(t)$ both exist. Moreover, with the closed-form of $\boldsymbol{\nu}(t)$, one can check that Theorem 3.4.4 for the case $\beta = 0$ coincide with Proposition IV.3 and IV.4 in [84].

2) β=1: Denote by $\hat{\boldsymbol{\nu}}(t)$ the solution to equation (3.14) in the other extreme case when $\beta = 1$. The iteration equation for $\hat{\boldsymbol{\nu}}(t)$ is nonlinear and written as

$$\begin{aligned}
\hat{\nu}_1(t+1) &= \hat{\nu}_1(t) + \hat{\nu}_3(t)\big(p_1(t+1) + \beta p_{\mathrm{nd}}(t+1)\hat{\nu}_1(t)\big),\\
\hat{\nu}_2(t+1) &= \hat{\nu}_2(t) + \hat{\nu}_3(t)\big(p_0(t+1) + \beta p_{\mathrm{nd}}(t+1)\hat{\nu}_2(t)\big),\\
\hat{\nu}_3(t+1) &= p_{\mathrm{nd}}(t+1)\hat{\nu}_3(t)^2.
\end{aligned} \tag{3.19}$$

One can deduce, from the third equation above, the closed form of $\hat{\nu}_3(t)$:

$$\hat{\nu}_3(t) = \prod_{\tau=1}^{t} p_{\mathrm{nd}}(\tau)^{2^{t-\tau}}.$$

Similar to the case when $\beta = 0$, we conclude that the limit of $\hat{\boldsymbol{\nu}}(t)$ exists, as t tends to infinity. Moreover, with the same IDPS, $\hat{\nu}_3(t)$ decays to zero faster than $\nu_3(t)$, that is, in the system with large n and $\beta = 1$, the expected decision time for the individual SDMs is no larger than in the case when $\beta = 0$.

3) Small β: We conduct the leading order analysis in β, for the expression of $\boldsymbol{\mu}(t)$, when β is very small. The following proposition is stated without proof.

Proposition 3.4.5 (Leading order analysis for small β) *Consider the iteration equation* (3.14) *for $\boldsymbol{\mu}(t)$ with β positive but close to 0. Let $\mu_r(t) = \nu_r(t) + g_r(t)\beta + O(\beta^2)$ for any $r \in \{1,2,3\}$, where $g_r(t)$ is the coefficient of the leading order in β and $\boldsymbol{\nu}(t) = \big(\nu_1(t), \nu_2(t), \nu_3(t)\big)$ is given by equation* (3.18)*. Then,* *(i. for any $r \in \{1, 2, 3\}$, $g_r(t)$ satisfies the following iteration formula:*

$$
\begin{aligned}
g_1(t+1) &= g_1(t) + p_1(t+1)g_3(t) + \nu_1(t)\nu_3(t)p_{\mathrm{nd}}(t+1),\\
g_2(t+1) &= g_2(t) + p_0(t+1)g_3(t) + \nu_2(t)\nu_3(t)p_{\mathrm{nd}}(t+1),\\
g_3(t+1) &= p_{\mathrm{nd}}(t+1)g_3(t) - p_{\mathrm{nd}}(t+1)\nu_3(t)\big(1-\nu_3(t)\big),
\end{aligned}
$$

and $g_1(t) + g_2(t) + g_3(t) = 0$ for any $t \in \mathbb{N}$;

(ii. the closed form of $g_r(t)$ is given by $g_1(1) = g_2(1) = g_3(1) = 0$, $g_1(2) = p_1(1)p_{\mathrm{nd}}(1)p_{\mathrm{nd}}(2)$, $g_2(2) = p_0(1)p_{\mathrm{nd}}(1)p_{\mathrm{nd}}(2)$, $g_3(2) = -p_{\mathrm{nd}}(1)p_{\mathrm{nd}}(2)\big(p_1(1)+p_0(1)\big)$, and, for any $t \geq 3$,

$$
\begin{aligned}
g_1(t) &= g_1(2) + \sum_{l=3}^{t} p_{\mathrm{nd}}(l)\nu_1(l-1)\nu_3(l-1)\\
&\qquad - \sum_{l=3}^{t} p_1(l) \sum_{s=2}^{l-1}\sum_{\tau=s}^{l-1} p_{\mathrm{nd}}(\tau)\nu_3(s-1)\big(1-\nu_3(s-1)\big),\\
g_2(t) &= g_2(2) + \sum_{l=3}^{t} p_{\mathrm{nd}}(l)\nu_2(l-1)\nu_3(l-1)\\
&\qquad - \sum_{l=3}^{t} p_0(l) \sum_{s=2}^{l-1}\sum_{\tau=s}^{l-1} p_{\mathrm{nd}}(\tau)\nu_3(s-1)\big(1-\nu_3(s-1)\big),\\
g_3(t) &= -\sum_{s=2}^{t}\prod_{\tau=s}^{t} p_{\mathrm{nd}}(\tau)\nu_3(s-1)\big(1-\nu_3(s-1)\big);
\end{aligned}
$$

(iii. for any $t \in \mathbb{N}$, $g_3(t) \leq 0$, and therefore $\mu_3(t)$ is non-increasing with β;

(iv. for any $t \in \mathbb{N}$, $g_1(t)$ ($g_2(t)$ resp.) is non-decreasing with $p_{\mathrm{nd}}(t)$ and non-increasing with $p_1(t)$ ($p_0(t)$ resp.), and $|g_3(t)|$ is non-decreasing with $p_{\mathrm{nd}}(t)$.

4) β close to 1: We present the following proposition on the leading order in $\delta = 1-\beta$ for small δ.

Proposition 3.4.6 (Leading order analysis for β close to 1) *Consider equation* (3.14) *for $\boldsymbol{\mu}(t)$ with β close to but less than 1. Let $\delta = 1 - \beta$ and $\mu_r(t) = \hat{\nu}_r(t) + \delta h_r(t) + O(\delta^2)$ for $r \in \{1, 2, 3\}$, where $H_r(t)$ is the coefficient of the leading order in δ and $\hat{\boldsymbol{\nu}}(t)$ is given by equation* (3.19)*. Then we have:*

(i. for $r \in \{1, 2, 3\}$, $h_r(t)$ satisfies the following iteration formula:

$$
\begin{aligned}
h_1(t+1) &= \big(1 + \hat{\nu}_3(t) p_{\mathrm{nd}}(t+1)\big) h_1(t) + p_1(t+1) h_3(t) \\
&\quad + p_{\mathrm{nd}}(t+1) \hat{\nu}_1(t) \big(h_3(t) - \hat{\nu}_3(t)\big), \\
h_2(t+1) &= \big(1 + \hat{\nu}_3(t) p_{\mathrm{nd}}(t+1)\big) h_2(t) + p_0(t+1) h_3(t) \\
&\quad + p_{\mathrm{nd}}(t+1) \hat{\nu}_2(t) \big(h_3(t) - \hat{\nu}_3(t)\big), \\
h_3(t+1) &= p_{\mathrm{nd}}(t+1) \hat{\nu}_3(t) \big(2h_3(t) + \hat{\nu}_1(t) + \hat{\nu}_2(t)\big),
\end{aligned}
$$

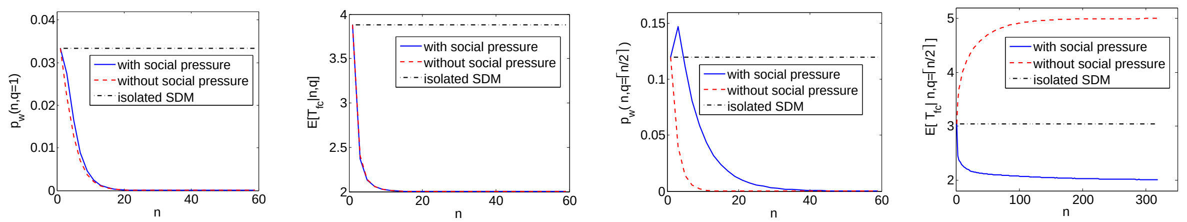
and $h_1(t) + h_2(t) + h_3(t) = 0$ for any $t \in \mathbb{N}$;

(ii. for any $t \in \mathbb{N}$, $h_3(t) \geq 0$, and therefore $\mu_3(t)$ is non-decreasing with β;

(iii. for any $t \in \mathbb{N}$, $h_1(t)$ ($h_2(t)$ resp.) are non-decreasing with $p_1(t)$ ($p_0(t)$ resp.), and $h_3(t)$ is non-decreasing with $p_{\mathrm{nd}}(t)$.

3.5 Further Simulation

1) Validation of the asymptotic performance: Simulation work has been conducted to validate the results of Theorems 3.4.1 and 3.4.4. In Figure 3.5(a) and 3.5(b), the IDPS has $\bar{t} = 2$ and $p_1(\bar{t}) > p_0(\bar{t})$. The simulation result indicates that, as n increases, the fusion center's accuracy, i.e., $1 - p_w(n, 1)$ gets close to 1 and the expected decision time converges to $\bar{t}$. In Figure 3.5(c) and 3.5(d), the IDPS satisfies $\mu_1(\infty) > 1/2 > \mu_2(\infty)$ and $t_{>\frac{1}{2}} = 2$ for $\beta = 1$; $\mu_1(\infty) > 1/2 > \mu_2(\infty)$ and $t_{>\frac{1}{2}} = 5$ for $\beta = 0$. The simulation result indicates

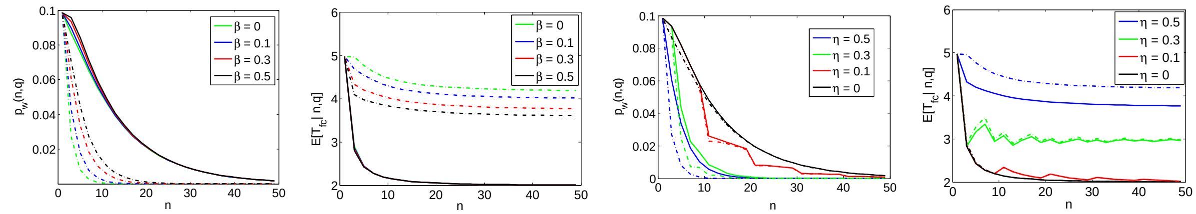


(a) error rate, fastest rule (b) expected decision time, fastest rule (c) error rate, majority rule (d) expected decision time, majority rule

Figure 3.5: The probability of making wrong global decision, and the expected decision time, for the fusion center in n-SDM systems with the fastest rule and the majority rule. The blue curves correspond to the n-SDM systems with $\beta = 1$. The red dash-dot curves represent the n-SDM systems with $\beta = 0$ and the black dotted lines correspond to the isolated SDM.

that, as n tends to infinity, the probability of making wrong global decision under the majority rule, i.e., the probability $p_w(n, \lceil n/2 \rceil)$, converges to 0 and the expected decision time converges to $t_{>\frac{1}{2}}$, as indicated by Theorem 3.4.4. Moreover, Figure 3.5(d) shows that, with the presence of social pressure, the expected decision time of the fusion center running the majority rule can be even less than the expected decision time of a single isolated SDM, while the expected decision time of the model without social pressure, as the red dash line in Figure 3.5(d) indicates, is much larger than the single isolated SDM's.

2) Comparison among different values of β: Simulation work has been conducted to compare the performances of systems with different values of the model parameter β. The IDPS shown in Figure 3.1 are used in the simulation work illustrated by Figure 3.6. Figure 3.6(a) and 3.6(b) are comparisons between the fastest rule and the majority rule with varying values of β. We can see that, for any fixed n and β, the fastest rule has less accuracy while faster decision speed than the majority. Moreover, the performance of the fastest rule is not sensitive to the value of β while, for the majority rule with fixed system size n, the probability of wrong global decision gets larger as β increases but the



(a) error rate, varying β (b) expected decision time, varying β (c) error rate, varying η (d) expected decision time, varying η

Figure 3.6: The probability of making wrong global decision and the expected decision time, as functions of the system size n respectively, for different values of the parameter β and different q-out-of-n rules. In Figure (a) and (b), the solid lines correspond to the fastest rules while the dash lines correspond to the majority rules. In Figure (c) and (d), the solid lines correspond to the systems with $\beta = 0.3$ while the dash lines correspond to the system with $\beta = 0$.

expected decision time decreases as β increases.

3) Comparison among different q-out-of-n rules: Refer to the *η-total rule* as the $\lceil \eta n \rceil$-out-of-n rule. The case $\eta = 0$ corresponds to the fastest rule while $\eta = 0.5$ is the majority rule. Figure 3.6(c) and 3.6(d) reveal that the system performance gets more sensitive to β as η increases. Moreover, for fixed n and β, the system's accuracy increases with the increase of η, at the cost of the higher expected decision time.

3.6 Conclusion and Discussion

This chapter proposes a sequential decision aggregation model that does not rely on the specific individual decision making policy and incorporates social pressure. Individuals in our model are sequential decision makers (SDMs) influenced by the decisions of other individuals. We present an algorithm to compute the system's decision probabilities, accuracy and expected decision time. Two specific group decision rules, the fastest rule and the majority rule, are analyzed in detail. We then focus on the case when the system size tends to infinity and, via a mean-field analysis, provide the exact expression of the asymptotic accuracy and expected decision time for both the fastest rule and the majority rule. These results relate the group's decision making behavior to the isolated SDM's. In addition to the theoretical analysis, we provide some simulation work to present the performance of our group decision making model and compare it to the sequential decision aggregation model without social pressure, first proposed in [84]. Within our model, we also compared the performance of different q-out-of-n aggregation rules.

This model could be extended to a generalized problem, in which the SDMs' IDPS are heterogeneous. Moreover, the connections between the SDMs might not necessarily be all-to-all. If both the heterogeneous SDMs and the network structure are taken into consideration, the group decision making policy becomes more complicated. The generalized model would help to explain how a group of decision makers with different information sources and confidence levels collaborate together and the optimization of the group decision making performance will be related to the network topology.

Part II

Dynamics of Interpersonal Appraisal Networks

Overview: dynamics of social networks

One of the main challenges in the research on social network dynamics is that, there are not only dynamical processes occurring over the networks, but the social networks themselves are also evolving with time. Models of dynamics of social networks aim to explain the evolution of the interconnections in social networks and the emergent global network structure. Examples of such models include the network formation games [101, 102], evolution of interpersonal influence along issue sequence [103, 104], and dynamics social balance [105, 106, 107].

In Part II of this thesis, we focus on the dynamics of interpersonal appraisal networks. Depending on the specific content of the "appraisal" and the underlying microscopic interaction mechanism, different models of dynamics of appraisal networks explains different social phenomena. In Chapter 4, the appraisal network represents individuals' mutual evaluations of their certain skill levels and the evolution of such appraisal dynamics lead to collective learning under some conditions; In Chapter 5, appraisal network refers to the sentiments network among a group of individuals and the evolution of such appraisal network via homophily or influence mechanism leads to a special network configuration called structural balance.

Chapter 4

Collective Learning via Assign/appraise/influence Dynamics

4.1 Introduction

4.1.1 Transactive memory system in applied psychology

Researchers in sociology, psychology, and organization science have long studied the inner functioning and performance of teams with multiple individuals engaged in tasks. Extensive qualitative studies, conceptual models and empirical studies in the laboratory and field reveal some statistical features and various phenomena of teams [108, 109, 110, 111], but only a few quantitative and mathematical models are available [112, 113].

Transactive memory system (TMS) is a conceptual model of team learning and performance well-established in organization science, see the seminal work by Wegner et al. [114] and other highly cited works [115, 108, 109, 116]. A TMS is a collective "memory" system that emerges in teams engaged in tasks, as the team members develop the collective knowledge on who possesses what expertise. TMS facilitates coordination and division of labor. Empirical research across a range of team types and settings [108, 117, 118], as well as some early simulation-based computational models [119, 120, 113], demonstrates a strong positive relationship between the development of a TMS and team performance. However, the mechanisms through which team members come to share an understanding of the distribution of expertise is typically treated as "black box" processes in TMS research. It remains an open problem how to mathematically characterize the TMS-related social and cognitive processes, such as the division of labor and the evolution of collective knowledge.

4.1.2 Problem description

In this chapter we propose a class of multi-agent dynamical systems as mathematical formalizations of some important aspects of the TMS theory. We consider a natural social process, in which a team of individuals, with unknown skill levels, is completing a sequence of tasks. Each task is completed by subdividing it into subtasks with different workloads and assigning one subtask to each team member. The team performance is maximized when the workload assignments are proportional to the individuals' underlying skill levels. We adopt the concept of *appraisal network*, or equivalently its corresponding row-stochastic appraisal matrix, to model the TMS of the team. The appraisal network represents how the team members evaluate each other's underlying skill level. The dynamics of the appraisal matrix is as follows: First, after completing the task, each individual receives a feedback signal equal to the deviation of her/his own performance from the weighted average performance of a subset of observed individuals. Second, based on the feedback signal, each individual adjusts her/his own appraisal and the appraisals of other team members. Third, the appraisal network may or may not be updated via an interpersonal influence process. Fourth, the workload division for the next tasks is computed as a function of the appraisal matrix. The evolution of the appraisal network corresponds to the development of a team's TMS. This chapter aims to mathematically formalize this four-step process and investigate the conditions under which (i) the team as an whole achieves asymptotically the optimal workload assignment; (ii) each individual learns asymptotically the true relative skill levels of all the team members; and (iii) the learning fails to occur. We refer to property (ii) as *collective learning.*

4.1.3 Literature review

To the best of our knowledge, this chapter is the first attempt to model the development of TMS as a multi-agent system and provide rigorous conditions for collective learning. To the best of our knowledge, the only related previous works are the computational models proposed by Palazzolo et al [119], Ren et al [120], and Anderson et al [113]. The model in [113] is a 2-dimension ODE and treats the collective knowledge as a scalar variable, while the models in [119] and [120] are multi-agent. Palazzolo et al [119] consider time-varying skill levels. Ren et al [120] consider multi-dimension skills and task requirements. Both models take into account numerous complicated and realistic individual/group actions, and the analysis of both models is based on simulation.

In our models, collective learning arises as the result of the co-evolution of interpersonal appraisals and influence networks. Related previous work includes social comparison theory [121], averaging-based social learning [122], opinion dynamics [2, 123, 124], reflected appraisal mechanisms [125, 103], and the combined evolution of interpersonal appraisals and influence networks [107].

In the modeling and analysis of the evolution of appraisal and influence networks, we build an insightful connection between our model and the well-known replicator dynamics in evolutionary game theory; see the textbook [126], some control and optimization applications [127, 128], and the recent contributions [129, 130].

Our models are also marginally related to distributed optimization, e.g. [131, 132]. But in this chapter we focus on modeling the natural social behavior of individuals. Moreover, the evolution of the decision variable, i.e., the workload assignment, is not directly modeled, but a byproduct of the dynamics for the appraisal network.

4.1.4 Contribution

Firstly, based on a few natural assumptions, we propose three novel models with increasing complexity for the dynamics of teams: the manager dynamics, the assign/appraise dynamics, and the assign/appraise/influence dynamics. Without loosing mathematical tractability and intuitive insights, our work integrates several natural processes in a single model: the division of workload, the update of interpersonal appraisals via observation, and the opinion dynamics over the influence network. To the best of our knowledge, this is the first time that such an integration has been proposed and leads to rigorous and intuitive results. For the baseline manager dynamics, the workload assignment is adjusted in a centralized manner: the increase rate of workload assigned to an individual is equal to the deviation of his/her performance from the average. Under this intuitive assumption, the evolution of the workload assignment obeys the well-established replicator dynamics with novel fitness functions as the individual performances. The assign/appraise dynamics provides an insightful perspective on the connection between team performance and the appraisal network, by assuming that, instead of by the manager, the workload assignment is determined by the appraisal network in a social and distributed manner. The update of the appraisals is driven by the individuals' heterogeneous performance feedback. In the assign/appraise/influence dynamics model, we further incorporate the co-evolution of appraisal and influence networks.

Secondly, we present comprehensive theoretical analysis on the dynamical properties of our models. For the assign/appraise dynamics and the assign/appraise/influence dynamics, we relate the models' asymptotic behavior with the connectivity property of the *observation network*, which defines the heterogeneous feedback signals each individual observes. Our theoretical results on the asymptotic behavior can be interpreted as the exploration of the most relaxed conditions for the emergence of asymptotic optimal workload assignment. Moreover, some theoretical results also reveal insightful interpretations that are consistent with the TMS theory studied in organization science. According to Lee et al. [133], in teams with well-developed TMS, members' agreements on the distribution of expertise facilitate high levels of coordination and division of labor, which a centralized manager might otherwise provide. In our paper, we prove that, along the assign/appraise dynamics and the assign/appraise/influence dynamics, the evolution of the workload assignment determined by the appraisal network does indeed satisfy the manager (a.k.a., replicator) dynamics in a generalized form. In addition, the assign/appraise/influence dynamics describes an emergence process by which team members' perception of "who knows what" become more similar over time, a fundamental feature of TMS [134, 133].

Thirdly, besides the models in which the team eventually learns the individuals' true relative skill levels, we propose one variation in each of the three phases of the assign/appraise/influence dynamics: the assignment rule, the update of appraisal network based on feedback signal, and the opinion dynamics for the interpersonal appraisals. The variations reflect some sociological and psychological mechanisms known to prevent the team from learning. We investigate by simulation numerous possible causes of failure to learn.

4.1.5 Organization

The rest of this chapter is organized as follows: In the next subsection, we present some preliminaries on evolutionary games and replicator dynamics. Section II proposes our problem set-up and centralized manager model; Section III introduces the assign/appraise dynamics; Section IV is the assign/appraise/influence model; Section V discusses some causes of failure to learn; Section VI provides some further discussions and conclusion.

4.1.6 Preliminaries

Evolutionary games apply game theory to evolving populations adopting different strategies. Consider a game with n pure strategies, denoted by the unit vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ respectively. A mixed strategy $\boldsymbol{w}$ is thereby a vector in the n-dimension simplex denoted by Δ_n. Denote by $\pi(\boldsymbol{v}, \boldsymbol{w})$ the expected payoff for any mixed strategy $\boldsymbol{v}$ against mixed strategy $\boldsymbol{w}$. A strategy $\boldsymbol{w}^*$ is a locally *evolutionarily stable strategy* (ESS) if there exists a deleted neighborhood $\check{U}(\boldsymbol{w}^*)$ in $\mathrm{int}(\Delta_n)$ such that $\pi(\boldsymbol{w}^*, \boldsymbol{w}) > \pi(\boldsymbol{w}, \boldsymbol{w})$ for any $\boldsymbol{w} \in \check{U}(\boldsymbol{w}^*)$, which implies that, in a population adopting strategy $\boldsymbol{w}$, a sufficiently small mutated subpopulation adopting strategy $\boldsymbol{w}^*$ gets more payoff than the majority population.

Replicator dynamics models the evolution of sub-populations adopting different strategies. The total population is divided into n sub-populations. Individuals in each sub-population i adopt the pure strategy $\boldsymbol{e}_i$. Denote by $w_i(t)$ the fraction of sub-population i in the total population at time t. The fitness of sub-population i, denoted by $\pi_i\big(\boldsymbol{w}(t)\big)$, depends on the sub-population distribution $\boldsymbol{w}(t) = \big(w_1(t), \dots, w_n(t)\big)^\top$ and is defined as the expected payoff $\pi\big(\boldsymbol{e}_i, \boldsymbol{w}(t)\big)$. The growth rate of sub-population i is equal to the deviation of its fitness from the population average. The replicator dynamics is given by:

$$\dot{w}_i = w_i\Big(\pi_i(\boldsymbol{w}) - \sum_{k=1}^{n} w_k \pi_k(\boldsymbol{w})\Big). \tag{4.1}$$

There is a simple connection between the locally ESS and the replicator dynamics [129]: Generally, a locally ESS in $\text{int}(\Delta_n)$ is a locally asymptotic equilibrium of the replicator dynamics; Specifically, if there exists a matrix A such that $\pi(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}^\top A \boldsymbol{w}$ for any $\boldsymbol{v}, \boldsymbol{w} \in \Delta_n$, then a locally ESS in $\text{int}(\Delta_n)$ is a globally asymptotic stable equilibrium of the replicator dynamics. In addition, the replicator dynamics is also a mean-field approximation of some stochastic population process, which is out of the scope of this chapter.

4.2 Problem Set-up and Manager Dynamics

In this section, we first mathematically formalize some concepts related to the social processes we aim to model, and illustrate them by a concrete example. Then we introduce a baseline centralized model for team learning dynamics. In this chapter, in order to distinguish between vectors and matrices, we let symbols in bold represent vectors.

4.2.1 Model assumptions and notations

a) Team, tasks and assignments: The basic assumption on the individuals and the tasks are given below.

Assumption 4.1 (Team, task type and assignment) *Consider a team of n individuals characterized by a fixed but unknown vector $\boldsymbol{x} = (x_1, \dots, x_n)^\top$ satisfying $\boldsymbol{x} \succ \mathbb{0}_n$ and $\boldsymbol{x}^\top \mathbb{1}_n = 1$, where each x_i denotes the* skill level *of individual i. The tasks being completed by the team are assumed to have the following properties:*

(i. The total workload of each task is characterized by a positive scalar and is fixed as 1 *in this chapter;*

(ii. The task can be arbitrarily decomposed into n sub-tasks according to the workload assignment $\boldsymbol{w} = (w_1, \dots, w_n)^\top$, *where each w_i is the sub-task workload assigned to individual i. The workload assignment satisfies $\boldsymbol{w} \succ \mathbb{0}_n$ and $\boldsymbol{w}^\top \mathbb{1}_n = 1$. The sub-tasks are executed simultaneously.*

The scalar skill levels can be interpreted in an abstract way as the individuals' overall abilities of contributing to the tasks, while the workload assignment corresponds to the individuals' relative responsibilities.

b) Individual performance: The measure of individual performance is defined below.

Assumption 4.2 (Individual performance) *Given fixed skill levels, each individual i's performance, with the assignment $\boldsymbol{w}$, is measured by $p_i(\boldsymbol{w}) = f(x_i/w_i)$, where $f : [0, +\infty) \to [0, +\infty)$ is strictly concave, continuously differentiable and monotonically increasing.*

The function f is assumed concave since it is widely adopted that the relation between the performance and individual ability obeys the power law, i.e., $f(x) \sim x^\gamma$, with $\gamma \in (0, 1)$ [113]. The specific form $f(\frac{x_i}{w_i})$ could be generalized by adopting different measures of x_i and w_i.

c) Optimal assignment: It is reasonable to claim that, in a well-functioning team, individuals' relative responsibilities, characterized by the workload assignment, should be proportional to their true relative abilities. We thereby refer to $\boldsymbol{w}^* = \boldsymbol{x}$ as the *optimal assignment*. There are various team performance models for which $\boldsymbol{w}^*$ is the unique optimal solution in Δ_n. For example, define the measure of the mismatch between workload assignment and individual's true skill levels as $\mathcal{H}_1(\boldsymbol{w}) = \sum_{i=1}^n |\frac{w_i}{x_i} - 1|$. This mismatch is minimized at $\boldsymbol{w}^*$. Alternatively, if we define the team performance as the weighted average individual performance, i.e., $\mathcal{H}_2(\boldsymbol{w}) = \sum_{i=1}^n w_i f(\frac{x_i}{w_i})$, then the strict concavity of f implies that $\mathcal{H}_2(\boldsymbol{w})$ is maximized at $\boldsymbol{w}^* = \boldsymbol{x}$.

We introduce a simple and concrete example to illustrate the mathematical formalization introduced above.

Example (intruder detection task): Consider a group of n individuals monitoring an environment. The environment is divided into numerous non-overlapping regions with equal areas. Each region is monitored by a CCTV camera connected to its respective screen. The aim of the group is to detect the locations of randomly-appearing intruders via monitoring the screens. The appearance of the intruders is uniformly random in space and is a homogeneous Poisson process. An intruder is successfully detected if it is observed on a screen by one of the individuals within a certain time period since its appearance. The team performance over a given task period is the fraction of successfully detected intruders. The task is conducted in the follows way: each individual i monitors w_i number of screens and each screen is monitored by one and only one individual. Here w_i is normalized such that $\sum_i w_i = 1$. Each individual i has an intrinsic but unknown normalized skill level x_i. Denote by $p_i(\boldsymbol{w})$ the probability that an intruder is successfully detected by individual i, given the division of cameras $\boldsymbol{w} \in \Delta_n$. This probability $p_i(\boldsymbol{w})$ increases with individual i's intrinsic skill level x_i and decreases with the number of screens monitored by i, i.e., w_i. A natural assumption is that $p_i(\boldsymbol{w}) = f(\frac{x_i}{w_i})$, where f is a concave and monotonically increasing function, with $f(0) = 0$ and $f(\infty) = 1$. One can check that the expected team performance is given by $\sum_i w_i f(\frac{x_i}{w_i})$, which is maximized at $\boldsymbol{w}^* = \boldsymbol{x}$.

4.2.2 Centralized manager dynamics

In this subsection we introduce a continuous-time centralized model on the evolution of workload assignment, referred to as the *manager dynamics*. The diagram illustration is given by Figure 4.1(a). Suppose that, at each time t, a team is completing a task based on the assignment $\boldsymbol{w}(t)$. An outside manager observes the individuals' performance $\boldsymbol{p}\big(\boldsymbol{w}(t)\big)$. We adopt the intuitive assumption that the manager increases the workload assigned to individual i if her/his performance is above the weighted team average and vice versa. In addition, the sum of all the individuals' workloads remains 1. The manager is assumed to adjust the workload assignment according to the replicator dynamics below, which is arguably the simplest model for the process described above.

$$\dot{w}_i = w_i\Big(p_i(\boldsymbol{w}) - \sum_{k=1}^{n} w_k p_k(\boldsymbol{w})\Big), \tag{4.2}$$

for any $i \in \{1, \dots, n\}$. Equation (4.2) takes the same form as the classic replicator dynamics from evolutionary game theory [126, 129], with the nonlinear fitness function $f(x_i/w_i)$. We refer to Section 4.1.6 for some preliminaries on evolutionary games and replicator dynamics.

Theorem 4.2.1 (Manager dynamics) *Consider equation* (4.2) *for the workload assignment as in Assumption 4.1 with performance as in Assumption 4.2. Then*

(i. the set int(Δ_n) *is invariant;*

(ii. for any $\boldsymbol{w}(0) \in$ *int*(Δ_n)*, the manager's assignment* $\boldsymbol{w}(t)$ *converges to* $\boldsymbol{w}^* = \boldsymbol{x}$*, as* $t \to \infty$*.*

Proof: The vector form of equation (4.2) is written as

$$\dot{\boldsymbol{w}} = \operatorname{diag}(\boldsymbol{w}) \left(\boldsymbol{p}(\boldsymbol{w}) - \boldsymbol{w}^\top \boldsymbol{p}(\boldsymbol{w}) \mathbb{1}_n\right). \tag{4.3}$$

Left multiply both sides by $\mathbb{1}_n^\top$. We get $d(\mathbb{1}_n^\top \boldsymbol{w})/dt = 0$. Moreover, since $\dot{w}_i = 0$ whenever $w_i = 0$, the n-dimension simplex Δ_n is an positively invariant set.

Since the function f is continuously differentiable, the right-hand side of equation (4.3) is continuously differentiable and locally Lipschitz in $\text{int}(\Delta_n)$. Define

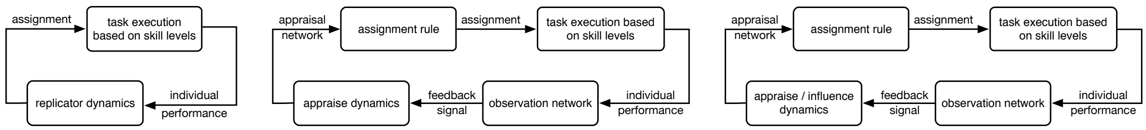
$$V(\boldsymbol{w}) = -\sum_{i=1}^{n} x_i \log \frac{w_i}{x_i}.$$

Due to the strict concavity of log function and $\mathbb{1}_n^\top \boldsymbol{w} = 1$, we have that $V(\boldsymbol{w}) \geq 0$ for any $\boldsymbol{w} \in \Delta_n$ and $V(\boldsymbol{w}) = 0$ if and only if $\boldsymbol{w} = \boldsymbol{x}$. Moreover, since $V(\boldsymbol{w})$ is continuously differentiable in $\boldsymbol{w}$, the level set $\{\boldsymbol{w} \in \text{int}(\Delta_n) \,|\, V(\boldsymbol{w}) = \xi\}$ is a compact subset of $\text{int}(\Delta_n)$. Since the function f is monotonically increasing, along the trajectory,

$$\frac{dV(\boldsymbol{w})}{dt} = -\sum_{i \in \theta_1(\boldsymbol{w})} (x_i - w_i) f(x_i/w_i) - \sum_{i \in \theta_2(\boldsymbol{w})} (x_i - w_i) f(x_i/w_i) < 0,$$

where $\theta_1(\boldsymbol{w}) = \{i \,|\, x_i \geq w_i\}$ and $\theta_2(\boldsymbol{w}) = \{i \,|\, x_i < w_i\}$. This concludes the proof for the invariant set and the asymptotic stability of $\boldsymbol{w}^* = \boldsymbol{x}$, and one can infer, from the inequality above, that $\boldsymbol{w}^* = \boldsymbol{x}$ is the ESS for the evolutionary game with the payoff function $\pi_i(\boldsymbol{w}) = f(x_i/w_i)$. Moreover, since $V(\boldsymbol{w}) \to +\infty$ as $\boldsymbol{w}$ tends to the boundary of Δ_n, the region of attraction is $\text{int}(\Delta_n)$. ■

In the proof above, we adopt the same Lyapunov function used for the asymptotic stability analysis of the replicator dynamics in [126, 129]. The fitness function in the manager dynamics is novel.



(a) manager (b) assign/appraise (c) assign/appraise/influence

Figure 4.1: Diagram illustrations of manager dynamics, assign/appraise dynamics, and assign/appraise/influence dynamics.

4.3 The Assign/Appraise Dynamics of the Appraisal Networks

Despite the desired property on the convergence of the workload assignment to optimality, the manager dynamics does not capture the evolution of the team's inner structures. In this section, we introduce a multi-agent system, in which workload assignments are determined by the team members' interpersonal appraisals, rather than any outside authority, and the appraisal network is updated in a decentralized manner, driven by the team members' heterogeneous feedback signals.

4.3.1 Model description and problem statement

Appraisal network: Denote by a_{ij} the individual i's evaluation of j's skill levels and refer to $A = (a_{ij})_{n\times n}$ as the *appraisal matrix.* Since the evaluations are in the relative sense, we assume $A \succeq \mathbb{0}_{n\times n}$ and $A\mathbb{1}_n = \mathbb{1}_n$. The directed and weighted graph $G(A)$, referred to as the *appraisal network*, reflects the team's collective knowledge on the distribution of its members' abilities.

Assign/appraise dynamics: This multi-agent model is illustrated by the diagram in Figure 4.1(b). We model three phases: the workload assignment, the feedback signal and the update of the appraisal network, specified by the following three assumptions respectively.

Assumption 4.3 (Assignment rule) *At any time $t \geq 0$, the task is assigned according to the left dominant eigenvector of the appraisal matrix, i.e., $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}\big(A(t)\big)$.*

Justification of Assumption 4.3 is given in Section 4.6.3. For now we assume $A(t)$ is row-stochastic and irreducible for all $t \geq 0$, so that $\boldsymbol{v}_{\text{left}}\big(A(t)\big)$ is always well-defined. This will be proved later in this section.

Assumption 4.4 (Feedback signal) *After executing the workload assignment $\boldsymbol{w}$, each individual i observes, with no noise, the difference between her own performance and the quality of some part of the whole task, given by $\sum_k m_{ik} p_k(\boldsymbol{w})$, in which m_{ik} denotes the fraction of workload individual k contributes to the part of task observed by i. The matrix $M = (m_{ij})_{n\times n}$ defines a directed and weighted graph $G(M)$, referred to as the* observation network*, and satisfies $M \succeq \mathbb{0}_{n\times n}$ and $M\mathbb{1}_n = \mathbb{1}_n$ by construction.*

The topology of the observation network defines the individuals' feedback signal structure. Notice that, the feedback signal for each individual i is only the deviation $p_i\big(\boldsymbol{w}(t)\big) - \sum_k m_{ik} p_k\big(\boldsymbol{w}(t)\big)$, while the matrix M is not necessarily known to the individuals.

Assumption 4.5 (Update of interpersonal appraisals) *With the performance feedback signal defined as in Assumption 4.4, each individual i increases her self appraisal and decreases the appraisals of all the other individuals, if $p_i(\boldsymbol{w}) > \sum_k m_{ik} p_k(\boldsymbol{w})$, and vice versa. In addition, the appraisal matrix $A(t)$ remains row-stochastic.*

The following dynamical system for the appraisal matrix, referred to as the *appraise* *dynamics*, is arguably the simplest model satisfying Assumptions 4.4 and 4.5:

$$\begin{cases} \dot{a}_{ii} = a_{ii}(1-a_{ii})\Big(p_i(\boldsymbol{w}) - \sum_{k=1}^{n} m_{ik}p_k(\boldsymbol{w})\Big), \\ \dot{a}_{ij} = -a_{ii}a_{ij}\Big(p_i(\boldsymbol{w}) - \sum_{k=1}^{n} m_{ik}p_k(\boldsymbol{w})\Big). \end{cases} \tag{4.4}$$

The matrix form of the appraise dynamics, together with the assignment rule as in Assumption 4.3, is given by

$$\begin{cases} \dot{A} = \operatorname{diag}\big(\boldsymbol{p}(\boldsymbol{w}) - M\boldsymbol{p}(\boldsymbol{w})\big)A_{\mathrm{d}}(I_n - A), \\ \boldsymbol{w} = \boldsymbol{v}_{\mathrm{left}}(A), \end{cases} \tag{4.5}$$

and collectively referred to as the assign/appraise dynamics. Here $A_{\mathrm{d}} = \operatorname{diag}(a_{11}, \dots, a_{nn})$.

Problem statement: In Section III.B, we investigate the asymptotic behavior of dynamics (4.5), including:

(i. convergence to the optimal assignment, which means that the team as an entirety eventually learns all its members' relative skill levels, i.e., $\lim_{t\to+\infty} \boldsymbol{w}(t) = \boldsymbol{x}$;

(ii. *appraisal consensus*, which means that the individuals asymptotically reach consensus on the appraisals of all the team members, i.e., $a_{ij}(t) - a_{kj}(t) \to 0$ as $t \to +\infty$, for any i, j, k.

Collective learning is the combination of the convergence to optimal assignment and appraisal consensus.

4.3.2 Dynamical behavior of the assign/appraise dynamics

We start by establishing that the appraisal matrix $A(t)$, as the solution to equation (4.5), is extensible to all $t \in [0, +\infty)$ and the assignment $\boldsymbol{w}(t)$ is well-defined, in that $A(t)$ remains row-stochastic and irreducible. Moreover, some finite-time properties are investigated.

Theorem 4.3.1 (Finite-time properties of assign/appraise dynamics) *Consider the assign/appraise dynamics* (4.5)*, based on Assumptions 4.3-4.5, describing a workload assignment as in Assumption 4.1, with performance as in Assumption 4.2. For any observation network $G(M)$, and any initial appraisal matrix $A(0)$ that is row-stochastic, irreducible and has strictly positive diagonal,*

(i. The appraisal matrix $A(t)$, as the solution to (4.5)*, is extensible to all $t \in [0, +\infty)$. Moreover, $A(t)$ remains row-stochastic, irreducible and has strictly positive diagonal for all $t \geq 0$;*

(ii. there exists a row-stochastic irreducible matrix $C \in \mathbb{R}^{n \times n}$ with zero diagonal such that

$$A(t) = \mathrm{diag}\big(\boldsymbol{a}(t)\big) + \big(I_n - \mathrm{diag}\big(\boldsymbol{a}(t)\big)\big)\, C, \tag{4.6}$$

for all $t \geq 0$, where $\boldsymbol{a}(t) = \big(a_1(t), \dots, a_n(t)\big)^\top$ and $a_i(t) = a_{ii}(t)$, for $i \in \{1, \dots, n\}$;

(iii. Define the reduced assign/appraise dynamics *as*

$$\begin{cases} \dot{a}_i = a_i(1 - a_i)\big(p_i(\boldsymbol{w}) - \displaystyle\sum_{k=1}^{n} m_{ik} p_k(\boldsymbol{w})\big), \\ w_i = \dfrac{c_i}{(1 - a_i)} \Big/ \displaystyle\sum_{k=1}^{n} \dfrac{c_k}{(1 - a_k)}, \end{cases} \tag{4.7}$$

where $\boldsymbol{c} = (c_1, \dots, c_n)^\top = \boldsymbol{v}_{\mathrm{left}}(C)$. This dynamics is equivalent to system (4.5) *in the following sense: The matrix $A(t)$'s each diagonal entry $a_{ii}(t)$ satisfies the dynamics* (4.7) *for $a_i(t)$, and, for any $t \geq 0$, $a_{ii}(t) = a_i(t)$ for any i, and $a_{ij}(t) = a_{ij}(0)\big(1 - a_i(t)\big)/\big(1 - a_i(0)\big)$ for any $i \neq j$;*

(iv. *The set* $\Omega = \big\{\boldsymbol{a} \in [0,1]^n \big| 0 \leq a_i \leq 1 - \zeta_i\big(\boldsymbol{a}(0)\big)\big\}$, *where* $\zeta_i\big(\boldsymbol{a}(0)\big) = \frac{c_i}{x_i} \min_k \frac{x_k}{c_k}\big(1 - a_k(0)\big)$, *is a compact positively invariant set for the reduced assign/appraise dynamics* (4.7)*;*

(v. *the assignment* $\boldsymbol{w}(t)$ *satisfies the* generalized replicator dynamics *with time-varying fitness function* $a_i(t)\Big(p_i\big(\boldsymbol{w}(t)\big) - \sum_l m_{il} p_l\big(\boldsymbol{w}(t)\big)\Big)$ *for each* i*:*

$$\dot{w}_i = w_i\Big(a_i\big(p_i(\boldsymbol{w}) - \sum_{l=1}^n m_{il}p_l(\boldsymbol{w})\big) - \sum_{k=1}^n w_k a_k\big(p_k(\boldsymbol{w}) - \sum_{l=1}^n m_{kl}p_l(\boldsymbol{w})\big)\Big). \quad (4.8)$$

Before the proof, we state a useful lemma summarized from Page 62-67 of [135].

Lemma 4.3.2 (Continuity of eigenvalue and eigenvector) *Suppose* $A, B \in \mathbb{R}^{n\times n}$ *satisfy* $|a_{ij}| < 1$ *and* $|b_{ij}| < 1$ *for any* $i, j \in \{1, \dots, n\}$*. For sufficiently small* $\epsilon > 0$*,*

(i. *the eigenvalues* λ *and* $\lambda^{'}$ *of* A *and* $(A + \epsilon B)$*, respectively, can be put in one-to-one correspondence so that* $|\lambda^{'} - \lambda| < 2(n+1)^2(n^2\epsilon)^{\frac{1}{n}}$*;*

(ii. *if* λ *is a simple eigenvalue of* A*, then the corresponding eigenvalue* $\lambda(\epsilon)$ *of* $A + \epsilon B$ *satisfies* $|\lambda(\epsilon) - \lambda| = O(\epsilon)$*;*

(iii. *if* $\boldsymbol{v}$ *is an eigenvector of* A *associated with a simple eigenvalue* λ*, then the eigenvector* $\boldsymbol{v}(\epsilon)$ *of* $A + \epsilon B$ *associated with the corresponding eigenvalue* $\lambda(\epsilon)$ *satisfies* $|v_i(\epsilon) - v_i| = O(\epsilon)$ *for any* $i \in \{1, \dots, n\}$*.*

Proof of Theorem 4.3.1: In this proof, we extend the definition of $\boldsymbol{v}_{\text{left}}(A)$ to the normalized entry-wise positive left eigenvector, associated with the eigenvalue of A with the largest magnitude, if such an eigenvector exists and is unique. According to Perron-Frobenius theorem and Lemma 4.3.2, vector $\boldsymbol{v}_{\text{left}}(A)$, as long as well-defined, depends continuously on the entries of A. Therefore, for system (4.5), there exists a sufficiently small $\tau > 0$ such that $A(t)$ and $\boldsymbol{w}(t)$ are well-defined and continuously differentiable at any $t \in [0, \tau]$, and, moreover, $p_i\big(\boldsymbol{w}(t)\big) - \sum_k m_{ik} p_k\big(\boldsymbol{w}(t)\big)$ remains finite. Therefore, for any $t \in [0, \tau]$ and $i, j \in \{1, \dots, n\}$, $a_{ij}(t) > 0$ if $a_{ij}(0) > 0$; $a_{ij}(t) = 0$ if $a_{ij}(0) = 0$, and thus $A(t)$ is row-stochastic and primitive for any $t \in [0, \tau]$.

For any $i \in \{1, \dots, n\}$, there exists $k \neq i$ such that $a_{ik}(0) > 0$. According to equation (4.4),

$$\frac{da_{ij}(t)}{da_{ik}(t)} = \frac{a_{ij}(t)}{a_{ik}(t)},\ \forall t \in [0, \tau],\ \forall j \in \{1, \dots, n\} \setminus \{i, k\},$$

which leads to $a_{ij}(t)/a_{ik}(t) = a_{ij}(0)/a_{ik}(0)$. Let C be an $n \times n$ matrix with the entries c_{ij} defined as: (i) $c_{ii} = 0$ for any $i \in \{1, \dots, n\}$; (ii) $c_{ij} = a_{ij}(0)/\big(1 - a_{ii}(0)\big)$ for any $j \neq i$. One can check that C is row-stochastic and $A(t)$ is given by equation (4.6), for any $t \in [0, \tau]$, where $\boldsymbol{a}(t) = \big(a_1(t), \dots, a_n(t)\big)^\top$ with $a_i(t) = a_{ii}(t)$. Since the digraph, with C as the adjacency matrix, has the same topology with the digraph associated with $A(0)$, matrix C is irreducible and $\boldsymbol{c} = \boldsymbol{v}_{\text{left}}(C)$ is well-defined.

Since the matrix $A(t)$ has the structure given by (4.6), according to Lemma 2.2 in [103], for any $t \in [0, \tau]$,

$$w_i(t) = \frac{c_i}{1 - a_i(t)} \Big/ \sum_k \frac{c_k}{1 - a_k(t)}.$$

Therefore, for any $t \in [0, \tau]$,

$$p_i\big(\boldsymbol{w}(t)\big) = f\left(\frac{x_i}{c_i}\big(1 - a_i(t)\big) \sum_k w_k(t) \frac{c_k}{1 - a_k(t)}\right).$$

According to equation (4.4), $\dot{a}_j(t) \leq 0$ for any $j \in \operatorname{argmin}_k \frac{x_k}{c_k}\big(1 - a_k(t)\big)$. Therefore, $\operatorname{argmin}_k \frac{x_k}{c_k}\big(1 - a_k(t)\big)$ is increasing, and similarly, $\operatorname{argmax}_k \frac{x_k}{c_k}\big(1 - a_k(t)\big)$ is decreasing with t, which implies that, the set

$$\Omega_A\big(A(0)\big) = \Big\{A \in \mathbb{R}^{n\times n} \,\Big|\, A = \operatorname{diag}(\boldsymbol{a}) + (I - \operatorname{diag}(\boldsymbol{a}))C,$$
$$0 \le a_i \le 1 - \frac{c_i}{x_i} \min_k \frac{x_k}{c_k}\big(1 - a_{kk}(0)\big), \forall i\Big\}$$

is a compact positive invariant set for system (4.5), as long as $A(0)$ is row-stochastic, irreducible and has strictly positive diagonal. Moreover, one can check that, for any $A \in \Omega_A\big(A(0)\big)$, $\boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A)$ is well-defined and strictly lower (upper resp.) bounded from 0 (1 resp.). Therefore, the solution $A(t)$ is extensible to all $t \in [0, +\infty)$ and equations (4.6) and (4.7) hold for any $t \in [0, +\infty)$. Moreover, since $p_i\big(\boldsymbol{w}(t)\big) - \sum_k m_{ik} p_k\big(\boldsymbol{w}(t)\big)$ remains bounded, we have $a_{ij} > 0$ if $a_{ij}(0) > 0$ and $a_{ij}(t) = 0$ if $a_{ij}(0) = 0$. This concludes the proof for (i) - (iv).

For statement (v), differentiate both sides of the equation $\boldsymbol{w}^\top(t)A(t) = \boldsymbol{w}^\top(t)$ and substitute equation (4.5) into the differentiated equation. We obtain

$$(\boldsymbol{w}^\top \operatorname{diag}(\boldsymbol{p}(\boldsymbol{w}) - M\boldsymbol{p}(\boldsymbol{w}))A_d - \frac{d\boldsymbol{w}^\top}{dt})(I_n - A) = \mathbb{0}_n^\top,$$

where time index t is omitted for simplicity. Equation (4.8) in statement (v) is obtained due to $\boldsymbol{w}^\top(t)\mathbb{1}_n = 1$.■

With the extensibility of $A(t)$ and the finite-time properties, we now present the main theorem of this section.

Theorem 4.3.3 (Asymptotic behavior of assign/appraise dynamics) *Consider the dynamics* (4.5)*, based on Assumptions 4.3-4.5, with the workload assignment as in Assumption 4.1 and the performance as in Assumption 4.2. Assume the observation network* $G(M)$ *is strongly connected. For any initial appraisal matrix* $A(0)$ *that is row-stochastic,* *irreducible and has positive diagonal,*

(i. *the solution* $A(t)$ *converges, i.e., there exists* $A^* \in \mathbb{R}^{n \times n}$ *such that* $\lim_{t\to\infty} A(t) = A^*$*;*

(ii. *the limit appraisal matrix* A^* *is row-stochastic and irreducible. Moreover, the workload assignment satisfies* $\lim_{t\to\infty} \boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}(A^*) = \boldsymbol{x}$*.*

Proof: We prove the theorem by analyzing the generalized replicator dynamics (4.8) for $\boldsymbol{w}(t)$, and the reduced assign/appraise dynamics (4.7) for $\boldsymbol{a}(t)$, given any constant, normalized and entry-wise positive vector $\boldsymbol{c}$. According to equation (4.7), the assignment $\boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A)$ can be considered as a function of the self appraisal vector $\boldsymbol{a}$, that is, $\boldsymbol{w}(t) = \boldsymbol{w}\big(\boldsymbol{a}(t)\big)$ for any $t \geq 0$. In this proof, let $\boldsymbol{\phi}(\boldsymbol{a}) = \boldsymbol{p}\big(\boldsymbol{w}(\boldsymbol{a})\big) - M\boldsymbol{p}\big(\boldsymbol{w}(\boldsymbol{a})\big)$ and denote by $\mathcal{D} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ the distance induced by the 2-norm in $\mathbb{R}^n$. For any $\boldsymbol{x} \in \mathbb{R}^n$ and subset S of $\mathbb{R}^n$, defined $\mathcal{D}(\boldsymbol{x}, S) = \inf_{\boldsymbol{y}\in S} \mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$.

First of all, for any given $\boldsymbol{a}(0) \in (0,1)^n$, we know that the set Ω, as defined in Theorem 4.3.1(iv), is a compact positively invariant set for dynamics (4.7), and $\boldsymbol{w}(t)$ is well-defined and entry-wise strictly lower (upper resp.) bounded from $\mathbb{0}_n$ ($\mathbb{1}_n$ resp.), for all $t \in [0, +\infty)$.

Secondly, for any $\boldsymbol{a} \in \Omega$, define a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(\boldsymbol{a}) = \log \frac{\max_k x_k / w_k(\boldsymbol{a})}{\min_k x_k / w_k(\boldsymbol{a})},$$

and the following index sets

$$\overline{\theta}(\boldsymbol{a}) = \Big\{ i \,\Big|\, \exists t_i > 0 \text{ s.t. } \frac{x_i}{w_i\big(\boldsymbol{a}(t)\big)} = \max_k \frac{x_k}{w_k\big(\boldsymbol{a}(t)\big)} \text{ for any } t \in [0, t_i], \text{ with } \boldsymbol{a}(0) = \boldsymbol{a} \Big\},$$

and

$$\underline{\theta}(\boldsymbol{a}) = \Big\{ j \,\Big|\, \exists t_j > 0 \text{ s.t. } \frac{x_j}{w_j\big(\boldsymbol{a}(t)\big)} = \min_k \frac{x_k}{w_k\big(\boldsymbol{a}(t)\big)} \text{for any } t \in [0, t_j], \text{ with } \boldsymbol{a}(0) = \boldsymbol{a} \Big\}.$$

Then the right time derivative of $V\big(\boldsymbol{a}(t)\big)$, along the solution $\boldsymbol{a}(t)$, is given by

$$\frac{d^+ V\big(\boldsymbol{a}(t)\big)}{dt} = a_j(t)\phi_j\big(\boldsymbol{a}(t)\big) - a_i(t)\phi_i\big(\boldsymbol{a}(t)\big),$$

for any $i \in \overline{\theta}\big(\boldsymbol{a}(t)\big)$ and $j \in \underline{\theta}\big(\boldsymbol{a}(t)\big)$. Define

$$\begin{aligned} E &= \big\{ \boldsymbol{a} \in \Omega \,\big|\, a_j\phi_j(\boldsymbol{a}) - a_i\phi_i(\boldsymbol{a}) = 0 \text{ for any } i \in \overline{\theta}(\boldsymbol{a}), j \in \underline{\theta}(\boldsymbol{a}) \big\}, \\ E_1 &= \big\{ \boldsymbol{a} \in E \,\big|\, \boldsymbol{\phi}(\boldsymbol{a}) = \mathbb{0}_n \big\}, \\ E_2 &= \big\{ \boldsymbol{a} \in E \,\big|\, \boldsymbol{\phi}(\boldsymbol{a}) \neq \mathbb{0}_n \big\}. \end{aligned}$$

One can check that E and E_1 are compact subsets of Ω, $E = E_1 \cup E_2$, and $E_1 \cap E_2$ is empty. Denote by $\hat{E}$ the largest invariant subset of E. Applying the LaSalle Invariance Principle, see Theorem 3 in [136], we have $\mathcal{D}\big(\boldsymbol{a}(t), \hat{E}\big) \to 0$ as $t \to +\infty$. Note that, $\lim_{t\to+\infty} \mathcal{D}\big(\boldsymbol{a}(t), \hat{E}\big) = 0$ does not necessarily leads to $\lim_{t\to+\infty} \boldsymbol{w}(t) = \boldsymbol{x}$. We need to further refine the result.

For set E_1, it is straightforward to see that $E_1 \subset \hat{E}$ and $\boldsymbol{w}(\boldsymbol{a}) = \boldsymbol{x}$ for any $\boldsymbol{a} \in E_1$. Now we prove by contradiction that, if $E_2 \cap \hat{E}$ is not empty, then, for any $\boldsymbol{a} \in E_2 \cap \hat{E}$, there exists $i \in \overline{\theta}(\boldsymbol{a})$ such that $a_i = 0$. Suppose $a_i > 0$ for any $i \in \overline{\theta}(\boldsymbol{a})$. Since the observation network $G(M)$ is strongly connected, there exists a directed path $i, k_1, \dots, k_q, j$ on $G(M)$, where $i \in \overline{\theta}(\boldsymbol{a})$ and $j \in \underline{\theta}(\boldsymbol{a})$. We have $k_1 \in \overline{\theta}(\boldsymbol{a})$, otherwise, starting with $\tilde{\boldsymbol{a}}(0) = \boldsymbol{a}$, there exists sufficiently small $\Delta t > 0$ such that $\phi_i\big(\tilde{\boldsymbol{a}}(t)\big) > 0$ and $\tilde{a}_i(t) > 0$, which contradicts the fact that $\boldsymbol{a}$ is in the largest invariant set of E. Repeating this argument, we have

$j \in \overline{\theta}(\boldsymbol{a})$, which contradicts $\boldsymbol{\phi}(\boldsymbol{a}) \neq \mathbb{0}_n$. Similarly, we have that, for any $\boldsymbol{a} \in E_2 \cap \hat{E}$, there exists $j \in \underline{\theta}(\boldsymbol{a})$ with $a_j = 0$.

If the fixed vectors $\boldsymbol{c}$ and $\boldsymbol{x}$ satisfy $\boldsymbol{c} = \boldsymbol{x}$, then there can not exist $\boldsymbol{a} \in E_2 \cap \hat{E}$ satisfying all the following three properties: i) there exists $i \in \overline{\theta}(\boldsymbol{a})$ such that $a_i = 0$; ii) there exists $j \in \underline{\theta}(\boldsymbol{a})$ such that $a_j = 0$; iii) $\boldsymbol{\phi}(\boldsymbol{a}) \neq \mathbb{0}_n$. In this case, $E_2 \cap \hat{E}$ is an empty set, which implies that $\boldsymbol{a}(t) \to \hat{E} = E_1$ and thus $\boldsymbol{w}(t) \to \boldsymbol{x}$ as $t \to +\infty$.

Before discussing the case when $\boldsymbol{c} \neq \boldsymbol{x}$, we present some properties of the individual performance measure:

P1: For any $k, l \in \{1, \dots, n\}$, $\frac{x_k}{c_k}(1 - a_k) \leq \frac{x_l}{c_l}(1 - a_l)$ leads to $p_k(\boldsymbol{a}) \leq p_l(\boldsymbol{a})$, and $\frac{x_k}{c_k}(1 - a_k) > \frac{x_l}{c_l}(1 - a_l)$ leads to $p_k(\boldsymbol{a}) > p_l(\boldsymbol{a})$;

P2: If there exists $\tau \geq 0$ such that $i \in \overline{\theta}\big(\boldsymbol{a}(\tau)\big)$ and $a_i(\tau) = 0$, then $i \in \overline{\theta}\big(\boldsymbol{a}(t)\big)$ for all $t \geq \tau$;

P3: $\boldsymbol{p}(\boldsymbol{a}(t))$ is finite and strictly bounded from 0, satisfying $f\big(\frac{x_i}{c_i}(1 - \zeta_i(\boldsymbol{a}(0)))\big) \leq p_i(\boldsymbol{a}(t)) \leq f\big(\frac{x_i}{c_i}\sum_k \frac{c_k}{\zeta_k(\boldsymbol{a}(0))}\big)$, with $\zeta_i(\boldsymbol{a})$ defined in Theorem 4.3.1(iv).

For the case when $\boldsymbol{c} \neq \boldsymbol{x}$, consider the partition $\varphi_1, \dots, \varphi_m$ of the index set $\{1, \dots, n\}$, with $m \leq n$, satisfying the following two properties:

(i. $x_k/c_k = x_l/c_l$ for any k, l in the same subset φ_r;

(ii. $x_k/c_k > x_l/c_l$ for any $k \in \varphi_r$, $l \in \varphi_s$, with $r < s$.

For any $\boldsymbol{a} \in E_2 \cap \hat{E}$, since there exists $j \in \underline{\theta}(\boldsymbol{a})$ with $a_j = 0$, we have $\varphi_m \subset \underline{\theta}(\boldsymbol{a})$. For any $i \in \cup_{r=1}^{m-1}\varphi_r$, let

$$\begin{aligned} E_{2,i} = \Big\{\boldsymbol{a} \in \Omega \,\Big|\, & a_i = 0,\ a_j = 0 \text{ for any } j \in \varphi_m, 1 - \frac{x_i}{c_i}\frac{c_k}{x_k} \leq a_k \leq 1 - \min_{l \in \{1,\dots,n\}} \frac{x_l}{c_l}\frac{c_k}{x_k}, \\ & \text{for any } k \in \varphi_1 \cup \dots \cup \varphi_{m-1} \setminus \{i\}\Big\}. \end{aligned}$$

With properties P1 and P2 of $\boldsymbol{p}(\boldsymbol{a})$, for any $\boldsymbol{a} \in E_{2,i}$, we have $i \in \overline{\theta}(\boldsymbol{a})$ and $a_i = 0$.

Moreover,

(i. $E_{2,i} \subset \mathbb{R}^n$ is compact for any $i \in \varphi_1 \cup \cdots \cup \varphi_{m-1}$;

(ii. $\cup_{i\in\varphi_1} E_{2,i}, \dots, \cup_{i\in\varphi_{m-1}} E_{2,i}$ are disjoint and compact subsets of $\mathbb{R}^n$;

(iii. $E_2 \cap \hat{E} \subset \bigcup_{i\in\varphi_1\cup\cdots\cup\varphi_{m-1}} E_{2,i}$.

For any $\boldsymbol{a} \in E_2 \cap \hat{E}$, since there exists $i \in \overline{\theta}(\boldsymbol{a})$ and $j \in \underline{\theta}(\boldsymbol{a})$ such that $a_i = a_j = 0$, on the observation network $G(M)$, there must exists a path $i, k_1, \dots, k_q$ satisfying: i) $i \in \overline{\theta}(\boldsymbol{a})$ and $a_i = 0$; ii) $a_{k_q} = 0$ and $x_{k_q}/c_{k_q} < x_i/c_i$; iii) $a_{k_l} > 0$ for any $l \in \{1, \dots, q-1\}$. Consider the trajectory $\tilde{\boldsymbol{a}}(t)$ with $\tilde{\boldsymbol{a}}(0) = \boldsymbol{a}$, we have

$$\begin{aligned}\dot{\tilde{a}}_{k_{q-1}} \geq \tilde{a}_{k_{q-1}}(1-\tilde{a}_{k_{q-1}})\Bigg(& f\Big(\frac{x_{k_{q-1}}}{c_{k_{q-1}}}(1-\tilde{a}_{k_{q-1}})\sum_{l=1}^{n}\frac{c_l}{1-\tilde{a}_l}\Big) \\ & - f\Big(\big(m_{k_{q-1}k_q}\frac{x_{k_q}}{c_{k_q}} + (1-m_{k_{q-1}k_q})\frac{x_i}{c_i}\big)\sum_{l=1}^{n}\frac{c_l}{1-\tilde{a}_l}\Big)\Bigg).\end{aligned}$$

The inequality is due to properties P1-P3 of $p_i(\boldsymbol{a})$ for $i \in \overline{\theta}(\boldsymbol{a})$ with $a_i = 0$, and the concavity of the function f. Moreover, since $\tilde{a}_{k_{q-1}}$ is strictly bounded from 1 and $\sum_l c_l/(1-\tilde{a}_l)$ is strictly lower bounded from 0, there exists $T_{k_{q-1}}(M, \boldsymbol{a}(0), \boldsymbol{a}) > 0$ such that

$$p_{k_{q-1}}\big(\tilde{\boldsymbol{a}}(t)\big) < \frac{2-m_{k_{q-1}k_q}}{2} p_i\big(\tilde{\boldsymbol{a}}(t)\big) + \frac{m_{k_{q-1}k_q}}{2} p_{k_q}\big(\tilde{\boldsymbol{a}}(t)\big).$$

Applying the same argument to $k_{q-2}, \dots, k_1$, we have that, there exists $T_{k_1}(M, \boldsymbol{a}(0), \boldsymbol{a}) > 0$ and $\eta_{ik_1\dots k_q}(M) \in (0,1)$ such that, for the solution $\tilde{\boldsymbol{a}}(t)$ with $\tilde{\boldsymbol{a}}(0) = \boldsymbol{a}$,

$$p_{k_1}\big(\tilde{\boldsymbol{a}}(t)\big) < \big(1-\eta_{ik_1\dots k_q}(M)\big) p_i\big(\tilde{\boldsymbol{a}}(t)\big) + \eta_{ik_1\dots k_q}(M) p_{k_q}\big(\tilde{\boldsymbol{a}}(t)\big),$$

for all $t \geq T_{k_1}(M, \boldsymbol{a}(0), \boldsymbol{a})$. This inequality implies that,

$$\begin{aligned}\phi_i\big(\tilde{\boldsymbol{a}}(t)\big) &\geq m_{ik_1}\eta_{ik_1\dots k_q}(M)\Big(p_i\big(\tilde{\boldsymbol{a}}(t)\big) - p_{k_q}\big(\tilde{\boldsymbol{a}}(t)\big)\Big) \\ &\geq m_{ik_1}\eta_{ik_1\dots k_q}(M) f'\left(\frac{x_i}{c_i}\right)\sum_{l=1}^{n}\frac{c_l}{1-\zeta_l\big(\boldsymbol{a}(0)\big)}\Big(\frac{x_i}{c_i} - \frac{x_{k_q}}{c_{k_q}}\Big) > 0.\end{aligned}$$

Since the choices of i and the paths $i, k_1, \dots, k_q$ are finite, there exists a constant $\eta > 0$ such that, for any $\boldsymbol{a} \in E_2 \cap \hat{E}$, there exists $T\big(\boldsymbol{a}(0), \boldsymbol{a}\big) > 0$ such that, for any $t \geq T\big(\boldsymbol{a}(0), \boldsymbol{a}\big) > 0$, the solution $\tilde{\boldsymbol{a}}(t)$, with $\tilde{\boldsymbol{a}}(0) = \boldsymbol{a}$, satisfies $i \in \overline{\theta}\big(\tilde{\boldsymbol{a}}(t)\big)$ and $\phi_i\big(\tilde{\boldsymbol{a}}(t)\big) \geq \eta > 0$.

For any $i \in \varphi_1 \cup \dots \cup \varphi_{m-1}$, define

$$\hat{E}_{2,i} = \Big\{\boldsymbol{a} \in E_{2,i} \,\Big|\, p_i(\boldsymbol{a}) - \sum_{k=1}^{n} m_{ik} p_k(\boldsymbol{a}) \geq \eta\Big\}.$$

We have: i) each $\hat{E}_{2,i}$ is a compact subset of $\mathbb{R}^n$; ii) $\cup_{i\in\varphi_1}\hat{E}_{2,i}, \dots, \cup_{i\in\varphi_{m-1}}\hat{E}_{2,i}$ are disjoint and compact subsets of $\mathbb{R}^n$. Let $\hat{E}_2 = \cup_{r=1}^{m-1}\big(\cup_{r\in\varphi_r}\hat{E}_{2,i}\big)$. For dynamics (4.7), due to the continuous dependency on the initial condition, for any $\boldsymbol{a} \in (E_2 \cap \hat{E}) \setminus (\hat{E}_2 \cap \hat{E})$, there exists $\delta > 0$ such that, for any $\tilde{\boldsymbol{a}}(0) \in \mathcal{U}(\boldsymbol{a}, \delta) \cap (E_2 \cap \hat{E})$, where $\mathcal{U}(\boldsymbol{a}, \delta) = \big\{\boldsymbol{b} \in \Omega \,\big|\, \mathcal{D}(\boldsymbol{b}, \boldsymbol{a}) \leq \delta\big\}$, $\tilde{\boldsymbol{a}}(t) \in \hat{E}_2 \cap \hat{E}$ for sufficiently large t. Therefore, $\boldsymbol{a}$ can not be an ω-limit point of $\boldsymbol{a}(0)$. We thus obtain that, the ω-limit set of $\boldsymbol{a}(0)$ is in the set $E_1 \cup (\hat{E}_2 \cap \hat{E})$. Moreover, since $E_1, \cup_{i\in\varphi_1}\hat{E}_{2,i}, \dots, \cup_{i\in\varphi_{m-1}}\hat{E}_{2,i}$ are disjoints compact subsets of $\mathbb{R}^n$, and the ω-limit set of $\boldsymbol{a}(0)$ is connected and compact, $\boldsymbol{a}(t)$ can only converge to one of the sets $E_1, \cup_{i\in\varphi_1}\hat{E}_{2,i}, \dots, \cup_{i\in\varphi_{m-1}}\hat{E}_{2,i}$.

Now we prove $\lim_{t\to+\infty}\mathcal{D}(\boldsymbol{a}(t), E_1) = 0$ by contradiction. Suppose $\omega\big(\boldsymbol{a}(0)\big) \in \cup_{i\in\varphi_r}\hat{E}_{2,i}$ for some $r \in \{1, \dots, m-1\}$. Since each $\hat{E}_{2,i}$ is a compact set, there exists $\epsilon > 0$ and $\eta(\epsilon) > 0$ such that $\phi_i(\boldsymbol{a}) \geq \eta(\epsilon) > 0$ for any $\boldsymbol{a} \in \mathcal{U}(\hat{E}_{2,i}, \epsilon)$. For this given $\epsilon > 0$, since $\omega\big(\boldsymbol{a}(0)\big) \in \cup_{i\in\varphi_r}\hat{E}_{2,i}$ leads to $\mathcal{D}\big(\boldsymbol{a}(t), \cup_{i\in\varphi_r}\hat{E}_{2,i}\big) \to 0$ as $t \to +\infty$, we conclude that, there exists $T > 0$ such that, for any $t \geq T$, $\boldsymbol{a}(t) \in \cup_{i\in\varphi_r}\mathcal{U}(\hat{E}_{2,i},\epsilon)$. Define $V_r(\boldsymbol{a}) = \min_{i\in\varphi_r} a_i$, for any $\boldsymbol{a} \in \cup_{i\in\varphi_r}\mathcal{U}(\hat{E}_{2,i},\epsilon)$. The function $V_r(\boldsymbol{a})$ satisfies that, $V_r(\boldsymbol{a}) \geq 0$ for any $\boldsymbol{a} \in \cup_{i\in\varphi_r}\mathcal{U}(\hat{E}_{2,i},\epsilon)$ and $V_r(\boldsymbol{a}) = 0$ if and only if $\boldsymbol{a} \in \cup_{i\in\varphi_r}\hat{E}_{2,i}$. Therefore, $\mathcal{D}\big(\boldsymbol{a}(t), \cup_{i\in\varphi_r}\hat{E}_{2,i}\big) \to 0$ leads to $V_r\big(\boldsymbol{a}(t)\big) \to 0$ as $t \to +\infty$. Moreover, since $\boldsymbol{a} \in \mathcal{U}(\hat{E}_{2,i},\epsilon)$ for any $i \in \operatorname{argmin}_{k\in\varphi_r} a_k$, we have

$$\frac{d^+V_r\big(\boldsymbol{a}(t)\big)}{dt} = \min_{i\in\operatorname{argmin}\limits_{k\in\varphi_r} a_k(t)} \dot{a}_i(t) \geq \delta a_i(t)\big(1 - a_i(t)\big).$$

According to Theorem 4.3.1(i), for any given $\boldsymbol{a}(0) \in (0,1)^n$, $\boldsymbol{a}(t) \in (0,1)^n$ for all $t \geq 0$. Therefore, $d^+V_r(\boldsymbol{a}(t))/dt > 0$ for all $t \geq T$, which contradicts $\lim_{t\to+\infty} V_r\big(\boldsymbol{a}(t)\big) = 0$. Therefore, we have $\lim_{t\to+\infty}\mathcal{D}(\boldsymbol{a}(t), E_1) = 0$ and $\lim_{t\to+\infty}\boldsymbol{w}(t) = \boldsymbol{x}$.

Since $\dot{A}(t) \to \mathbb{0}_{n\times n}$ as $\boldsymbol{\phi}\big(\boldsymbol{a}(t)\big) \to \mathbb{0}_n$, there exists an entry-wise non-negative and irreducible matrix A^*, depending on $A(0)$ and satisfying $\boldsymbol{v}_{\text{left}}(A^*) = \boldsymbol{x}$, such that $A(t) \to A^*$ as $t \to +\infty$. This concludes the proof. ■

Theorem 4.3.3 indicates that, the teams obeying the assign/appraise dynamics asymptotically achieves the optimal workload assignment, but do not necessarily reach appraisal consensus. Figure 4.2 gives a visualized illustration of the asymptotic behavior of the assign/appraise dynamics.

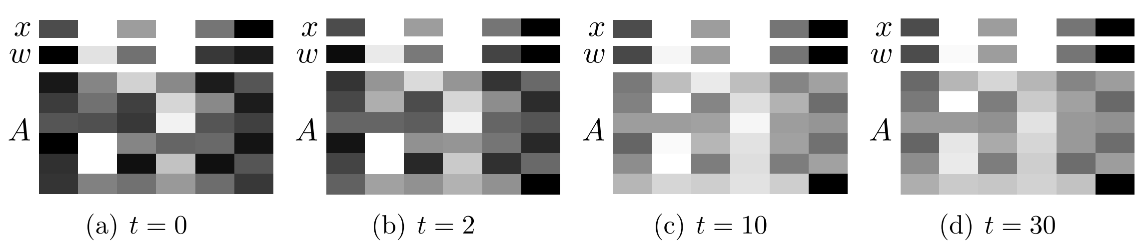


Figure 4.2: Visualization of the evolution of $A(t)$ and $\boldsymbol{w}(t)$ obeying the assign/appraise dynamics with $n = 6$. The observation network is strongly connected. In these visualized matrices and vectors, the darker the entry, the higher value it has.

Remark 4.3.4 *From the proof for Theorem 4.3.3 we know that, the teams obeying the following dynamics*

$$\begin{cases} \dot{a}_{ii} = \gamma_i(t) a_{ii}(1 - a_{ii})\big(p_i(\boldsymbol{w}) - \sum_k m_{ik} p_k(\boldsymbol{w})\big), \\ \dot{a}_{ij} = -\gamma_i(t) a_{ii} a_{ij}\big(p_i(\boldsymbol{w}) - \sum_k m_{ik} p_k(\boldsymbol{w})\big), \end{cases}$$

also asymptotically achieve the optimal assignment, if each $\gamma_i(t)$ remains strictly bounded from 0. This result indicates that our model can be generalized to the case of heterogeneous sensitivities to performance feedback.

4.4 The Assign/appraise/influence Dynamics of the Appraisal Networks

In this section we further elaborate the assign/appraise dynamics by assuming that the appraisal network is updated via not only the performance feedback, but also the influence process inside the team.

4.4.1 Model description

The new model, named the *assign/appraise/influence dynamics*, is defined by three components: the assignment rule as in Assumption 4.3, the appraise dynamics based on Assumptions 4.4 and 4.5, and the *influence dynamics*, which is the opinion exchanges among individuals on the interpersonal appraisals. Denote by w_{ij} the weight individual i assigns to j (including self weight w_{ii}) in the opinion exchange. The matrix $W = (w_{ij})_{n\times n}$ defines a directed and weighted graph, referred to as the *influence network*, is row-stochastic and possibly time-varying.

The diagram illustration of assign/appraise/influence dynamics is presented in Figure 4.1(c), and the general form is given as follows:

$$\begin{cases} \dot{A} &= \frac{1}{\tau_{\text{ave}}} F_{\text{ave}}(A, W) + \frac{1}{\tau_{\text{app}}} F_{\text{app}}(A, \boldsymbol{w}), \\ \boldsymbol{w} &= \boldsymbol{v}_{\text{left}}(A). \end{cases} \tag{4.9}$$

The time index t is omitted for simplicity. The term $F_{\text{app}}(A, \boldsymbol{w})$ corresponds to the appraise dynamics given by the right-hand side of the first equation in (4.5), while the term $F_{\text{ave}}(A, W)$ corresponds to the influence dynamics specified by the assumption below. Parameters τ_{ave} and τ_{app} are positive, and relate to the time scales of influence dynamics and appraise dynamics respectively.

Assumption 4.6 (Influence dynamics) *For the assign/appraise/influence dynamics, assume that, at each time $t \geq 0$, the influence network is identical to the appraisal network, i.e., $W(t) = A(t)$. Moreover, assume that the individuals obey the classic DeGroot opinion dynamics [2] for the interpersonal appraisals, i.e., $F_{\text{ave}}(W, A) = -(I_n - W)A$.*

Based on equation (4.9) and Assumptions 4.3-4.6, the assign/appraise/influence dynamics is written as

$$\begin{cases} \dot{A} = \frac{1}{\tau_{\text{ave}}}(A^2 - A) \\ \qquad + \frac{1}{\tau_{\text{app}}} \operatorname{diag}\big(\boldsymbol{p}(\boldsymbol{w}) - M\boldsymbol{p}(\boldsymbol{w})\big) A_{\text{d}}(I_n - A), \\ \boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A), \end{cases} \tag{4.10}$$

In the next subsection, we relate the topology of the observation network $G(M)$ to the asymptotic behavior of the assign/appraise/influence dynamics, i.e., the convergence to optimal assignment and the appraisal consensus.

4.4.2 Dynamical behavior of the assign/appraise/influence dynamics

The following lemma shows that, for the assign/appraise/influence dynamics, we only need to consider the all-to-all initial appraisal network.

Lemma 4.4.1 (entry-wise positive for initial appraisal) *Consider the assign/appraise/influence dynamics* (4.10) *based on Assumptions 4.3-4.6, with the workload assignment and performance as in Assumptions 4.1 and 4.2 respectively. For any initial appraisal matrix* $A(0)$ *that is primitive and row-stochastic, there exists* $\Delta t > 0$ *such that* $A(t) \succ \mathbb{0}_{n\times n}$ *for any* $t \in (0, \Delta t]$.

Proof: Since $A(0)$ is primitive and row-stochastic, following the same argument in the proof for Theorem 4.3.1(i), we have that, there exists $\Delta\tilde{t} > 0$ such that, for any $t \in [0, \Delta\tilde{t}]$: i) $\boldsymbol{w}(t)$ is well-defined and $\boldsymbol{w}(t) \succ \mathbb{0}_n$; ii) $A(t)$ is bounded, continuously differentiable to t, and satisfies $A(t)\mathbb{1}_n = \mathbb{1}_n$; iii) $\boldsymbol{p}\big(\boldsymbol{w}(t)\big) - M\boldsymbol{p}\big(\boldsymbol{w}(t)\big)$ is bounded. Therefore, for any $t \geq 0$, there exists μ, depending on t and $A(0)$, such that $\dot{A}(t) \succeq \frac{1}{\tau_{\text{ave}}}A^2(t) - (\frac{1}{\tau_{\text{ave}}} + \mu)A(t)$.

Consider the equation $\dot{B}(t) = \frac{1}{\tau_{\text{ave}}}B^2(t) - (\frac{1}{\tau_{\text{ave}}} + \mu)B(t)$, with $B(0) = A(0)$. According to the comparison theorem, $A(t) \succeq B(t)$ for any $t \geq 0$. Let $\boldsymbol{b}_i(t)$ be the i-th column of $B(t)$ and let $\boldsymbol{y}_k(t) = e^{(\frac{1}{\tau_{\text{ave}}}+\mu)t}\boldsymbol{b}_k(t)$. We obtain $\dot{\boldsymbol{y}}_k(t) = \frac{1}{\tau_{\text{ave}}}B(t)\boldsymbol{y}_k(t)$.

Denote by $\Phi(t, 0)$ the state transition function for the equation $\dot{\boldsymbol{y}}_k(t) = \frac{1}{\tau_{\text{ave}}}B(t)\boldsymbol{y}_k(t)$, which is written as $\Phi(t, 0) = I_n + \sum_{k=1}^{\infty}\Phi_k(t)$, where $\Phi_1(t) = \int_0^t B(\tau_1)d\tau_1$ and $\Phi_l(t) = \int_0^t B(\tau_1)\int_0^{\tau_1}\dots B(\tau_{l-1})\int_0^{\tau_{l-1}} B(\tau_l)d\tau_l$ for $l \geq 2$. By computing the MacLaurin expansion for each $\Phi_k(t)$ and summing them together, we obtain that

$$\Phi(t, 0) = I_n + h_1(t)B(0) + h_2(t)B^2(0) + \dots + h_{n-1}(t)B^{n-1}(0) + O(t^n),$$

where $h_k(t)$ is a polynomial with the form $h_k(t) = \eta_{k,k}t^k + \eta_{k,k+1}t^{k+1} + \dots$, and, moreover, $\eta_{k,k} > 0$ for any $k \in \mathbb{N}$. Therefore, for t sufficiently small, we have $h_k(t) > 0$ for any $k \in \{1, \dots, n-1\}$. Moreover, since $B^k(0) \succeq \mathbb{0}_{n\times n}$ for any $k \in \mathbb{N}$ and $B(0)+\dots+B^{n-1}(0) \succ \mathbb{0}_{n\times n}$, there exists $\Delta t \leq \Delta\tilde{t}$ such that $\Phi(t,0) \succ \mathbb{0}_{n\times n}$ for any $t \in [0, \Delta t]$. ■

Before discussing the asymptotic behavior, we state a technical assumption.

Conjecture 4.4.2 (Strict lower bound of the interpersonal appraisals) *Consider the assign/appraise/influence dynamics* (4.10) *based on Assumptions 4.3-4.6, with the workload assignment and performance as in Assumptions 4.1 and 4.2 respectively. For any $A(0)$ that is entry-wise positive and row-stochastic, there exists $a_{\min} > 0$, depending on $A(0)$, such that $A(t) \succ a_{\min}\mathbb{1}_n\mathbb{1}_n^\top$ for any time $t \geq 0$, as long as $A(\tau)$ and $\boldsymbol{w}(\tau)$ are well-defined for all $\tau \in [0, t]$.*

Monte Carlo validation and a sufficient condition for Conjecture 4.4.2 are presented in Section 4.6.4. Now we state the main results of this section.

Theorem 4.4.3 (Assign/appraise/influence dynamical behavior) *Consider the assign/appraise/influence dynamics* (4.10) *based on Assumptions 4.3-4.6, with the task assignment and performance as in Assumptions 4.1 and Assumption 4.2 respectively. Suppose that Conjecture 4.4.2 holds. Assume that the observation network $G(M)$ contains a globally reachable node. For any initial appraisal matrix $A(0)$ that is entry-wise positive and row-stochastic, and any time scales $\tau_{ave} > 0$ and $\tau_{app} > 0$ in equation* (4.10)*,*

(i. the solution $A(t)$ exists and $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}\big(A(t)\big)$ is well-defined for all $t \in [0, +\infty)$. Moreover, $A(t) \succ \mathbb{0}_{n\times n}$ and $A(t)\mathbb{1}_n = \mathbb{1}_n$ for any $t \geq 0$;

(ii. the assignment $\boldsymbol{w}(t)$ obeys the generalized replicator dynamics (4.8)*, and $\xi_0\mathbb{1}_n \preceq$*

$\boldsymbol{w}(t) \preceq \big(1-(n-1)\xi_0\big)\mathbb{1}_n$, *where*

$$\xi_0 = \left(1+(n-1)\frac{\max_k x_k}{\min_l x_l}\gamma_0\right)^{-1}, \quad \textit{and } \gamma_0 = \frac{\max_k x_k/w_k(0)}{\min_l x_l/w_l(0)};$$

(iii. as $t \to +\infty$, $A(t)$ converges to $\mathbb{1}_n \boldsymbol{x}^\top$ and thereby $\boldsymbol{w}(t)$ converges to $\boldsymbol{x}$.

Proof: Statement (i) is proved following the same argument in the proof for Theorem 4.3.1 (i). For any given $A(0)$ that is row-stochastic and entry-wise positive, the closed and bounded invariant set Ω for $A(t)$ is given by $\Omega = \big\{A \in \mathbb{R}^{n\times n} \,\big|\, A \succ a_{\min}\mathbb{1}_n\mathbb{1}_n^\top, A\mathbb{1}_n = \mathbb{1}_n\big\}$, where $a_{\min} > 0$ is given by Conjecture 4.4.2.

Since $\boldsymbol{w}^\top(t)\big(A^2(t) - A(t)\big) = \mathbb{0}_n^\top$ for all $t \geq 0$, we conclude that, $\boldsymbol{w}(t)$ in the assign/appraise/influence dynamics also obeys the generalized replicator dynamics (4.8). Consider $\boldsymbol{w}(t)$ as a function of $A(t)$. Define $\boldsymbol{\phi}(A) = \boldsymbol{p}\big(\boldsymbol{w}(A)\big) - M\boldsymbol{p}\big(\boldsymbol{w}(A)\big)$ and

$$V(A) = \log\frac{\max_k x_k/w_k(A)}{\min_k x_k/w_k(A)}.$$

For any $t \in [0,+\infty)$, there exists $i \in \operatorname{argmax}_k x_k/w_k\big(A(t)\big)$ and $j \in \operatorname{argmin}_k x_k/w_k\big(A(t)\big)$ such that $V\big(A(t)\big) = \log\Big(x_i w_j\big(A(t)\big)/x_j w_i\big(A(t)\big)\Big)$, and $\frac{d^+V(A)}{dt} = a_{jj}\phi_j(A) - a_{ii}\phi_i(A) \leq 0$. Therefore, $V\big(A(t)\big)$ is non-increasing with t, which in turn implies

$$\frac{x_i}{x_j}\frac{w_j(t)}{w_i(t)} \leq \frac{\max_k x_k/w_k(0)}{\min_k x_k/w_k(0)} = \gamma_0,$$

for any $i, j \in \{1,\dots,n\}$. This inequality, combined with the fact that $\sum_k w_k(t) = 1$ for any $t \geq 0$, leads to the inequalities in statement (ii).

Similar to the proof for Theorem 4.3.3, define

$$\overline{\theta}(A) = \Big\{ i \,\Big|\, \exists\, t_i > 0 \text{ s.t. } \frac{x_i}{w_i\big(A(t)\big)} = \max_k \frac{x_k}{w_k\big(A(t)\big)} \text{ for any } t \in [0, t_i] \text{ with } A(0) = A \Big\},$$

$$\underline{\theta}(A) = \Big\{ j \,\Big|\, \exists\, t_j > 0 \text{ s.t. } \frac{x_j}{w_j\big(A(t)\big)} = \min_k \frac{x_k}{w_k\big(A(t)\big)} \text{ for any } t \in [0, t_j] \text{ with } A(0) = A \Big\},$$

and let $E = \big\{ A \in \Omega \,\big|\, d^+V(A)/dt = 0 \big\}$. For any $A \in E$, since $A \succeq a_{\min} \mathbb{1}_n \mathbb{1}_n^\top$, we have $\phi_i(A) = \phi_j(A) = 0$ for any $i \in \overline{\theta}(A)$ and $j \in \underline{\theta}(A)$. Suppose individual s is a globally reachable node in the observation network. There exists a directed path $i, k_1, \dots, k_q, s$. Without loss of generality, suppose $q \geq 1$. For any A in the largest invariant subset of E, we have $k_1 \in \overline{\theta}(A)$ and therefore $\phi_{k_1}(A) = 0$. This iteration of argument leads to $s \in \overline{\theta}(A)$. Following the same line of argument, we have $s \in \underline{\theta}(A)$. Therefore, for any given $A(0) \succ \mathbb{0}_{n\times n}$ that is row-stochastic, the solution $A(t)$ converges to $\hat{E} = \{ A \in \Omega \,|\, \boldsymbol{\phi}(A) = \mathbb{0}_n \} = \{ A \in \Omega \,|\, \boldsymbol{v}_{\text{left}}(A) = \boldsymbol{x} \}$.

Let $\tilde{A} = \max_j \big(\max_k a_{kj} - \min_k a_{kj} \big)$. One can check that $d^+\tilde{V}(A)/dt$ along the dynamics (4.10) is a continuous function of A for any $A \in \Omega$. Define $\hat{E}_{\epsilon/2} = \big\{ A \in \hat{E} \,\big|\, \| A - \mathbb{1}_n \boldsymbol{x}^\top \|_2 \geq \epsilon/2 \big\}$. Since $\hat{E}$ is compact, $\hat{E}_{\epsilon/2}$ is also a compact set. For any $A \in \hat{E}_{\epsilon/2}$, since $d^+\tilde{V}(A)/dt$ is strictly negative and depends continuously on A, there exists a neighborhood $\mathcal{U}(A, r_A) = \{ \tilde{A} \in \Omega \,|\, \|\tilde{A} - A\|_2 \leq r_A \}$ such that $d^+\tilde{V}(\tilde{A})/dt < 0$ for any $\tilde{A} \in \mathcal{U}(A, r_A)$. Due to the compactness of $\hat{E}_{\epsilon/2}$ and according to the Heine-Borel finite cover theorem, there exists $K \in \mathbb{N}$ and $\{A_k, r_k\}_{k \in \{1,\dots,K\}}$, where $A_k \in \hat{E}_{\epsilon/2}$ and $r_k > 0$ for any $k \in \{1, \dots, K\}$, such that $\hat{E}_{\epsilon/2} \subset \cup_{k=1}^K \mathcal{U}(A_k, r_k)$.

Define the distance $\mathcal{D} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as in the proof for Theorem 4.3.3. Let

$\delta = \min\{r_1, \dots, r_k, \epsilon/2\}$ and

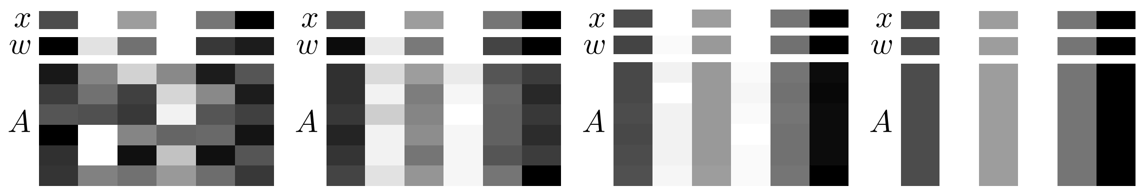
$$B_1 = \big\{A \in \Omega \,\big|\, \mathcal{D}(A, \hat{E}) \le \delta, \mathcal{D}(A, \hat{E}_{\epsilon/2}) > \delta\big\},$$
$$B_2 = \big\{A \in \Omega \,\big|\, \mathcal{D}(A, \hat{E}) \le \delta, \mathcal{D}(A, \hat{E}_{\epsilon/2}) \le \delta\big\}.$$

We have $B_1 \cap B_2$ is empty. For any $A \in B_1$, since $\mathcal{D}(A, \hat{E}) \le \delta$, $\mathcal{D}(A, \hat{E}_{\epsilon/2}) > \delta$, there exists $\tilde{A} \in \hat{E}_{\epsilon/2}$ such that $\mathcal{D}(A, \tilde{A}) \le \delta$. Since $\mathcal{D}(\tilde{A}, \mathbb{1}_n \boldsymbol{x}^\top) < \epsilon/2$, we have $\mathcal{D}(A, \mathbb{1}_n \boldsymbol{x}^\top) \le \mathcal{D}(A, \tilde{A}) + \mathcal{D}(\tilde{A}, \mathbb{1}_n \boldsymbol{x}^\top) < \epsilon$. Therefore, $B_1 \subset \mathcal{U}(\mathbb{1}_n \boldsymbol{x}^\top, \epsilon)$. Moreover, since B_2 is compact, $\tilde{V}(A)$ is lower bounded and $d^+\tilde{V}(A)/dt$ is strictly upper bounded from 0 in B_2. Since $\lim_{t\to+\infty} \mathcal{D}(A(t), \hat{E}) = 0$, there exists $t_0 > 0$ such that $A(t) \in B_1 \cup B_2$ for any $t \ge 0$. Therefore, for any $t \ge t_0$, there exists $t_1 \ge t$ such that $A(t_1) \in B_1$. This argument is valid for any $\epsilon > 0$, which implies that $\mathbb{1}_n \boldsymbol{x}^\top$ is an ω-limit point for any given $A(0)$.

Since $\hat{E}$ is compact, $\mathcal{D}(A, \tilde{E})$ is strictly positive. Since $\lim_{t\to+\infty} \mathcal{D}\big(A(t), \hat{E}\big) = 0$, any $A \in \Omega \setminus \hat{E}$ can not be an ω-limit point of $A(0)$. For any $A \in \hat{E} \setminus \{\mathbb{1}_n \boldsymbol{x}^\top\}$, since the solution passing through A asymptotically converges to $\mathbb{1}_n \boldsymbol{x}^\top$, $A \in \hat{E} \setminus \{\mathbb{1}_n \boldsymbol{x}^\top\}$ can not be an ω-limit point of $A(0)$ either. Therefore, the ω-limit set of $A(0)$ is $\{\mathbb{1}_n \boldsymbol{x}^\top\}$. This concludes the proof. ■

As Theorem 4.4.3 indicates, the team obeying the assign/appraise/influence dynamics achieves collective learning. A visualized illustration of the dynamics is given by Figure 4.3.

Theorem 4.4.3 indicates that the asymptotic behavior of the assign/appraise/influence dynamics is independent of the time scales τ_{ave} and τ_{app}. The following argument adds some intuition to this observation. The assign/appraise/influence dynamics can be regarded a combination of the assign/appraise dynamics (4.5) and the influence dynamics $\dot{A} = A^2$. As shown in Section III, for an appraisal matrix $A(t)$ obeying the assign/appraise dynamics (4.5), the left dominant eigenvector $\boldsymbol{v}_{\text{left}}(A(t))$ converges to the optimal assign-



(a) $t = 0$ (b) $t = 2$ (c) $t = 10$ (d) $t = 30$

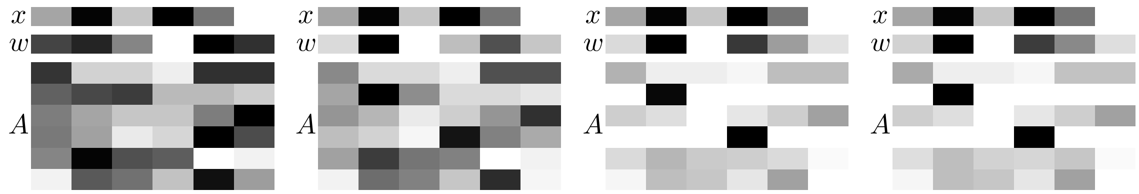
Figure 4.3: Visualization of the evolution of $A(t)$ and $\boldsymbol{w}(t)$ obeying the assign/appraise/influence dynamics with $n = 6$. The observation network contains a globally reachable node. In these visualized matrices and vectors, the darker the entry, the higher value it has.

ment $\boldsymbol{x}$. Moreover, along the dynamics $\dot{A} = A^2$, the eigenvector $\boldsymbol{v}_{\text{left}}(A(t))$ remains unchanged. Theorem 4.4.3 states that the introduction of the influence dynamics does not affect the convergence of the left dominant eigenvector of $A(t)$ to $\boldsymbol{x}$.

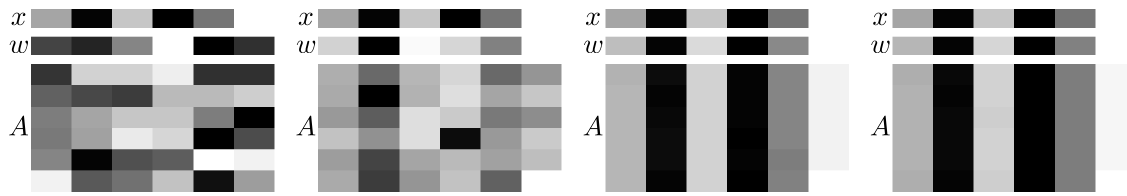
4.5 Model Variations: Causes of Failure to Learn

The assign/appraise/influence dynamics (4.10) consists of three phases: the assignment rule, the appraise dynamics, and the influence dynamics. In this section, we propose one variation in each of the three phases, based on some socio-psychological mechanisms that may cause a failure in team learning. We investigate the behavior of each model variation by numerical simulation.

a) Variation in the assignment rule: workload assignment based on degree centrality: In Assumption 4.3, the workload assignment is based on the individuals' eigenvector centrality in the appraisal network. If we assume instead that the assignment is based on the individuals' normalized in-degree centrality in the appraisal network, i.e., $\boldsymbol{w}(t) = A^{\top}(t)\mathbb{1}_n/\mathbb{1}_n^{\top}A(t)\mathbb{1}_n$, then the numerical simulation, see Figure 4.4, shows the following



(a) no influence dynamics, $t = 0$ (b) no influence dynamics, $t = 2$ (c) no influence dynamics, $t = 30$ (d) no influence dynamics, $t = 50$



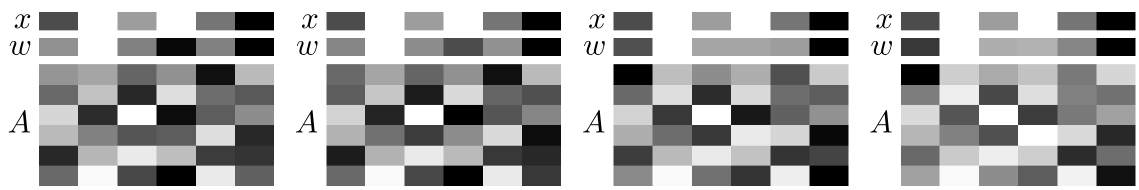
(e) with influence dynamics, $t = 0$ (f) with influence dynamics, $t = 2$ (g) with influence dynamics, $t = 30$ (h) with influence dynamics, $t = 50$

Figure 4.4: Examples of the assign/appraise (first row) and the assign/appraise/influence (second row) dynamics in which the assignment is based on the individuals' in-degree centrality. The assign/appraise dynamics does not achieve the collective learning, while the assign/appraise/influence dynamics does.

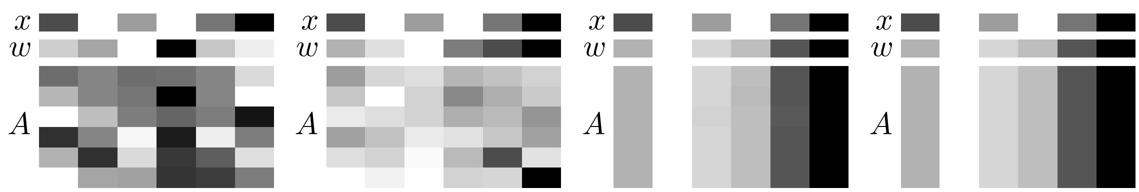
results: the team obeying the assign/appraise dynamics does not necessarily achieve collective learning, while the team obeying the assign/appraise/influence dynamics still achieves collective learning.

b) Variation in the appraise dynamics: partial observation of performance feedback: According to Assumption 4.4, the observation network $G(M)$ determines the feedback signals received by each individual. If the observation network does not have the desired connectivity property, the individuals do not have sufficient information to achieve collective learning. Simulation results in Figure 4.5 shows that, if $G(M)$ is not strongly connected for the assign/appraise dynamics, or if $G(M)$ does not contain a globally reachable node for the assign/appraise/influence dynamics, the team does not necessarily achieve collective learning.

c) Variation in the influence dynamics: prejudice model: In Assumption 4.6, we assume that the individuals obey the DeGroot opinion dynamics. If we instead adopt



(a) no influence dynamics, $t = 0$ (b) no influence dynamics, $t = 1$ (c) no influence dynamics, $t = 5$ (d) no influence dynamics, $t = 10$



(e) with influence dynamics, $t = 0$ (f) with influence dynamics, $t = 5$ (g) with influence dynamics, $t = 50$ (h) with influence dynamics, $t = 60$

Figure 4.5: Examples of failure to learn with partial observation for a six-individual team. The figures in the first row correspond to the assign/appraise dynamics, in which the observation network is not strongly connected but contains a globally reachable node. The figures in the second row correspond to the assign/appraise/influence dynamics, in which the observation network does not contain a globally reachable node. In both cases, $A(t)$ converges but $\lim_{t\to+\infty} \boldsymbol{w}(t) \neq \boldsymbol{x}$.

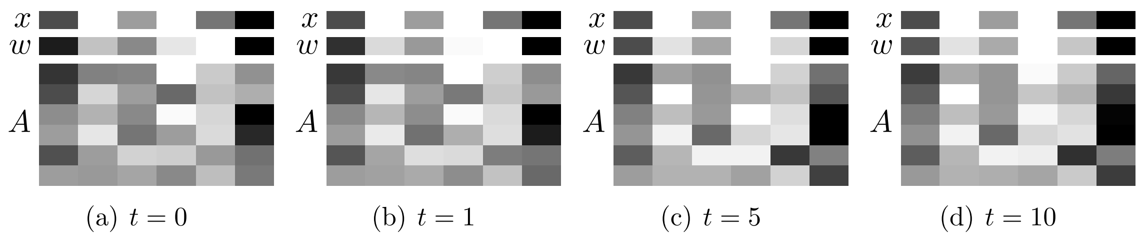


Figure 4.6: Example of the evolution of $A(t)$ and $\boldsymbol{w}(t)$ in the prejudice model with $n = 6$. The darker the entry, the higher value it has. The simulation result shows that $A(t)$ converges but $\boldsymbol{w}(t) = \boldsymbol{v}_{\mathrm{left}}\big(A(t)\big)$ does not necessarily converges to $\boldsymbol{x}$.

the Friedkin-Johnsen opinion dynamics, given by

$$F_{\mathrm{ave}}(A, W) = -\Lambda(I_n - W)A + (I_n - \Lambda)(A(0) - A),$$

where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and each λ_i characterizes individual i's attachment to her initial appraisals. Numerical simulation, see Figure 4.6, shows that the team does not necessarily achieve collective learning. The Friedkin-Johnsen model captures the social-psychological mechanism in which individuals show an attachment to their initial opinions, which causes the failure to learn.

4.6 Further Discussion and Conclusion

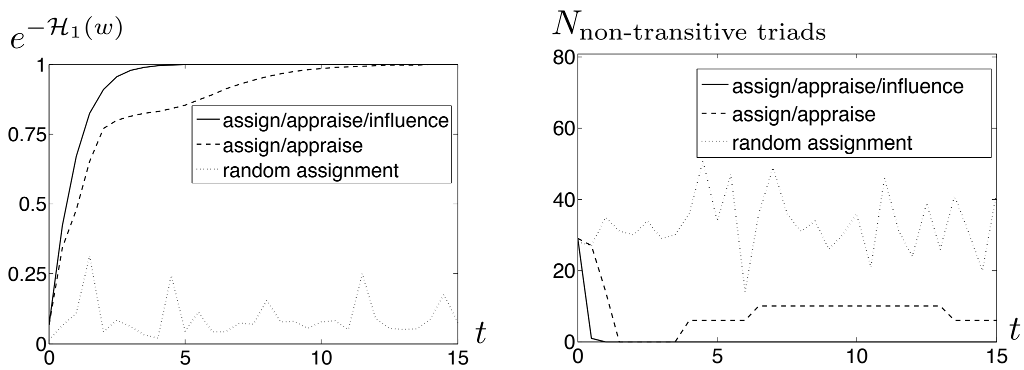
4.6.1 Connections with TMS theory

TMS structure: As discussed in the introduction, one important aspect of TMS is the members' shared understanding about who possess what expertise. For the case of one-dimension skill, TMS structure is approximately characterized by the appraisal matrix and thus the development of TMS corresponds to the collective learning on individuals' true skill levels. Simulation results in Figure 4.7 compare the evolution of some features among the teams obeying the assign/appraise/influence model, the assign/appraise model, and the team that randomly assigns the sub-tasks, respectively. Figure 4.7(a) shows that, for both the assign/appraise/influence dynamics and the assign/appraise dynamics, the team performance measure $\mathcal{H}_1(\boldsymbol{w})$, defined by the mismatch between workload assignment and individual skill levels, converges to 0, which exhibits the advantage of a developing TMS.

Transitive triads: As Palazzolo [134] points out, transitive triads are indicative of a well-organized TMS. The underlying logic is that inconsistency of interpersonal appraisals lowers the efficiency of locating the expertise and allocating the incoming information. In order to reveal the evolution of triad transitivity in our models, we define an unweighted and directed graph, referred to as the *comparative appraisal graph* $\widetilde{G}(A) = (V, E)$, with $V = \{1, \dots, n\}$, as follows: for any $i, j \in V$, $(i, j) \in E$ if $a_{ij} \geq a_{ii}$, i.e., if individual i thinks j has no lower skill level than i herself. We adopt the standard notion of triad transitivity and use the number of non-transitive triads as the indicator of inconsistency in a team. Figure 4.7(b) shows that, the non-transitive triads vanish along the assign/appraise/influence dynamics, but persist along the assign/appraise dynamics or the random assignments.

4.6.2 Observation network structure and learning speed

Simulation results illustrate how the structure of the observation network affects the convergence speeds of our models, characterized by the *convergence time* $T_c = \min\big\{t \geq 0 \,\big|\, e^{-\mathcal{H}_1(\boldsymbol{w}(t))} \geq 0.99\big\}$. T_c is a function of the skill level $\boldsymbol{x}$, the initial condition $A(0)$, and the observation network. We run 100 independent realizations of the assign/appraise dynamics for a team with 7 individuals. In each realization, we first randomly generate $\boldsymbol{x}$ and $A(0)$, and then randomly generate 9 strongly connected observation net-



(a) $e^{-\mathcal{H}_1(\boldsymbol{w},\boldsymbol{x})}$ (b) Number of non-transitive triads

Figure 4.7: Evolution of the measure of mismatch between assignment and individual skill levels, and the number of non-transitive triads in the comparative appraisal graph. The solid curves represent the team obeying the assign/appraise/influence dynamics. The dash curves represent the team obeying the assign/appraise dynamics. The dotted curves represent the team that randomly assign sub-task workloads.

works, $G_1, \dots, G_9$, where each G_i is an Erdős-Rényi graph with the link probability $p_{\text{link},i} = 0.2 + 0.1(i-1)$ and the individuals' out-degrees normalized to 1. With the same $\boldsymbol{x}$ and $A(0)$, we run the assign/appraise dynamics with the observation networks $G_1, \dots, G_9$ respectively, and denote by $T_{c,i}$ the convergence time with respect to the observation network G_i. In each realization, $T_{c,1}, \dots, T_{c,9}$ are scaled by dividing them by $\max_i T_{c,i}$. For the 100 realizations, we compute the mean value of each $T_{c,i}$ and plot it as a function of $p_{\text{link},i}$, see Figure 4.8(a). The same simulation study has also been done for the assign/appraise/influence dynamics, see Figure 4.8(b). Simulation results clearly indicate that, for both the assign/appraise and the assign/appraise/influence dynamics with Erdős-Rényi observation network, the convergence speed increases with the link probability.

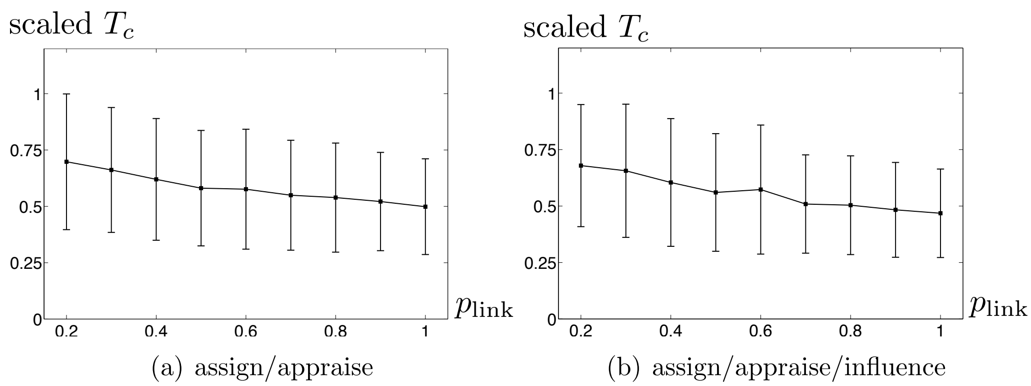


Figure 4.8: The error bar plots for the mean convergence time of 100 random realizations, as a function of the link probability of the Erdős-Rényi observation network. The errors are set to be the standard deviation of the convergence time. Figure 4.8(a) depicts the realizations for the assign/appraise dynamics, while Figure 4.8(b) depicts the realizations for the assign/appraise/influence dynamics.

4.6.3 Justifications of Assumption 4.3

We provide some justification of Assumption 4.3 on the workload assignment rule $\boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A)$. Firstly, the entries of $\boldsymbol{v}_{\text{left}}(A)$ correspond to the individuals' eigenvector centrality in the appraisal network and thus reflect how much each individual is appraised by the team. Secondly, each row i of $A(t)$ can be considered as individual i's opinion on how to divide the workload for the task at time t. Suppose the group of individuals exchange their opinions over the influence network defined by $W = A(t)$ and eventually reach consensus on the workload assignment. We have that the consensus workload assigned to any individual j, denoted by $w_j(t)$, satisfies $w_j(t) = \lim_{k\to\infty} W^k \boldsymbol{A}_j(t) = \mathbb{1}_n \boldsymbol{v}_{\text{left}}(A(t))^\top \boldsymbol{A}_j(t)$, where $\boldsymbol{A}_j(t)$ denotes the j-th column of $A(t)$. Therefore, $\boldsymbol{w}^\top(t) = \boldsymbol{v}_{\text{left}}(A(t))^\top A(t)$, which leads to $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}(A(t))$. Thirdly, our eigenvector assignment rule is consistent with the following natural property: in a team without performance feedback, , due to the lack of information inflow, the team's task assignment does not change. These arguments justify Assumption 4.3; recall also Section 4.5 a) with a numerical evaluation of a different assignment rule.

4.6.4 Discussion on Conjecture 4.4.2

The Monte Carlo method [137] is adopted to estimate the probability that Conjecture 4.4.2 holds. For any randomly generated $A(0) \in \mathrm{int}(\Delta_n)$, define the random variable $Z : \mathrm{int}(\Delta_n) \to \{0, 1\}$ as

(i. $Z\big(A(0)\big) = 1$ if there exists $a_{\min} > 0$ such that $A(t) \succeq a_{min}\mathbb{1}_n\mathbb{1}_n^\top$ for all $t \in [0, 1000]$;

(ii. $Z\big(A(0)\big) = 0$ otherwise.

Let $p = \mathbb{P}\big[Z\big(A(0)\big) = 1\big]$. For N independent random samples $Z_1, \dots, Z_N$, in each of which $A(0)$ is randomly generated in $\mathrm{int}(\Delta_n)$, define $\hat{p}_N = \sum_{i=1}^N Z_i/N$. For any accuracy $1 - \epsilon \in (0, 1)$ and confidence level $1 - \xi \in (0, 1)$, $|\hat{p}_N - p| < \epsilon$ with probability greater than $1 - \xi$ if

$$N \geq \frac{1}{2\epsilon^2} \log \frac{2}{\xi}. \tag{4.11}$$

For $\epsilon = \xi = 0.01$, the Chernoff bound (4.11) is satisfied by $N = 27000$. We run 27000 independent MATLAB simulations of the assign.appraise/influence dynamics with $n = 7$ and find that $\hat{p}_N = 1$. Therefore, for any $A(0) \in \mathrm{int}(\Delta_n)$, with 99% confidence level, there is at least 0.99 probability that $A(t)$ is entry-wise strictly lower bounded from $\mathbb{0}_{n\times n}$ for all $t \in [0, 10000]$.

Moreover, we present in the following lemma a sufficient condition for Conjecture 4.4.2 on the initial appraisal matrix $A(0)$ and the parameters τ_{ave}, τ_{app}.

Lemma 4.6.1 (Strictly positive lower bound of appraisals) *Consider the assign/appraise/influence dynamics* (4.10)*, based on Assumptions 4.3-4.6, with the assignment* $\boldsymbol{w}(t)$ *and performance* $\boldsymbol{p}(\boldsymbol{w})$ *as in Assumptions 4.1 and 4.2 respectively. For any initial appraisal matrix* $A(0)$ *that is entry-wise positive and row-stochastic, as long as*

$$\frac{\tau_{\mathrm{app}}}{\tau_{\mathrm{ave}}} \geq \frac{1-\xi_0}{\xi_0}\left(f\left(\frac{x_{\max}}{\xi_0}\right) - f\left(\frac{x_{\min}}{1-(n-1)\xi_0}\right)\right),$$

where the constant ξ_0 *is defined as in Theorem 4.4.3 (ii), then there exists* $a_{\min} > 0$ *such that* $A(t) \succeq a_{\min}\mathbb{1}_n\mathbb{1}_n^\top$.

Proof: First of all, by definition we have $w_s(t) = \sum_k w_k(t)a_{ks}(t)$. The right-hand side of this equation is a convex combination of $\{a_{1s}(t), \dots, a_{ns}(t)\}$. Therefore, $\max_k a_{ks}(t) \geq w_s(t) \geq \xi_0$ for all $t \in [0, +\infty)$.

At any time $t \geq 0$, for any pair (i,j) such that $a_{ij}(t) = \min_{k,l} a_{kl}(t)$, the dynamics for $a_{ij}(t)$ is

$$\dot{a}_{ij}(t) = \frac{1}{\tau_{\text{ave}}}\Bigg(\sum_k a_{ik}(t)a_{kj}(t) - a_{ij}(t)\Bigg) - \frac{1}{\tau_{\text{app}}}a_{ii}(t)a_{ij}(t)\Big(p_i\big(\boldsymbol{w}(t)\big) - \sum_{k=1}^n m_{ik}p_k\big(\boldsymbol{w}(t)\big)\Big).$$

For simplicity, in this proof, denote $\phi_i = p_i\big(\boldsymbol{w}(t)\big) - \sum_{k=1}^n m_{ik}p_k\big(\boldsymbol{w}(t)\big)$. Suppose $a_{mj}(t) = \max_k a_{kj}(t)$. We have

$$\dot{a}_{ij}(t) \geq \frac{1}{\tau_{\text{ave}}}a_{ij}(t)a_{mj}(t) - \frac{1}{\tau_{\text{ave}}}a_{ij}^2(t) - \frac{1}{\tau_{\text{app}}}a_{ii}(t)a_{ij}(t)\phi_i.$$

Therefore,

$$\frac{\dot{a}_{ij}}{a_{ij}} \geq \frac{1}{\tau_{\text{ave}}}\xi_0 - \frac{1}{\tau_{\text{app}}}(1-\xi_0)\Big(f\big(\frac{x_{\max}}{\xi_0}\big) - f\big(\frac{x_{\min}}{1-(n-1)\xi_0}\big)\Big).$$

The condition on $\frac{1}{\tau_{\text{ave}}}/\frac{1}{\tau_{\text{app}}}$ in Lemma 4.6.1 guarantees that $\dot{a}_{ij}(t)/a_{ij}(t)$ is positive if $a_{ij}(t) = \min_{k,l} a_{kl}(t)$. This concludes the proof. ■

4.6.5 Conclusion

This chapter proposes a class of models closely connected with the TMS theory in organization science. We generalize from qualitative TMS theory the following two arguments, as the staring point of the mathematical modeling: (1) Team performance depends on whether the team members' relative responsibilities are proportional to their relative abilities in the team; (2) The team members' relative responsibilities are determined by how they evaluate each other's relative ability. Theoretical analysis of the assign/appraise dynamics and the assign/appraise influence dynamics can be interpreted as the exploration of the most relaxed condition for the convergence to optimal workload assignment, concluded as follows: (i) Each individual only needs to know, as feedback, the difference between her own performance and the average performance of some subgroup of individuals, but do not need to know exactly whom she is compared with; (ii) The individuals can have heterogeneous but strictly positive sensitivities to the performance feedback; (iii) With opinion exchange, the observation network with one globally reachable node is sufficient for the convergence to optimal assignment; (iv) Without opinion exchange, strongly connected observation network is sufficient for the convergence to optimal assignment.

The theoretical results in this chapter can be broadly interpreted as follows. First, we note that the connectivity requirement on the observation network for asymptotic optimal assignment is more relaxed in the assign/appraise/influence model than in the assign/appraise model. Therefore, our models lend credence to the argument that opinion exchanges inside the group can compensate for the lack of sufficiently-rich observation of performance feedback. Second, the numerical comparison between the assignment simply by the average appraisals, i.e., $\boldsymbol{w}(t) = \mathbb{1}_n^\top A(t)/n$, and the assignment by the appraisal centrality, i.e., $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}(A(t))$, shows that the former does not always leads to asymptotic optimal assignment in the assign/appraise dynamics, while the latter does. The main difference between these two assignment rules is that, for the assignment by the appraisal centrality, the opinions of the "highly-appraised individuals" on how the workload should be assigned are more important than those of the "lowly appraised," whereas, for the assignment by the average appraisals, all the individuals' opinions are equally important. The interpretation of this observation is that, in a well-functioning team, the individuals with higher appraisal should have higher weights in decision-making processes. Third, as illustrated by the numerical study of the causes of failure to learn, our models indicate that individuals' persistent attachment to their initial appraisals, i.e., prejudice, generally impedes collective learning and thus should be avoided in team tasks.

Future research directions might include more realistic models considering noisy observation and finite individual memory.

Chapter 5

Dynamics Structural Balance via Homophily and Influence Mechanisms

5.1 Introduction

Motivation and problem description

Social systems involving friendly/antagonistic relationships between their members are often modeled as signed networks. *Social balance* (also referred to as *structural balance*) theory, which originated from several seminal works by Heider [138, 139], characterizes the stable configurations of signed social networks. According to the classic social balanced theory [138, 139], in a balanced network, the interpersonal relationships satisfy the four famous Heider's axioms: "The friend of my friend is my friend; the friend of my enemy is my enemy; the enemy of my friend is my enemy; the enemy of my enemy is my friend." While classic studies of social balance focus mainly on the static theory (i.e., the local and global configurations of balanced networks), dynamic social balance theory has attracted much recent interest. In short, dynamic social balance theory aims to explain if and how an initially unbalanced network evolves to a balanced network. Despite recent progress, it remains a valuable open problem to propose dynamic models that are based on natural assumptions and that enjoy desirable boundedness and convergence properties. Such models make it possible for researchers to formulate and study meaningful predictions and control/intervention strategies for the evolution of the social network.

In this chapter, we propose two novel discrete-time dynamic models describing the evolution of the interpersonal appraisals towards social balance. For both models, we consider a group of individuals who repeatedly update their interpersonal appraisals via two socio-psychological mechanisms respectively: the homophily mechanism and the influence mechanism. The homophily mechanism means that individuals in a group tend to be friendly to each other if their appraisals of the group members are in agreement (in the sense of signs), and vice versa. On the other hand, the influence mechanism defines an influence process, in which each individual updates its appraisals by assigning positive or negative influences to all the group members. The interpersonal influences are assumed proportional to the corresponding interpersonal appraisals. For both models, our objectives are to characterize their fixed points and their dynamical behavior, with a special emphasis on boundedness and convergence properties.

The homophily and influence mechanisms are both well established in the social sciences literature; they have been studied separately in different contexts, e.g., see the seminal work by Lazarsfeld and Merton [140] on the homophily mechanism and the award-winning book by Friedkin and Johnsen on the influence mechanism [141], respectively. These two mechanisms are not necessarily mutually exclusive: in reality, it can be argued that they simultaneously play a role in shaping the evolution of a social network, through surely to varying and distinct degree. It is an open question beyond the scope of this chapter to determine conditions under which one phenomenon dominates the other.

Literature review

Following the early works by Heider [138, 139], static social balance theory has been extensively studied in the last seven decades. Theoretical studies include the characterization of the structurally balanced configurations for both complete networks [142, 143] and arbitrary networks [144, 37]; the measure of the degree of balance [145, 146]; the concept of clustering and its relation to balance [147, 148]; as well as the partitioning algorithms that cut a signed network into multiple clusters [149, 150, 151]. In addition to the theoretical contributions, numerous empirical studies have been conducted for different types of social systems, including social systems at the national level [152, 153], at the group level [154, 155], and at the individual level [156].

In the last decade, researchers have started to incorporate dynamical systems into the social balance theory, aiming to explain how a signed network evolves to a structurally balanced state. Early works include the discrete-time *local triad dynamics* and *constrained triad dynamics* on complete graphs, proposed by Antal et al. [157, 158]. These two models do not always converge to social balance as they suffer from the existence of so-called *jammed states*, i.e., unbalanced equilibria. Radicchi et al. [159] extend the LTD model to arbitrary graphs. Van De Rijt [160] proposes a network game model in which each individual minimizes the number of the unbalanced triads involving itself, by changing the signs of its out-links; this model evolves to social balance if each individual is allowed to change the signs of multiple links simultaneously. A similar network game model, allowing the adding and deleting of links, is proposed by Malekzadeh et al. [161]. In all the models introduced in this paragraph, the link weights in the signed networks are assumed to only take values from the set $\{-1, 0, 1\}$.

Our models are related to the continuous-time dynamic social balance models studied by Kułakowski et al. [105], Marvel et al. [106], and Traag et al. [162], as well as the discrete-time model proposed by Jia et al. [107]. In these models, the link weights can take arbitrary real values. The model proposed by Kułakowski et al. [105] is based on an influence-like mechanism. The theoretical analysis of this model by Marvel et al. [106] and Traag et al. [162] reveals that, from a specific set of initial conditions, the system first reaches a structurally balanced state and then diverges to unbounded interpersonal appraisals in finite time. In [105], the authors also modify their original model by imposing a predetermined upper bound R of the interpersonal appraisals so that the evolution of the system remains bounded. Rigorous analysis by Wongkaew et al. [163] shows that, in the modified model, the interpersonal appraisals achieve social balance in finite time and the magnitudes of all appraisals converge to the predetermined upper bound R, if the initial appraisals are all lower bounded from $-R$. Traag et al. [162] propose and analyze a continuous-time model based on the homophily mechanism. Similar to the first model proposed by Kułakowski et al. [105], the homophily-based model also reaches social balance and then diverges to unbounded interpersonal appraisals in finite time. Recently, Jia et al. [107] propose and analyze a discrete-time model based on the relaxation of the classic Heider's social balance theory and a modified influence mechanism with convergence to a generalized notion of structural stability.

Contributions

The contributions of this chapter are manifold. Firstly, we propose two novel discrete-time dynamic social balance models and establish their well-posedness and bounded evolution properties. These two models explain the evolution of the interpersonal appraisal networks towards the classic Heider's social balance [138] via two sociologically-grounded processes respectively: the homophily mechanism and the influence mechanism. In both models, the appraisal networks are represented by their associated adjacency matrices, i.e., the appraisal matrices. For the homophily-based model, we prove that, the appraisal matrix is well-defined and uniformly bounded at any time, if each row of the initial appraisal matrix has at least one nonzero item. For the influence-based model, we prove that the well-posedness and bounded evolution are guaranteed if the initial appraisal matrix is a symmetric matrix left multiplied by the diagonal matrix with positive diagonal entries. In addition, both our models are invariant under scaling, i.e., if a solution is scaled by a constant, then it remains a solution.

Secondly, we fully characterize the equilibrium sets and the asymptotic behavior of both models. The analyses of the two models are performed in analogous ways. We prove that, for both models, any appraisal network in the equilibrium state is composed of an arbitrary number of isolated subgraphs, each of which satisfies social balance. Moreover, we prove that, for the homophily-based model, in each of such subgraphs, all individuals' appraisals have the same magnitude, while, for the influence-based model, in each subgraph of the equilibrium appraisal network, the individuals reach consensus, in the sense of magnitude, on the appraisals of each individual. Finally, for both models, under a technical condition, we establish the convergence of the appraisal networks to structurally balanced complete graphs.

Thirdly, in addition to the comprehensive theoretical analysis, we further investigate our models by numerical simulations. We provide numerical evidence that our technical condition for the convergence of the appraisal networks to balanced complete graphs holds for generic initial conditions. Moreover, for the homophily-based model, numerical results on the emergence of multi-clique social balance states and their behavior under perturbations reveals some realistic and strategic insights on the escalation and mediation of conflicts. Finally, we numerically investigate the effect of the initial appraisal distribution on the formation of factions, that is, whether an appraisal network eventually evolves to two antagonistic factions or an all-friendly network.

In summary, our paper is the first to propose discrete-time dynamic social balance models, for both the homophily and influence mechanisms, and to establish, through a comprehensive theoretical analysis, that the evolution of appraisals is bounded and convergent from generic initial conditions in appropriate sets. Compared with the continuous-time homophily-based and influence-based models analyzed in [162] and [106] respectively, our models enjoy the desirable property of convergent appraisals, (as opposed to the undesirable property of finite-time divergence). Compared with the model proposed in [105] with bounded evolution, (1) our models do not rely on any predetermined bound to prevent divergence and (2) the asymptotic appraisals in our models are determined by the initial condition rather than the predetermined bound.

Organization

The rest of the paper is organized as follows. Section 5.2 introduces some notations and basic concepts. Section 5.3 and Section 5.4 are the theoretical analyses of our homophily-based and influence-based models respectively. Section 5.5 provides further discussions and numerical results. Section 5.6 gives the conclusion.

5.2 Notations and basic concepts

5.2.1 Notations

Let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers, respectively. For any $X \in \mathbb{R}^{m\times n}$, denote by X_{ij} the (i,j)-th entry of X. Let $|X|$ denote the entry-wise absolute value of X, i.e., each (i, j)-th entry of $|X|$ is equal to $|X_{ij}|$. Let $\text{sign}(X) \in \{-1, 0, +1\}^{m\times n}$ denote the entry-wise sign of X, i.e., for any i and j, $\text{sign}(X_{ij}) = +1$ when $X_{ij} > 0$, $\text{sign}(X_{ij}) = -1$ when $X_{ij} < 0$ and $\text{sign}(X_{ij}) = 0$ when $X_{ij} = 0$. Define the *max norm* of X by $\|X\|_{\max} = \max_{i,j} |X_{ij}|$. Let X_{i*} (X_{*i} resp.) denote the row (column resp.) vector corresponding to the i^{th} row (column resp.) of the matrix X. Let $G(X)$ denote the directed and weighted graph associated with the adjacency matrix X.

Note that, unlike in the traditional definition of weighted graphs, in this chapter we allow the presence of links with negative weights. That is, if $X_{ij} < 0$ for some i and j, then the directed link (i, j) in graph $G(X)$ has negative weight equal to X_{ij}. We assume that there is no link from i to j whenever $X_{ij} = 0$. The terms graph and network are assumed interchangeable.

The following sets will be used throughout this chapter:

$$\mathcal{S}_{\text{nz-row}} = \{X \in \mathbb{R}^{n\times n} \mid \text{for every } i, X_{i*} \neq \mathbb{0}_n^\top\}, \tag{5.1}$$

$$\mathcal{S}^+_{\text{s-symm}} = \{X \in \mathbb{R}^{n\times n} \mid \text{sign}(X) = \text{sign}(X)^\top \text{ and } X_{ii} > 0 \text{ for every } i\}, \tag{5.2}$$

$$\mathcal{S}^+_{\text{rs-symm}} = \{X \in \mathcal{S}^+_{\text{s-symm}} \mid \text{there exists } \gamma \succ \mathbb{0}_n \text{ such that } \text{diag}(\gamma)X = X^\top \text{diag}(\gamma)\}, \tag{5.3}$$

$$\mathcal{S}^+_{\text{symm}} = \{X \in \mathcal{S}^+_{\text{s-symm}} \mid X = X^\top\}. \tag{5.4}$$

In other words, $\mathcal{S}_{\text{nz-row}}$ is the set of matrices with at least one non-zero entry in each row, while $\mathcal{S}^+_{\text{s-symm}}$ is the set of sign-symmetric matrices with positive diagonals. Simple calculations show that any $X \in \mathcal{S}^+_{\text{rs-symm}}$ can be written as the product of a diagonal matrix with positive diagonal entries with a symmetric matrix with positive diagonal entries. By definition, we have

$$\mathcal{S}^+_{\text{symm}} \subset \mathcal{S}^+_{\text{rs-symm}} \subset \mathcal{S}^+_{\text{s-symm}} \subset \mathcal{S}_{\text{nz-row}}.$$

In addition, one can easily check by definition that, the sets $\mathcal{S}^+_{\text{s-symm}}$, $\mathcal{S}^+_{\text{rs-symm}}$, and $\mathcal{S}^+_{\text{symm}}$ are all invariant under permutations. That is, given any $X \in \mathcal{S}^+_{\text{s-symm}}$ (or $X \in \mathcal{S}^+_{\text{rs-symm}}$ and $X \in \mathcal{S}^+_{\text{symm}}$ resp.) and a permutation matrix P, we have $PXP^\top \in \mathcal{S}^+_{\text{s-symm}}$ (or $PXP^\top \in \mathcal{S}^+_{\text{rs-symm}}$ and $PXP^\top \in \mathcal{S}^+_{\text{symm}}$ resp.).

5.2.2 Appraisal matrices and social balance

Given a group of n agents, the network of interpersonal appraisals among the agents is given by the *appraisal matrix* $X \in \mathbb{R}^{n\times n}$. The sign of X_{ij} determines whether agent i's appraisal of j is positive, i.e., i "likes" j, or negative, i.e., i "dislikes" j. The magnitude of X_{ij} represents the intensity of the sentiment. When $X_{ij} = 0$, the appraisal is one of indifference. The diagonal entry X_{ii} represents agent i's self-appraisal. The directed and weighted graph $G(X)$ associated to X as the adjacency matrix is referred to as the *appraisal network*.

Social balance is a specific property of complete appraisal networks, defined as follows.

Definition 5.2.1 (social balance [142, 139]) *An appraisal network $G(X)$ satisfies social balance, or, equivalently, is structurally balanced, if the associated appraisal matrix is such that all of its entries are non-zero and the following conditions are satisfied for all $i, j, k \in \{1, \dots, n\}$:*

(i. positive self-appraisals: $X_{ii} > 0$,

(ii. positive triads: $\operatorname{sign}(X_{ij}) \operatorname{sign}(X_{jk}) \operatorname{sign}(X_{ki}) = 1$.

Proposition 5.2.2 (Equivalent conditions for social balance) *For any $X \in \mathbb{R}^{n\times n}$ such that all of its entries are non-zero, $G(X)$ satisfies social balance if and only if it satisfies (i and* *(iii.* $\mathrm{sign}(X_{i*}) = \pm\,\mathrm{sign}(X_{j*})$*, for all* $i, j \in \{1, \dots, n\}$ *.*

Moreover, for $G(X)$ *satisfying social balance,* X *is sign-symmetric, i.e.,* $\mathrm{sign}(X) = \mathrm{sign}(X)^\top$*.*

Proof: Suppose that (i and (iii hold. For any $i, j \in \{1, \dots, n\}$, we have $\mathrm{sign}(X_{i*}) = \delta\,\mathrm{sign}(X_{j*})$, where δ is either -1 or 1. Therefore, $\mathrm{sign}(X_{ij})\,\mathrm{sign}(X_{ji}) = \delta^2\,\mathrm{sign}(X_{jj})\,\mathrm{sign}(X_{ii}) = 1$, i.e., $\mathrm{sign}(X_{ij}) = \mathrm{sign}(X_{ji})$. Moreover, for any k, since $\mathrm{sign}(X_{ij}) = \delta\,\mathrm{sign}(X_{jj})$ and $\mathrm{sign}(X_{jk}) = \delta\,\mathrm{sign}(X_{ik})$, we have

$$\mathrm{sign}(X_{ij})\,\mathrm{sign}(x_{jk})\,\mathrm{sign}(X_{ki}) = \delta^2\,\mathrm{sign}(X_{jj})\,\mathrm{sign}(X_{ik})\,\mathrm{sign}(X_{ki}) = 1.$$

Therefore, (i and (iii imply (i and (ii, as well as the sign symmetry of X.

Now suppose (i and (ii hold. The sign symmetry of X is obtained by letting $k = j$ in (ii. Moreover, due to the sign symmetry and (ii, we obtain $\mathrm{sign}(X_{ij})\,\mathrm{sign}(X_{jk})\,\mathrm{sign}(X_{ik}) = 1$, which in turn implies that $\mathrm{sign}(X_{ik})\,\mathrm{sign}(X_{jk})$ does not depend on k and is equal to $\mathrm{sign}(X_{ij}) \in \{-1, 1\}$. Therefore, $\mathrm{sign}(X_{i*}) = \pm\,\mathrm{sign}(X_{j*})$ for any i and j. This concludes the proof.

■

According to [142], a structurally balanced appraisal network either has only one *faction* in which the interpersonal appraisals are all positive, or is composed of two antagonistic factions such that individuals in the same faction positively appraise each other while all the inter-faction appraisals are negative.

5.3 Homophily-based Model

In this and the next sections, we propose and analyze two dynamic social balance models respectively. These two models are distinct in the microscopic individual interaction mechanisms. In this section, we propose our first model: the *homophily-based model* (HbM), and analyze its dynamical behavior.

Definition 5.3.1 (Homophily-based model) *Given an initial appraisal matrix* $X(0) \in \mathcal{S}^+_{\text{s-symm}} \subset \mathbb{R}^{n\times n}$*, the homophily-based model is defined by:*

$$X(t+1) = \operatorname{diag}(|X(t)|\mathbb{1}_n)^{-1}X(t)X^\top(t). \tag{5.5}$$

Remark 5.3.2 (Interpretation) *Equation* (5.5) *updates the appraisals based on what can be considered as the homophily mechanism. For any* $i, j \in \{1, \dots, n\}$*, agent* i*'s appraisal of agent* j *at time step* $t+1$ *depends on to what extend they are in agreement with each other on the appraisals of all the agents in the group. For any* $k \in \{1, \dots, n\}$*, if* $\operatorname{sign}(X_{ik}(t)) = \operatorname{sign}(X_{jk}(t))$*, then the term* $X_{ik}(t)X_{jk}(t)$ *contributes positively to* $X_{ij}(t+1)$*, and vice versa.*

The proposition below presents some useful results on the finite-time behavior of the homophily-based model.

Proposition 5.3.3 (Invariant set and finite-time behavior of HbM) *Consider the dynamical system* (5.5) *and define* $f_{\text{homophily}}(X) = \operatorname{diag}(|X|\mathbb{1}_n)^{-1}XX^\top$*. Pick* $X_0 \in \mathcal{S}^+_{\text{s-symm}}$*. The following statements hold:*

(i. the map $f_{\text{homophily}}$ *is well-defined for any* $X \in \mathcal{S}_{\text{nz-row}}$ *and maps* $\mathcal{S}^+_{\text{s-symm}}$ *to* $\mathcal{S}^+_{\text{s-symm}}$*;*

(ii. the solution $X(t)$*,* $t \in \mathbb{Z}_{\geq 0}$*, to equation* (5.5) *from initial condition* $X(0) = X_0$ *is unique and well-defined;*

(iii. the max norm of any solution $X(t)$ *satisfies*

$$\|X(t+1)\|_{\max} \leq \|X(t)\|_{\max} \leq \|X(0)\|_{\max};$$

(iv. *for any* $c \neq 0$*, the trajectory* $cX(t)$ *is the solution to equation* (5.5) *from initial condition* $X(0) = cX_0$.

Proof: For simplicity, denote $X^+ = f_{\text{homophily}}(X)$. For any $X \in \mathcal{S}_{\text{nz-row}}$, since, for any i and j, $X^+_{ij} = \frac{1}{\|X_{i*}\|_1} \sum_k X_{ik} X_{jk}$ and $\|X_{i*}\|_1 > 0$, $f_{\text{homophily}}(X)$ is well-defined. Moreover,

$$X^+_{ii} = \frac{1}{\|X_{i*}\|_1} \sum_k X_{ik} X_{ik} = \frac{\|X_{i*}\|_2^2}{\|X_{i*}\|_1} > 0, \quad \text{and}$$
$$X^+_{ij} = \frac{\|X_{j*}\|_1}{\|X_{i*}\|_1} X^+_{ji}, \quad \text{for any } i \text{ and } j.$$

Therefore, $f_{\text{homophily}}$ maps $\mathcal{S}^+_{\text{s-symm}}$ to $\mathcal{S}^+_{\text{s-symm}}$. This concludes the proof of statement (i. Statements (ii is a direct consequence of statement (i. In addition,

$$|X^+_{ij}| \leq \frac{1}{\|X_{i*}\|_1} \sum_{k=1}^{n} \big|X_{ik} X_{jk}\big| \leq \frac{1}{\|X_{i*},\|_1} \sum_{k=1}^{n} \big|X_{ik}\big|\big|X_{jk}\big| \leq \max_k |X_{jk}| \leq \|X\|_{\max}$$

immediately leads to statement (iii. Finally, statement (iv is obtained by replacing $X(t)$ with $cX(t)$ on the right-hand side of equation (5.5). ■

In fact, for any $X(0) \in \mathcal{S}_{\text{nz-row}}$, we have $X(1) \in \mathcal{S}^+_{\text{s-symm}}$ and, thus, $X(t) \in \mathcal{S}^+_{\text{s-symm}}$ for any $t \geq 1$. Therefore, the set of initial conditions of the system can be extended to the set $\mathcal{S}_{\text{nz-row}}$. However, without loss of generality, we still consider $\mathcal{S}^+_{\text{s-symm}}$ as the domain of system (5.5). In addition, according to Proposition 5.3.3, for any $X(0) \in \mathcal{S}^+_{\text{s-symm}}$, the solution $X(t)$ to equation (5.5) is uniformly upper bounded for all $t \in \mathbb{Z}_{\geq 0}$. This is a desired property compared with some previous models, in which $X(t)$ diverge in finite time [106, 162].

The theorem below characterizes the set of fixed points of system (5.5).

Theorem 5.3.4 (Fixed points and balance) *Consider the dynamical system* (5.5) *in* *domain $\mathcal{S}^+_{\text{s-symm}}$. Define*

$$Q_{\text{homophily}} = \Big\{ PYP^\top \in \mathcal{S}^+_{\text{s-symm}} \,\Big|\, P \text{ is a permutation matrix,} \\ Y \text{ is a block diagonal matrix with blocks of} \\ \text{the form } \alpha bb^\top,\ \alpha > 0, b \in \{-1,+1\}^m,\ m \le n \Big\}.$$

Then we have that,

(i. $Q_{\text{homophily}}$ is the set of all the fixed points of (5.5)*,*

(ii. for any $X \in Q_{\text{homophily}}$, $G(X)$ is composed by isolated complete subgraphs that satisfy social balance.

Proof: We first prove that any $X^* \in Q_{\text{homophily}}$ is a fixed point of system (5.5). For any $\alpha > 0$ and $b \in \{-1,+1\}^n$, the matrix $Y = \alpha bb^\top$ satisfies

$$f_{\text{homophily}}(Y) = \text{diag}(n\alpha\mathbb{1}_n)^{-1}\alpha^2 bb^\top bb^\top = \alpha bb^\top = Y.$$

Now suppose that Y is a block diagonal matrix, i.e., $Y = \text{diag}(Y^{(1)}, \dots, Y^{(K)})$, where each $Y^{(i)}$ is a $n_i \times n_i$ matrix of the form $\alpha_i b^{(i)} {b^{(i)}}^\top$, with $\alpha_i > 0$, $b^{(i)} \in \{-1,+1\}^{n_i}$, and $n_1 + \dots + n_K = n$. One can check that, as long as

$$Y^{(i)} = \text{diag}(|Y^{(i)}|\mathbb{1}_n)^{-1} Y^{(i)} {Y^{(i)}}^\top \tag{5.6}$$

for any $i \in \{1, \dots, K\}$, Y is a fixed point of system (5.5). Since $Y^{(i)} = \alpha_i b^{(i)} {b^{(i)}}^\top$, we know that equation (5.6) is satisfied for any i. Therefore, Y is a fixed point of system (5.5). Moreover, given any fixed point Y of system (5.5), for any permutation matrix $P \in \mathbb{R}^{n\times n}$, we have

$$\begin{aligned} PYP^\top &= P\,\mathrm{diag}(|Y|\mathbb{1}_n)^{-1}YY^\top P^\top \\ &= \mathrm{diag}(|PYP^\top|\mathbb{1}_n)^{-1}(PYP^\top)(PYP^\top)^\top \\ &= f_{\text{homophily}}(PYP^\top). \end{aligned}$$

Therefore, any $X^* \in Q_{\text{homophily}}$ is a fixed point of (5.5).

Now we prove by induction that $Q_{\text{homophily}}$ is the set of all the fixed points of system (5.5). For the trivial case of $n = 1$, $Q_{\text{homophily}}$ represents the set of all the positive scalars and one can easily check that any positive scalar X is a fixed point of system (5.5) with $n = 1$. Suppose statement (i holds for any system with dimension $\tilde{n} < n$.

For system (5.5) with dimension n, suppose X is a fixed point, i.e., $X = f_{\text{homophily}}(X)$, which implies that,

$$X_{ij} = \frac{1}{\|X_{i*}\|_1}\sum_{k=1}^{n} X_{ik}X_{jk} = \frac{\|X_{j*}\|_1}{\|X_{i*}\|_1}X_{ji}, \text{ for any } i \neq j.$$

Therefore, for any $i, j \in \{1, \dots, n\}$ and $j \neq i$, we have $\mathrm{sign}(X_{ij}) = \mathrm{sign}(X_{ji})$. In addition, since $X_{ii} = \sum_{k=1}^{n} X_{ik}^2/\|X_{i*}\|_1$, we have $X_{ii} > 0$ for any i.

Let X^+ denote $f_{\text{homophily}}(X)$ for simplicity. Since $X = X^+$, we have $\|X\|_{\max} = \|X^+\|_{\max}$, which implies that there exists i, j such that $|X^+_{ij}| = \|X\|_{\max}$. We discuss two cases which together include all the possible X's.

Case 1: there exists i such that $|X_{ii}| = \|X\|_{\max}$ and $|X_{ij}| = 0$ for any $j \neq i$. Since $\mathrm{sign}(X_{ij}) = \mathrm{sign}(X_{ji})$, we have $X_{ji} = 0$ for any $j \neq i$. In addition, since $X_{ii} > 0$,

$X_{ii} = \|X\|_{\max}$. Therefore, there exists a permutation matrix P such that

$$PXP^\top = \begin{bmatrix} \|X\|_{\max} & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{X}_{(n-1)\times(n-1)} \end{bmatrix},$$

Since $PXP^\top$ is also a fixed point of system (5.5), one can check that $\tilde{X}$ satisfies $\tilde{X} = \operatorname{diag}(|\tilde{X}|\mathbb{1}_n)^{-1}\tilde{X}\tilde{X}^\top$. Therefore, $\tilde{X}$ is a fixed point of system (5.5) with dimension $n-1$. Since we have assumed that statement (i holds for dimension $\tilde{n} < n$, there exists an $(n-1)\times(n-1)$ permutation matrix $\tilde{P}$ and a block diagonal $\tilde{Y}$, with blocks of the form $\alpha bb^\top$, where $\alpha > 0$, $b \in \{-1,+1\}^m$, $m < n-1$, such that $\tilde{X} = \tilde{P}\tilde{Y}\tilde{P}^\top$. Therefore,

$$X = P^\top \begin{bmatrix} 1 & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{P} \end{bmatrix} \begin{bmatrix} \|X\|_{\max} & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{Y} \end{bmatrix} \begin{bmatrix} 1 & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{P} \end{bmatrix}^\top P.$$

The matrix

$$P^\top \begin{bmatrix} 1 & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{P} \end{bmatrix}$$

is also a permutation matrix. Therefore $X \in Q_{\text{homophily}}$.

Case 2: there exists $j \neq i$ such that $|X^+_{ij}| = \|X\|_{\max}$. We first define some notations used in the following proof: For any k, let $\theta_k = \{\ell \mid X_{k\ell} \neq 0\}$ and $|\theta_k|$ be the cardinality of the set θ_k. Note that, since $X = f_{\text{homophily}}(X) \in \mathcal{S}^+_{\text{s-symm}}$, k is always in θ_k and $X_{kk} > 0$. Let $X_{\ell *,\theta_k} \in \mathbb{R}^{1\times|\theta_k|}$ be the ℓ-th row vector of X with all the $X_{\ell p}$ entries such that $p \notin \theta_k$ removed.

We point out a general result that, for any k and ℓ, if

$$|X^+_{k\ell}| = \frac{1}{\|X_{k*}\|_1}\Big|\sum_{p=1}^{n} X_{kp}X_{\ell p}\Big| = \|X\|_{\max},$$

then, for the second equality to hold, X must satisfy that: 1) $\theta_k \subset \theta_l$; 2) $|X_{\ell p}| = \|X\|_{\max}$ for any $p \in \theta_k$; 3) $\operatorname{sign}(X_{\ell *, \theta_k}) = \pm \operatorname{sign}(X_{k*,\theta_k})$. Therefore, for the i, j indexes such that $|X^+_{ij}| = \|X\|_{\max}$ and $i \neq j$, we have: $|X_{jk}| = \|X\|_{\max}$, for any $k \in \theta_i$; $\theta_i \subset \theta_j$; and $\operatorname{sign}(X_{j*,\theta_i}) = \pm \operatorname{sign}(X_{i*,\theta_i})$. Since $i \in \theta_i$ and $X^+ = X$, we obtain $|X^+_{ji}| = |X_{ji}| = \|X\|_{\max}$. Therefore, $|X^+_{ik}| = |X_{ik}| = \|X\|_{\max}$, for any $k \in \theta_j$, and $\theta_j \subset \theta_i$, which in turn leads to $\theta_i = \theta_j$ and $|X^+_{ik}| = |X_{ik}| = \|X\|_{\max}$ for any $k \in \theta_i$. Therefore, for any $k \in \theta_i$, $|X^+_{ik}| = \|X\|_{\max}$, which implies $|X_{k\ell}| = \|X\|_{\max}$ for any $l \in \theta_i$. Since $|X^+_{k\ell}| = |X_{k\ell}|$, we further obtain that $\theta_k \subset \theta_l$ and $\operatorname{sign}(X_{k*,\theta_k}) = \pm \operatorname{sign}(X_{\ell *,\theta_k})$. Moreover, due to the fact that the indexes k and l are interchangeable, we conclude that, for any $k, l \in \theta_i$: a) $\theta_k = \theta_l = \theta_i$; b) $|X_{k\ell}| = \|X\|_{\max}$; c) $\operatorname{sign}(X_{k*}) = \pm \operatorname{sign}(X_{\ell *})$.

If $|\theta_i| = n$, let $\alpha = X_{11}$ and $b = \operatorname{sign}(X_{1*})^\top$, then we have $X = \alpha b b^\top$. If $|\theta_i| < n$, there exists a permutation matrix P such that

$$PXP^\top = \begin{bmatrix} X^{(\theta_i)} & \mathbb{0}_{|\theta_i| \times (n - |\theta_i|)} \\ \mathbb{0}_{(n-|\theta_i|)\times|\theta_i|} & \tilde{X} \end{bmatrix},$$

where $X^{(\theta_i)}$ is a $|\theta_i| \times |\theta_i|$ matrix. Moreover, $X^{(\theta_i)} = \|X\|_{\max} b b^\top$, where $b = \operatorname{sign}(X_{i*,\theta_i})^\top$. Following the same line of argument for Case 1, we know that $\tilde{X}$ is of the form $\tilde{P}\tilde{Y}\tilde{P}^\top$ and thereby $X \in Q_{\text{homophily}}$. This concludes the proof for statement (i.

For any $X^* \in Q_{\text{homophily}}$, there exists a permutation matrix P and a block diagonal matrix $Y = \operatorname{diag}(Y^{(1)}, \dots, Y^{(K)})$ such that $X^* = PYP^\top$. Note that $G(Y)$ has exactly the same topology as $G(X)$, but with the nodes reindexed. Therefore, we only need to analyze the structure of $G(Y)$. One can easily check that Y has positive diagonals (and is also sign-symmetric). Moreover, $G(Y)$ is made up of K isolated complete subgraphs. Therefore, for any triad (j, k, ℓ) in $G(Y)$, there exists $i \in \{1, \dots, K\}$ such that nodes j, k, ℓ are all in the subgraph $G(Y^{(i)})$ with the adjacency matrix $Y^{(i)} = \big(Y^{(i)}_{jk}\big)_{n_i \times n_i}$. Suppose

$Y^{(i)} = \alpha_i b^{(i)} b^{(i)\top}$, where $b^{(i)} = (b_1^{(i)}, \dots, b_{n_i}^{(i)})^\top$. We have $Y_{jk}^{(i)} Y_{k\ell}^{(i)} Y_{\ell j}^{(i)} = \alpha_i^3 {b_j^{(i)}}^2 {b_k^{(i)}}^2 {b_\ell^{(i)}}^2 > 0$. Therefore, every triad is positive in graph $G(Y)$. We conclude that any $X^* \in Q_{\text{homophily}}$ is associated with a graph $G(X^*)$ composed by isolated complete subgraphs that satisfy social balance. This concludes the proof for statement (ii. ■

Remark 5.3.5 *Since X being a fixed point of $f_{\text{homophily}}$ implies that X is sign-symmetric and has positive diagonal, $Q_{\text{homophily}}$ is actually the set of all the fixed points of $f_{\text{homophily}}$ in $\mathcal{S}_{\text{nz-row}}$.*

Now we present the main results on the convergence of the appraisal matrix to social balance.

Theorem 5.3.6 (Convergence and social balance in HbM) *Consider the homophily-based model given by equation* (5.5)*. The following statements hold:*

(i) Each fixed point of rank one in $Q_{\text{homophily}}$ is locally stable.

For any $X(0) \in \mathcal{S}^+_{\text{s-symm}}$ such that $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$, we have that:

(ii) there exists $X^* \in Q_{\text{homophily}}$ of rank one such that $\lim_{t\to\infty} X(t) = X^*$, and

(iii) there exists $T > 0$ such that $G(X(t))$ satisfies social balance for all $t \geq T$.

Proof: We start by proving the following two claims. For any given $t_0 \geq 0$, if all the entries of $X(t_0)$ are non-zero and $G(X(t_0))$ satisfies social balance, then,

C.1) for any $t \geq t_0$, $G(X(t))$ satisfies social balance and $\text{sign}(X(t)) = \text{sign}(X(t_0))$;

C.2) there exists $\alpha > 0$ and $b \in \{-1, +1\}^n$, depending on $X(t_0)$, such that $X(t)$ converges to $\alpha b b^\top$ as $t \to \infty$.

To prove claim C.1), it suffices to prove that $G(X(t_0+1))$ satisfies social balance and $\mathrm{sign}(X(t_0+1)) = \mathrm{sign}(X(t_0))$, as the cases for $t \geq t_0+1$ follow by induction. For any i and j, since $G(X(t_0))$ satisfies social balance, according to Proposition 5.2.2, we have $\mathrm{sign}(X_{i*}(t_0)) = \pm\,\mathrm{sign}(X_{j*}(t_0))$. In addition, we have $X_{jj}(t_0) > 0$ for any j. Therefore,

$$\begin{aligned}\mathrm{sign}\big(X_{ij}(t_0+1)\big) &= \mathrm{sign}\Big(\frac{1}{\|X_{i*}(t_0)\|_1}\sum_{k=1}^{n} X_{ik}(t_0)X_{jk}(t_0)\Big)\\ &= \mathrm{sign}\big(X_{ij}(t_0)X_{jj}(t_0)\big) = \mathrm{sign}\big(X_{ij}(t_0)\big),\end{aligned}$$

for any i and j. This concludes the proof for claim C.1).

For any $t \geq t_0$, since $G(X(t))$ satisfies social balance,

$$|X_{ij}(t+1)| = \frac{1}{\|X_{i*}(t)\|_1}\sum_{k=1}^{n} |X_{ik}(t)||X_{jk}(t)| \text{ for any } i,j,$$

which leads to the following two inequalities:

$\min_{k,\ell}|X_{k\ell}(t{+}1)| \geq \min_{k,\ell}|X_{k\ell}(t)|$; $\max_{k,\ell}|X_{k\ell}(t{+}1)| \leq \max_{k,\ell}|X_{k\ell}(t)|$. Therefore, $\min_{k,\ell}|X_{k\ell}(t)|$ is non-decreasing and upper bounded by $\max_{k,\ell}|X_{k\ell}(t_0)|$, while $\max_{k,\ell}|X_{k\ell}(t)|$ is non-increasing and lower bounded by $\min_{k,\ell}|X_{k\ell}(t_0)|$, which in turn implies that there exists $0 < \underline{\alpha} \leq \overline{\alpha}$, depending on $X(t_0)$, such that

$$\lim_{t\to\infty}\min_{k,\ell}|X_{k\ell}(t)| = \underline{\alpha}, \quad \text{and } \lim_{t\to\infty}\max_{k,\ell}|X_{k\ell}(t)| = \overline{\alpha}.$$

Moreover, suppose $\max_{k,\ell}|X_{k\ell}(t)| > \min_{k,\ell}|X_{k\ell}(t)|$, for some $t \geq t_0$, and $|X_{pq}(t)| =$

$\min_{k,\ell} |X_{k\ell}(t)|$. We have

$$|X_{jp}(t+1)| = \frac{1}{\|X_{j*}(t)\|_1} \sum_{k=1}^{n} |X_{jk}(t)||X_{pk}(t)| < \max_{k,\ell} |X_{k\ell}(t)|, \quad \text{and}$$
$$|X_{ij}(t+2)| = \frac{1}{\|X_{i*}(t+1)\|_1} \sum_{k=1}^{n} |X_{ik}(t+1)||X_{jk}(t+1)| < \max_{k,\ell} |X_{k\ell}(t)|, \tag{5.7}$$

for any i and j. Let $V_1 : \mathbb{R}^{n\times n} \to \mathbb{R}_{\geq 0}$ be defined as:

$$V_1(X) = \max_{k,\ell} |X_{k\ell}| - \min_{k,\ell} |X_{k\ell}|.$$

Due to inequality (5.7), for any $t \geq t_0$, $0 \leq V_1(X(t+2)) < V_1(X(t))$ as long as $V_1(X(t)) > 0$. Therefore, $V_1(X(t))$ converges to 0 as $t \to \infty$, which implies $\underline{\alpha} = \overline{\alpha} = \alpha > 0$. For any i, j and any $t \geq t_0$, since $\min_{k,\ell} |X_{k\ell}(t)| \leq |X_{ij}(t)| \leq \max_{k,\ell} |X_{k\ell}(t)|$, we conclude that $\lim_{t\to\infty} |X_{ij}(t)| = \alpha$. Moreover, since $\text{sign}(X(t)) = \text{sign}(X(t_0))$ for any $t \geq t_0$, we have $\lim_{t\to\infty} X(t) = \alpha bb^\top$, where $b = \text{sign}(X_{1*}(t_0))^\top$. This concludes the proof for claim C.2).

Now we prove statement (i), i.e., each $X^* \in Q_{\text{homophily}}$ with rank 1 is locally stable. Let $X^* = \alpha bb^\top$, where $\alpha > 0$ and $b \in \{-1, +1\}^n$. For any matrix $\Delta \in \mathbb{R}^{n\times n}$ such that $\delta = \max_{i,j} |\Delta_{ij}| < \alpha$, we have $\text{sign}(X^* + \Delta) = \text{sign}(X^*)$. Due to claim C.1) and the proof of claim C.2), we know that, for $X(0) = X^* + \Delta$, $X(t)$ satisfies that, for any $t \geq 0$: (1) $\text{sign}(X(t)) = \text{sign}(X(0)) = \text{sign}(X^*)$; (2) $\alpha - \delta \leq \min_{i,j} |X_{ij}(t)| \leq \max_{i,j} |X_{ij}(t)| \leq \alpha + \delta$. Therefore, for any i and j, $X_{ij}(t)$ is of the form $\alpha_{ij}(t)\,\text{sign}(X^*_{ij})$, where $0 < \alpha - \delta \leq \alpha_{ij}(t) \leq \alpha + \delta$. We thereby have

$$\|X(t) - X^*\|_{\max} = \max_{ij} \left|\alpha_{ij}(t)\,\text{sign}(X^*_{ij}) - \alpha\,\text{sign}(X^*_{ij})\right| = \max_{ij} |\alpha_{ij}(t) - \alpha| \leq \delta.$$

Therefore, for any $\epsilon > 0$, there exists $\delta = \min\{\frac{\alpha}{2}, \frac{\epsilon}{2}\}$ such that, for any $X(0)$ satisfying $\|X(0) - X^*\|_{\max} < \delta$, $\|X(t) - X^*\|_{\max} < \epsilon$ for any $t \geq 0$, i.e., X^* is locally stable.

For the rest of the proof, we proceed to prove the statements (ii) and (iii) of the theorem. For simplicity, denote $X^+ = f_{\text{homophily}}(X)$. Firstly, one can easily check that $f_{\text{homophily}}(X)$ is continuous for any $X \in \mathcal{S}^+_{\text{s-symm}}$. Secondly, for any given $X(0) \in \mathcal{S}^+_{\text{s-symm}}$, according to Proposition 5.3.3, $\|X(t)\|_{\max} \leq \|X(0)\|_{\max}$ for any $t \in \mathbb{Z}_{\geq 0}$. In addition, $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| = \delta$ for some $\delta > 0$ implies that there exists $\tilde{t} \in \mathbb{Z}_{\geq 0}$ such that $\min_{i,j} |X_{ij}(t)| \geq \delta/2$ for any $t \geq \tilde{t}$. Therefore, the set

$$G_c = \left\{ X \in \mathcal{S}^+_{\text{s-symm}} \;\middle|\; \min_{i,j} |X_{ij}| \geq \delta/2,\ \|X\|_{\max} \leq \|X(0)\|_{\max} \right\}$$

is a compact subset of $\mathcal{S}^+_{\text{s-symm}}$ and $X(t) \in G_c$ for any $t \geq \tilde{t}$. Thirdly, define $V_2(X) = \|X\|_{\max}$. The function V_2 is continuous on $\mathcal{S}^+_{\text{s-symm}}$ and, according to Proposition 5.3.3, satisfies $V_2(X^+) - V_2(X) \leq 0$ for any $X \in \mathcal{S}^+_{\text{s-symm}}$. According to the extended LaSalle invariance principle presented in Theorem 2 of [164], $X(t)$ converges to the largest invariant set M of the set $E = \{X \in G_c \mid V_2(X^+) - V_2(X) = 0\}$.

Now we characterize the largest invariant set M. For any $X \in M \subset E$, $V_2(X^+) = V_2(X) = \|X\|_{\max}$. Suppose $|X^+_{ij}| = \max_{k,\ell} |X^+_{k\ell}|$. Since

$$|X^+_{ij}| = \frac{1}{\|X_{i*}\|_1} \left| \sum_{\ell=1}^{n} X_{i\ell} X_{j\ell} \right| \leq \frac{1}{\|X_{i*}\|_1} \sum_{\ell=1}^{n} |X_{i\ell}||X_{j\ell}| \leq \max_{\ell} |X_{j\ell}|, \tag{5.8}$$

we need all of these inequalities to hold with equality and $\max_{\ell} |X_{j\ell}| = \|X\|_{\max}$. Since $X \in G_c$ implies $|X_{k\ell}| > 0$, for any $k, \ell \in \{1, \dots, n\}$, X must satisfy that

(a) X_{i*} and X_{j*} have the same or opposite sign pattern, i.e., $\operatorname{sign}(X_{i*}) = \pm \operatorname{sign}(X_{j*})$,

(b) All entries of X_{j*} have the magnitude $\|X\|_{\max}$.

Therefore, for any $X \in E$, there exist some i and j such that the aforementioned conditions (a) and (b) hold. Moreover, since the set M is invariant, $X \in M$ implies $X^+ \in M \subset E$, which in turn implies that there exists a $\tilde{j}$ such that, for any p, $|X^+_{\tilde{j}p}| = \|X^+\|_{\max} = \|X\|_{\max}$. Following the same argument on the conditions such that the inequalities (5.8) become equalities, we know that, for any p, $\operatorname{sign}(X_{\tilde{j}*}) = \pm \operatorname{sign}(X_{p*})$ and $|X_{pk}| = \|X\|_{\max}$ for any k. As these relationships hold for any p, we conclude that for any $i, j \in \{1, \dots, n\}$, X_{i*} and X_{j*} must have the same or the opposite sign pattern. Let $\alpha = \|X\|_{\max}$ and $b = \operatorname{sign}(X^\top_{1*})$. Each row of X is thereby equal to either $\alpha b^\top$ or $-\alpha b^\top$. Therefore, X is of the form $X = \alpha c b^\top$, where $c \in \{-1, 1\}^n$. Moreover, since all the diagonal entries of X are positive, the column vector c must satisfy $c_i b_i = 1$ for any i, which implies $c = b$. Therefore, $X = \alpha b b^\top$. In addition, since any matrix X of the form $\alpha b b^\top$, with $\alpha > 0$ and $b \in \{-1, 1\}^n$, is a fixed point of system (5.5), we conclude that

$$M = \Big\{ X = \alpha b b^\top \mid \frac{\delta}{2} \leq \alpha \leq \|X(0)\|_{\max}, b \in \{-1, 1\}^n \Big\},$$

which is a compact subset of $\mathcal{S}^+_{\text{s-symm}}$.

For any $\hat{X} \in M$, since $\hat{X}$ satisfies social balance (see Theorem 5.3.4) and $\min_{i,j} \hat{X}_{ij} \geq \delta/2 > 0$, there exists an open neighbor set defined as $\mathcal{U}(\hat{X}) = \{X = \hat{X} + \Delta \mid \|\Delta\|_{\max} < \min_{i,j} \hat{X}_{ij}\}$ such that any $X \in \mathcal{U}(\hat{X})$ satisfies social balance. According to Hein-Borel theorem, there exists a finite set $\{\hat{X}_1, \dots, \hat{X}_K\} \subset M$ such that $M \subset \cup_{k=1}^K \mathcal{U}(\hat{X}_k)$. Since $\cup_{k=1}^K \mathcal{U}(\hat{X}_k)$ is an open set, there exists $\epsilon > 0$ such that the neighbor set of M, defined as

$$\mathcal{U}(M, \epsilon) = \{X \in \mathcal{S}^+_{\text{s-symm}} \mid \|X - M\|_{\max} < \epsilon\},$$

satisfies that $\mathcal{U}(M, \epsilon) \subset \cup_{k=1}^K \mathcal{U}(\hat{X}_k)$ and thereby any $X \in \mathcal{U}(M, \epsilon)$ satisfies social balance.

Since $X(t) \to M$ as $t \to \infty$, there exists $T \in \mathbb{Z}_{\geq 0}$ such that $X(t) \in \mathcal{U}(M, \epsilon)$ for any $t \geq T$. Therefore, $X(t)$ satisfies social balance for any $t \geq T$, which proves statement (iii).

Moreover. Statement (ii) follows from claim C.2) and Theorem 5.3.4. ■

Extensive simulation results indicate that, under generic initial conditions $X(0) \in \mathcal{S}_{\text{nz-row}}$, every entry of the solution $|X(t)|$ is uniformly lower bounded by a positive number for all $t > 0$. This numerical result will be discussed in details in Section 5.5.

5.4 Influence-based Model

In this section, we propose and analyze our second model: the *influence-based model* (IbM).

Definition 5.4.1 (Influence-based model) *Given an initial appraisal matrix $X(0) \in \mathcal{S}^{+}_{\text{rs-symm}} \subset \mathbb{R}^{n\times n}$, the influence-based model is defined by:*

$$X(t+1) = \operatorname{diag}(|X(t)|\mathbb{1}_n)^{-1} X(t) X(t). \tag{5.9}$$

Remark 5.4.2 (Interpretation) *The evolution of the appraisal matrix given by equation* (5.9) *can be interpreted as an influence process. The associated time-varying influence matrix $W(t)$ is constructed by $W(t) = \operatorname{diag}(|X(t)|\mathbb{1}_n)^{-1} X(t)$. That is, the influence any agent i assigns to agent j is assumed to be proportional to i's appraisal of j. We allow negative influences. For any i and k, if agent i is has a positive appraisal of agent k, then agent k's positive (negative resp.) appraisal of j contributes positively (negatively resp.) to agent i's appraisal of j at the next time step, and vice versa.*

Next, we present some results on the invariant set and finite-time behavior of the influence-based model.

Proposition 5.4.3 (Finite-time Properties of the IbM) *Consider the dynamical system* (5.9) *and define $f_{\text{influence}}(X) = \operatorname{diag}(|X|\mathbb{1}_n)^{-1} X X$. Pick any $X_0 \in \mathcal{S}^{+}_{\text{rs-symm}}$. The following statements hold:*

(i. the map $f_{\text{influence}}$ is well-defined for any $X \in \mathcal{S}_{\text{nz-row}}$ and maps $\mathcal{S}^{+}_{\text{rs-symm}}$ to $\mathcal{S}^{+}_{\text{rs-symm}}$;

(ii. the solution $X(t)$, $t \in \mathbb{Z}_{\geq 0}$, to equation (5.9) from initial condition $X(0) = X_0$ is unique and well-defined;

(iii. the max norm of $X(t)$ satisfies

$$\|X(t+1)\|_{\max} \leq \|X(t)\|_{\max} \leq \|X(0)\|_{\max};$$

(iv. for any $c \neq 0$, the trajectory $cX(t)$ is the solution to equation (5.9) from initial condition $X(0) = cX_0$.

Proof: Denote $X^{+} = f_{\text{influence}}(X)$ for simplicity. Following the same argument as in the proof of Proposition 5.3.3, we know that $f_{\text{influence}}$ is well-defined for any $X \in \mathcal{S}_{\text{nz-row}}$. For any $X \in \mathcal{S}^{+}_{\text{rs-symm}}$, there exists $\gamma \succ \mathbb{0}_n$ such that $\operatorname{diag}(\gamma)X = X^{\top}\operatorname{diag}(\gamma)$. Therefore,

$$\begin{aligned}
X^{+}_{ii} &= \frac{1}{\|X_{i*}\|_1}\sum_k X_{ik}X_{ki} = \frac{1}{\|X_{i*}\|_1}\sum_k \frac{\gamma_i}{\gamma_k}X^2_{ik} > 0, \text{ and} \\
X^{+}_{ij} &= \frac{1}{\|X_{i*}\|_1}\frac{\gamma_j}{\gamma_i}\sum_k X_{jk}X_{ki} = \frac{\|X_{j*}\|_1\gamma_j}{\|X_{i*}\|_1\gamma_i}X^{+}_{ji}.
\end{aligned}$$

Let $\tilde{\gamma} = \operatorname{diag}\big(|X|\mathbb{1}_n\big)\gamma$, then we have $\operatorname{diag}(\tilde{\gamma})X = X^{\top}\operatorname{diag}(\tilde{\gamma})$. Therefore, $X^{+} = f_{\text{influence}}(X) \in \mathcal{S}^{+}_{\text{rs-symm}}$. This concludes the proof of statement (i. Statements (ii is a direct consequence of statement (i. Moreover,

$$\big|X^{+}_{ij}\big| = \frac{1}{\|X_{i*}\|_1}\Big|\sum_k X_{ik}X_{kj}\Big| \leq \frac{1}{\|X_{i*}\|_1}\sum_k |X_{ik}||X_{kj}| \leq \max_k |X_{kj}| \leq \|X\|_{\max}$$

immediately lead to statement (iii. Statement (iv is a straightforward observation obtained from equation (5.9). ■

Notice that, unlike the homophily-based model, $f_{\text{influence}}$ is not well-defined for all $X \in \mathcal{S}_{\text{nz-row}}$. For example,

$$X(0) = \begin{bmatrix} 1 & 2 \\ -0.5 & -1 \end{bmatrix} \in \mathcal{S}_{\text{nz-row}}$$

leads to $X(1) \notin \mathcal{S}_{\text{nz-row}}$ and $f_{\text{influence}}(X(1))$ is not defined. Instead, we consider $\mathcal{S}^{+}_{\text{rs-symm}}$ as the domain of system (5.9). According to Proposition 5.4.3, for any $X(0) \in \mathcal{S}^{+}_{\text{rs-symm}}$, the solutions $X(t)$ to equation (5.9) is uniformly upper bounded, which is a desired property, that the previous models in [106, 162] do not have.

The following theorem characterizes the set of fixed points of the map $f_{\text{influence}}$ in $\mathcal{S}^{+}_{\text{rs-symm}}$.

Theorem 5.4.4 (Fixed points and social balance) *Consider system* (5.9) *in domain* $\mathcal{S}^{+}_{\text{rs-symm}}$. *Define*

$$\begin{aligned} Q_{\text{influence}} = \Big\{ PYP^{\top} \in \mathcal{S}^{+}_{\text{rs-symm}} \,\Big|\, & P \text{ is a permutation matrix,} \\ & Y \text{ is a block diagonal matrix with blocks of the} \\ & \text{form } \operatorname{sign}(w)w^{\top},\ w \in \mathbb{R}^m \textit{ and } |w| \succ \mathbb{0}_m, m \leq n \Big\}. \end{aligned}$$

Then the following statements hold:

(i. $Q_{\text{influence}}$ *is the set of all the fixed points of system* (5.9) *in domain* $\mathcal{S}^{+}_{\text{rs-symm}}$*,*

(ii. for any $X \in Q_{\text{influence}}$*,* $G(X)$ *is composed by isolated complete subgraphs that satisfy social balance.*

Proof: We first prove that any $X^{*} \in Q_{\text{influence}}$ is a fixed point of system (5.9). For any $w \in \mathbb{R}^n$ such that $|w| \succ \mathbb{0}_n$, the matrix $Y = \operatorname{sign}(w)w^\top$ satisfies

$$f_{\text{influence}}(Y) = \operatorname{diag}(|\operatorname{sign}(w)w^\top|\mathbb{1}_n)^{-1}(\operatorname{sign}(w)w^\top)(\operatorname{sign}(w)w^\top) = \operatorname{sign}(w)w^\top = Y.$$

Therefore, $Y = \operatorname{sign}(w)w^\top$ is a fixed point of system (5.9). Now suppose that Y is a block diagonal matrix, i.e., $Y = \operatorname{diag}(Y^{(1)}, \dots, Y^{(K)})$, where each $Y^{(i)}$ is a $n_i \times n_i$ matrix of the form $\operatorname{sign}(w^{(i)}){w^{(i)}}^\top$, with $|w^{(i)}| \succ \mathbb{0}_{n_i}$, and $n_1 + \cdots + n_K = n$. One can check that, as long as

$$Y^{(i)} = \operatorname{diag}(|Y^{(i)}|\mathbb{1}_n)^{-1}Y^{(i)}Y^{(i)} \tag{5.10}$$

for any $i \in \{1, \dots, K\}$, Y is a fixed point of system (5.9). Since $Y^{(i)} = \operatorname{sign}(w^{(i)}){w^{(i)}}^\top$, following the same line of argument for the case in which Y only has one block, we know that equation (5.10) holds for any i. Therefore, Y is a fixed point of system (5.9). Moreover, given any fixed point Y, for any permutation matrix $P \in \mathbb{R}^{n\times n}$, since

$$\begin{aligned} PYP^\top &= P\operatorname{diag}(|Y|\mathbb{1}_n)^{-1}YYP^\top = \operatorname{diag}(|PYP^\top|\mathbb{1}_n)^{-1}(PYP^\top)(PYP^\top) \\ &= f_{\text{influence}}(PYP^\top), \end{aligned}$$

any $X^* \in Q_{\text{influence}}$ is a fixed point of $f_{\text{influence}}$.

For any $X^* \in Q_{\text{influence}}$, there exists a permutation matrix P and a block diagonal matrix Y in the form $\operatorname{diag}(Y^{(1)}, \dots, Y^{(K)})$ such that $X^* = PYP^\top$. One can easily check that Y has positive diagonals and is sign-symmetric. Moreover, $G(Y)$ is made up of K isolated complete subgraphs. Therefore, for any triad (j, k, l) in $G(Y)$, there exists $i \in \{1, \dots, K\}$ such that nodes j, k, l are all in the subgraph $G(Y^{(i)})$ with the adjacency matrix $Y^{(i)} = \big(Y^{(i)}_{jk}\big)_{n_i\times n_i}$. Suppose $Y^{(i)} = \operatorname{sign}(w^{(i)}){w^{(i)}}^\top$, where $w^{(i)} = (w^{(i)}_1, \dots, w^{(i)}_{n_i})^\top$. We have

$$Y^{(i)}_{jk}Y^{(i)}_{k\ell}Y^{(i)}_{\ell j} = |w^{(i)}_j||w^{(i)}_k||w^{(i)}_l| > 0.$$

Therefore, every triad is positive in graph $G(Y)$. Since $G(Y)$ has exactly the same topology as $G(X)$, but just with the nodes reindexed. We conclude that any $X^* \in Q_{\text{influence}}$ is associated with a graph $G(X^*)$ composed by isolated complete subgraphs that satisfy social balance. This concludes the proof for statement (ii.

Now we prove that $Q_{\text{influence}}$ contains all the fixed points of system (5.9) in $\mathcal{S}^+_{\text{rs-symm}}$. We the notations θ_i and X_{j*,θ_i} in the same way as defined in the proof of Theorem 5.3.4, and, in addition, define X_{*j,θ_i} as the j-th column of X with all the k-th entry such that $k \notin \theta_i$ removed.

Now we prove by induction that $Q_{\text{influence}}$ is actually the set of all the fixed point of system (5.9). One can check that the trivial case of $n = 1$ is true. Suppose statement (i holds for any system with dimension $\tilde{n} < n$.

For system (5.9) with dimension n, suppose $X \in \mathcal{S}^+_{\text{rs-symm}}$ is a fixed point of the system (5.9), i.e., $X = f_{\text{influence}}(X)$. For any given j,

$$|X_{ij}| = \frac{1}{\|X_{i*}\|_1}\Big|\sum_k X_{ik}X_{kj}\Big| \leq \frac{1}{\|X_{i*}\|_1}\sum_k |X_{ik}||X_{kj}| \leq \max_k |X_{kj}|, \quad \text{for any } i,$$

and there exists some i such that $|X_{ij}| = \max_k |X_{kj}|$. Now we discuss two cases that cover all the possible X's.

Case 1: $|X_{jj}| = \max_k |X_{kj}|$ and $|X_{ij}| < \max_k |X_{kj}|$ for any $i \neq j$. Since $X \in \mathcal{S}^+_{\text{rs-symm}}$, X is sign-symmetric,

$$|X_{jj}| = \frac{1}{\|X_{j*}\|_1}\sum_k |X_{jk}||X_{kj}| = \max_k |X_{kj}|.$$

Due to the second equality in the equations above, $|X_{kj}| = \max_\ell |X_{\ell j}|$ for any $k \in \theta_j$. Therefore, in Case 1, $i \notin \theta_j$ for any $i \neq j$, which in turn implies that $X_{ji} = X_{ij} = 0$ for any $i \neq j$. As the consequence, there exists a permutation matrix P such that

$$PXP^\top = \begin{bmatrix} X_{jj} & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{X} \end{bmatrix},$$

where $\tilde{X}$ is an $(n-1)\times(n-1)$ matrix. Following the same line of argument in the Case 1 of the proof of Theorem 5.3.4, we conclude that $X \in Q_{\text{influence}}$.

Case 2: there exists $i \neq j$ such that $|X_{ij}| = \max_k |X_{kj}|$. For such i, we have $j \in \theta_i$. In addition, the equality below

$$|X_{ij}| = \frac{1}{\|X_{i*}\|_1}\Big|\sum_k X_{ik}X_{kj}\Big| = \max_k |X_{kj}|$$

leads to the following two results:

R.1) $\operatorname{sign}(X_{i*,\theta_i}) = \pm\operatorname{sign}(X_{*j,\theta_i}^\top)$;

R.2) $|X_{kj}| = \max_\ell |X_{\ell j}|$ for any $k \in \theta_i$.

Result R.2) and $j \in \theta_i$ lead to $|X_{jj}| = \max_\ell |X_{\ell j}|$. Therefore, for any $k \in \theta_j$, $|X_{kj}| = \max_\ell |X_{\ell j}|$. Moreover, since X is sign-symmetric, for any $k \notin \theta_j$, $X_{jk} = X_{kj} = 0$.

For any $i \in \theta_j$, since $|X_{ij}| = \frac{1}{\|X_{i*}\|_1}\Big|\sum_k X_{ik}X_{kj}\Big|$, $|X_{ij}| = \max_\ell |X_{\ell j}|$ implies that $|X_{kj}| = \max_\ell |X_{\ell j}|$ for any $k \in \theta_i$. Since $k \notin \theta_j$ leads to $k \notin \theta_i$, we have $\theta_i \subset \theta_j$.

For any given $i \in \theta_j$, since $X \in \mathcal{S}^+_{\text{rs-symm}}$, we know that $X_{ii} > 0$ and $X_{ji} > 0$. Apply the same argument for the j-th column in Case 2 to the i-th column, we conclude that $X_{ii} = \max_\ell |X_{\ell i}|$ and $|X_{ji}| = \max_\ell |X_{\ell i}|$, the latter of which in turn implies that $|X_{ki}| = \max_\ell |X_{\ell i}|$ for any $k \in \theta_j$. Moreover, since $X_{ii} = \max_\ell |X_{\ell i}|$ leads to $|X_{ki}| = \max_\ell |X_{\ell i}|$ for any $k \in \theta_i$ and $X_{ik} = X_{ki} = 0$ for any $k \notin \theta_i$, we have $\theta_j \subset \theta_i$. Since we already get $\theta_i \subset \theta_j$, we conclude that $\theta_j = \theta_i$ for any $i \in \theta_j$. In addition, due to Result R.1) and the facts that $\theta_i = \theta_j$ and X is sign-symmetric, we obtain that $\mathrm{sign}(X_{i*}, \theta_j) = \mathrm{sign}(X^{\top}_{*i,\theta_j}) = \pm\,\mathrm{sign}(X^{\top}_{*j,\theta_j})$ for all $i \in \theta_j$.

Taking together all the results we have obtained for Case 2, we conclude that, for any given j in Case 2: (1) $|X_{kj}| = \max_\ell |X_{\ell j}|$ for any $k \in \theta_j$ and $X_{kj} = X_{jk} = 0$ for any $k \notin \theta_j$; (2) For any $i \in \theta_j$, $\theta_i = \theta_j$. In addition, $|X_{ki}| = \max_\ell |X_{\ell i}|$ for any $k \in \theta_j$ and $X_{ki} = X_{ik} = 0$ for any $k \notin \theta_j$; (3) For any $i \in \theta_j$, $\mathrm{sign}(X_{*i}) = \mathrm{sign}(X_{*j})$. Denote by $|\theta_j|$ the cardinality of θ_j and define the $|\theta_j| \times |\theta_j|$ matrix $X^{(\theta_j)} = \mathrm{sign}(w^{(\theta_j)}) {w^{(\theta_j)}}^{\top}$, where $w^{(\theta_j)} = X^{\top}_{j*,\theta_j}$. There exists a permutation matrix P such that

$$PXP^{\top} = \begin{bmatrix} X^{(\theta_j)} & \mathbb{0}_{|\theta_j|\times(n-|\theta_j|)} \\ \mathbb{0}_{(n-|\theta_j|)\times|\theta_j|} & \tilde{X} \end{bmatrix}.$$

Following the line of argument in Case 1, we have $X \in Q_{\mathrm{influence}}$. This concludes the proof for statement (i. ■

Remark 5.4.5 *By carefully examining the proof for Theorem 5.4.4, one can observe that $Q_{\mathrm{influence}}$ is actually the set of all the fixed point of the map $f_{\mathrm{influence}}$ in $\mathcal{S}^{+}_{\text{s-symm}}$. However, the set $Q_{\mathrm{influence}}$ does not contain all the fixed points in $\mathcal{S}_{\text{nz-row}}$. For example, let $X = \alpha bb^{\top}$ for some $\alpha > 0$ and $b \in \{-1,+1\}^n$. Then, pick one $i \in \{1,\dots,n\}$ and set $X_{*i} = \mathbb{0}_n$. It can be easily verified that $X = f_{\mathrm{influence}}(X)$ but $X \notin Q_{\mathrm{influence}}$.*

Now we present the main results on the convergence of the appraisal network to social balance.

Theorem 5.4.6 (Convergence and social balance in the IbM) *Consider the influence-based model given by equation* (5.9)*. The following statements hold:*

(i. Each fixed point of rank one in $Q_{\mathrm{influence}}$ is locally stable.

For any $X(0) \in \mathcal{S}^{+}_{\text{rs-symm}}$ *such that* $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$,

(ii) *there exists* $X^* \in Q_{\text{influence}}$ *of rank one such that* $\lim_{t\to\infty} X(t) = X^*$, *and*

(iii) *there exists* $T > 0$ *such that* $G(X(t))$ *satisfies social balance for all* $t \geq T$.

Proof: We start by proving the following two claims. For any given $t_0 \geq 0$, if all the entries of $X(t_0)$ are non-zero and $G(X(t_0))$ satisfies social balance, then,

C.1) for any $t \geq t_0$, $G(X(t))$ satisfies social balance and $\operatorname{sign}(X(t)) = \operatorname{sign}(X(t_0))$;

C.2) there exists $w \in \mathbb{R}^n \setminus \{\mathbb{0}_n\}$, depending on $X(t_0)$, such that $X(t)$ converges to $\operatorname{sign}(w)w^\top$ as $t \to \infty$.

Claim C.1) is proved in the same way as in the proof of Theorem 5.3.6. For any $t \geq t_0$, since $G(X(t))$ satisfies social balance,

$$|X_{ij}(t+1)| = \frac{1}{\|X_{i*}(t)\|_1} \sum_{\ell=1}^{n} |X_{i\ell}(t)||X_{\ell j}(t)|,$$

for any i and j. Therefore, for a given j, the previous expression leads to the following two inequalities:
$\min_\ell |X_{\ell j}(t+1)| \geq \min_\ell |X_{\ell j}(t)|$; $\max_\ell |X_{\ell j}(t+1)| \leq \max_\ell |X_{\ell j}(t)|$. Therefore, $\min_\ell |X_{\ell j}(t)|$ is non-decreasing and upper bounded by $\max_\ell |X_{\ell j}(t_0)|$, while $\max_\ell |X_{\ell j}(t)|$ is non-increasing and lower bounded by $\min_\ell |X_{\ell j}(t_0)|$, which in turn implies that there exists $0 < \underline{\omega} \leq \overline{\omega}$, depending on $X(t_0)$, such that

$$\lim_{t\to\infty} \min_\ell |X_{\ell j}(t)| = \underline{\omega}, \quad \text{and } \lim_{t\to\infty} \max_\ell |X_{\ell j}(t)| = \overline{\omega}.$$

Suppose, at some time $t \geq t_0$, $\min_\ell |X_{\ell j}(t)| < \max_\ell |X_{\ell j}(t)|$ and $|X_{pj}(t)| = \max_\ell |X_{\ell j}(t)|$.

For any i,

$$|X_{ij}(t+1)| \geq \frac{\sum_{k \neq p} |X_{ik}(t)| \min_{\ell} |X_{\ell j}(t)| + |X_{ip}(t)||X_{pj}(t)|}{\|X_{i*}(t)\|_1} > \min_{\ell} |X_{\ell j}(t)|.$$

Therefore, $\min_{\ell} |X_{\ell j}(t+1)| > \min_{\ell} |X_{\ell j}(t)|$ and, similarly, $\max_{\ell} |X_{\ell j}(t+1)| < \max_{\ell} |X_{\ell j}(t)|$. As the consequence, $\max_{\ell} |X_{\ell j}(t)| - \min_{\ell} |X_{\ell j}(t)|$ is strictly decreasing as long as $\min_{\ell} |X_{\ell j}(t)| < \max_{\ell} |X_{\ell j}(t)|$. Therefeore, $\underline{\omega} = \overline{\omega} > 0$, which implies that $|X_{ij}|$ converges for any i and j, and the magnitude of the entries of X in the same column converge to the same value. In addition, since $\text{sign}(X(t)) = \text{sign}(X(t_0))$ for all $t \geq t_0$, we conclude that $X(t)$ converges to a matrix in the form $\text{sign}(w)w^\top$. This concludes the proof for claim C.2).

Now we prove statement (i, i.e., each $\hat{X} \in Q_{\text{influence}}$ with rank 1 is locally stable. Let $\hat{X} = \text{sign}(w)w^\top$, where $|w| \succ \mathbb{0}_n$. For any matrix $\Delta \in \mathbb{R}^{n\times n}$ such that for any $k \in \{1, \dots, n\}$, $\delta_k = \max_i |\Delta_{ik}| < |w_k|$, we have $\text{sign}(\hat{X}_{*k} + \Delta_{*k}) = \text{sign}(\hat{X}_{*k})$. Due to claim C.1) and the proof of claim C.2), we know that, for $X(0) = \hat{X} + \Delta$, $X(t)$ satisfies that, for any $t \geq 0$,

(1) $\text{sign}(X(t)) = \text{sign}(X(0)) = \text{sign}(\hat{X})$;

(2) $|w_k| - \delta_k \leq \min_i |X_{ik}(t)| \leq \max_i |X_{ik}(t)| \leq |w_k| + \delta_k$.

Therefore, for any i, $X_{ik}(t)$ is of the form $\alpha_{ik}(t)\text{sign}(\hat{X}_{ik})$, where $0 < |w_k| - \delta_k \leq \alpha_{ik}(t) \leq |w_k| + \delta_k$. We have

$$\left\| X(t) - \hat{X} \right\|_{\max} = \max_{ij} \left| \alpha_{ij}(t)\, \text{sign}(\hat{X}_{ij}) - |w_j|\, \text{sign}(\hat{X}_{ij}) \right| = \max_{ij} \left| \alpha_{ij}(t) - |w_j| \right| \leq \delta,$$

where $\delta = \max_{k} \delta_k$. Therefore, for any $\epsilon > 0$, there exists $\delta = \min\{\frac{\max_k |w_k|}{2}, \frac{\epsilon}{2}\}$ such that, for any $X(0)$ satisfying $\|X(0) - X^*\|_{\max} < \delta$, $\|X(t) - X^*\|_{\max} < \epsilon$ for any $t \geq 0$. That is, $\hat{X}$ is locally stable.

Now we proceed to prove the statements (ii) and (iii) of the theorem. For simplicity, denote $X^+ = f_{\text{influence}}(X)$. Firstly, one can easily check that $f_{\text{influence}}(X)$ is continuous for any $X \in \mathcal{S}^+_{\text{rs-symm}}$. Secondly, for any $X(0) \in \mathcal{S}^+_{\text{rs-symm}}$ and any $k \in \{1, \dots, n\}$, according to the proof of Proposition 5.4.3, $\|X_{*k}(t)\|_{\max} \leq \|X_{*k}(0)\|_{\max}$ for any $t \in \mathbb{Z}_{\geq 0}$. In addition, $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$ implies that there exists $\delta > 0$ and $\tilde{t} \in \mathbb{Z}_{\geq 0}$ such that $\min_{i,j} |X_{ij}(t)| \geq \delta/2$ for any $t \geq \tilde{t}$. Therefore, the set

$$G_c = \Big\{ X \in \mathcal{S}^+_{\text{rs-symm}} \,\Big|\, \min_{i,j} |X_{ij}| \geq \delta/2, \text{ and, for any } k, \|X_{*k}\|_{\max} \leq \|X_{*k}(0)\|_{\max} \Big\}$$

is a compact subset of $\mathcal{S}^+_{\text{rs-symm}}$ and $X(t) \in G_c$ for any $t \geq \tilde{t}$. Thirdly, define $V_2(X_{*k}) = \|X_{*k}\|_{\max}$. The function V_2 is continuous on $\mathcal{S}^+_{\text{rs-symm}}$ and, according to the proof of Proposition 5.4.3, satisfies $V_2(X^+_{*k}) - V_2(X_{*k}) \leq 0$ for any $X \in \mathcal{S}^+_{\text{rs-symm}}$. According to the extended LaSalle invariance principle presented in Theorem 2 of [164], we conclude that, given any $X(0) \in \mathcal{S}^+_{\text{symm}}$ such that $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| = \delta$, $X(t)$ converges to the largest invariantset M of the set $E = \{X \in G_c \mid V_2(X^+_{*k}) - V_2(X_{*k}) = 0 \text{ for any } k\}$.

Now we characterize the largest invariant set M. For any $X \in M \subset E$ and $k \in \{1, \dots, n\}$, $V_2(X^+_{*k}) = V_2(X_{*k}) = \|X_{*k}\|_{\max}$. Suppose $|X^+_{ik}| = \max_{\ell} |X^+_{\ell k}|$. Since

$$|X^+_{ik}| = \frac{1}{\|X_{i*}\|_1} \left| \sum_{\ell=1}^{n} X_{i\ell} X_{\ell k} \right| \leq \frac{1}{\|X_{i*}\|_1} \sum_{\ell=1}^{n} |X_{i\ell}||X_{\ell k}| \leq \max_{\ell} |X_{\ell k}|, \tag{5.11}$$

we need all of these inequalities to hold with equality Since $X \in G_c \subset \mathcal{S}^+_{\text{rs-symm}}$ implies $|X_{j\ell}| > 0$, for any $j, \ell \in \{1, \dots, n\}$, X must satisfy that

(a) X_{i*} and X_{*k} have the same or opposite sign pattern, i.e., $\operatorname{sign}(X_{*k}) = \operatorname{sign}(X_{k*}) = \pm \operatorname{sign}(X_{i*})$,

(b) All entries of X_{*k} have magnitude $\|X_{*k}\|_{\max}$.

Therefore, for any $X \in E$ and k, there exist some i such that the aforementioned conditions (a) and (b) hold. Moreover, since the set M is invariant, $X \in M$ implies $X^+ \in M \subset E$, which in turn implies that, for any p, $|X^+_{pk}| = \left\|X^+_{*k}\right\|_{\max} = \|X_{*k}\|_{\max}$. Following the same argument on the conditions such that the inequalities (5.11) become strict equalities, we know that, for any p, $\operatorname{sign}(X_{p*}) = \pm \operatorname{sign}(X^\top_{*k})$ and $|X_{pk}| = \|X_{*k}\|_{\max}$ for any k. Using these relationships, we conclude that for any i and j, X_{i*} and X_{j*} must have the same or the opposite sign pattern, and that $|X_{ij}| = \|X_{*j}\|_{\max}$. Let $w = X^\top_{1*}$. Each row of X is thereby equal to either $w^\top$ or $-w^\top$. Therefore, X is of the form $X = cw^\top$, where $c \in \{-1, 1\}^n$. Moreover, since all the diagonal entries of X are positive, the column vector c must satisfy $c_i w_i = 1$ for any i, which implies $c = \operatorname{sign}(w)$. Therefore, $X = \operatorname{sign}(w)w^\top$. Thus, since any matrix X of the form $\operatorname{sign}(w)w^\top$, with $|w| \succ \mathbb{0}_n$, is a fixed point of system (5.9), we conclude that

$$M = \{X = \operatorname{sign}(w)w^\top \mid \delta/2 \leq w_i \leq \|X(0)\|_{\max}, w \in \mathbb{R}^n \setminus \{\mathbb{0}_n\}, \text{ for any } i \in \{1, \dots, n\}\},$$

which is a compact subset of $\mathcal{S}^+_{\text{rs-symm}}$. Following the same line of argument in the proof of Theorem 5.3.6, we conclude that there exists $\epsilon > 0$ such that any X in the neighbor set $\mathcal{U}(M, \epsilon)$ satisfies social balance.

Since $X(t) \to M$ as $t \to \infty$, there exists $T \in \mathbb{Z}_{\geq 0}$ such that $X(t) \in \mathcal{U}(M, \epsilon)$ for any $t \geq T$. Therefore, $X(t)$ satisfies social balance for any $t \geq T$, which proves statement (iii). Moreover, according to claim C.2) and Theorem 5.4.4, there exists $X^* = \operatorname{sign}(w)w^\top$, which is a matrix in the set $Q_{\text{influence}}$ with rank one, such that $X(t) \to X^*$ as $t \to \infty$, concluding the proof for statement (ii). ■

Extensive simulation results indicate that, under generic initial conditions $X(0) \in \mathcal{S}^+_{\text{rs-symm}}$, every entry of $|X(t)|$ is uniformly strictly lower bounded from 0 for all $t > 0$. This numerical result is further discussed in Section 5.5.

5.5 Further discussion and numerical simulations

5.5.1 Generic convergence to rank-one appraisal matrix

According to Theorem 5.3.6, for any $X(0) \in \mathcal{S}^{+}_{\text{s-symm}}$ such that $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$, in the homophily-based model, the solution $X(t)$ converges to some rank-one matrix of the form $\alpha bb^\top$. In this subsection, we use the Monte Carlo method to numerically verify that $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$ holds for generic initial conditions in $\mathcal{S}_{\text{nz-row}}$. By generic initial condition, we mean each of $X(0)$'s entries is selected independently and uniformly at random from a support of positive measure. We consider the support to be $[-a, a]$, where $a > 0$. For any randomly generated $X(0) \in \mathcal{S}_{\text{nz-row}}$, define the random variable $Z : \mathcal{S}_{\text{nz-row}} \to \{0, 1\}$ as

(i. There exists $\delta > 0$ such that $\min_{i,j} |X_{ij}(t)| \geq \delta$ for any $t \in \{100, \dots, 10000\}$;

(ii. $Z(X(0)) = 0$ otherwise.

Let $p = \mathbb{P}[Z(X(0)) = 1]$. For N independent random samples $Z_1, \dots, Z_N$, in each of which $X(0) \in \mathcal{S}_{\text{nz-row}}$ is a generic initial condition, define $\hat{p}_N = \sum_{i=1}^{N} Z_i/N$. For any accuracy $1 - \varepsilon \in (0, 1)$ and confidence level $1 - \xi \in (0, 1)$, $|\hat{p}_N - p| < \varepsilon$ with probability greater than $1 - \xi$ if the Chernoff bound is satisfied: $N \geq \frac{1}{2\varepsilon^2} \log \frac{2}{\xi}$. For $\varepsilon = \xi = 0.01$, the bound is satisfied by $N = 27000$. We ran the 27000 independent simulations of the homophily-based model with $n = 8$ and $a = 20$, and found that $\hat{p} = 1$. Then, we conclude that for any generic initial condition $X(0) \in \mathcal{S}_{\text{nz-row}}$, with 99% confidence level, there is at least 0.99 probability that every entry of $|X(t)|$ is lower bounded by a positive scalar for all $t \in \{100, \dots, 10000\}$.

The Monte Carlo method under the same settings is applied to the influence-based model, except that now the generic initial conditions $X(0) \in \mathcal{S}^{+}_{\text{rs-symm}} \subset \mathbb{R}^{n\times n}$ is generated by the following steps: 1) Randomly and independently generate the diagonal and the upper triangular entries of a matrix $\hat{X} \in \mathbb{R}^{n\times n}$, according to some uniform distribution; 2) Let $\hat{X}_{ij} = \hat{X}_{ji}$ for any $i > j$; 3) Randomly and independently generate the entries of a $n \times 1$ vector γ, according to some uniform distribution with some positive interval as the support; 4) Let $X(0) = \mathrm{diag}(\gamma)\hat{X}$. Not surprisingly, we obtained the same results as the homophily-based model. That is, for any generic initial condition $X(0) \in \mathcal{S}_{\text{nz-row}}$, with 99% confidence level, there is at least 0.99 probability that every entry of $|X(t)|$ is uniformly strictly lower bounded from 0 for all $t \in \{100, \dots, 10000\}$.

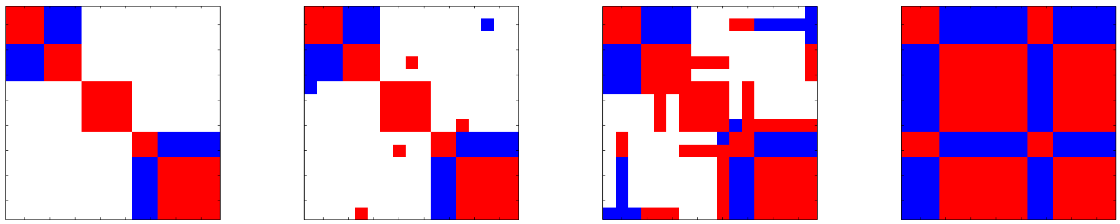
5.5.2 Multi-clique social balance and perturbation

1) The initial conditions leading to multi-clique social balance: Despite the generic convergence to complete graphs, for both the homophily-based and the influence-based models, there exists some special initial conditions leading to the *multi-clique social balance*. By *clique* we mean an isolated subgraph (Negative links are counted as links), and by multi-clique social balance we mean that the appraisal network consists of multiple cliques and each of them satisfies social balance. For example, let

$$X(0) = \begin{bmatrix} 1 & 1 & 1 \\ 0.5 & -1 & 0.5 \\ -0.5 & 1 & -0.5 \end{bmatrix}, \quad \hat{X}(0) = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & -2 \\ 0 & -2 & -1 \end{bmatrix}.$$

For the homophily-based model, the initial condition $X(0)$ eventually results in the formation of two isolated cliques with node sets $\{1\}$ and $\{2, 3\}$ respectively. For both the homophily-based and the influence-based models, the initial condition $\hat{X}(0)$ results in the formation of two isolated cliques with node sets $\{2\}$ and $\{1, 3\}$.

2) Multi-clique social balance under perturbation: For the homophily model, extensive simulation observations indicate that the multi-clique social balance is unstable



(a) no link added (b) t=0 (c) t=1 (d) t=6

Figure 5.1: Visualization of the evolution of the appraisal matrix under perturbations. For each entry, the red color indicates a positive appraisal, while the blue color indicates a negative one. The white color indicate no appraisal. The appraisal network has 17 nodes and is initially in a multi-clique structurally balanced state with three isolated balanced cliques. With 6 links (4 positive and 2 negative links) added to the network, the appraisal network evolves to a single-clique structurally balanced state after 6 iterations.

under perturbations. With some links added to the multi-clique structurally balanced network, the perturbed network eventually converges to a single-clique structurally balanced state, see Fig. 5.1 as a concrete example. The following two examples illustrate the behavior of multi-clique social balance under perturbations.

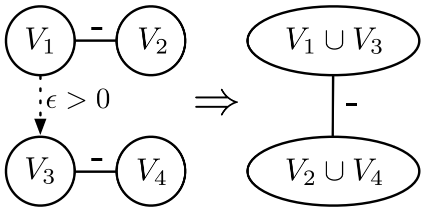
Example 1: (Globalization of local conflicts) Consider the appraisal network with two isolated cliques. Each clique is made up of two antagonistic factions. Clique 1 has two factions, with node sets V_1 and V_2 respectively, and Clique 2 also has two factions, with node sets V_3 and V_4 respectively. Suppose one link with weight ϵ is added from one node in V_1 to one node in V_3. We find that the perturbed appraisal network always recovers to a complete and structurally balanced network such that:

(i. It is composed of two antagonistic factions;

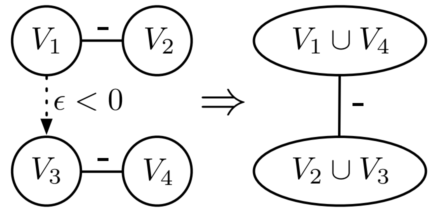
(ii. If $\epsilon > 0$, the two factions are $V_1 \cup V_3$ and $V_2 \cup V_4$;

(iii. If $\epsilon < 0$, the two factions are $V_1 \cup V_4$ and $V_2 \cup V_3$.

Figure 5.2 visualizes the behavior described above. In reality, such behavior could be interpreted as the escalation of local conflicts. In the example above, the two original



(a) $\epsilon > 0$



(b) $\epsilon < 0$

Figure 5.2: Visual illustration of the behavior of the multi-clique social balance with the addition of one inter-clique link.

conflicting relations, i.e., V_1 v.s. V_2 and V_3 v.s. V_4, are escalated into global conflicts between two reunified factions $V_1 \cup V_2$ and $V_3 \cup V_4$, once a node in V_1 builds a connection with V_3. One real example of such phenomena is the formation of the globalized conflicts between the Axis and the Ally in World War II, after the Nazi German allied with the Imperial Japan.

Example 2: (Competition for ally and mediation of conflicts) Consider an appraisal network with two isolated cliques: Clique 1 with two antagonistic factions $V_1 = \{1, \dots, n_1\}$ and $V_2 = \{n_1 + 1, \dots, n_1 + n_2\}$, and Clique 2 with only one faction $V_3 = \{n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3\}$. Suppose the appraisal matrix associated with Clique 1 is given by $\alpha bb^\top$, where $b = (\mathbb{1}_{n_1}^\top, -\mathbb{1}_{n_2}^\top)^\top$, and $\alpha > 0$ represents the sentiment strength inside Clique 1. Similarly, the appraisal matrix associated with Clique 2 is given by $\hat{\alpha}\hat{b}\hat{b}^\top$, where $\hat{b} = \mathbb{1}_{n_3}$ and $\hat{\alpha} > 0$ represents the sentiment strength inside Clique 2. Imagine then that both cliques V_1 and V_2 would aim to ally with V_3 in order to grow the number of their members. Accordingly, suppose that, in order to ally with V_3, each node in V_1 builds a bilateral link with each node in V_3, with link weight $\epsilon_1 > 0$, while each node in V_2 builds a bilateral link with each node in V_3 with weight $\epsilon_2 > 0$. With all these links added, the associated appraisal matrix takes the following form:

$$X(0) = \begin{bmatrix} \alpha \mathbb{1}_{n_1}\mathbb{1}_{n_1}^\top & -\alpha \mathbb{1}_{n_1}\mathbb{1}_{n_2}^\top & \epsilon_1 \mathbb{1}_{n_1}\mathbb{1}_{n_3}^\top \\ -\alpha \mathbb{1}_{n_2}\mathbb{1}_{n_1}^\top & \alpha \mathbb{1}_{n_2}\mathbb{1}_{n_2}^\top & \epsilon_2 \mathbb{1}_{n_2}\mathbb{1}_{n_3}^\top \\ \epsilon_1 \mathbb{1}_{n_3}\mathbb{1}_{n_1} & \epsilon_2 \mathbb{1}_{n_3}\mathbb{1}_{n_1}^\top & \hat{\alpha} \mathbb{1}_{n_3}\mathbb{1}_{n_3}^\top \end{bmatrix}.$$

Along the evolution of $X(t)$ determined by $X(0)$, we obtain the following numerical results.

(i) If $\epsilon_1 n_1 > \epsilon_2 n_2$, i.e., faction V_1 takes greater effort than V_2 in allying with V_3, then faction V_1 gains at least one ally, either V_2 or V_3, which is a situation more in favor of V_1 than V_2. Moreover, the following conditions:

$$\epsilon_1 n_1 - \epsilon_2 n_2 \geq \hat{\alpha}\epsilon_2 n_3/\alpha \quad \text{and} \quad \epsilon_1\epsilon_2 n_3 \leq \alpha^2(n_1+n_2)$$

guarantee that V_1 ally with V_3; This argument also holds when all the subscripts 1 and 2 are switched;

(ii) If $\epsilon_1\epsilon_2 n_3 \leq \alpha^2(n_1+n_2)$, then V_3 eventually gains at least one ally. That is, V_3 avoids the situation in which V_1 and V_2 end up allying with each other against V_3;

(iii) Any of the following conditions guarantees the non-existence of any negative link in the asymptotic state of the appraisal network: (1) $\epsilon_1\epsilon_2 n_3 \geq \alpha^2(n_1+n_2)$ and $\epsilon_1 n_1 - \epsilon_2 n_2 = 0$; (2) $\epsilon_1\epsilon_2 n_3 \geq \alpha^2(n_1+n_2)$ and $0 < \epsilon_1 n_1 - \epsilon_2 n_2 \leq \epsilon_2\hat{\alpha} n_3$; (3) $\epsilon_1\epsilon_2 n_3 \geq \alpha^2(n_1+n_2)$ and $0 < \epsilon_2 n_2 - \epsilon_1 n_1 \leq \epsilon_1\hat{\alpha} n_3$. Notice that the inequality

$$\epsilon_1\epsilon_2 n_3 \geq \alpha^2(n_1+n_2)$$

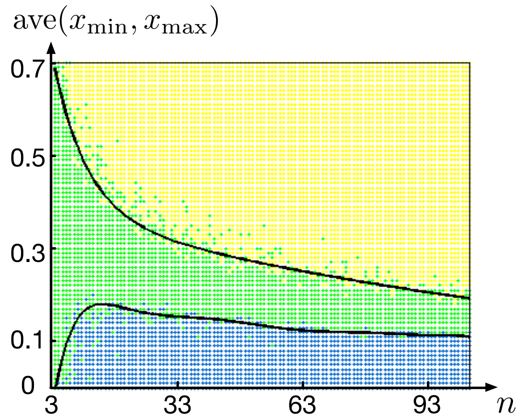
is required for all the three sufficient conditions. The right-hand side of the inequality above reflects the "scale" of the conflicts between factions V_1 and V_2, while the left-hand side is V_1 and V_2's average efforts in allying with V_3, multiplied by the size of V_3. From the three sufficient conditions, we learn that, the larger the size of V_3, the more capable it is of mediating the conflicts between V_1 and V_2. In addition, V_1 and V_2's strong willingness to ally with V_3, as well as the sentiment strength inside V_3, i.e., $\hat{\alpha}$, also help mediate the conflicts.

5.5.3 Distribution of initial conditions and formation of factions in the homophily-based model

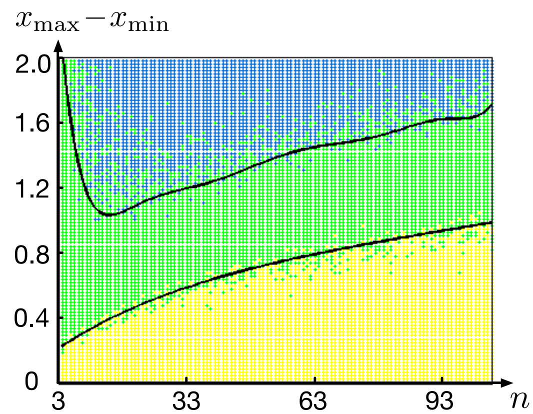
We investigate numerically, for the homophily-based model, the relation between the initial condition distribution and the formation of factions. The question of interest is whether the appraisal network evolves to only one faction or two antagonistic factions. We randomly and independently sample the entries of $X(0)$ from the uniform distribution with support $[x_{\min}, x_{\max}]$. The quantity $x_{\max} - x_{\min}$ indicates how spread out are the possible values taken by the initial appraisals, while $\text{ave}(x_{\min}, x_{\max}) = (x_{\max} + x_{\min})/2$ indicates how the initial appraisals are biased towards being positive. Given $[x_{\min}, x_{\max}]$, we independently generate 30 random samples of the initial condition $X(0)$ and count how many factions appear at $X(500)$. The simulations are conducted under two different set-ups:

Case 1: We set $x_{\max} - x_{\min} = 2$ and change the values of $\text{ave}(x_{\min}, x_{\max})$ and the number of agents. Since any $X(0)$ and $-X(0)$ lead to the same $X(1)$ and $X(t)$ thereafter, we only consider different values of $\text{ave}(x_{\min}, x_{\max}) \geq 0$. Figure 5.3(a) shows that, for fixed network size, the smaller the value of $\text{ave}(x_{\min}, x_{\max})$, the more likely is to find two antagonistic factions; for fixed value of $\text{ave}(x_{\min}, x_{\max})$, the larger the network size, the more likely that only one faction emerges.

Case 2: We set $\text{ave}(x_{\min}, x_{\max}) = 1$ and change $x_{\max} - x_{\min}$. Figure 5.3(b) shows that,



(a) Case 1



(b) Case 2

Figure 5.3: Formation of factions under different initial condition distributions. The blue color indicates the presence of two factions in all the 30 random samples, while the yellow color indicates the presence of one factions in all of the samples. The green color indicates any other case.

for fixed network size n, the larger $x_{\max} - x_{\min}$, the more likely to find two antagonistic factions; For fixed $x_{\max} - x_{\min}$, the larger the network size, the more likely that only one faction emerges.

5.6 Conclusion

This chapter proposes two novel discrete-time dynamical models for the bounded evolution of interpersonal appraisal networks towards social balance. Under a technical condition, theoretical analysis shows that both models exhibit asymptotic convergence to structurally balanced networks. Each model uses different social updating mechanisms for updating the appraisals and, as a result, the asymptotic balanced states are qualitatively different between the two models. Numerical study indicates how the final emergence of factions in the social network is sensitive to the initial distribution of appraisals among its agents. Moreover, our models admits the existence of two or more isolated cliques in the final structure of the evolved social network, and simulation results reveal interesting sociological phenomena when they are under certain classes of perturbations. Possible future research directions include a better understanding of the influence-based model for arbitrary initial conditions, a validation of the proposed models with laboratory and/or field data, the study of asynchronous models with pairwise updates, and the study of conditions and cases in which one socio-psychological mechanism dominates the other.

Part III

Future Research Directions

In this part we briefly introduce some potentially interesting general future research directions regarding social network dynamics.

Opinion evolution as the best-response dynamics

Consensus in opinion dynamics has been widely studied. The classic DeGroot model predicts consensus when the influence network contains a globally reachable node [2]. However, in reality, one can observe that, opinion consensus usually occurs only in groups with relatively small sizes, while persistent disagreement and even opinion polarization are common phenomena in large-scale social networks. Most of the current models explain the presence of persistent disagreement by incorporating some additional assumptions into the DeGroot model, such as the lack of network connectivity [2], the presence of negative weights in the influence network [165], the existence of stubborn nodes [166], the individuals' bounded confidence [167], and the individuals' persistent attachment to their initial opinions [168]. These additional assumptions do not fully capture the roles network structure and network size play in the emergence of opinion disagreement and polarization. In fact, if we consider the evolution of individuals' opinions as a best response dynamics, see the recent work by Bindel et al. [169], we can find that, the cost function for DeGroot model takes the quadratic form, which implies the individuals' unrealistically high sensitivity to opinions distant from their own. The high sensitivity to distant opinions forces the individual opinions to converge to consensus. One could expect that, by only modifying the cost function, a novel opinion dynamics model would be derived. The new model will reveal some interesting sociological insights and bifurcation behavior, and serves as a unified framework connecting different opinion dynamics such as the averaging protocols and the voter models. Moreover, it is also interesting to study the particularly interesting case in which individuals in the network have heterogeneous forms of cost functions.

Dynamic social balance as network games

Structural balance theory is a classic highly-successful framework from mathematical sociology [139, 170] aiming to describe allowable and forbidden configurations of topologies of signed directed social sentiment networks. A key research trend in the last decade, see [157, 106, 162], has been the development of dynamic balance theories to describe how a network moves towards structural balance from an initially unbalanced configuration. Only very limited attention [160, 171] has been paid so far on game theoretical models of balanced networks and on network games informed by cognitive dissonances in interpersonal relations. Important progress can be made by a synergistic integration of strategic interests and psychological response to cognitive dissonances. Specifically, the game-theoretic dynamic structural balance model could be mathematically formalized based on the following two mechanisms, corresponding to individuals' strategic interests and psychological response to cognitive dissonance respectively. The strategic mechanism assumes that individuals benefit from friendly social links. The utility obtained from each friendly link is proportional to the link weight, i.e., the strength of friendship, which in turn increases with the number of common friends and common enemies. The psychological mechanism is relevant to the concept of ego-network, i.e., the subgraph made up of the individual itself, all its social neighbors and all the social links among these nodes. Individuals change the signs of their own social links to minimize the number of unbalanced triads in their ego-networks. In addition, it remains an open problem to study the noisy settings and characterize appropriate notions of approximate balance, and stochastic stability.

Further mathematical formalization of transactive memory systems (TMS)

The assign/appraise/influence dynamics proposed in [172] applies to the class of team tasks that are completed by allocating resources or assigning workloads to the team members. The individuals interact with each other in a decentralized manner and the system asymptotically achieve optimal team performance. Based on the previous work, it would be of significant research value to further investigate the role of TMS in two other types of tasks: the role-playing tasks and the group decision-making tasks. In a role-playing task, a team need to assign its members to multiple roles requiring different sets of skills, which can be mathematically modeled as an assignment problem [173]. On the other hand, in a group decision-making task, a team consisting of individual members with heterogeneous information sets is required to make a single or a sequence of collective decisions/motions. This process naturally involves the influence systems [141, 174], which have been extensively studied in mathematical sociology. For both types of tasks, there is currently a dearth of mathematical models that characterize how the TMS facilitates the development of performance and how certain network structures inside a team evolve along the task sequence. Potentially, theoretical analysis of such models might lead to some well-motivated "socio-inspired" distributed optimization algorithms.

Novel model connecting team performance, evolutionary dynamics, and structural balance

Researchers in sociology and management science have long been interested in investigating the relation between team performance and structural balance. Evolutionary games might provide an innovative tool to approach this problem mathematically. In evolutionary games, matrix games with symmetric payoff matrices are referred to as *partnership games* [175]. For such games, the corresponding replicator equation is written as,

$$\dot{x} = \mathrm{diag}(x)\Big(Ax - (x^\top Ax)\Big),$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and x is in the n-simplex. It is known that for any initial condition in the simplex, along the replicator dynamics, $x^\top(t)Ax(t)$ is non-decreasing.

If we interpret this system as the dynamics of team performance, in which x denotes the resource allocation or workload assignment among the team members and each A_{ij} denotes the productivity of the collaboration between team member i and j, we could observe that, along the dynamics, the team productivity (performance), characterized by $x^\top(t)Ax(t)$ is non-decreasing. Furthermore, if we allow the entries of A to be negative, it would be an interesting problem to investigate how the sign pattern of the graph $\mathcal{G}(A)$ associated with the adjacency matrix A influence the asymptotic team productivity. It remains an open question whether and in what sense a structurally balanced network $\mathcal{G}(A)$ leads to better asymptotic team productivity.

Mathematical connections between random graph models and network dynamical systems

One of the main difficulties in the research on large-scale networks is the lack of complete and well-quantified local details. Generally speaking, there are currently two widely-adopted approaches to deal with this difficulty. The first approach is to model the large-scale networks as random graphs. The advantage of this approach is that, some of the global characteristics of the random graphs, e.g., the degree distribution and the diameter, are relatively easy to be statistically estimated using the rapidly growing technology of data mining. The second approach is to first assume that the network has an arbitrary topology with all the connections well-quantified and known, then derive the theoretical results that do not depend on all the local details of the network. The advantage of the second approach is that, more sophisticated and realistic dynamical processes can be modeled and understood based on the well-established mathematical tools in dynamical systems, matrix analysis and algebraic graph theory. These two approaches are both powerful. However, they often lead to different and inconsistent conclusions for the same dynamical process being modeled. Take the epidemic spreading model as a concrete example. The epidemic threshold obtained by the random graph approach is determined by the nodes' degree distribution, while the second approach gives the threshold as a function of the spectral radius of the contact network's adjacency matrix. It is still unknown how this spectral radius relates to the degree distribution. Due to the inconsistency between these two approaches, it is of great scientific value to investigate the fundamental mathematical connections between the random graph models and the network dynamics built on algebraic graph theory.

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