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# Opinion Dynamics with Heterogeneous Interactions and Information Assimilation

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[2] P. Jia, A. MirTabatabaei, N. E. Friedkin, and F. Bullo. Opinion dynamics, reflected appraisal, and the evolution of social influence networks. March 2013. Submitted.

[3] A. Mirtabatabaei, P. Jia, and F. Bullo. Eulerian opinion dynamics with bounded confidence and exogenous inputs. July 2013. Submitted.

**Book Chapters and Conference Articles**

[1] P. Jia, A. Mirtabatabaei, N. E. Friedkin, and F. Bullo. On the dynamics of influence networks via reflected appraisal. Washington, DC, USA, June 2013. To appear

[2] A. MirTabatabaei, P. Jia, and F. Bullo. Eulerian opinion dynamics with bounded confidence and exogenous inputs. In IFAC Workshop on Distributed Estimation and Control in Networked Systems, Santa Barbara, CA, USA, September 2012. To appear

[3] A. MirTabatabaei, F. Bullo, and M. Khammash. Flow cytometry based state aggregation of a stochastic model of protein expression. In IEEE Conf. on Decision and Control and European Control Conference, pages 4383-4388, Orlando, FL, USA, December 2011

[4] A. MirTabatabaei and F. Bullo. On opinion dynamics in heterogeneous networks. In American Control Conference, pages 2807-2812, San Francisco, CA, USA, June 2011

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# Abstract

## Opinion Dynamics with Heterogeneous Interactions and Information Assimilation

Anahita MirTabatabaei

In any modern society, individuals interact to form opinions on various topics, including economic, political, and social aspects. Opinions evolve as the result of the continuous exchange of information among individuals and of the assimilation of information distributed by media. The impact of the individuals' opinions on each other forms a network, and as the time progresses, their opinions change as a function of structure of such network. It is a central question whether this interaction and assimilation process leads to a socially beneficial aggregation of information. Considering a large population allows approximation of the decision making rules with Non-Bayesian "rule of thumb" methods without relying on detailed social psychological findings. This thesis mainly addresses complex problems in the analysis of opinion evolution in (a) heterogeneous societies and (b) societies with large population under the influence of exogenous events.

In the study of opinion dynamics in heterogeneous social networks, we modeled the system as follows: the neighbors of each agent can be defined as either (1) those agents whose opinions are in its confidence range, or (2) those agents whose

influence range contain the agents opinion. The former definition is employed in Hegselmann and Krauses bounded confidence model, and the latter is novel in our work. As the confidence and influence ranges are distinct for each agent, the heterogeneous state-dependent interconnection topology leads to a poorly-understood complex dynamic behavior. For both models, we have classified the agents via their interconnection topology and, accordingly, characterized the equilibria of the system. Additionally we have introduced a notion of a positive-invariant set centered at each equilibrium point: if a trajectory enters one such set, it converges to a steady state with constant interconnection topology. This result gives us a novel sufficient condition for both models to establish convergence.

In the study of opinion dynamics in societies with large population, we analyze the behavior of an Eulerian bounded confidence model of opinion dynamics with time-varying input. In this model, a population is distributed over an opinion set and updates its opinion via 1) the opinion of the population inside the confidence range, and 2) the information from an exogenous input in that range. First, we prove some fundamental properties of this system's dynamics with time-varying input. We derive a simple sufficient condition for opinion consensus, and prove the convergence of population's distribution under time-invariant input to a sum of Dirac Delta functions. We compute an empirical upper bound on the largest range of opinions that a fixed Gaussian distributed input can attract to its center.

Finally, we define the *attraction range* of an input, and for a normally distributed input and uniformly distributed initial population, respectively, we conjecture a linear relation between this range's length, population's confidence bound, and input's variance. Accordingly to the limited attraction range of manipulator, we compare different manipulation strategies.

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# Chapter 1

# Introduction

After recent cases of bankruptcy in the western countries, many have been interested in anticipating the final decision made by individuals in a society or the directors of the largest corporations of a country. The impact of the individuals' opinions on each other forms a network, and as the time progresses, their opinions change as a function of structure of such network. One important question is that how the topological properties of the network can affect the individuals' final decisions, and how long it takes them to reach final decisions. Decision-making is a complex process, which is led to the final state by *endogenous* and *exogenous* factors. The interaction of people via in person meetings or online social networks is an endogenous factor. One of the most influential exogenous factors is the mainstream media that acts as a real-time input owing to its easy access to the public. "After introduction and expansion of Fox News, between 1996 and 2000, it is estimated that 3-28 percent of the audiences was persuaded to vote Republican" [18]. Owing to the media's easy access to the public, they can quickly get out their message and hence act as a real-time input in the opinion evolution of decision makers. Models of social networks and opinion dynamics are structures made up of individuals that are tied based on their interdependency. Such models explain the confidence or influence flow in populations without relying on detailed social psychological findings.

## 1.1 Literature Review

In this section we give a brief literature review of the main references of the various topics or tools mentioned in this thesis.

### 1.1.1 Opinion Dynamics: A Brief History

In the field of social networks, opinion dynamics is of high interest in many areas including: politics, as in voting prediction [3]; physics, as in spinning particles [4]; sociology, as in the diffusion of innovation [52], the electronic exchange of personal information [40], and language change [17, 50]; and finally economics, as in price change [49]. The study of opinion dynamics and social networks goes back to the early work by J.R.P. French [20] on "a formal theory of social power." This work explores the patterns of interpersonal relations and agreements that can explain the influence process in groups of agents. Subsequently, F. Harary provides a necessary and sufficient condition to reach a consensus in French's model of power networks [24]. Later in 1959, Harary and Cartwright introduced signed graphs to handle a problem in social psychology. Modeling of "continuous opinion dynamics", in which opinions are represented by real positive numbers, is initially studied in [15, 28, 48]. In contrast to the classical case of "binary opinion dynamics" [22, 54], the continuous case deals with the problem of what happens to the worthiness of a choice or the probability of choosing one decision over another.

### 1.1.2 Opinion Evolution of a Finite Population

A popular opinion update rule is the non-Bayes method of averaging neighbors' opinions [1]. This "rule of thumb" method provides a good approximation to the behavior of a large population without relying on detailed social psychological findings. In social interaction networks, a common way of defining neighboring relation is based on *bounded confidence* (BC), a label coined by Krause in 1998 [29]. BC models are models of continuous opinion dynamics in which an individual only interacts with those whose opinions are close enough to its own. This idea reflects: 1) *filter bubbles*, a phenomenon in which websites use algorithms to show users only information that agrees with their past viewpoints [44]; and 2) *selective exposure*, a psychological concept broadly defined as "behaviors that bring the communication content within reach of one's sensory apparatus" [39, 59].

Recently, Hegselmann and Krause formulated a bounded confidence model (HK model) where agents synchronously update their opinions by averaging all opinions in their confidence bound [25]. In the online world, individuals gather information from their friends synchronously as opposed to a pairwise gossip manner. Another popular version of BC models was developed and investigated by Deffuant and Weisbuch [55], called DW model. The HK and DW models are very similar, they differ in their update rule: in the DW model a pairwise-sequential updating procedure is employed instead of the synchronized one. In the HK model, the set of neighbors of the $i$th agent is defined as those agents whose opinions differ from the $i$th opinion by less than the $i$th confidence bound. Hence, this model is dealing with *endogenously* changing topologies, that is, state dependent or changing from inside, in contrast to the *exogenously* changing topologies. For instance, [14, 17, 27, 42] study a synchronized linear averaging model with time-dependent exogenously changing topologies. The HK models are classified based on various factors: a model is called agent- or density-based if its number of agents is finite or infinite, respectively; and a model is called homogeneous or heterogeneous if its confidence bounds are uniform or agent-dependent, respectively. The convergence of both agent- and density-based homogeneous HK models are discussed in [6]. The agent-based homogeneous HK system is proved to reach a fixed state in finite time [19], the time complexity of this convergence is discussed in [37], its stabilization theorem is given in [31], and its rate of convergence to a global consensus is studied in [43]. The heterogeneous HK model is studied by Lorenz who reformulated the HK dynamics as an interactive Markov chain [32], analyzed the effects of heterogeneous confidence bounds [36], and characterized the sets of fixed points [35]. The convergence of the agent-based heterogeneous HK systems is experimentally observed, but its proof is still an open problem.

### 1.1.3 Opinion Evolution of an Infinite Population

According to [13, 26], models of continuous opinion dynamics can be described by either a *Lagrangian* or an *Eulerian* method. "In fluid dynamics, the fluid motion is described in a Lagrangian way if the observer follows an individual fluid parcel as it moves through space and time. On the other hand, the fluid motion is described in an Eulerian way if the observer is is fixed at one location through which the fluid flows as time passes" [57]. A Lagrangian description focuses on changes in each agent's opinion; however, an Eulerian description focuses on changes in population or the probability distribution associated to each opinion interval. A Lagrangian model of opinion dynamics is defined over a continuous [6] or discrete state space [6, 25, 36, 43] if the number of agents are infinite or finite, respectively. An Eulerian model of opinion dynamics is defined over a continuous [13, 16, 34, 35] or discrete state space [2] depending on whether the opinion grid size is converging to zero or not, respectively. The behavior of both discrete and continuous Lagrangian models of opinion dynamics under homogeneous bounds of confidence is discussed in [5].

Similar to the Lagrangian HK model of opinion dynamcis, an Eulerian HK model is defined over a continuous state space [13, 34], where a *mass distribution* over an opinion set is being updated by a *flow map*. The flow map for any opinion value is a function of the average opinion of the population inside the confidence range of that opinion value. Previously, (Canuto et al, 2008) proved the convergence of a variation of Eulerian HK model both in discrete and continuous time. In their model, the weights that two opinion values assign to each other are equal, and this symmetry preserves the global average during the evolution. In this context, we consider a more general Eulerian HK model where the symmetric weight constraint has been relaxed. Specifically, the weight an opinion assigns to other opinions is a function of the integral of the mass distribution (and of the exogenous input measure in our model) in that opinion's confidence bound. Since, the measures on different opinions' confidence bounds are not necessarily equal, the weights assigned to different opinions are generally asymmetric, and thus the global average is not preserved.

## 1.2 Contribution of the Thesis

The organization of the thesis, and the contributions of each chapter can be summarized as follows.

**Chapter 2 - Finite Homogeneous Population:** We first establish some fundamental properties of the Lagrangian homogeneous HK model. Previously, (Martínez et al. 2007) proved that the Lagrangian homogeneous HK model of opinion dynamics has a time complexity of order $O(n^5)$ [38], where $n$ is the number of agents. Here, we prove that this time complexity is of the lower order $O(n^4)$.

**Chapter 3 - Finite Heterogeneous Population:** We focus on heterogeneous HK models, and in order to distinguish between the HK and DW models, we call a discrete-time agent-based heterogeneous HK model a *synchronized bounded confidence* (*SBC*) model. Additionally, we introduce a model similar to the SBC model and call it the *synchronized bounded influence* (*SBI*) model. The difference is that in an SBI model the set of neighbors of the $i$th agent is defined as those agents $j$ whose influence range contain the $i$th agent's opinion. We analyze SBC and SBI models with heterogeneous bounds of confidence or influence, respectively. Indeed, if the SBC and SBI models have agents with homogeneous bounds, then both models are equivalent to the homogeneous HK model. These heterogeneous models of opinion dynamics, in spite of the considerable complexity of their dynamics, describe features of a real society that cannot be explained by homogeneous models. Specifically, the behavior of a heterogeneous opinion dynamics model is richer than that of a homogeneous model in the following ways: (1) an agent may trust an individual but not be trusted back by that same individual; (2) in steady state, an agent can keep its own opinion constant while listening to dissimilar opinions, (whereas, in the steady state of a homogeneous society, any two agents are either disconnected or in consensus); (3) one can observe *pseudo-stable* configurations, i.e., "configurations that have a subset of agents that is stationary, and the rest are the reason for further dynamics" [51]; (4) it is possible for two disconnected agents to reconnect, and an agent with a large bound of confidence or influence can pull clusters of agents towards or away from each other; (5) the order of opinions is not preserved along the system evolution; (6) "given an average level of confidence, diversity of bounds of confidence enhances the chances for consensus" [36]; and (7) convergence in infinite time is possible.

Based on numerical evidence, we formulate our main conjecture that along the evolution of an SBC or SBI system there exists a finite time, after which the topology of the interconnection network remains unchanged, and as a result, the trajectory converges to a limiting opinion vector. We also observe that each trajectory either reaches a fixed state in finite time or exhibits a pseudo-stable behavior. This observation is verified assuming that the main conjecture is true. Furthermore, the following results put together partly prove our main conjecture: (1) We design an appropriate classification of agents in both SBC and SBI systems. This classification is a function of state-dependent interconnection topology of the system, and can explain the observed pseudo-stable behavior. (2) We introduce the new notion of *final value at constant topology*, and based on our classification, we formulate the map under which this value is an image of the current opinion vector. The set of final values at constant topology is a superset of the equilibria of the system. We derive necessary and sufficient conditions for the final value at constant topology to be an equilibrium vector. (3) For each equilibrium opinion vector, we define its *equi-topology neighborhood* and *invariant equi-topology neighborhood*. We show that if a trajectory enters the invariant equi-topology neighborhood of an equilibrium vector, then it remains confined to its equi-topology neighborhood, and sustains an interconnection topology equal to that of the equilibrium vector. This fact establishes a novel and simple sufficient condition under which: the initial opinion vector converges to a steady state; the topology of the interconnection network remains unchanged; and the limiting opinion vector is equal to the final value at constant topology of the initial opinion vector. (4) We define a rate of convergence as a function of final value at constant topology. Based on the direction of convergence and the defined rate, we derive a sufficient condition under which the trajectory *monotonically* converges to a steady state, and the topology of the interconnection network remains unchanged. (5) We explore some interesting behavior of classes of agents when they update their opinions under fixed interconnection topology for infinite time. For instance, we compute agents' rates and directions of convergence, and show the existence of a leader group for each group of agents that determines the follower's rate and direction of convergence. In our extensive simulation results, we observe that for uniformly randomly generated initial opinion vector and bounds vector, the SBC and SBI trajectories eventually satisfy our novel sufficient condition for convergence with probability one. We give some intuitive explanation for this observation. Finally, we conjecture that the SBI trajectories reach a fixed state in finite time more often than the SBC trajectories. To substantiate this conjecture, we present sufficient conditions for SBC and SBI systems that guarantee convergence in finite time.

**Chapter 4 - Infinite Population:** We study the opinion evolution among a large population via the Eulerian HK model of opinion dynamics. First, we derive a simple sufficient condition for the system to reach opinion consensus. Second, we establish some important properties of the Eulerian HK model. Finally, this analysis also leads to a convergence proof of the mass distribution to a sum of Dirac Delta functions.

**Chapter 5 - Opinion Manipulation in Infinite Population:** The contributions are mainly four-fold. 1) We propose a reasonable model for exogenous inputs in the Eulerian HK opinion-dynamics model. We derive a simple sufficient condition for the system to reach opinion consensus. 2) We establish some important properties of the Eulerian HK model with a time-varying input. Under mild technical assumptions (the initial opinion is a finite and absolutely continuous mass distribution over the opinion set), we show that the opinion update via an Eulerian flow map has the following properties: i) the mass distribution on opinions remains finite and absolutely continuous; ii) the flow map preserves opinion order, due to the homogeneity of confidence bounds; and iii) the flow map is bi-Lipschitz. 3) We represent the exogenous input by a background Gaussian distribution centered at the advertised opinion. We introduce the *attraction range* of an input, which is the largest range of opinions that the input can attract to its center. We conjecture a linear relation between attraction range, input's variance, and confidence bound. Accordingly, we compare two different manipulation strategies that aim to increase the size of population who votes positively in finite time. 4) We present a real world example of decision making in a committee of experts, whose interconnection network is constructed via their meetings transcripts.

In [10], the medical device advisory panel in the US Food and Drug Administration is analyzed, and a novel method in construction of experts' interconnection network via employing the *Author-Topic* model to their meetings documents is presented. However, the dynamics of the process of decision making by the committee is not analyzed in [10]. Here, we introduce a Lagrangian mutli-dimensional HK model whose interconnection network is strongly correlated with the real example's network. We highlight common features between the dynamics of the introduced model and the presented real world example. We approximate the Lagrangian model with an Eulerian HK model, and then we discuss manipulation strategies that can alter the final state of opinion evolution process.

Finally, Chapter 6 presents summaries of the results and suggested future research directions.

# Chapter 2

# Finite Homogeneous Population

In the study of *Lagrangian homogeneous HK* models of opinion dynamics, recently introduced by Hegselmann and Krause, the focus is on the changes in each agent's opinion as agents synchronously update their opinions by averaging all opinions in their neighborhood. The set of neighbors of the $i$th agent is defined as those agents whose opinions differ from the $i$th opinion by less than its confidence bound, where the confidence bounds are uniform. Previously, (Martínez et al. 2007) proved that the Lagrangian homogeneous HK model of opinion dynamics has a time complexity of order $O(n^5)$, where $n$ is the number of agents. "The time complexity of an algorithm is the minimum number of communication rounds required by the agents to achieve the task" [38]. For the Lagrangian homogeneous HK models of opinion dynamics, time complexity is the minimum iterations for the agents to achieve either consensus or *local rendezvous*. By local rendezvous we mean separate agent clusters rendezvous at multiple final opinions while preserving disconnectivity. In this chapter, we prove that the time complexity of the Lagrangian homogeneous HK model is of the lower order $O(n^4)$.

## 2.1 Homogeneous HK Model

Given the confidence bound $r \in \mathbb{R}_{>0}$, we associate to each opinion vector $x(t) = y \in \mathbb{R}^n$ the *proximity graph* $G_r(y)$ with nodes $\{1, \ldots, n\}$ and edge set defined as follows: the set of out-neighbors of node $i$ is $\mathcal{N}_i(y) = \{j \in \{1, \ldots, n\} : |y_i - y_j| \leq r\}$. The Lagrangian homogeneous HK model of opinion dynamics updates $x(t)$ according to

$$x(t+1) = A(x(t))x(t), \tag{2.1}$$

where, denoting the cardinality of $\mathcal{N}_i(y)$ by $|\mathcal{N}_i(y)|$, the $i, j$ entry of $A(x(t) = y)$ is defined by

$$a_{ij}(y) = \begin{cases} \frac{1}{|\mathcal{N}_i(y)|}, & \text{if } j \in \mathcal{N}_i(y), \\ 0, & \text{if } j \notin \mathcal{N}_i(y). \end{cases}$$

Notice that since each agent is its own out-neighbor, $\mathcal{N}_i(y)$'s are nonempty. From here on, we call the set of $x_i(t)$'s with $i \in \{1, \ldots, n\}$ the *opinion profile* at time $t$, and the opinion difference between minimum and maximum values of the opinion profile the *opinion range*. An *opinion sub-profile* is a subset of the main profile, associated with a connected sub-graph of the main proximity graph.

**Lemma 2.1.1** (Properties of homogeneous HK). *For any $y \in \mathbb{R}^n$, the trajectory $x(t)$ of a homogeneous HK system (2.1) with $x(0) = y$, satisfies the following properties for all $t \geq 0$:*

*(i) Given the initial profile is ordered in the sense that $x_1(0) \leq x_2(0) \leq, \dots, x_n(0)$, then the order will be preserved over time.*

*(ii) If two immediate neighbors split, they will no longer interact.*

*(iii) The maximum convergence time and the minimum change in the opinion of agents 1 and $n$ are increasing and decreasing functions of number of agents, respectively.*

*Proof.* Regarding part (i), by induction, assume that the opinion profile is ordered at time $t$. Then, for any $m \in \{1, \dots, n\}$, let $U_1, U_2$, and $U_3$ denote the three sets $\mathcal{N}_m(x(t)) - \mathcal{N}_{m+1}(x(t))$, $\mathcal{N}_m(x(t)) \cap \mathcal{N}_{m+1}(x(t))$, and $\mathcal{N}_{m+1}(x(t)) - \mathcal{N}_m(x(t))$, respectively. Under ordered opinion profile, for nonempty sets $U_{1,2,3}$,

$$\frac{\sum_{j \in U_1} x_i(t)}{|U_1|} < \frac{\sum_{j \in U_2} x_i(t)}{|U_2|} < \frac{\sum_{j \in U_3} x_i(t)}{|U_3|}.$$

Hence, for any two nonempty sets $U_{i,j}$, where $i < j$, it holds that

$$|U_j| \sum_{k \in U_i} x_k(t) < |U_i| \sum_{k \in U_j} x_k(t).$$

Consequently, knowing that at least two out of the three sets $U_i$'s are nonempty, the following inequality holds

$$\begin{aligned}
&|U_2| \sum_{k \in U_1} x_k(t) + |U_3| \sum_{k \in U_1} x_k(t) + |U_2| \sum_{k \in U_2} x_k(t) + |U_3| \sum_{k \in U_2} x_k(t) \\
&\quad \leq |U_1| \sum_{k \in U_2} x_k(t) + |U_2| \sum_{k \in U_2} x_k(t) + |U_1| \sum_{k \in U_3} x_k(t) + |U_2| \sum_{k \in U_3} x_k(t), \\
&\quad \Rightarrow \frac{\sum_{k \in U_1} x_k(t) + \sum_{k \in U_2} x_k(t)}{|U_1| + |U_2|} \leq \frac{\sum_{k \in U_2} x_k(t) + \sum_{k \in U_3} x_k(t)}{|U_2| + |U_3|}, \\
&\qquad \Rightarrow x_m(t+1) \leq x_{m+1}(t+1).
\end{aligned}$$

Regarding part (ii), according to [25], "the *extreme opinions* of a split opinion (sub-)profile are under one sided influences and converge toward the center of the (sub-)profile. As a consequence the range of the (sub-)profile shrinks."

Regarding part (iii), refer to (Martínez et al. 2007). □

**Corollary 2.1.2.** *It can be concluded from Lemma 2.1.1 part (i) that for any two opinions $x_i(t) < x_j(t)$ we have:*

$$|\mathcal{N}_j(x(t))| \sum_{k \in \mathcal{N}_i(x(t))} x_k(t) < |\mathcal{N}_i(x(t))| \sum_{k \in \mathcal{N}_j(x(t))} x_k(t).$$

## 2.2 Time Complexity

In this section, we compute the time complexity of the Lagrangian homogeneous HK models in achieving consensus or local rendezvouses.

**Theorem 2.2.1.** *The time complexity of homogeneous HK system (2.1) is of order $O(n^4)$, where $n$ is the number of agents.*

*Proof.* We follow a similar approach to the proof by (Martínez et al. 2007), that is, knowing the maximum opinion range, we compute a lower bound on the reduction rate of opinion range. Assuming that the initial profile is connected, the opinion range is less than $nr$. Otherwise, according to Lemma 2.1.1 part (ii), separated sub-profiles will converge toward the stable state synchronously with smaller number of agents, thus in fewer time steps, and total convergence time will be lower. Consider one of sub-profiles at any time $t$, whose opinion range is less than $nr$. Let $x_1(t)$ denote the smallest opinion of the opinion (sub-)profile under discussion, $\alpha_{i+1}$ denote the cardinality of $\mathcal{N}_1(x(t+i))$, $k_{i+1}$ denote the cardinality of the set of agent 1's neighbors that are not agent $\alpha_{i+1}+1$'s neighbors at time $t+i$ for $i=0,\ldots$. One can show that the initial opinion range is less than or to equal $(n-2\alpha_1)r+2r$, however, substituting this upper bound with $nr$ does not affect the order of complexity.

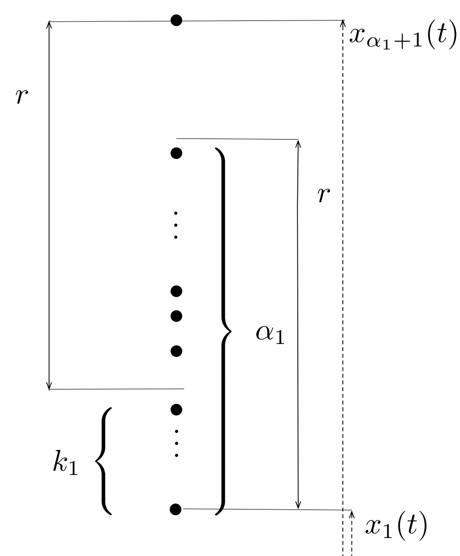



**Figure 2.1:** An illustration of the lower extreme agent in an opinion profile with opinion $x_1(t)$, where $\alpha_1 = |\mathcal{N}_1(x(t))|$ and $k_1$ is the number of agent 1's neighbors that are not agent $\alpha_1 + 1$'s neighbors.

Here, we discuss the behavior of the system after every $m$ iterations, where $m \leq n$. At any time step, if the sub-profile undergoes separation, then the opinion range shrinks $r$ units. Therefore, the reduction rate of opinion range will be at least $r/n$. This value is larger than the minimum rate obtained in the following by assuming that the sub-profile is connected. Under the assumption of continuity of the opinion sub-profile, it holds that $\alpha_i > k_i$ for all $1 \leq i \leq m$. According to Lemma 2.1.1 part (i),

$$x_i(t+1) \geq x_{k_1+1}(t+1) \geq \frac{x_1(t) + \cdots + x_{\alpha_1+1}(t)}{\alpha_1 + 1} \qquad\qquad i \geq k_1 + 1.$$

Now, two possible cases exist:

1) If $\alpha_2 > k_1$, then for the next iteration we can write:

$$\begin{aligned} x_1(t+2) \qquad &= \quad \frac{x_1(t+1) + \cdots + x_{k_1}(t+1) + x_{k_1+1}(t+1) + \cdots + x_{\alpha_2}(t+1)}{\alpha_2} \\ &\geq \quad \frac{1}{\alpha_2}\Big(k_1 \frac{x_1(t) + \cdots + x_{\alpha_1}(t)}{\alpha_1} + (\alpha_2 - k_1)\frac{x_1(t) + \cdots + x_{\alpha_1+1}(t)}{\alpha_1 + 1}\Big) \end{aligned}$$

Our goal is to find a lower bound on agent 1's opinion change after two time steps. Opinions $x_1(t), \ldots, x_{\alpha_1}(t)$ are lower bounded by $x_1(t)$, and since $\alpha_1 + 1$ is not a neighbor of agent 1 at time $t$, $x_{\alpha_1+1}(t)$ is lower bounded by $x_1(t) + r$. Therefore

$$\begin{aligned} x_1(t+2) \qquad &\geq \quad \frac{1}{\alpha_2}\Big(k_1 \frac{\alpha_1 x_1(t)}{\alpha_1} + (\alpha_2 - k_1)\frac{(\alpha_1 + 1)x_1(t) + r}{\alpha_1 + 1}\Big) \\ &= \quad \frac{1}{\alpha_2}\Big(k_1 x_1(t) + (\alpha_2 - k_1)x_1(t) + \frac{(\alpha_2 - k_1)r}{\alpha_1 + 1}\Big) \\ &= \quad x_1(t) + \frac{(\alpha_2 - k_1)r}{(\alpha_1 + 1)\alpha_2}. \end{aligned}$$

According to our assumption that $\alpha_2 > k_1$, $\alpha_2 - k_1$ is lower bounded by 1, and $\alpha_1$ and $\alpha_2$ are upper bounded by $n$, hence,

$$x_1(t+2) - x_1(t) \geq \frac{r}{n^2}.$$

2) If $\alpha_2 \leq k_1$, then $|\mathcal{N}_1(x(t+1))| \leq k_1 < |\mathcal{N}_1(x(t))|$, and thus number of neighbors of agent 1 strictly decreases in one iteration. Clearly, case (2) can hold for at most $n$ iterations, after which, under the continuity assumption, case (1) holds. Therefore, minimum decrease in opinion range in $n$ time steps will be equal to $\frac{r}{n^2}$.

So far, we have shown that for all sub-profiles, the opinion range reduces at least $\frac{r}{n^2}$ units in every $n$ iteration, and this reduction occurs synchronously. Knowing that the opinion range of any sub-profile is upper bounded by $nr$, the time needed for the system to convergence to a steady state is upper bounded by

$$\frac{nr}{r/n^3} = n^4$$

□

## 2.3 Summary

We analyzed a model of opinion dynamics recently introduced by Hegselmann and Krause: each agent in a group maintains a real number describing its opinion;

and each agent updates its opinion by averaging all other opinions that are within some given confidence range. HK models can be classified into heterogeneous and homogeneous models, if the confidence bounds are uniform or agent-dependent, respectively. Previously, (Martínez et al. 2007) proved that the Lagrangian homogeneous HK model of opinion dynamics has a time complexity of order $O(n^5)$, where $n$ is the number of agents. In this chapter, we proved that this time complexity is of the lower order $O(n^4)$.

# Chapter 3

# Finite Heterogeneous Population

In this chapter, we analyze heterogeneous models of opinion dynamics with finite number of agents, in which opinions are described by real numbers, and agents update their opinions synchronously by averaging their neighbors' opinions. The neighbors of each agent can be defined as either (1) those agents whose opinions are in its "confidence range," or (2) those agents whose "influence range" contain the agent's opinion. The former definition is employed in bounded confidence models, and the latter is novel here. As the confidence and influence ranges are distinct for each agent, the heterogeneous state-dependent interconnection topology leads to a poorly-understood complex dynamic behavior.

Mathematical models of opinion dynamics under bounded confidence have been presented independently by: Hegselmann and Krause (HK model) [25], where agents synchronously update their opinions by averaging all opinions in their confidence bound, and by Deffuant and Weisbuch and others (DW model) [55], where a pairwise-sequential updating procedure is employed instead of the synchronized one. Here, we focus on HK models, and in order to distinguish between the HK and DW models, we call a discrete-time agent-based heterogeneous HK model a *synchronized bounded confidence* (*SBC*) model. Additionally, we introduce a model similar to the SBC model and call it the *synchronized bounded influence* (*SBI*) model. The difference is that in an SBI model the set of neighbors of the $i$th agent is defined as those agents $j$ whose influence range contain the $i$th agent's opinion. We analyze SBC and SBI models with heterogeneous bounds of confidence or influence, respectively. Indeed, if the SBC and SBI models have agents with homogeneous bounds, then both models are equivalent to the homogeneous HK model. These heterogeneous models of opinion dynamics, in spite of the considerable complexity of their dynamics, describe features of a real society that cannot be explained by homogeneous models. Specifically, the behavior of a heterogeneous opinion dynamics model is richer than that of a homogeneous model in the following ways: (1) an agent may trust an individual but not be trusted back by that same individual; (2) in steady state, an agent can keep its own opinion constant while listening to dissimilar opinions, (whereas, in the steady state of a homogeneous society, any two agents are either disconnected or in consensus);

(3) one can observe *pseudo-stable* configurations, i.e., “configurations that have a subset of agents that is stationary, and the rest are the reason for further dynamics” [51]; (4) it is possible for two disconnected agents to reconnect, and an agent with a large bound of confidence or influence can pull clusters of agents towards or away from each other; (5) the order of opinions is not preserved along the system evolution; (6) ”given an average level of confidence, diversity of bounds of confidence enhances the chances for consensus” [36]; and (7) convergence in infinite time is possible.

Based on numerical evidence, we formulate our main conjecture that along the evolution of an SBC or SBI system there exists a finite time, after which the topology of the interconnection network remains unchanged, and as a result, the trajectory converges to a limiting opinion vector. We also observe that each trajectory either reaches a fixed state in finite time or exhibits a pseudo-stable behavior. This observation is verified assuming that the main conjecture is true. Furthermore, the following results put together partly prove our main conjecture: (1) We design an appropriate classification of agents in both SBC and SBI systems. This classification is a function of state-dependent interconnection topology of the system, and can explain the observed pseudo-stable behavior. (2) We introduce the new notion of *final value at constant topology*, and based on our classification, we formulate the map under which this value is an image of the current opinion vector. The set of final values at constant topology is a superset of the equilibria of the system. We derive necessary and sufficient conditions for the final value at constant topology to be an equilibrium vector. (3) For each equilibrium opinion vector, we define its *equi-topology neighborhood* and *invariant equi-topology neighborhood*. We show that if a trajectory enters the invariant equi-topology neighborhood of an equilibrium vector, then it remains confined to its equi-topology neighborhood, and sustains an interconnection topology equal to that of the equilibrium vector. This fact establishes a novel and simple sufficient condition under which: the initial opinion vector converges to a steady state; the topology of the interconnection network remains unchanged; and the limiting opinion vector is equal to the final value at constant topology of the initial opinion vector. (4) We define a rate of convergence as a function of final value at constant topology. Based on the direction of convergence and the defined rate, we derive a sufficient condition under which the trajectory *monotonically* converges to a steady state, and the topology of the interconnection network remains unchanged. (5) We explore some interesting behavior of classes of agents when they update their opinions under fixed interconnection topology for infinite time. For instance, we compute agents rates and directions of convergence, and show the existence of a leader group for each group of agents that determines the follower's rate and direction of convergence.

In our extensive simulation results, we observe that for uniformly randomly generated initial opinion vector and bounds vector, the SBC and SBI trajectories eventually satisfy our novel sufficient condition for convergence with probability one. We give some intuitive explanation for this observation. Finally, we conjecture that the SBI trajectories reach a fixed state in finite time more often than the SBC trajectories. To substantiate this conjecture, we present sufficient conditions for SBC and SBI systems that guarantee convergence in finite time.

This chapter is organized as follows. In Section 3.1, the mathematical models, conjectures, agents classification, and spectral properties of the adjacency matrices are presented. In Section 3.2, the final value at constant topology is introduced and characterized. Section 3.3 contains novel sufficient conditions for constant topology and convergence, for constant topology and monotonic convergence, and for convergence in finite time. In Section 3.4, the simulation results and intuitive explanations are presented. In Section 3.5, the behavior of the system assuming that its interconnection topology remains unchanged in a long run is analyzed. Finally, Section 6 contains the conclusion and open questions.

## 3.1 SBC and SBI Models

Consider $n$ interacting agents and assume that each agent's opinion is expressed by a real number, say $y_i$ for agent $i \in \{1, \dots, n\}$. In *bounded confidence interaction*, the opinion $y_i$ is affected by the opinion $y_j$ if $|y_i - y_j| \leq r_i$, where the positive number $r_i$ is the *confidence bound* of agent $i$. In *bounded influence interaction*, the opinion $y_i$ is affected by the opinion $y_j$ if $|y_i - y_j| \leq r_j$, where the positive number $r_j$ is the *influence bound* of agent $j$. The *opinion vector* $y \in \mathbb{R}^n$ and the *bounds vector* $r \in \mathbb{R}^n_{>0}$ are obtained by stacking all $y_i$'s and $r_i$'s, respectively. We associate to each opinion vector $y$ two digraphs, both with nodes $\{1, \dots, n\}$ and edge set defined as follows: denoting the set of out-neighbors of node $i$ by $\mathcal{N}_i(y)$

- in a *synchronized bounded confidence* (SBC) digraph, $\mathcal{N}_i(y) = \{j \in \{1, \dots, n\} : |y_i - y_j| \leq r_i\}$; and
- in a *synchronized bounded influence* (SBI) digraph, $\mathcal{N}_i(y) = \{j \in \{1, \dots, n\} : |y_i - y_j| \leq r_j\}$.

We let $G_r(y)$ denote one of the two *proximity digraphs*, its precise meaning being clear from the context.

We associate to the SBC and SBI digraphs two dynamical systems, called the *SBC* and *SBI systems* respectively. Both dynamical systems update a trajectory

$x : \mathbb{N} \to \mathbb{R}^n$ according to the discrete-time and continuous-state rule

$$x(t+1) = A(x(t))x(t), \tag{3.1}$$

where the $i, j$ entry of the *adjacency matrix* $A(y) \in \mathbb{R}^{n\times n}$ for any $y \in \mathbb{R}^n$ is defined by

$$a_{ij}(y) = \begin{cases} \frac{1}{|\mathcal{N}_i(y)|}, & \text{if } j \in \mathcal{N}_i(y), \\ 0, & \text{if } j \notin \mathcal{N}_i(y), \end{cases}$$

and $|\mathcal{N}_i(y)|$ is the cardinality of $\mathcal{N}_i(y)$. Note that $i \in \mathcal{N}_i(y)$, in other words, every agent has some self-confidence or self-influence. This assumption is a key factor in the convergence of infinite products of adjacency matrices [33]. In the following, we present some interesting conjectures on SBC and SBI systems, and the trajectories of Figure 3.1 support these conjectures.

**Conjecture 3.1.1** (Existence of a limiting opinion vector)**.** *Every trajectory of an SBC or SBI system converges to a limiting opinion vector.*

**Conjecture 3.1.2** (Constant-topology in finite time)**.** *For any trajectory $x(t)$ of an SBC or SBI system, there exists a finite time $\tau$ after which the state-dependent interconnection topology, or equivalently $G_r(x(t))$, remains constant.*

Before proceeding, let us define a term borrowed from [51]. A trajectory $x(t) \in \mathbb{R}^n$ that is converging to limiting opinion vector $x_\infty \in \mathbb{R}^n$ is said to have a *pseudo-stable* behavior after $\tau$, if the node set $\mathcal{V} = \{1, \dots, n\}$ is composed of two non-empty subsets $\mathcal{V}_{\text{fixed}}$ and $\mathcal{V}_{\text{converging}}$ such that, for all $t \geq \tau$,

$$\begin{cases} x_i(t) = x_{\infty,i}, & \text{if } \ i \in \mathcal{V}_{\text{fixed}}, \\ x_i(t) < x_i(t+1) < x_{\infty,i} \ \ \textit{or} \ \ x_i(t) > x_i(t+1) > x_{\infty,i}, & \text{if } \ i \in \mathcal{V}_{\text{converging}}. \end{cases} \tag{3.2}$$

**Conjecture 3.1.3** (Pseudo-stable behavior)**.** *For any SBC or SBI trajectory, there exists a finite time after which the trajectory either reaches a fixed state or exhibits a pseudo-stable behavior.*

**Conjecture 3.1.4** (Convergence of SBI systems versus SBC systems)**.** *For any uniformly-randomly generated initial opinion vector and bounds vector (in compact sets), the SBI system converges in finite time with higher probability than the SBC system.*

In the remainder of this chapter, we are going to analyze the trajectories of SBI and SBC systems and establish some of their convergence properties. In Section 3.3 we establish convergence under some necessary conditions, but we will not fully establish the conjectures. In Section 3.4 we verify our theoretical results and provide some numerical evidence in support of the conjectures. Finally, in the concluding Section 6 we discuss the relationships among the conjectures and the convergence properties we rigorously establish.

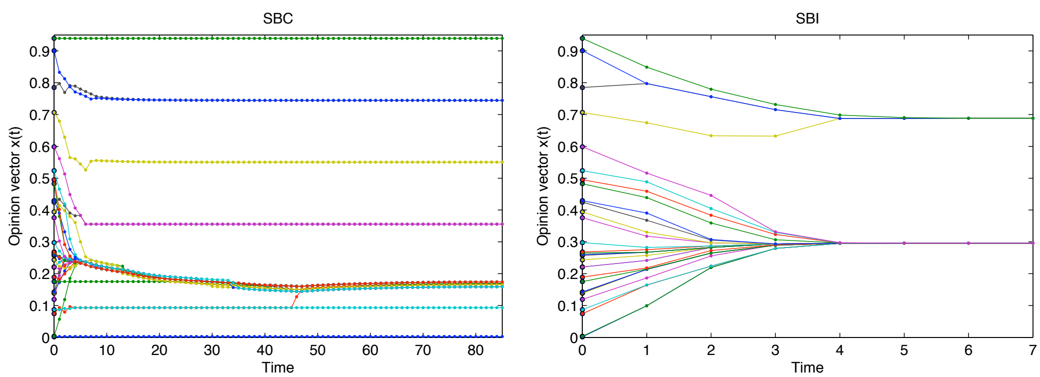



**Figure 3.1:** The trajectory of an SBC system (left) and an SBI system (right) are illustrated. Both systems have the same initial opinion vector and bounds vectors that are randomly generated. However, the SBI trajectory reaches a fixed state in six time steps, while the SBC trajectory converges in infinite time. The interconnection topology of the agents in the SBC system remains constant after $t = 64$, and hence its trajectory exhibits a pseudo-stable behavior.

### 3.1.1 Agents Classification

In this section, we introduce a classification of agents for both SBC and SBI systems based on their state-dependent interaction topology at each time step. This classification is used later to find the limiting opinion vector and explain the pseudo-stable behavior. First, let us quote some relevant definitions from graph theory, e.g. see [12]. A node of a digraph is *globally reachable* if it can be reached from any other node by traversing a directed path. A digraph is *strongly connected* if every node is globally reachable. A digraph is *weakly connected* if replacing all of its directed edges with undirected edges produces a connected undirected graph. A maximal subgraph which is strongly or weakly connected forms a *strongly connected component* (SCC) or a *weakly connected component* (WCC), respectively. Every digraph $G$ can be decomposed into either its SCC's or WCC's. Accordingly, the *condensation digraph* of $G$, denoted $C(G)$, is defined as follows: the nodes of $C(G)$ are the SCC's of $G$, and there exists a directed edge in $C(G)$ from node $H_1$ to node $H_2$ if and only if there exists a directed edge in $G$ from a node of $H_1$ to a node of $H_2$. A node with out-degree zero is named a *sink*. Knowing that the condensation digraphs are acyclic, each WCC of $C(G)$ is also acyclic and thus has at least one sink. In a digraph, $i$ is a *predecessor* of $j$ and $j$ is a *successor* of $i$ if there exists a directed path from node $i$ to node $j$.

For opinion vector $y \in \mathbb{R}^n$, let $G_r(y)$ denote either of its SBC or SBI digraphs. We classify the SCC's of $G_r(y)$ into three classes. An SCC of $G_r(y)$ is called a *closed-minded component* if it is a complete subgraph of $G_r(y)$ and corresponds to a sink of $C(G_r(y))$. An SCC of $G_r(y)$ is called a *moderate-minded component* if it is a non-complete subgraph of $G_r(y)$ and corresponds to a sink of $C(G_r(y))$. The rest of SCC's of $G_r(y)$ are called *open-minded SCC's*. Now, the *open-minded subgraph* of $G_r(y)$ is the remaining subgraph after removing $G_r(y)$'s closed- and moderate-minded components and their edges. A WCC of the open-minded subgraph of $G_r(y)$ will be called an *open-minded WCC*, see Figure 3.2.

**Remark 3.1.5.** *Previously, (Lorenz, 2006) classified the agents of an SBC system into two classes of* essential *and* inessential*. An agent is essential if any of its successors is also a predecessor, and an agent is inessential if it has a successor who is not a predecessor [33]. This classification is similar to the one used for Markov chains [47, Chapter 1.2]. It is easy to see that the closed- and moderate-minded components are in essential class, and the open-minded components are inessential.*

### 3.1.2 Spectral Properties of Adjacency Matrix

For any opinion vector $y \in \mathbb{R}^n$ in an SBC or SBI system (3.1), the adjacency matrix $A(y)$ is a non-negative row-stochastic matrix, and its nonzero diagonal

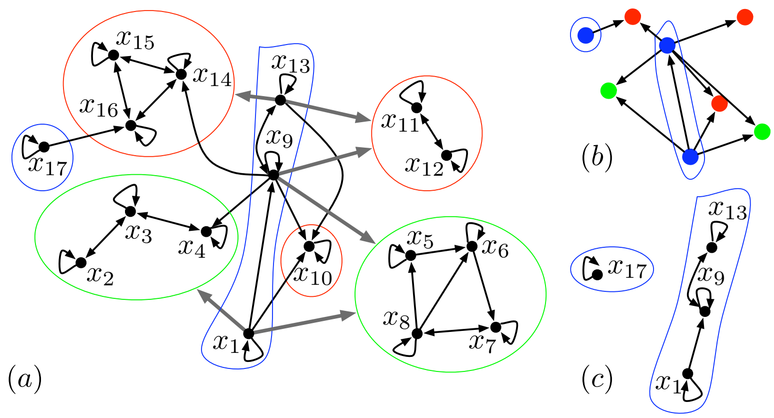



**Figure 3.2:** Consider the opinion vector $x = [0.1\ 0.24\ 0.27\ 0.3\ 0.34\ 0.37\ 0.39\ 0.4\ 0.5\ 0.6\ 0.67\ 0.68\ 0.75\ 0.85\ 0.86\ 0.87\ 1]^T$ and bounds vector $r = [0.5\ 0.04\ 0.04\ 0.04\ 0.031\ 0.021\ 0.011\ 0.061\ 0.25\ 0.01\ 0.04\ 0.03\ 0.3\ 0.07\ 0.07\ 0.07\ 0.135]^T$: (a) shows the SBC digraph of $x$, $G_r(x)$, with its closed- (red), moderate- (green), and open-minded (blue) components, and each thick gray edge represents multiple edges to all agents in one component; (b) shows the condensation digraph of $G_r(x)$; and (c) shows the open-minded subgraph of $G_r(x)$ that is composed of two open-minded WCC's.

establishes its aperiodicity. Since $C(G_r(y))$ is an acyclic digraph, its adjacency matrix is lower-triangular in an appropriate ordering [12]. In such ordering, the adjacency matrix of $G_r(y)$ is lower block triangular. Based on the classification of the SCC's in $G_r(y)$, we put $A(y)$ into the *canonical form* $\overline{A}(y)$, by an appropriate *canonical permutation matrix* $P(y)$,

$$\overline{A}(y) = P(y)A(y)P^T(y) = \begin{bmatrix} C(y) & 0 & 0 \\ 0 & M(y) & 0 \\ \Theta_C(y) & \Theta_M(y) & \Theta(y) \end{bmatrix}. \tag{3.3}$$

The submatrices $C(y)$, $M(y)$, and $\Theta(y)$ are block diagonal. Each diagonal block $C_i(y)$, with size $n_i(y)$, is the adjacency matrix of the $i$th closed-minded component, and is equal to $C_i(y) = \mathbf{1}_{n_i(y)}\mathbf{1}^T_{n_i(y)}/n_i(y)$. Let us call a matrix with such structure a *complete consensus matrix*, whose spectrum is found to be $\{1, 0, \ldots, 0\}$. Similarly, each diagonal block $M_i(y)$ is the adjacency matrix of the $i$th moderate-minded component. Each entry in $\Theta_C(y)$ or $\Theta_M(y)$ represents an edge from an open-minded node to a closed- or moderate-minded node, respectively. Finally, in the submatrix $\Theta(y)$, each diagonal block $\Theta_i(y)$ corresponds to one open-minded WCC, and is block lower triangular and strictly row-substochastic. By strictly row-substochastic we mean a square matrix with nonnegative entries so that every row adds up to at most one, and there exists at least one row whose sum is strictly less than one. Note that the adjacency matrix of each SCC in $G_r(y)$ is a diagonal block of $\overline{A}(y)$ and is row-stochastic, nonnegative and primitive. On account of the properties of the open-minded class, the following lemma is proved.

**Lemma 3.1.6.** *For any row $k$ of the submatrix $\Theta(y)$, there exists $p_k \in \mathbb{N}$ such that the $k$th row sum of $\Theta(y)^{p_k}$ is strictly less than 1.*

*Proof.* Every WCC of $C(G_r(y))$ contains at least one sink. Hence, from any open-minded agent $k$, there exists a directed path of length $p_k$ to an agent $s$ in either a closed- or moderate-minded component. Now, consider the canonical adjacency matrix to the power $p_k$,

$$\overline{A}(y)^{p_k} = \begin{bmatrix} C(y)^{p_k} & 0 & 0 \\ 0 & M(y)^{p_k} & 0 \\ \Theta_C^{(p_k)}(y) & \Theta_M^{(p_k)}(y) & \Theta(y)^{p_k} \end{bmatrix}.$$

Existence of such directed path, by [12, Lemma 1.32], implies that the $(k, s)$ entry of $\overline{A}(y)^{p_k}$, which belongs to either of the submatrices $\Theta_C^{(p_k)}(y)$ or $\Theta_M^{(p_k)}(y)$, is nonzero. Consequently, the $k$th row sum of $\Theta(y)^{p_k}$ is strictly less than 1. $\square$

It follows from Lemma 3.1.6 that $\lim_{t\to\infty} \Theta(y)^t = \mathbf{0}$, for a proof of which refer to [47, Theorem 4.3]. Therefore, the spectral radius of $\Theta(y)$ is strictly less than one.

**Example 3.1.7.** *Consider the SBC system of Figure 3.2 with the permuted opinion vector $x = [x_2\ \ x_3\ \ x_4\ \ x_5\ \ x_6\ \ x_7\ \ x_8\ \ x_{10}\ \ x_{11}\ \ x_{12}\ \ x_{14}\ \ x_{15}\ \ x_{16}\ \ x_{17}\ \ x_9\ \ x_1]^T$.*

*Then, the canonical form of the adjacency matrix $A(x)$ contains the following submatrices:*

$$
C(x) = \begin{bmatrix}
1 & & & & & \\
& \frac{1}{2} & \frac{1}{2} & & & \\
& \frac{1}{2} & \frac{1}{2} & & & \\
& & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
& & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
& & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}, \quad
M(x) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & & & & & \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & & & \\
& \frac{1}{2} & \frac{1}{2} & & & & \\
& & & \frac{1}{2} & \frac{1}{2} & & \\
& & & & \frac{1}{2} & \frac{1}{2} & \\
& & & & & \frac{1}{2} & \frac{1}{2} \\
& & & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{bmatrix},
$$

$$
\Theta(x) = \begin{bmatrix}
\frac{1}{2} & & & \\
& \frac{1}{8} & \frac{1}{8} & \\
& \frac{1}{11} & \frac{1}{11} & \\
& & \frac{1}{10} & \frac{1}{10}
\end{bmatrix}, \quad
\Theta_C(x) = \begin{bmatrix}
& & & & & \frac{1}{2} \\
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & & \\
\frac{1}{10} & & & & &
\end{bmatrix},
$$

$$
\Theta_M(x) = \begin{bmatrix}
0 & & & \dots & & & 0 \\
0 & & & \dots & & & 0 \\
& & & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} \\
\frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10}
\end{bmatrix}.
$$

## 3.2 Equilibria and Final Value at Constant Topology

An opinion vector $y_0$ is an *equilibrium opinion vector* of the dynamical system (3.1) if and only if $y_0$ is an eigenvector of the adjacency matrix $A(y_0)$ for the eigenvalue one or, equivalently, $y_0 = A(y_0)y_0$. Next, based on Conjecture 3.1.2, we introduce the following definition.

**Definition 3.2.1** (Final value at constant topology)**.** *For any opinion vector $y \in \mathbb{R}^n$ we define its* final value at constant topology $\mathrm{fvct} : \mathbb{R}^n \to \mathbb{R}^n$ *to be the limiting opinion vector of an SBC or SBI system whose initial opinion vector is $y$ and the interconnection topology of its agents remains unchanged for all $t \geq 0$. That is,*

$$\mathrm{fvct}(y) = \lim_{t\to\infty} A(y)^t y \in \mathbb{R}^n.$$

The final value at constant topology of any equilibrium opinion vector is equal to itself, that is, $\mathrm{fvct}(y_0) = \lim_{t\to\infty} A(y_0)^t y_0 = y_0$. Therefore, the set of final values at constant topology is a superset of the equilibria of the system and the limiting opinion vectors. However, not all limiting opinion vectors are the equilibria of the system, see Example 3.3.8. The condition under which a final value at constant topology is an equilibrium is discussed as follows.

**Proposition 3.2.2** (Properties of the final value at constant topology)**.** *For any opinion vector $y \in \mathbb{R}^n$ in an SBC or SBI system, whose adjacency matrix can be found from equation (3.3):*

*(i)* $\mathrm{fvct}(y)$ *is well defined, and is equal to*

$$\mathrm{fvct}(y) = P^T(y) \begin{bmatrix} C & 0 & 0 \\ 0 & M^* & 0 \\ (I-\Theta)^{-1}\Theta_C C & (I-\Theta)^{-1}\Theta_M M^* & 0 \end{bmatrix} (y)P(y)y, \tag{3.4}$$

*where the submatrix $M^*(y)$ is set equal to $\lim_{t\to\infty} M(y)^t$ and is well defined.*

*(ii) If the two networks of agents with opinion vectors $y$ and* $\mathrm{fvct}(y)$ *have the same interconnection topology or, equivalently, $G_r(y) = G_r(\mathrm{fvct}(y))$, then*

*(a)* $\mathrm{fvct}(y)$ *is an equilibrium opinion vector,*

*(b) $G_r(y)$ contains no moderate-minded component, and*

*(c) in any WCC of $G_r(\mathrm{fvct})$, the maximum and minimum opinions belong to its closed-minded components.*

*Proof.* Let us first drop the $y$ argument for matrices for readability. Regarding part (i),

$$\operatorname{fvct}(y) = \lim_{t\to\infty} A^t y = P^T \lim_{t\to\infty} \overline{A}^t P y = P^T \lim_{t\to\infty} \begin{bmatrix} C^t & 0 & 0 \\ 0 & M^t & 0 \\ \Theta_C^{(t)} & \Theta_M^{(t)} & \Theta^t \end{bmatrix} P y.$$

From Section 3.1.2, $C^t = C$ for any $t \geq 1$, $\lim_{t\to\infty} \Theta^t = \mathbf{0}$, and each diagonal block $M_i$, with size $n_i$, is a row-stochastic primitive nonnegative matrix. For such matrices the Perron-Frobenius Theorem tells us that the spectral radius is equal to one, and the essential spectral radius (i.e., the second largest eigenvalue) is strictly less than one. Thus, if we let $\nu_i \in \mathbb{R}^{n_i}$ be a left eigenvector of $M_i$ for the eigenvalue one, then from [12, Remark 1.69],

$$M_i^* = \lim_{t\to\infty} M_i^t = (\nu_i \mathbf{1}_{n_i})^{-1} \mathbf{1}_{n_i} \nu_i. \tag{3.5}$$

Using the solution to the infinite products of transition matrices of a Markov chain, given in [23, Chapter 5], it can be shown that $\lim_{t\to\infty} \Theta_C^{(t)} = (I-\Theta)^{-1}\Theta_C C$, and $\lim_{t\to\infty} \Theta_M^{(t)} = (I-\Theta)^{-1}\Theta_M M^*$.

Regarding part (ii)a, $G_r(y) = G_r(\operatorname{fvct}(y))$ results in $A(y) = A(\operatorname{fvct}(y))$, and hence

$$A(\operatorname{fvct}(y)) \operatorname{fvct}(y) = A(y) \lim_{t\to\infty} A(y)^t y = \lim_{t\to\infty} A(y)^t y = \operatorname{fvct}(y).$$

Regarding part (ii)b, by contradiction assume that $G_r(y)$ contains at least one moderate-minded component with the opinion vector $y_{M_1}$ and the adjacency matrix $M_1$. The trajectory of each sink in $C(G_r(y))$ is independent of other nodes, hence $\mathrm{fvct}_{M_1}(y) = \lim_{t\to\infty} M_1^t y_{M_1}$, and by equation (3.5), $\mathrm{fvct}_{M_1}(y) = (\nu_1 \mathbf{1}_{n_1})^{-1}\mathbf{1}_{n_1}\nu_1 y_{M_1}$. Since $(\nu_1\mathbf{1}_{n_1})^{-1}\nu_1 y_{M_1}$ is a scalar, all agents in one moderate-minded component are in consensus in final value at constant topology, and their adjacency matrix is no longer $M_1$, but rather a complete consensus matrix, which contradicts the assumption of $G_r(y) = G_r(\mathrm{fvct}(y))$.

Regarding part (ii)c, let $i$ denote the agent with the minimum final value at constant topology in one WCC of $G_r(\mathrm{fvct}(y))$. By contradiction, assume that $i$ is open-minded. Granted that the confidence or influence bounds are strictly greater than zero, in the set of out-neighbors of each open-minded agent there exists at least one agent with distinct opinion. Since $i$ has the smallest opinion among its neighbors, $\mathrm{fvct}_i(y)$ increases after taking an average of $i$'s out-neighbors opinions. In other words, the $i$th entry in the vector $A(\mathrm{fvct}(y))\,\mathrm{fvct}(y)$ is strictly larger than $\mathrm{fvct}_i(y)$, which contradicts the fact that $\mathrm{fvct}(y)$ is an equilibrium opinion vector and invariant under matrix $A(\mathrm{fvct}(y))$. Same can be proved for the agent with the maximum opinion. $\square$

## 3.3 Convergence Analysis

In this section, motivated by Conjectures 3.1.1 and 3.1.2, we drive two separate set of sufficient conditions that guarantee that an SBC or SBI trajectory converges to a limiting opinion vector. Next, to explain Conjecture 3.1.4, we study sufficient conditions for SBC and SBI systems separately that guarantee reaching a fixed state in finite time.

### 3.3.1 Convergence and Constant Topology

Our sufficient condition is based on specific neighborhoods of each opinion vector, which is introduced in the following.

**Definition 3.3.1** (Equi-topology distances and neighborhoods). *Consider an SBC or SBI system with opinion vector $z \in \mathbb{R}^n$.*

*(i) The* equi-topology distance *of $z$ is a non-negative vector $\epsilon(z) \in \mathbb{R}^n_{\geq 0}$ whose entries are defined by*

$$\epsilon_i(z) = 0.5 \min\{||z_i - z_j| - R| : j \in \{1, \dots, n\} \setminus \{i\}, R \in \{r_i, r_j\}\}, \quad (3.6)$$

*and the* equi-topology neighborhood *of $z$ is the set $\mathcal{B}_{\mathrm{et}}(z)$ of opinion vectors $y \in \mathbb{R}^n$ such that*

$$|y_i - z_i| < \epsilon_i(z), \textit{ for all } i \in \{1, \dots, n\} \textit{ with } \epsilon_i(z) > 0, \textit{ and}$$

$$|y_i - z_i| = \epsilon_i(z), \textit{ for all } i \in \{1, \dots, n\} \textit{ with } \epsilon_i(z) = 0.$$

*(ii) The* invariant equi-topology distance *of $z$ is a non-negative vector $\delta(z) \in \mathbb{R}^n_{\geq 0}$ whose entries are defined by*

$$\delta_i(z) = \min\{\epsilon_j(z) : j \textit{ is a predecessor of } i \textit{ in the graph } G_r(z)\}, \tag{3.7}$$

*and the* invariant equi-topology neighborhood *of $z$ is the set $\mathcal{B}_{\mathrm{iet}}(z)$ of opinion vectors $y \in \mathbb{R}^n$ such that*

$$|y_i - z_i| < \delta_i(z), \textit{ for all } i \in \{1, \dots, n\} \textit{ with } \delta_i(z) > 0, \textit{ and}$$

$$|y_i - z_i| = \delta_i(z), \textit{ for all } i \in \{1, \dots, n\} \textit{ with } \delta_i(z) = 0.$$

Note that in any SBC or SBI digraph, each node has a self-loop, and hence each agent is a predecessor of itself. Therefore, for any opinion vector $z \in \mathbb{R}^n$ and for all $i \in \{1, \dots, n\}$, we have $\delta_i(z) \leq \epsilon_i(z)$, which results in $\mathcal{B}_{\mathrm{iet}}(z) \subset \mathcal{B}_{\mathrm{et}}(z)$.

**Lemma 3.3.2** (Sufficient condition for equal topologies)**.** *Consider an SBC or SBI system with opinion vectors $y, z \in \mathbb{R}^n$. If $y$ belongs to the equi-topology neighborhood of $z$, then the two networks of agents with opinion vectors $y$ and $z$ have the same interconnection topology, or equivalently $G_r(y) = G_r(z)$.*

*Proof.* For any $i, j \in \{1, \dots, n\}$ two cases exists:

1. $j$ is an out-neighbor of $i$ in $G_r(z)$, hence $|z_i - z_j| \leq r$, where in an SBC system $r = r_i$ and in an SBI system $r = r_j$. In either system, since $y \in \mathcal{B}_{\text{et}}(z)$,

$$|y_i - y_j| \leq |z_i - z_j| + \epsilon_i(z) + \epsilon_j(z) \leq |z_i - z_j| + ||z_i - z_j|| - r| = r.$$

2. $j$ is not an out-neighbor of $i$ in $G_r(z)$, hence $|z_i - z_j| > r$, with $r$ defined above. If both $\epsilon_i(z)$ and $\epsilon_j(z)$ are zero, then $y \in \mathcal{B}_{\text{et}}(z)$ gives us

$$|y_i - y_j| = |z_i - z_j| > r,$$

and if at least one is nonzero, then

$$|y_i - y_j| > |z_i - z_j| - \epsilon_i(z) - \epsilon_j(z) \geq |z_i - z_j| - ||z_i - z_j| - r| = r.$$

Therefore, the neighboring relation of agents in $G_r(z)$ is preserved in $G_r(y)$. One can also prove that any neighboring relation in $G_r(y)$ is preserved in $G_r(z)$. $\square$

**Remark 3.3.3.** *For any $y \in \mathbb{R}^n$, if $y \in \mathcal{B}_{\text{et}}(\text{fvct}(y))$, then by Lemma 3.3.2 we have $G_r(y) = G_r(\text{fvct}(y))$. Hence, by Proposition 3.2.2,* fvct$(y)$ *is an equilibrium opinion vector, and $G_r(y)$ contains no moderate-minded component.*

**Theorem 3.3.4** (Sufficient condition for constant topology and convergence)**.** *Consider a trajectory $x(t)$ of an SBC or SBI system. Assume that there exists an equilibrium opinion vector $z \in \mathbb{R}^n$ for the system such that $x(0) \in \mathbb{R}^n$ belongs to the invariant equi-topology neighborhood of $z$. Then, for all $t \geq 0$:*

*(i)* $x(t)$ *takes value in the equi-topology neighborhood of* $z$*, and hence* $G_r(z) = G_r(x(t))$*;*

*(ii)* $G_r(x(t))$ *contains no moderate-minded component; and*

*(iii)* $x(t)$ *converges to* $\mathrm{fvct}(x(0))$ *as time goes to infinity.*

**Remark 3.3.5** (Interpretation of Theorem 3.3.4)**.** *This theorem tells us that if the trajectory of an SBC or SBI system enters a specific ball around any equilibrium opinion vector of that system, then it remains in some larger ball around that vector for all future iterations. Moreover, the proximity digraph of the trajectory and the equilibrium opinion vector remain equal.*

**Remark 3.3.6.** *Under the condition of Theorem 3.3.4, a trajectory* $x(t)$ *converges to its final value at constant topology* $\mathrm{fvct}(x(t)) = \mathrm{fvct}(x(0))$*. However,* $\mathrm{fvct}(x(t))$ *is not necessarily equal to the equilibrium opinion vector* $z$*, and the proximity digraphs* $G_r(\mathrm{fvct}(x(t)))$ *and* $G_r(x(t))$ *can be different, see Example 3.3.8.*

**Remark 3.3.7.** *One special case of Theorem 3.3.4 is when* $x(0) \in \mathcal{B}_{\mathrm{iet}}(\mathrm{fvct}(x(0)))$*, which implies that* $\mathrm{fvct}(x(0))$ *is an equilibrium opinion vector, again see Example 3.3.8.*

**Example 3.3.8.** *Consider an SBC trajectory with* $x(0) = [0 \;\; 0.6 \;\; 1]^T$ *and confidence bounds* $r = [0.5 \;\; 1 \;\; 0.25]^T$*, which converges with constant topology after*

$t = 0$. *It can be computed that* $\mathrm{fvct}(x(0)) = [0 \quad 0.5 \quad 1]^T$ *and* $\delta(\mathrm{fvct}(x(0))) = [0.25 \ \ 0.25 \ \ 0.25]^T$. *Clearly,* $x(0) \in \mathcal{B}_{\mathrm{iet}}(\mathrm{fvct}(x(0)))$, *i.e., the initial vector satisfies the special case of Theorem 3.3.4 stated in Remark 3.3.7. Hence,* $x(t)$ *converges to* $\mathrm{fvct}(x(0))$, *and their proximity digraphs are equal. However, if the confidence bounds are equal to* $r = [0.5 \quad 1 \quad 0.25]^T$, *then* $\delta(\mathrm{fvct}(x(0))) = [0 \quad 0 \quad 0]^T$, *and* $x(t) \notin \mathcal{B}_{\mathrm{iet}}(\mathrm{fvct}(x(0)))$ *for all* $t \geq 0$. *Therefore,* $x(t)$ *converges to* $\mathrm{fvct}(x(0))$, *while their proximity digraphs are different. Both trajectories with the two confidence bounds vectors are the same, see Figure 3.3.*

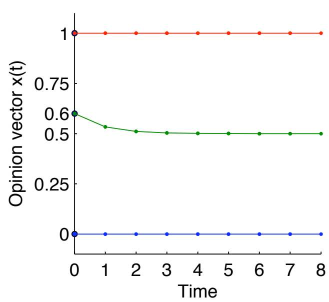



**Figure 3.3:** The SBC trajectory of Example 3.3.8 is illustrated.

*Proof of Theorem 3.3.4.* Regarding statement (i), by induction we prove that $x(t) \in \mathcal{B}_{\mathrm{et}}(z)$ for all $t \geq 0$, which by Lemma 3.3.2 results in $G_r(x(t)) = G_r(z)$. The first induction step is $x(0) \in \mathcal{B}_{\mathrm{et}}(z)$, which is true knowing that $x(0) \in \mathcal{B}_{\mathrm{iet}}(z)$ and $\mathcal{B}_{\mathrm{iet}}(z) \subset \mathcal{B}_{\mathrm{et}}(z)$. To complete the induction argument, assume that the statement (i) holds at times $t = 0, \ldots, \tau$, which implies that $A(z) = A(x(t))$. The equilibrium opinion vector $z$ satisfies $z = A(z)z$, thus we have $x(t+1) - z =$

$A(x(t))x(t) - A(z)z = A(z)(x(t) - z)$ or, equivalently

$$x_i(t+1) - z_i = \frac{1}{|\mathcal{N}_i(z)|} \sum_{j \in \mathcal{N}_i(z)} (x_j(t) - z_j). \tag{3.8}$$

One can see that

$$\begin{aligned} |x_i(\tau+1) - z_i| \leq \max_{j \in \mathcal{N}_i(z)} |x_j(\tau) - z_j| \leq \max_{\ell \in \mathcal{N}_j(z), j \in \mathcal{N}_i(z)} |x_\ell(\tau-1) - z_\ell| \\ \leq \cdots \leq \max_{k \in \mathcal{M}} |x_k(0) - z_k|, \end{aligned} \tag{3.9}$$

where $\mathcal{M}$ is a subset of successors of $i$ in $G_r(z)$, and thus for any $k \in \mathcal{M}$, equation (3.7) tells us that $\delta_k(z) \leq \epsilon_i(z)$. Here again two cases exists: First, if for all $k \in \mathcal{M}$, $\delta_k(z) = 0$, then the condition $x(0) \in \mathcal{B}_{\text{iet}}(z)$ implies that $x_k(0) - z_k = 0$, and it follows from inequality (3.9) that $x_i(\tau+1) - z_i = 0$. Second, if there exists $\ell \in \mathcal{M}$ such that $\delta_\ell(z) > 0$, then $\epsilon_i(z) > 0$ and

$$|x_i(\tau+1) - z_i| \leq \max_{k \in \mathcal{M}} |x_k(0) - z_k| < \max_{k \in \mathcal{M}} \delta_k(z) \leq \epsilon_i(z).$$

Therefore, $x(\tau+1) \in \mathcal{B}_{\text{et}}(z)$.

Regarding statement (ii), according to Section 3.2, an equilibrium opinion vector is equal to its own final value at constant topology. Hence $G_r(z) = G_r(z^*(z))$, and by Proposition 3.2.2, $G_r(z)$ and thus $G_r(x(t))$ contain no moderate-minded component.

Regarding statement (iii), according to the definition of the final value at constant topology, if the topology remains constant for all $t \geq 0$, then $x(t)$ converges to $\text{fvct}(x(0))$. □

Motivated by Conjecture 3.1.1, the existence of a limiting opinion vector is required in the following lemma.

**Lemma 3.3.9** (Sufficient condition for a limiting opinion vector to be an equilibrium)**.** *Pick a trajectory $x(t)$ of an SBC or SBI system that is convergent. Denote the limiting opinion vector of $x(t)$ by $x_\infty$. If $\min_{i\in\{1,\dots,n\}} \epsilon_i(x_\infty) > 0$, where $\epsilon(x_\infty)$ is the equi-topology distance of $x_\infty$ , then there exists time $T$ such that for all $t \geq T$:*

*(i)* $G_r(x_\infty) = G_r(x(t))$*, and*

*(ii)* $x_\infty = \mathrm{fvct}(x(t))$*, and is an equilibrium opinion vector.*

*Proof.* According to the definition of convergence, for any $\delta \in \mathbb{R}_{>0}$, there exists $T$ such that for all $t \geq T$, $\|x(t) - x_\infty\|_\infty < \delta$. Now, if we let $\delta$ be equal to $\min_{i\in\{1,\dots,n\}} \epsilon_i(x_\infty)$, then $\|x(t) - x_\infty\|_\infty < \min_{i\in\{1,\dots,n\}} \epsilon_i(x_\infty)$ for all $t \geq T$, and it follows from Lemma 3.3.2 that $G_r(x_\infty) = G_r(x(t))$. Under fixed topology, $x(t)$ converges to its final value at constant topology, thus $x_\infty = \mathrm{fvct}(x(t))$. Moreover, the equality $G_r(x(t)) = G_r(\mathrm{fvct}(x(t)))$ tells us that $\mathrm{fvct}(x(t))$, and hence $x_\infty$, is an equilibrium opinion vector. □

**Corollary 3.3.10.** *An equilibrium opinion vector $z$ is a Lyapunov stable equilibrium vector for the system if $\min_{i\in\{1,\dots,n\}} \epsilon_i(z) > 0$. In other words, there exists a neighborhood around $z$ where any solution of the dynamical system whose initial* *condition belongs to that neighborhood, stays in that neighborhood for all future iterations.*

### 3.3.2 Monotonic Convergence and Constant Topology

The second sufficient condition for convergence is based on the rate and direction of convergence of an SBI or SBC trajectory in one time step. If a trajectory satisfies this condition, then any two opinions will either monotonically converge to each other or diverge from each other for all future iterations.

**Definition 3.3.11** (Agent's per-step convergence factor)**.** *In an SBC or SBI system with trajectory $x(t) \in \mathbb{R}^n$, we define the* per-step convergence factor *of an agent $i$ whose $x_i(t) - \mathrm{fvct}_i(x(t))$ is nonzero to be*

$$k_i(x(t)) = \frac{x_i(t+1) - \mathrm{fvct}_i(x(t))}{x_i(t) - \mathrm{fvct}_i(x(t))}.$$

The per-step convergence factor of a network of agents was previously introduced in [58] to measure the overall speed of convergence toward consensus.

**Remark 3.3.12** (Monotonic convergence)**.** *If an SBC or SBI trajectory $x(t)$ monotonically converges toward $\mathrm{fvct}(x(t))$ in one time step, that is, for any $i \in$*

$\{1, \ldots, n\}$,

$$\begin{cases} x_i(t) \leq x_i(t+1) \leq \mathrm{fvct}_i(x(t)), & \text{if } x_i(t) < \mathrm{fvct}_i(x(t)), \\ x_i(t) \geq x_i(t+1) \geq \mathrm{fvct}_i(x(t)), & \text{if } x_i(t) > \mathrm{fvct}_i(x(t)), \\ x_i(t) = x_i(t+1) = \mathrm{fvct}_i(x(t)), & \text{if } x_i(t) = \mathrm{fvct}_i(x(t)), \end{cases}$$

*then*

$$\begin{cases} 0 \leq k_i(x(t)) \leq 1, & \text{if } k_i(x(t)) \text{ exists,} \\ x_i(t) = x_i(t+1) = \mathrm{fvct}_i(x(t)), & \text{otherwise.} \end{cases}$$

Before proceeding, let us define the *distance to final value* of any $y \in \mathbb{R}^n$ to be $\Delta(y) = y - \mathrm{fvct}(y)$. For any open-minded agent $i$, let $k_{max_i}(y)$ and $k_{min_i}(y)$ denote the maximum and minimum per-step convergence factors over all $i$'s open-minded successors with nonzero distance to final value. Also, for any open-minded agents $i$ and $j$, let $k_{max_{i,j}}(y) = \max\{k_{max_i}(y), k_{max_j}(y)\}$ and $k_{min_{i,j}}(y) = \min\{k_{min_i}(y), k_{min_j}(y)\}$.

**Lemma 3.3.13** (Bound on per-step convergence factor)**.** *If in an SBC or SBI system with opinion vector $y \in \mathbb{R}^n$,*

*(i) $G_r(y)$ contains no moderate-minded component, and*

*(ii) for any open-minded agent $i$ and any of its open-minded child $j$, $\Delta_i(y)\Delta_j(y) \geq 0$,*

*then $k_i(A(y)y)$ is in the convex hull of $k_j(y)$'s.*

*Proof.* From here on, we often drop $y$ argument from proofs, and for any $y \in \mathbb{R}^n$, we denote $A(y)y$ by $y^+$ and $\mathrm{fvct}(y)$ by $y^*$. If there is no moderate-minded component in $G_r(y)$, then $y_\Theta^+ - y_\Theta^* = \Theta(y)(y_\Theta - y_\Theta^*)$, where $y_\Theta$ is the opinion vector of the open-minded class whose adjacency matrix is $\Theta(y)$, see [41, Theorem 6.4]. Consider an open-minded agent $i$ whose children belong to the set $\{1, \ldots, m\}$, and denote the entries of $A(y)$ by $a_{ij}$, then

$$k_i(y^+) = \frac{a_{i1}(y_1^+ - y_1^*) + \cdots + a_{im}(y_1^+ - y_m^*)}{a_{i1}(y_1 - y_1^*) + \cdots + a_{im}(y_1 - y_m^*)}$$
$$= \frac{a_{i1}k_1(y)\Delta_1(y) + \cdots + a_{im}k_m(y)\Delta_m(y)}{a_{i1}\Delta_1(y) + \cdots + a_{im}\Delta_m(y)}. \quad (3.10)$$

Under condition (ii), all $\Delta_j(y)$'s have the same sign, and hence all the terms in the right hand side are positive. Therefore, $k_i(y^+)$ is in the convex hull of $k_j(y)$'s. □

**Theorem 3.3.14** (Sufficient condition for constant topology and monotonic convergence)**.** *Assume that in an SBC or SBI system, the opinion vector $y \in \mathbb{R}^n$ satisfies the following:*

*(i) the networks of agents with opinion vectors $y$ and $\mathrm{fvct}(y)$ have the same interconnection topology, that is, $G_r(y) = G_r(\mathrm{fvct}(y))$;*

*(ii) for any two agents $i$ and $j$, if $y_i \geq y_j$, then $\mathrm{fvct}_i(y) \geq \mathrm{fvct}_j(y)$;*

*(iii) $y$ monotonically converges toward $\mathrm{fvct}(y)$ in one iteration;*

*(iv) for any open-minded neighbors $i$ and $j$, $\Delta_i(y)\Delta_j(y) \geq 0$; and*

*(v) any open-minded agents $i$ and $j$ that belong to the same WCC of $G_r(y)$ and that have nonzero $\Delta_i(y)$ and $\Delta_j(y)$ have the following property:*

*a) if the sets of open-minded children of $i$ and $j$ are identical, then $k_i(y) = k_j(y)$,*

*b) otherwise, assuming that $\Delta_i(y) \geq \Delta_j(y)$,*

$$k_{max_{i,j}}(y) - k_{min_{i,j}}(y) \leq \min\{1 - k_{max_{i,j}}(y), k_{min_{i,j}}(y)\}$$
$$\min_{m\in\mathbb{Z}_{\geq 0}}\{\left|1 - \frac{\alpha^m \Delta_j(y)}{\beta^m \Delta_i(y)}\right| : \alpha \in [k_{min_j}(y), k_{max_j}(y)], \beta \in [k_{min_i}(y), k_{max_i}(y)]\}$$

*Then the solution $x(t)$ from the initial condition $x(0) = y$ has the following properties: the proximity digraph $G_r(x(t))$ is equal to $G_r(y)$ for all time $t$, and the solution $x(t)$ monotonically converges to* $\mathrm{fvct}(y)$ *as $t$ goes to infinity.*

A justification of sufficient conditions of Theorem 3.3.14 is presented in Remark 3.5.5.

*Proof.* Here, we show that if $x(0) = y$ satisfies all the theorem's conditions, then $y^+$ also satisfies them, and similarly they hold for all subsequent times. Note that condition (iii) guarantees entrywise monotonic convergence, and condition (i) guarantees constant topology. Let us start by proving that $G_r(y) = G_r(y^+)$. On account of Proposition 3.2.2 part (ii) and under condition (i), there are no moderate-minded component in $G_r(y)$, thus, for any $i, j \in \{1, \dots, n\}$, four cases are possible:

1. $i, j$ are open-minded and weakly connected in $G_r(y)$.

a) If $\Delta_i \Delta_j > 0$, then without loss of generality we assume that $\Delta_i \geq \Delta_j > 0$, since otherwise we can multiply the opinion vector by $-1$. Hence, the monotonic convergence of the two opinions toward each other, or equivalently,

$$y_i^* - y_j^* \leq y_i^+ - y_j^+ \leq y_i - y_j, \tag{3.11}$$

should be proved. Under condition (v), it is true that $|k_i - k_j| \leq (1 - \frac{\Delta_j}{\Delta_i}) \min\{1 - k_j, k_j\}$. On the other hand,

$$\begin{aligned}(y_i^+ - y_j^+) - (y_i^* - y_j^*) &= (k_i - k_j)\Delta_i + k_j(\Delta_i - \Delta_j) \\ &\leq (1 - k_j)(\Delta_i - \Delta_j) + k_j(\Delta_i - \Delta_j) = \Delta_i - \Delta_j,\end{aligned}$$

which implies that $y_i^+ - y_j^+ \leq y_i - y_j$. Furthermore,

$$\begin{aligned}(y_i^+ - y_j^+) - (y_i^* - y_j^*) &\geq -|k_i - k_j|\Delta_i + k_j(\Delta_i - \Delta_j) \\ &\geq -k_j(\Delta_i - \Delta_j) + k_j(\Delta_i - \Delta_j) = 0,\end{aligned}$$

which implies that $y_i^+ - y_j^+ \geq y_i^* - y_j^*$. Now, we can show that the neighboring relation between $i$ and $j$ in the digraph $G_r(y^+)$ is equal to that of $G_r(y)$. We let $r$ denote either $r_i$ or $r_j$. The sign of $|y_i - y_j| - r$, $|y_i^+ - y_j^+| - r$, and $|y_i^* - y_j^*| - r$ govern the neighboring relations between $i$ and $j$ in the digraphs $G_r(y)$, $G_r(y^+)$, and $G_r(y^*)$, respectively. Using inequalities (3.11) and condition (ii)

$$\begin{cases} 0 < y_i^* - y_j^* \leq y_i^+ - y_j^+ \leq y_i - y_j & \text{if } \; y_i \geq y_j, \\ y_j^* - y_i^* \geq y_j^+ - y_i^+ \geq y_j - y_i > 0 & \text{if } \; y_i \leq y_j, \end{cases} \tag{3.12}$$

subtracting $r$ from above inequalities gives

$$\begin{cases} |y_i^* - y_j^*| - r \leq |y_i^+ - y_j^+| - r \leq |y_i - y_j| - r \text{ if } y_i \geq y_j, \\ |y_j^* - y_i^*| - r \geq |y_j^+ - y_i^+| - r \geq |y_j - y_i| - r \text{ if } y_i \leq y_j. \end{cases}$$

Hence, $|y_i^+ - y_j^+| - r$ is bounded between the two other values, which have the same sign by condition (i). Therefore, $i$ and $j$'s neighboring relation is preserved in $G_r(y^+)$.

b) If $\Delta_i \Delta_j \leq 0$, then for instance assume that $\Delta_i \geq 0 \geq \Delta_j$. By condition (iii), it is easy to see that

$$y_i - y_i^* \geq y_i^+ - y_i^* \geq 0 \geq y_j^+ - y_j^* \geq y_j - y_j^*.$$

Using above inequalities and under condition (ii), inequalities (3.12) hold, which again proves that $i$ and $j$'s neighboring relation is preserved in $G_r(y^+)$.

2. $i$ and $j$ are open-minded and belong to two separate WCC's of $G_r(y)$, whose agent sets are $\mathcal{V}_1$ and $\mathcal{V}_2$. Since $G_r(y) = G_r(y^*)$, by Proposition 3.2.2 part (ii)c, the minimum and maximum opinions of a separate WCC in both $G_r(y)$ and $G_r(y^*)$ belong to closed-minded components. Define the *opinion range* of any subgraph to

be a real interval between the minimum and maximum opinions of its agents and its *sensing range* to be the union of closed intervals in the confidence bounds of its agents around their opinions. Therefore, the sensing range of $\mathcal{V}_1$ is separated from the opinion range of $\mathcal{V}_2$ in both $G_r(y)$ and $G_r(y^*)$. Due to monotonic convergence toward $y^*$ in one step, the sensing range of $\mathcal{V}_1$ in $G_r(y^+)$ lies in the union of its sensing ranges in $G_r(y)$ and $G_r(y^*)$. The boundary closed-minded component of $\mathcal{V}_1$ in $G_r(y)$ keeps the sensing range of $\mathcal{V}_1$ away from the opinion range of $\mathcal{V}_2$ in $G_r(y^+)$, see Figure 3.4 (a). Thus, two separate WCC's in $G_r(y)$ remain separate in $G_r(y^+)$.

3. $i$ and $j$ are both closed-minded in $G_r(y)$, hence, $y_i^+ = y_i^*$ and $y_j^+ = y_j^*$. The equality $y_i^+ - y_j^+ = y_i^* - y_j^*$ tells us that neighboring relation between $i$ and $j$ in $G_r(y^+)$ is same as in $G_r(y^*)$, and consequently in $G_r(y)$.

4. $i$ is open-minded and $j$ is closed-minded in $G_r(y)$. Since agents in one closed-minded component reach consensus in $G_r(y^*)$, $i$'s neighboring relation with $j$ in $G_r(y)$ is the same as its relation with other agents in $j$'s component. Assume that $y_i - y_j \leq r$, where $r$ denotes either $r_i$ or $r_j$, see Figure 3.4 (b), then $y_i - y_k \leq r$ for all $k$ in $j$'s component. The average of the latter inequalities gives $y_i - y_j^+ \leq r$, and from $G_r(y) = G_r(y^*)$ we have $y_i^* - y_j^* \leq r$, where for closed-minded $j$, $y_j^* = y_j^+$. Therefore, $y_i^+$, which under monotonic convergence is bounded between $y_i$ and $y_i^*$,

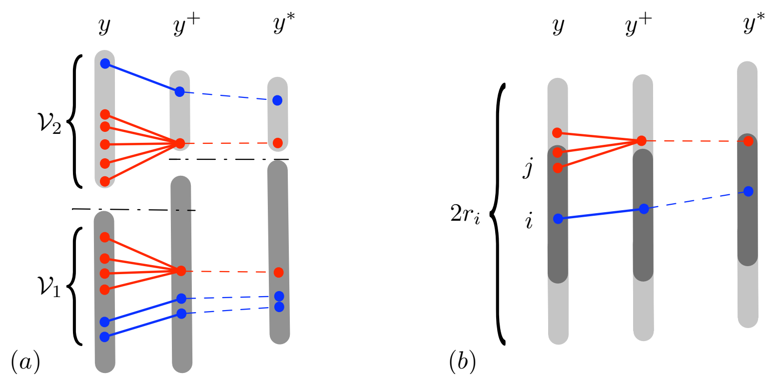



**Figure 3.4:** For the proof of Theorem 3.3.14: (a) illusterates the sets of agents in two separate WCC's of $G_r(y)$, $\mathcal{V}_1$ and $\mathcal{V}_2$. If $\mathcal{V}_1$'s sensing range (dark gray) is separated from $\mathcal{V}_2$'s opinion range (light gray) in $G_r(y)$ and $G_r(y^*)$, owing to boundary closed minded components (red), these ranges can not overlap in $G_r(y^+)$; and (b) shows that open-minded $i$ under light gray bound of confidence listens to closed-minded $j$ and its component in $G_r(y)$ and $G_r(y^*)$. Since $G_r(y) = G_r(y^*)$, closed-minded components reach consensus in $G_r(y^*)$. Otherwise, $i$ could listen to $j$ under dark gray bound of confidence, and get disconnected in $G_r(y^+)$.

also satisfies $y_i^+ - y_j^+ \leq r$. Similarly, one can show that the neighboring relation is preserved in $G_r(y^+)$ for the case when $y_i - y_j > r$.

So far, we have proved that $G_r(y) = G_r(y^+)$, hence condition (i) holds for $y^+$. Due to monotonic convergence in one time step under opinion vector $y$, opinion order and direction of convergence toward final value is preserved in $y^+$, that is conditions (ii) and (iv) are true for $y^+$. To prove the last two conditions for $y^+$, we should find $k_i(y^+)$'s. Regarding part (a), if the two open-mindeds $i$ and $j$ have the same set of open-minded children, then equation (3.10) tells us that $k_i(y^+) = k_j(y^+)$. Regarding part (b), clearly, both conditions of Lemma 3.3.13 hold for $G_r(y)$, hence for any open-minded $i$, $k_i(y^+)$ lies in the convex hull of $k_j(y)$'s, where $j$'s are its open minded children. This fact tells us that: $0 \leq k_i(y^+) \leq 1$, $k_{max_i}(y^+) \leq k_{max_i}(y)$, and $k_{min_i}(y^+) \geq k_{min_i}(y)$. Therefore, for any open-minded agents $i$ and $j$ with different sets of open-minded children,

$$\begin{aligned} k_{max_{i,j}}(y^+) - k_{min_{i,j}}(y^+) \leq \min_{m,\alpha_1,\beta_1} \left|1 - \frac{\alpha_1^m \Delta_j(y)}{\beta_1^m \Delta_i(y)}\right| \\ \times \min\{1 - k_{max_{i,j}}(y^+), k_{min_{i,j}}(y^+)\}, \end{aligned}$$

where $\alpha_1 \in [k_{min_j}(y), k_{max_j}(y)]$, $m \in \mathbb{Z}_{\geq 0}$, and $\beta_1 \in [k_{min_i}(y), k_{max_i}(y)]$. Knowing that $k_i(y) \in [k_{min_i}(y), k_{max_i}(y)]$,

$$\min_{m,\alpha_1,\beta_1} \left|1 - \frac{\alpha_1^m \Delta_j(y)}{\beta_1^m \Delta_i(y)}\right| \leq \min_{m,\alpha_1,\beta_1} \left|1 - \frac{\alpha_1^m k_j(y) \Delta_j(y)}{\beta_1^m k_i(y) \Delta_i(y)}\right|.$$

The right hand side of the above inequality is equal to

$$\min_{m,\alpha_1,\beta_1}\left|1-\frac{\alpha_1^m\Delta_j(y^+)}{\beta_1^m\Delta_i(y^+)}\right|\leq\min_{m,\alpha_2,\beta_2}\left|1-\frac{\alpha_2^m\Delta_j(y^+)}{\beta_2^m\Delta_i(y^+)}\right|,$$

where $\alpha_2$ and $\beta_2$, respectively, belong to smaller intervals of $[k_{min_j}(y^+), k_{max_j}(y^+)]$ and $[k_{min_i}(y^+), k_{max_i}(y^+)]$. Hence, part (b) holds for $y^+$, which completes the proof. □

**Example 3.3.15.** *An SBC system with the following initial opinion vector*

$$x_0 = [0\ \ 2.2\ \ 4\ \ 4\ \ 4\ \ 0.64\ \ 3*\mathbf{1}_{200}^T]^T$$

*and bounds of confidence* $r = [0.01\ \ 0.01\ \ 0.01\ \ 0.01\ \ 0.01\ \ 1.9254\ \ 2{*}\mathbf{1}_{200}^T]^T$ *satisfies all conditions but (v) of Theorem 3.3.14 at time steps* $t = 0, \ldots, 5$*. The proximity digraph* $G_r(x(0))$ *contains two open-minded SCC's* $\{x_6\}$ *and* $\{x_7, \ldots, x_{206}\}$*, who are two open-minded WCC's and weakly connected in* $G_r(x(0))$*. The per-step convergence factors of their agents, which is approximately equal to the spectral radius of the adjacency matrices of their SCC's (0.3333 and 0.9804), do not satisfy the boundary condition (v). Therefore, the monotonic convergence of opinion vector, or equivalently equation* (3.11)*, does not hold, see Figure 3.5.*

**Example 3.3.16.** *An SBC system with* $x_0 = [0\ \ 0\ \ 0\ \ 1\ \ 3*\mathbf{1}_{10}^T\ \ 6]^T$ *and* $r = [0.01\ \ 0.01\ \ 0.01\ \ 2\ \ 20*\mathbf{1}_{10}^T\ \ 0.01]^T$ *at time zero satisfies conditions (i) and (ii) of Theorem 3.3.14, however, condition (iii) does not hold. The per-step convergence*

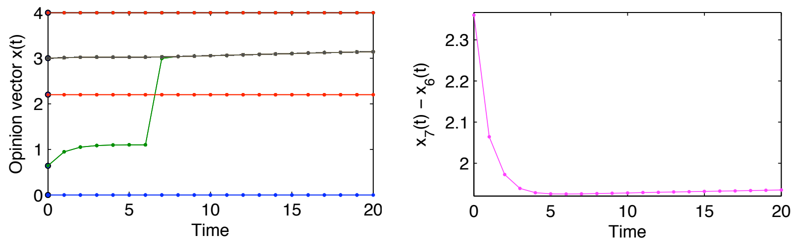



**Figure 3.5:** The evolution of opinions in Example 3.3.15 (left), and the non-monotonic evolution of opinion difference $x_7(t) - x_6(t)$ under fixed topology of $G_r(x(0))$, which is due to the large difference $k_6(x(t)) - k_7(x(t))$ (right).

*factor of open-minded agent $x_4$ at time zero is equal to -12.35, which results in a large jump at the subsequent time step, consequently the interconnection topology changes.*

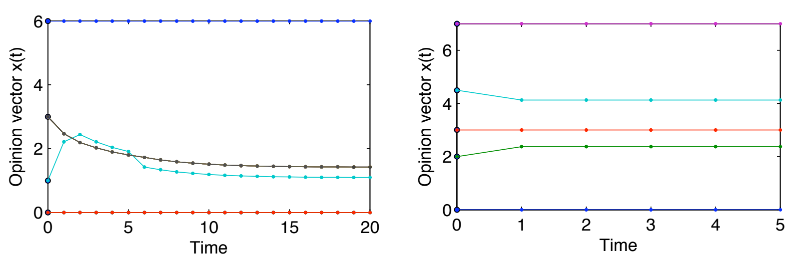



**Figure 3.6:** The evolution of opinions in Examples 3.3.16 (left), and 3.3.20 (right).

### 3.3.3 Convergence in Finite Time and Consensus

In this subsection, we discuss the sufficient conditions for SBC and SBI systems to converge to an *agreement opinion vector*. In an agreement opinion vector, any two agents are either disconnected or in consensus. Reaching global consensus, in which all agents hold the same opinion, is a special case of convergence to an agreement opinion vector. Note that it is possible for an SBC or SBI system to reach an opinion vector that contains two neighbor agents with separate opinions in finite time, see Example 3.3.20.

**Proposition 3.3.17** (Properties of agreement opinion vectors)**.** *For any agreement opinion vector $\tilde{y} \in \mathbb{R}^n$ in an SBC or SBI system:*

*(i)* $\min_{i\in\{1,\dots,n\}} \epsilon_i(\tilde{y}) > 0$*, where* $\epsilon(\tilde{y})$ *is the equi-topology distance of* $\tilde{y}$*; and*

*(ii) if* $\tilde{y}$ *is the limiting opinion vector of a trajectory, then the trajectory reaches* $\tilde{y}$ *in finite time.*

*Proof.* Regarding statement (i), by contradiction assume that $\min_{i\in\{1,\dots,n\}} \epsilon_i(\tilde{y}) = 0$. Then based on equation (3.6), there exist agents $i$ and $j$ such that $|\tilde{y}_i - \tilde{y}_j| = r_i$. The latter equation tells us that $j \in \mathcal{N}_i(\tilde{y})$ in an SBC digraph or $i \in \mathcal{N}_j(\tilde{y})$ in an SBI digraph, while their opinions are different from each other by $r_i$, which contradicts the definition of agreement opinion vectors. Regarding statement (ii), consider trajectory $x(t)$ that converges to $\tilde{y}$. Then, previous statement shows that the limiting opinion vector of $x(t)$ satisfies the condition of Lemma 3.3.9. Therefore, there exists time step $\tau$ such that the proximity digraphs $G_r(\tilde{y})$ and $G_r(x(t))$ are equal for all $t \geq \tau$. On the other hand, the proximity digraph of an agreement opinion vector contains only closed-minded components. Hence, the agents in each WCC of $G_r(x(\tau))$ reach consensus at the next iteration. $\square$

One sufficient condition that guarantees asymptotic consensus in "agreement algorithms", which includes SBC and SBI systems, is given in [43, Theorem 2.4] and is as follows. Take a trajectory $x(t)$ of an SBC or SBI system with proximity digraph $G_r(x(t)) = \big(V, E(x(t))\big)$, where $V$ and $E(x(t))$ are the sets of nodes and edges of the digraph, respectively. If there exists $\tau$ such that the graph $\big(V, E(x(k\tau)) \cup E(x(k\tau+1)) \cup \cdots \cup E(x((k+1)\tau - 1))\big)$ is strongly connected for all $k \in \mathbb{Z}_{\geq 0}$, then all entries of $x(t)$ converge to one real number. However, this sufficient condition requires knowledge of the system for infinite time, and is the same for both SBC and SBI systems. Hence, we derive sufficient conditions that are required to hold in one time step, and also make it possible to compare SBC and SBI systems in support of Conjecture 3.1.4. Let us first define the *opinion interval* of any subgraph of an SBC or SBI digraph be a closed interval in $\mathbb{R}$ between that subgraph's minimum and maximum opinions.

**Proposition 3.3.18** (Sufficient conditions for convergence to an agreement opinion vector)**.** *Consider the opinion vector $y \in \mathbb{R}^n$ in an SBC or SBI system with the following properties:*

*(i) the opinion intervals of any two WCC's of the proximity digraph are separated from each other by a distance strictly larger than the maximum confidence or influence bounds of the agents in those WCC's; and*

*(ii) it is true that:*

- *for any WCC of $y$'s SBC digraph, with $m$ agents, at least $m-1$ agents have confidence bounds larger than that WCC's opinion interval; and*
- *for any WCC of $y$'s SBI digraph, at least one agent has influence bound larger than that WCC's opinion interval.*

*Then, the trajectories of both SBC and SBI systems with the initial opinion vector $y$ converge to agreement opinion vectors in finite time. Moreover, in every WCC of either of the SBC or SBI digraphs, at least one node is an out-neighbor of all nodes in that WCC for all $t \geq 0$.*

**Remark 3.3.19.** *Any trajectory of an SBC or SBI system that converges to an agreement opinion vector will eventually satisfy the conditions of Proposition 3.3.18.*

*Proof of Proposition 3.3.18.* Let us denote either of the SBC or SBI digraphs of $y$ by $G_r(y)$. In an SBC or SBI system, the smallest and largest opinions in a separate WCC of the proximity digraph are, respectively, non-decreasing and non-increasing in one iteration [6]. This fact tells us that for all $t \geq 0$: first, under the condition (i), the two sets of nodes of two separate WCC's in $G_r(x(0))$ remain separate in $G_r(x(t))$; second, if the condition (ii) holds for $G_r(x(0))$, then it also holds for $G_r(x(t))$. Now, under condition (ii) for both SBC and SBI systems, any WCC of $G_r(x(t))$ contains at least one agent that is an out-neighbor of all agents in that WCC for all $t \geq 0$. Denote one such agent in a WCC by $s$, then that WCC's agents with maximum and minimum opinions update their opinions by taking an average of their out-neighbors, including $s$. Hence, at the next iteration, their opinions will converge to $s$'s opinion, which results in an strict decrease in the opinion interval of the WCC. Since the confidence or influence bounds are strictly greater than zero, there exists a time step after which the opinion interval of the WCC is larger than the minimum confidence or influence bound. Consequently, all agents become each others out-neighbors, and the WCC becomes one closed-minded component. $\square$

**Example 3.3.20.** *The trajectory of an SBC system with initial opinions* $[0 \quad 2 \quad 3 \quad 4.5 \quad 7]^T$ *and bounds vector* $[0.01 \quad 3 \quad 0.01 \quad 3 \quad 0.01]^T$ *exhibits convergence to a fixed profile in finite time. While, the SBC digraph of the limiting opinion vector contains open-minded agents, see Figure 3.6.*

## 3.4 Numerical Analysis

In this section, we provide extensive simulation results that demonstrate the results of Section 3.3 and are consistent with our conjectures. We performed 2000 simulations: 100 simulations of both SBC and SBI systems for ten different agent numbers. In each simulation, the initial opinion vector and bounds vector are generated randomly and uniformly distributed on [0, 1] and [0, 0.3], respectively. The time steps $\tau$ at which trajectories satisfied the condition of Theorem 3.3.4 are plotted in Figures 3.7 and 3.8. All the 2000 SBC and SBI trajectories eventually satisfied the special case of the sufficient condition of Theorem 3.3.4, stated in Remark 3.3.7. In other words, for each trajectory $x(t)$, there exists time $\tau$ such that $x(\tau)$ belongs to the invariant equi-topology neighborhood of its own final value at constant topology $\mathrm{fvct}(x(\tau))$. Thus, $\mathrm{fvct}(x(\tau))$ is an equilibrium opinion vector, and is equal to the limiting opinion vector of $x(t)$. The frequency of occurrence of this special case is intuitively explained by the following statements: First, by Conjecture 3.1.1, for each trajectory a limiting opinion vector $x_\infty$ exists. Second, for any randomly generated opinion vector $y$ and bounds vector $r$, the probability of having $\min_{i\in\{1,\dots,n\}} \epsilon_i(y) = 0$, where $\epsilon(y)$ is the equi-topology distance of $y$, is equal to zero, and one can assume that the same holds for any limiting opinion vector. Third, Lemma 3.3.9 tells us that if the limiting opinion

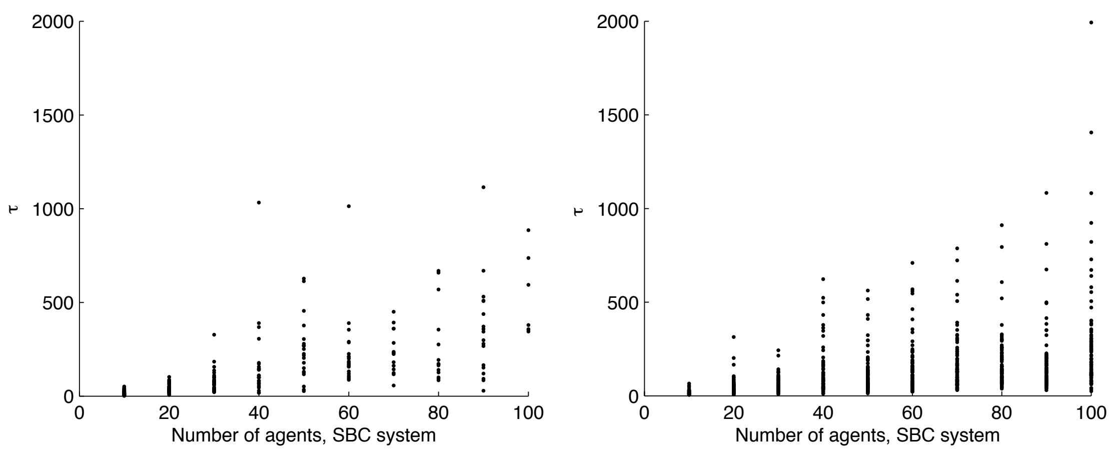
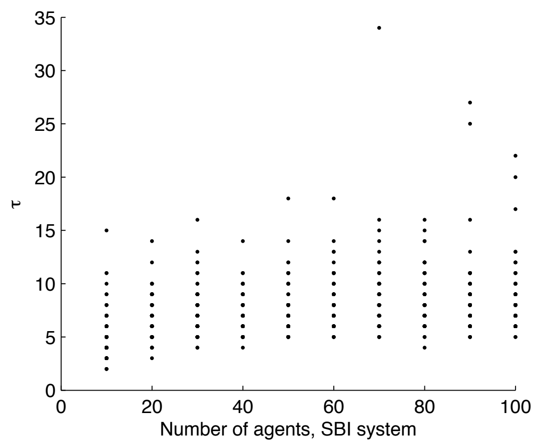



**Figure 3.7:** In one thousand simulations of SBC systems, the time step $\tau$ at which the trajectory of each system satisfied the sufficient condition of Theorem 3.3.4 is plotted versus the number of agents in that system. The left and right plots, respectively, illustrate time $\tau$ for trajectories that converged in finite time and infinite time. Each initial opinion vector and bounds vector are generated randomly and uniformly distributed on [0, 1] and [0, 0.3], respectively. For each agent number hundred simulations are performed. All trajectories satisfied the special case of sufficient condition of Theorem 3.3.4, stated in Remark 3.3.7 in finite time.



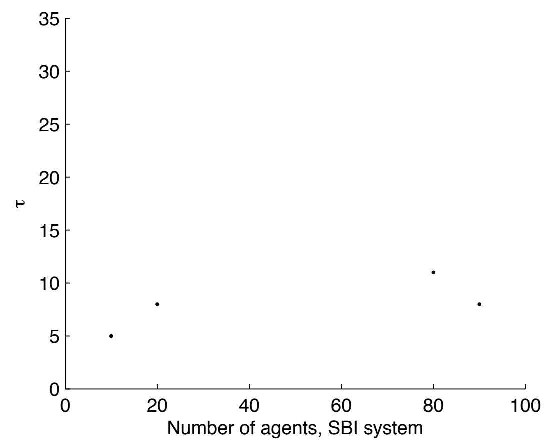




**Figure 3.8:** In one thousand simulations of SBI systems, the time step $\tau$ at which the trajectory of each system satisfied the sufficient condition of Theorem 3.3.4 is plotted versus the number of agents in that system. The left and right plots, respectively, illustrate time $\tau$ for trajectories that converged in finite time and infinite time. As shown, only four SBI trajectories converged in infinite time. Each initial opinion vector and bounds vector are generated randomly and uniformly distributed on [0, 1] and [0, 0.3], respectively. For each agent number hundred simulations are performed. All trajectories satisfied the special case of sufficient condition of Theorem 3.3.4, stated in Remark 3.3.7 in finite time.

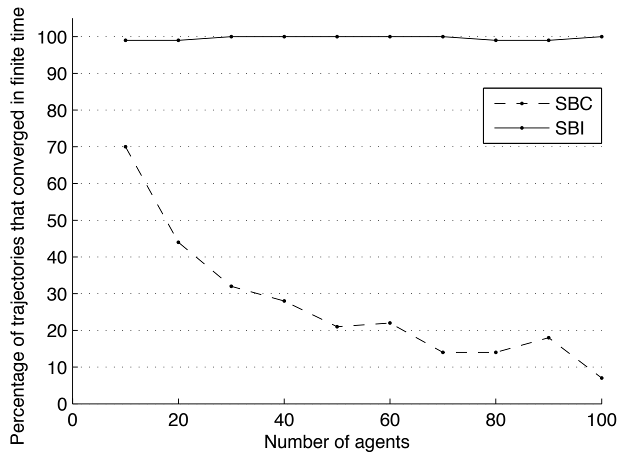



**Figure 3.9:** For each agent number 100 SBC systems and 100 SBI systems are simulated, as explained in Figures 3.7 and 3.8. The percentage of SBC (blue) and SBI (green) trajectories that reached their limiting opinion vector, which is an agreement opinion vector, in finite time is plotted versus agent number.

vector satisfies $\min_{i\in\{1,\ldots,n\}} \epsilon_i(x_\infty) > 0$, then the trajectory eventually satisfies the mentioned special case of condition of Theorem 3.3.4.

In above mentioned simulations, for each agent number, the percentage of SBC and SBI trajectories that reached a fixed profile in finite time are plotted in Figure 3.9. Clearly, Figure 3.9 supports Conjecture 3.1.4. To explain this frequency of convergence of SBI trajectories in finite time as compared with SBC trajectories, we use the results of Subsection 3.3.3. For uniformly randomly generated opinion vector and bounds vector, an SBI digraph is more likely to satisfy condition (ii) of Proposition 3.3.18 than an SBC digraph. One can assume that the same holds for a trajectory with uniformly randomly generated initial opinion vector and bounds vector.

In the next section, based on our "constant topology in finite time" conjecture, we assume that the interconnection topology in an SBC or SBI system remains constant for infinite time and address the following questions: How the three classes of agents behave? How groups of agents affect each other? And can one explain the observed pseudo-stable behavior of trajectories, as stated in Conjecture 3.1.2?

## 3.5 Evolution Under Constant Topology

Motivated by Conjecture 3.1.2, we investigate the rates and directions of convergence of separate classes of agents in the SBC and SBI systems under fixed interconnection topology as time goes to infinity. This analysis proves that the system shows a pseudo-stable behavior under fixed topology.

**Remark 3.5.1.** *Consider a converging trajectory $x(t)$ whose limiting opinion vector is equal to $\operatorname{fvct}(x(t))$ for all $t \geq 0$. Then, $x(t)$ exhibits a pseudo-stable behavior, see equation (3.2), if and only if for all $i \in \{1, \ldots, n\}$*

$$\begin{cases} 0 < k_i(x(t)) < 1, & \textit{if } k_i(x(t)) \textit{ exists}, \\ x_i(t) = x_i(t+1) = \operatorname{fvct}_i(x(t)), & \textit{otherwise}. \end{cases}$$

**Definition 3.5.2** (Leader SCC)**.** *For opinion vector $y \in \mathbb{R}^n$, let $G_r(y)$ denote either its SBC or SBI digraph. Consider an SBC or SBI system with opinion vector $y \in \mathbb{R}^n$. For any open-minded SCC, $S_k(y)$, of $G_r(y)$, denote the set of its open-minded successor SCC's, including $S_k(y)$, by $\mathcal{M}(S_k(y))$. We define $S_k(y)$'s* leader SCC *to be the SCC whose adjacency matrix has the largest spectral radius among all SCC's of $\mathcal{M}(S_k(y))$.*

Note that in SBC and SBI digraphs, the adjacency matrix of a large SCC has a large spectral radius, hence that SCC tends to become a leader SCC for its predecessors.

**Theorem 3.5.3** (Evolution under constant topology)**.** *Consider an SBC or SBI system, denote its trajectory by $x(t)$ and proximity digraph by $G_r(x(t))$. Assume that there exists a time $\tau$ after which $G_r(x(t))$ remains unchanged, that is, $G_r(x(t)) = G_r(x(\tau))$. Then, the following statements hold:*

*(i)* $\mathrm{fvct}(x(t)) = \mathrm{fvct}(x(\tau))$ *for all $t \geq \tau$.*

*(ii)* $G_r(x(\tau))$ *contains no moderate-minded component.*

*(iii) Consider any open-minded SCC $S_k(x(t))$ of $G_r(x(t))$ and its leader SCC $S_m(x(t))$, with adjacency matrices denoted by $\Theta_k$ and $\Theta_m$, respectively. Then,*

*(a) for any $i \in S_k(x(t))$, either $x_i(t) - \mathrm{fvct}_i(x(t)) = 0$ or its per-step convergence factor converges to the spectral radius of $\Theta_m$ as $t \to \infty$, and*

(b) *if the spectral radius of* $\Theta_m$ *is strictly larger than that of other SCC's in* $\mathcal{M}(S_k(y))$*, then there exists* $t_1 \geq \tau$ *such that for all* $i \in S_k(x(t))$*,* $j \in S_m(x(t))$*, and* $t \geq t_1$*,*

$$
\begin{aligned}
x_j(t_1) < \mathrm{fvct}_j(x(t_1)) \quad &\Longrightarrow \quad x_i(t) \leq \mathrm{fvct}_i(x(t)),\\
x_j(t_1) > \mathrm{fvct}_j(x(t_1)) \quad &\Longrightarrow \quad x_i(t) \geq \mathrm{fvct}_i(x(t)).
\end{aligned}
$$

(iv) *There exists time* $t_2 \geq \tau$ *such that for all* $t \geq t_2$*,* $x(t)$ *exhibits a pseudo-stable behavior, see equation* (3.2)*.*

**Remark 3.5.4** (Interpretation of statement (iii) in Theorem 3.5.3)**.** *Parts (a) and (b) tell us, respectively, that the rates and directions of convergence of opinions in an open-minded SCC toward the final value at constant topology are governed by the direction and rate of convergence of its leader SCC. It is easy to see that the per-step convergence factor has an inverse relation with the rate of convergence to the final value at constant topology. Therefore, Theorem 3.5.3 implies that under fixed interconnection topology, individuals converge to a final decision as slow as the slowest group of agents whom they listen to.*

*Proof of Theorem 3.5.3.* Statement (i) is a direct consequence of $A(x(t)) = A(x(\tau))$ for all $t \geq \tau$. Statement (ii) can be proved similar to part (ii)b of Proposition 3.2.2. It was shown that under fixed interconnection topology, all agents in one moderate-minded SCC of an SBC or SBI digraph reach consensus as time goes to infinity. Since, the bounds vector is strictly greater than zero, there exists a time step after which the adjacency matrix of one moderate-minded SCC transforms into a complete consensus matrix, which contradicts the assumption of having fixed topology for infinite time. Before proving statement (iii), since the canonical permutation matrix remains unchanged, let us assume that the opinions in $x(\tau)$ are ordered such that $A(x(\tau)) = \overline{A}(x(\tau))$. Furthermore, owing to the fixed interconnection topology, we drop the $x(t)$ argument for simplicity. Therefore, by equation (3.3),

$$A(x(t)) = \begin{bmatrix} C & 0 \\ \Theta_C & \Theta \end{bmatrix}.$$

Now, for all $t > \tau$ we have

$$x(t) - \text{fvct}(x(\tau)) = \begin{bmatrix} Cx_C(\tau) \\ x_\Theta(t) \end{bmatrix} - \begin{bmatrix} Cx_C(\tau) \\ \text{fvct}_\Theta(x(\tau)) \end{bmatrix} = \begin{bmatrix} 0 \\ x_\Theta(t) - \text{fvct}_\Theta(x(\tau)) \end{bmatrix},$$

where $x_C(t)$ and $x_\Theta(t)$ are the opinion vectors of agents in closed- and open-minded classes respectively. Using $\text{fvct}(x(\tau)) = A(x(\tau))\,\text{fvct}(x(\tau))$, the following recurrence relation holds

$$x_\Theta(t+1) - \text{fvct}_\Theta(x(\tau)) = \Theta(x_\Theta(t) - \text{fvct}_\Theta(x(\tau))) \quad \forall\, t \geq \tau. \tag{3.13}$$

Consider an open-minded WCC of $G_r(x(t))$, denoted by $W_1$. Let $\Theta_1$ denote $W_1$'s adjacency matrix, and $x_1(t)$ denote the trajectory of nodes of $W_1$. Under fixed interconnection topology, the trajectory of each WCC is independent of others, thus for all $t \geq 0$, $x_1(t+\tau) - \mathrm{fvct}_1(x(\tau)) = \Theta_1^t(x_1(\tau) - \mathrm{fvct}_1(x(\tau)))$. According to the block lower triangular from of $\Theta_1$,

$$\Theta_1^t = \begin{bmatrix} \Theta_{11}^t & & 0 \\ \Theta_{21}^{(t)} & \Theta_{22}^t & \\ \vdots & & \ddots \end{bmatrix},$$

where each $\Theta_{ii}$ is the adjacency matrix of an SCC, denoted by $S_{ii}$, of $W_1$. Let $x_{ii}(t)$ be the opinion trajectory of nodes of $S_{ii}$. Clearly, $S_{11}$ is one of the sink SCC's in $W_1$, and $\Theta_{ii}$'s are ordered in $\Theta_1$ according to the distance of $S_{ii}$'s to the sinks. For simplicity, we prove statement (iii) for $S_{11}$ and an SCC that is the direct predecessor of $S_{11}$. Without loss of generality, let $S_{22}$ be one such SCC. The proof for the rest of open-minded SCC's is similar.

Each block $\Theta_{ii}$ is nonnegative and primitive. By Perron-Frobenius Theorem: the spectral radius of $\Theta_{ii}$, denoted by $\lambda_i$, is positive and a simple eigenvalue of $\Theta_{ii}$; and there exists a positive eigenvector $\nu_i$ for $\Theta_{ii}$ associated to $\lambda_i$. Any $\Theta_{ii}$ can be written in Jordan normal form by some similarity transformation

$$\Theta_{ii} = QJQ^{-1} = \left[\begin{array}{c|c} \nu_i & Q_e \end{array}\right] \begin{bmatrix} \lambda_i & \mathbf{0} \\ \mathbf{0} & J_e \end{bmatrix} \left[\begin{array}{c} w_i \\ \hline Q_e^{(-1)} \end{array}\right],$$

where $w_i$ is the first row of $Q^{-1}$. Consequently,

$$\lim_{t\to\infty} \lambda_i^{-t}\Theta_{ii}^t = \lim_{t\to\infty}(\lambda_i^{-t}\lambda_i^t\nu_i w_i + \lambda_i^{-t}Q_e J_e^t Q_e^{(-1)}) = \nu_i w_i, \tag{3.14}$$

the term $\lambda_i^{-t} Q_e J_e^t Q_e^{(-1)}$ converges to zero owing to the fact that the norms of all eigenvalues in Jordan matrix $J_e$ are strictly less than $\lambda_i$. In any open-minded WCC, the sink SCC is not affected by other SCC's, hence a sink SCC is its own leader. Therefore, for all $t \geq 0$,

$$x_{11}(t+\tau) - \mathrm{fvct}_{11}(x(\tau)) = \Theta_{11}^t (x_{11}(\tau) - \mathrm{fvct}_{11}(x(\tau))).$$

In the interest of simplicity, let us denote the vector $x_{ii}(t) - \mathrm{fvct}_{ii}(x(\tau))$ by $\Delta_{ii}(t)$, then we have

$$\lim_{t\to\infty} \lambda_1^{-t} \Delta_{11}(t) = \nu_1 w_1 \Delta_{11}(\tau). \tag{3.15}$$

Regarding part (iii)a for $S_{11}$, for the per-step convergence factor of any $i \in S_{11}$ we have

$$\lim_{t\to\infty} k_i(x(t)) = \lim_{t\to\infty} \frac{[\Delta_{11}(t+1)]_i}{[\Delta_{11}(t)]_i} = \lim_{t\to\infty} \lambda_1 \frac{\lambda_1^{-t-1}[\Delta_{11}(t+1)]_i}{\lambda_1^{-t}[\Delta_{11}(t)]_i} = \lambda_1 \frac{\nu_{1i} w_1 \Delta_{11}(\tau)}{\nu_{1i} w_1 \Delta_{11}(\tau)} = \lambda_1$$

Regarding part (iii)b for $S_{11}$, since $\lambda_1^t w_1 \Delta_{11}(\tau)$ is a scalar, $\lambda_1$ is positive, and $\nu_1$ is a positive vector, all entries of vector $\nu_1 w_1 \Delta_{11}(\tau) \lambda_1^t$ have the same sign. Therefore, there exists time $T \geq \tau$ after which all entries of $\Delta_{11}(t)$ have the same sign for all $t \geq T$.

Here, we prove the two statements for $S_{22}$. It can be computed that for all $t \geq 0$

$$\Delta_{22}(t+\tau) = \sum_{i=0}^{t-1} \Theta_{22}^i \Theta_{21} \Theta_{11}^{t-i-1} \Delta_{11}(\tau) + \Theta_{22}^t \Delta_{22}(\tau). \tag{3.16}$$

Now, to find $\lim_{t\to\infty} \Delta_{22}(t)$, we consider three cases:

1) If $\lambda_1 > \lambda_2$, then $S_{11}$ is $S_{22}$'s leader. From Section 3.1.2 it is known that $\lambda_1$ and $\lambda_2$ are strictly less than one. Then, according to the transient analysis of the reducible Markov chains from [23, Section 5.6], the limit of the first term on the right hand side of equation (3.16) as $t \to \infty$ can be computed:

$$\lim_{t\to\infty} \lambda_1^{-t} \sum_{i=0}^{t-1} \Theta_{22}^i \Theta_{21} \Theta_{11}^{t-i-1} = \big(\lim_{t\to\infty} \sum_{i=0}^{t-1} \Theta_{22}^i\big) \Theta_{21} \lim_{t\to\infty} \lambda_1^{-t} \Theta_{11}^t$$
$$= (I - \Theta_{22})^{-1} \Theta_{21} \nu_1 w_1,$$

and the limit of the second term is equal to

$$\lim_{t\to\infty} \lambda_1^{-t} \Theta_{22}^t = \lim_{t\to\infty} \lambda_1^{-t} \lambda_2^t \lambda_2^{-t} \Theta_{22}^t = \nu_2 w_2 \lim_{t\to\infty} \lambda_1^{-t} \lambda_2^t = 0.$$

Therefore,

$$\lim_{t\to\infty} \lambda_1^{-t} \Delta_{22}(t) = (I - \Theta_{22})^{-1} \Theta_{21} \nu_1 w_1 \Delta_{11}(\tau). \tag{3.17}$$

Regarding part (iii)a for $S_{22}$, for any $i \in S_{22}$ we have

$$\lim_{t\to\infty} k_i(x(t)) = \lim_{t\to\infty} \frac{[\Delta_{22}(t+1)]_i}{[\Delta_{22}(t)]_i} = \lambda_1 \frac{w_1 \Delta_{11}(\tau)[(I - \Theta_{22})^{-1} \Theta_{21} \nu_1]_i}{w_1 \Delta_{11}(\tau)[(I - \Theta_{22})^{-1} \Theta_{21} \nu_1]_i} = \lambda_1.$$

Regarding part (iii)b for $S_{22}$, we first prove that $(I - \Theta_{22})^{-1} \Theta_{21} \nu_1$ is a nonnegative matrix. As stated in Section 3.1.2, the spectral radius of $\Theta_{22}$ is strictly less than one. Consequently, $(I - \Theta_{22})$ is invertible, $\lim_{t\to\infty} \Theta_{22}^t = 0$, the series $I + \Theta_{22} + \Theta_{22}^2 + \dots$ is convergent, and the following product can be computed

$$(I - \Theta_{22})(I + \Theta_{22} + \Theta_{22}^2 + \dots) = I.$$

Therefore,

$$(I-\Theta_{22})^{-1} = I + \Theta_{22} + \Theta_{22}^2 + \dots,$$

which states that $(I-\Theta_{22})^{-1}$ is a nonnegative matrix. Now, since $(I-\Theta_{22})^{-1}\Theta_{21}\nu_1$ is a nonnegative vector, all entries of the vector on the right hand side of equations (3.15) and (3.17) have the same sign as the scalar $w_1\Delta_{11}(\tau)$.

2) If $\lambda_1 < \lambda_2$, then $S_{22}$ is its own leader. Similarly, for the limit of the first term on the right hand side of equation (3.16) as $t \to \infty$ we have:

$$\begin{aligned}\lim_{t\to\infty} \lambda_2^{-t} \sum_{i=0}^{t-1} \Theta_{22}^i \Theta_{21} \Theta_{11}^{t-i-1} &= \lim_{t\to\infty} \lambda_2^{-t} \Theta_{22}^t \Theta_{21} \big( \lim_{t\to\infty} \sum_{i=0}^{t-1} \Theta_{11}^i \big) \\ &= \nu_2 w_2 \Theta_{21} (I-\Theta_{22})^{-1}.\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} \lambda_2^{-t} \Delta_{22}(t) = w_2\big(\Theta_{21}(I-\Theta_{11})^{-1}\Delta_{11}(\tau) + \Delta_{22}(\tau)\big)\nu_2, \tag{3.18}$$

where $w_2\big(\Theta_{21}(I-\Theta_{11})^{-1}\Delta_{11}(\tau) + \Delta_{22}(\tau)\big)$ is a scalar, $\lambda_2$ is positive, and $\nu_2$ is a positive vector. Regarding part (iii)a for $S_{22}$, for any $i \in S_{22}$ we have

$$\lim_{t\to\infty} k_i(x(t)) = \lim_{t\to\infty} \frac{[\Delta_{22}(t+1)]_i}{[\Delta_{22}(t)]_i} = \lambda_2 \frac{w_2(\Theta_{21}(I-\Theta_{11})^{-1}\Delta_{11}(\tau) + \Delta_{22}(\tau))\nu_{2i}}{w_2(\Theta_{21}(I-\Theta_{11})^{-1}\Delta_{11}(\tau) + \Delta_{22}(\tau))\nu_{2i}} = \lambda_2.$$

Regarding part (iii)b for $S_{22}$, all entries of the limiting vector in equation (3.18) have the same sign.

3) if $\lambda_1 = \lambda_2 = \lambda$, then for the limit of the first term on the right hand side of

equation (3.16) as $t \to \infty$ we have:

$$\lim_{t\to\infty} \lambda^{-t} \sum_{i=0}^{t-1} \Theta_{22}^{i} \Theta_{21} \Theta_{11}^{t-i-1} = \lim_{t\to\infty} \lambda_2^{-t} \Theta_{22}^{t} \Theta_{21} \big( \lim_{t\to\infty} \sum_{i=0}^{t-1} \Theta_{11}^{i} \big)$$
$$+ \big( \lim_{t\to\infty} \sum_{i=0}^{t-1} \Theta_{22}^{i} \big) \Theta_{21} \lim_{t\to\infty} \lambda_1^{-t} \Theta_{11}^{t} = \nu_2 w_2 \Theta_{21} (I - \Theta_{22})^{-1} + (I - \Theta_{22})^{-1} \Theta_{21} \nu_1 w_1.$$

Therefore,

$$\lim_{t\to\infty} \lambda^{-t} \Delta_{22}(t) = \big( \alpha \nu_2 + w_1 \Delta_{11}(\tau) (I - \Theta_{22})^{-1} \Theta_{21} \nu_1 \big),$$

where $\alpha = w_2 \big( \Theta_{21} (I - \Theta_{11})^{-1} \Delta_{11}(\tau) + \Delta_{22}(\tau) \big)$. Regarding part (iii)a for $S_{22}$, for any $i \in S_{22}$ we have

$$\lim_{t\to\infty} k_i(x(t)) = \lambda \frac{(\alpha \nu_{2i} + w_1 \Delta_{11}(\tau) [(I - \Theta_{22})^{-1} \Theta_{21} \nu_1]_i)}{(\alpha \nu_{2i} + w_1 \Delta_{11}(\tau) [(I - \Theta_{22})^{-1} \Theta_{21} \nu_1]_i)} = \lambda.$$

Regarding part (iii)b for $S_{22}$, notice that the theorem does not discuss the case with equal spectral radii.

Finally, statement (iv) is proved utilizing previous statements. For any $i \in G_r(x(t))$ two cases exists. First, if $i$ belongs to a closed-minded SCC, then $x_i(t) = x_i(\tau+1)$ for all $t > \tau$, and hence $k_i(x(t))$ does not exist. Second, if $i$ belongs to an open-minded SCC $S_k(x(t))$, then according to part (iii)a, either $x_i(t) = \text{fvct}_i(t)$ or $k_i(x(t))$ converges to the spectral radius of the adjacency matrix of $S_k(x(t))$'s leader SCC. This spectral radius is proved in Section 3.1.2 to be strictly larger than zero and strictly smaller than one. In other words, there exists time $t_2$ such that for all $t \geq t_2$, $0 < k_i(x(t)) < 1$. Therefore, according to Remark 3.5.1, $x(t)$ exhibits pseudo-stable behavior. $\square$

**Remark 3.5.5** (Justification of the sufficient condition for monotonic convergence)**.** *We justify the conditions of Theorem 3.3.14 employing Conjecture 3.1.2 and Theorem 3.5.3. Note that these conditions are sufficient but not necessary for monotonic convergence. Based on our conjecture, we assume that the topology of an SBC or SBI trajectory $x(t)$ remains unchanged after time $\tau$, thus condition (i) of Theorem 3.3.14 is satisfied. Regarding conditions (ii)* and *(iii), by statement (iii)a of Theorem 3.5.3, there exist a time step $t_1 \geq \tau$, after which the per-step convergence factor of all agents belong to $[0, 1]$. Therefore, the opinion vector converges toward its final value at constant topology monotonically in one step. Moreover, since the opinion vector is discrete, this monotonic convergence results in existence of a time step $t_2 \geq \tau$, after which condition (ii) of the Theorem 3.3.14 holds. Regarding condition (iv), statement (iii) of Theorem 3.5.3 shows that there exists time step $t_3 \geq \tau$, after which for any open-minded $i$ and $j$ it is true that: if they both belong to one SCC, then $\Delta_i(x(t))\Delta_j(x(t)) \geq 0$; and if they belong to two separate SCC's with adjacency matrices $\Theta_1$ and $\Theta_2$, respectively, while $j$ is a successor of $i$, then when $\rho(\Theta_1) < \rho(\Theta_2)$, often it is true that $\Delta_i(x(t))\Delta_j(x(t)) \geq 0$, and when $\rho(\Theta_1) > \rho(\Theta_2)$, $\Delta_j(x(t))$ converges to zero faster than $\Delta_i(x(t))$ and hence $\Delta_i(x(t))\Delta_j(x(t)) \simeq 0$. Regarding condition (v)* *part (a), if $i$ and $j$ have the same set of open-minded children at time $t$, then $k_i(x(t+1)) = k_j(x(t+1))$, see proof of Theorem 3.3.14. Finally, we explain why the upper bound in condition (v) part (b) is less restrictive as time goes to infinity. Since, for such agent $i$, the distance to final values of all successors with smaller per-step convergence factors converge to zero, the interval $[k_{min_i}(x(t)), k_{max_i}(x(t))]$ reduces to one value, that is $k_{max_i}(x(t))$ to which $k_i(x(t))$ converges. Consequently, for large $t$, $k_{max_{i,j}}(x(t)) = \max\{k_i(x(t)), k_j(x(t))\}$ , $\alpha = k_j(x(t))$, and $\beta = k_i(x(t))$. Also, if $\Delta_i(x(t)) \geq \Delta_j(x(t))$, then $k_i(x(t)) \geq k_j(x(t))$, and hence*

$$\min_{m,\alpha,\beta} \left|1 - \frac{\alpha^m \Delta_j(x(t))}{\beta^m \Delta_i(x(t))}\right| \simeq 1 - \frac{\Delta_j(x(t))}{\Delta_i(x(t))}.$$

*A system may monotonically converge under fixed topology while condition (v) of Theorem 3.3.14 is not satisfied. However, Example 3.3.15 illustrates the sufficiency of this condition.*

In the following, we provide numerical examples that facilitate the understanding of the conditions and results of Theorem 3.5.3.

**Example 3.5.6.** *Consider an SBC system with the initial opinion vector $x(0) = [0\ 1.5\ 3.5\ 5\ 1\ 1\ 4\ 2.1]^T$ and confidence bounds $r = [0.01\ 0.01\ 0.01\ 0.01\ 1\ 1\ 1\ 3]^T$. For all $t \geq 0$, the SBC digraph $G_r(x(t))$ remains unchanged and contains three open-minded SCC's: $\{x_5, x_6\}$, $\{x_7\}$, and $\{x_8\}$. The adjacency matrix of open-*

*minded subgraph equals*

$$\Theta = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & & \\ \frac{1}{4} & \frac{1}{4} & & \\ & & \frac{1}{3} & \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix}.$$

*The spectral radii of the adjacency matrices of the three SCC's are* $0.5$*,* $0.333$*, and* $0.125$*, respectively. The two SCC's* $\{x_5, x_6\}$ *and* $\{x_7\}$ *are successors of* $\{x_8\}$*, and based on their spectral radii,* $\{x_5, x_6\}$ *is* $\{x_8\}$*'s leader SCC. We can see that the per-step convergence factor of* $x_8$ *converges to* $0.5$*. Furthermore, the sign of its direction of convergence toward the final value, i.e, the sign of* $x_8(t) - x_8^*$*, is the same as the leader's after* $t = 1$*. These facts support Theorem 3.5.3, see Figure 3.10.*

**Example 3.5.7.** *The initial opinion vector and confidence bounds vector of an SBC system are generated randomly. This system satisfies the condition of Theorem 3.3.4 at* $t = 50$*. Moreover, since* $x(50) \in \mathcal{B}_{\text{iet}}(\text{fvct}(x(50)))$*, the SBC digraph* $G_r(x(t))$ *is equal to* $G_r(\text{fvct}(x(50)))$ *for all* $t \geq 50$*. The digraph* $G_r(\text{fvct}(x(50)))$ *contains two open-minded SCC's, whose limiting opinions lie: 1) in interval [0.39, 0.5], denoted by* $S_1$*, and in interval [0.3, 0.34], denoted by* $S_2$*. According to the topology of* $G_r(\text{fvct}(x(50)))$*,* $S_2$ *is a predecessor of* $S_1$*. The spectral radii of the adjacency matrices of* $S_1$ *and* $S_2$ *are equal to 0.6667 and 0.8381, respectively.*

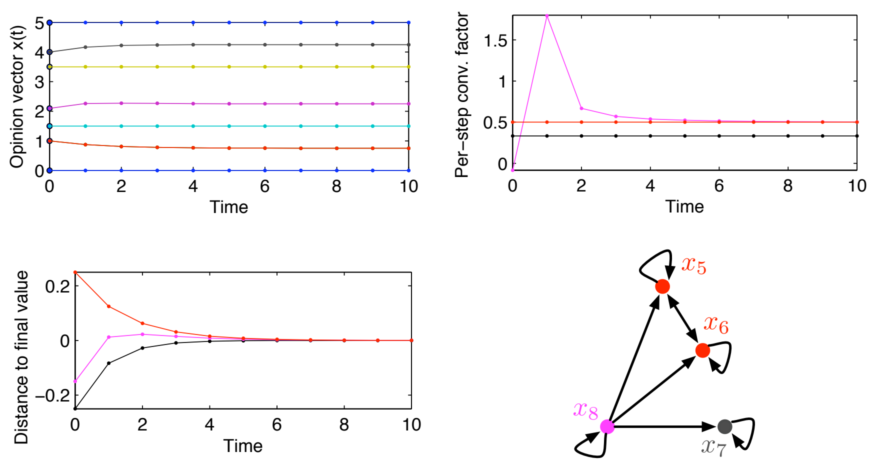



**Figure 3.10:** The SBC trajectory $x(t)$ of Example 3.5.6 is plotted on the top left, the open-minded agents per-step convergence factors on the top right, the open-minded agents distances to their final values at constant topology $x_i(t) - x_i^*(x(t))$ on the bottom left, and the open-minded subgraph of $G_r(x(t))$ is illustrated on the bottom right.

*Therefore, both $S_1$ and $S_2$ are their own leader SCC's, and by Theorem 3.5.3 the per-step convergence factors of their agents converge to 0.6667 and 0.8381, respectively. The right plot verifies that the per-step convergence factors of all open-minded agents converge to those two values, see Figure 3.11.*

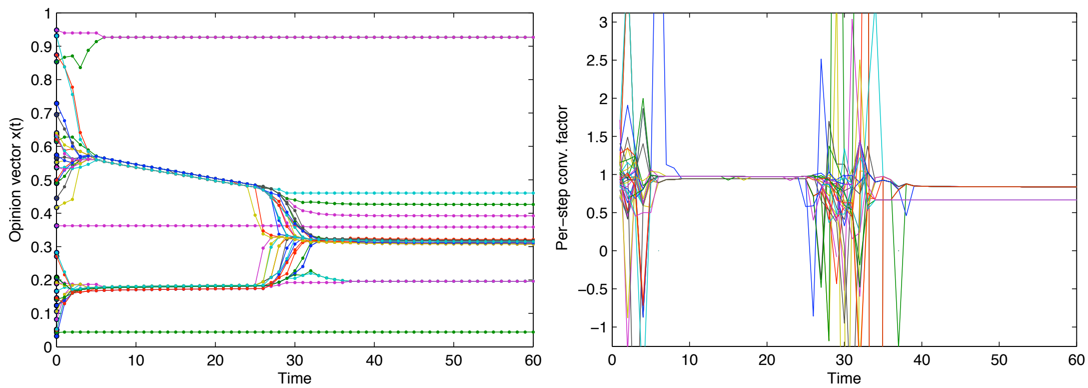



**Figure 3.11:** The trajectory of the SBC system in Example 3.5.7 (left) and the per-step convergence factor of its open-minded agents (right) are illustrated.

**Example 3.5.8.** *Another example in which topology remains fixed along the evolution is a system with* $x_0 = [0 \quad 0 \quad 3 \quad 4.3 \quad 4.3 \quad 4.3 \quad 4.3 \quad 3.4 \quad 1]^T$ *and* $r = [0.01 \quad 0.01 \quad 0.01 \quad 0.01 \quad 0.01 \quad 0.01 \quad 0.01 \quad 1.2 \quad 2.5]^T$*. The proximity digraph of this system contains two open-minded SCC's, $\{x_8\}$ which is the successor of the other SCC $\{x_9\}$. The spectral radii of adjacency matrices of these SCC's are $\frac{1}{6}$ and $\frac{1}{5}$ respectively. By Theorem 3.5.3, the per-step convergence factor of $\{x_9\}$ converges to the spectral radius of its own adjacency matrix, see Figure 3.12.*

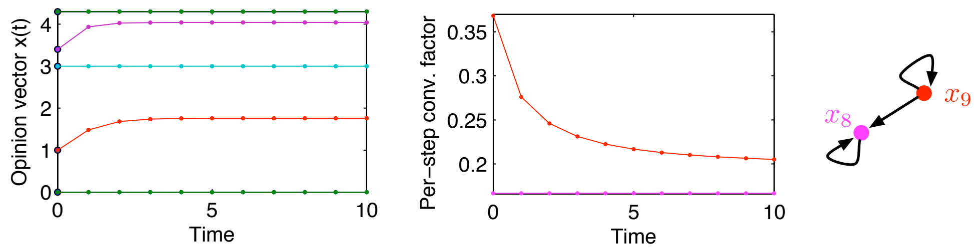



**Figure 3.12:** The evolution of opinions in Example 3.5.8 (left), $k_i(x(t))$ of open-minded agents (center), and the open-minded proximity digraph (right).

## 3.6 Summary

This chapter introduced a synchronized bounded influence (SBI) model of opinion dynamics, which is similar to the heterogeneous bounded confidence model introduced by Hegselmann and Krause, which we called synchronized bounded confidence (SBC) model. First, we conjectured that in both SBC and SBI systems, for each trajectory there exists a finite time, after which the topology of the interconnection network remains unchanged, hence, the trajectory converges to a limiting opinion vector. Second, we conjectured that if a trajectory does not reach a fixed profile in finite time, then it eventually shows a pseudo-stable behavior. We partly proved these conjectures through the following analysis. We designed a classification of agents that is employed in computing the equilibria of the system. We introduced the equi-topology neighborhood and the invariant equi-topology neighborhood of the equilibria of the system. Based on these neighborhoods, we derived sufficient condition for both SBC and SBI systems to guarantee that the interconnection topology remains unchanged for infinite time in a trajectory, and therefore, the trajectory converges to a steady state. In our simulation results, it is observed that for uniformly randomly generated initial opinion vector and bounds vector, the trajectories of both systems eventually satisfy the mentioned sufficient condition with probability one. However, the eventual convergence of every trajectory of the SBC and SBI systems to a steady state is still an open problem. Third, we conjectured that, for uniformly randomly generated initial opinion vector and bounds vector, the simulations of SBI systems converge in fewer time steps and more often in finite time than SBC systems. We derived sufficient conditions for convergence in finite time for SBC and SBI systems separately that intuitively explains our third conjecture. Finally, we studied the trajectory of both SBC and SBI systems when they update their opinions under fixed interconnection topology for infinite time. We showed the existence of a leader group for each group of agents that determines the follower's rate and direction of convergence.

It is useful to conclude with some formal statements regarding our conjectures. Assume that for any limiting opinion vector $x_\infty$, the probability that $\min_{i\in\{1,\dots,n\}} \epsilon_i(x_\infty) = 0$ is equal to zero. Then, according to Lemma 3.3.9, Conjecture 3.1.1 implies the following weak version of Conjecture 3.1.2: for almost all trajectories $x(t)$ of an SBC or SBI system, there exists a finite time $\tau$ after which the state-dependent interconnection topology, or equivalently $G_r(x(t))$, remains constant. Furthermore, if Conjecture 3.1.2 holds, then by Theorem 3.5.3, Conjecture 3.1.3 also holds. Finally, Conjecture 3.1.4 is related to and partly established in Proposition 3.3.18.

The main future challenge is to prove that all trajectories of SBC and SBI systems converge to steady states. One approach is to prove that, in each system, any trajectory is eventually confined to the invariant equi-topology neighborhood of an equilibrium opinion vector. The existence of such equilibrium vector is partly proved in the work by Lorenz [33], who establishes that the product of an infinite number of row-stochastic matrices with positive diagonals converges to a partly fixed matrix, with complete consensus matrices on the diagonal. Consequently, the structure of a limiting adjacency matrix can distinguish the closed- and open-minded components of the system as time goes to infinity, and also can determine the limiting opinions of the closed-minded components. In an equilibrium opinion vector, knowing the closed-minded's opinions, there are only a finite number of possible values for the open-minded's opinions. Therefore, the results in [33] leaves us with a finite number of limiting opinion vectors for each SBC and SBI system, and equivalently, a finite number of invariant equi-topology neighborhoods to which the trajectory can be asymptotically confined. Another future challenge is a probability analysis on the topology of the proximity digraphs, to prove the conjecture on higher probability of convergence of SBI trajectories in finite time.

# Chapter 4

# Infinite Population

Modeling the formation of opinions in a large population allows approximation of the decision making rules with Non-Bayesian "rule of thumb" methods. Models of opinion dynamics can be described by either a *Lagrangian* or an *Eulerian* method. A Lagrangian description focuses on changes in each agent's opinion; however, an Eulerian description focuses on the changes in agents population in one opinion interval as time progresses. A Lagrangian model of opinion dynamics is defined over a continuous or discrete state space if the number of agents are infinite or finite, respectively. An Eulerian model of opinion dynamics is defined over a continuous or discrete state space depending on whether the opinion grid size is converging to zero or not, respectively. In previous sections we focused on discrete Lagrangian models of opinion dynamics, mainly the bounded-confidence model called HK model, that involved finite number of agents. This chapter focuses on a discrete-time continuous-space HK model of opinion dynamics described by an Eulerian view point. In this model, a population is distributed over an opinion set and updates its opinion via the opinion of the population inside the confidence range. Previously, (Canuto et al, 2008) proved the convergence of a variation of Eulerian HK model both in discrete and continuous time. In their model, the weights that two opinion values assign to each other are equal, and this symmetry preserves the global average during the evolution. In other words, each agent updates its opinion to a sum of its neighbors' opinions rather than the average of those opinions. In this context, we consider a more general Eulerian HK model where the symmetric weight constraint has been relaxed. Specifically, the weight an opinion assigns to other opinions is a function of the integral of the mass distribution in that opinion's confidence bound. Since the measures on different opinions' confidence bounds are not necessarily equal, the weights assigned to different opinions are generally asymmetric, and thus the global average is not preserved.

The contributions of this chapter can be summarized as follows. We derive a simple sufficient condition for the system to reach opinion consensus. Second, we establish some important properties of the Eulerian HK model. Under mild technical assumptions (the initial opinion is a finite and absolutely continuous mass distribution over the opinion set), we show that the opinion update via an Eulerian flow map has the following properties: i) the mass distribution on opinions remains finite and absolutely continuous; ii) the flow map preserves opinion order, due to the homogeneity of confidence bounds; and iii) the flow map is bi-Lipschitz. Finally, this analysis also leads to a convergence proof of the mass distribution to a sum of Dirac Delta functions.

## 4.1 Eulerian HK Model

Here, we describe the process of opinion exchange in a large population at discrete times via a sequence of finite Borel measures. This approach is inspired by [13], where the *mass distribution* of agents over the opinion set is represented by $\mu_t : \mathbb{R} \to \mathbb{R}_{\geq 0}$ at discrete time steps $t$. The opinion set belongs to a continuous state space in $\mathbb{R}$ (one dimensional opinions), and each opinion value is denoted by independent variable $x$. Since the state space is continuous by definition, the mass distribution $\mu_t(x)$, whose sum over opinion space is preserved over time, is assumed to be a finite Borel probability measure on $R$. The value $\mu_t(dz)$ at $x$, denoted by $d\mu_t(x)$, represents the infinitesimal population whose opinion is equal to $x$ at time $t$. At $t+1$, this population updates its opinion to $\gamma_t(x)$, defined as the *flow map* of mass distribution $\gamma_t : \text{supp } \mu_t \subseteq \mathbb{R} \to \mathbb{R}$, where supp $\mu_t$ denotes the *support* of the measure $\mu_t$, that is, the set of all points $x \in R$ for which every open neighborhood of $x$ has positive measure. Here, the flow map is defined in compliance with the Lagrangian HK rules,

$$\gamma_t(x) = \frac{\int_{[x-r,x+r]} z d\mu_t(z)}{\int_{[x-r,x+r]} d\mu_t(z)}, \tag{4.1}$$

where $r$ is the confidence bound of agents. Now, the mass distribution can be tracked by the following recurrence relation,

$$\mu_{t+1} = \gamma_t \# \mu_t, \tag{4.2}$$

where $\gamma_t\#$ denotes the *push forward* of a measure via the flow map $\gamma_t$ [13]. Moreover, for every Borel set $E \in \mathbb{R}$,

$$\mu_{t+1}(E) = \mu_t(\gamma_t^{-1}(E)), \tag{4.3}$$

where $\gamma_t^{-1}(E)$ is the preimage of set $E$ under flow map $\gamma_t$ (not necessarily invertible).

**Definition 4.1.1** (Eulerian HK System)**.** *We call the dynamical system in which a mass distribution $\mu_t$ defined over a continuous state space is being pushed forward with flow map* (4.1) *an* Eulerian HK system with Input.

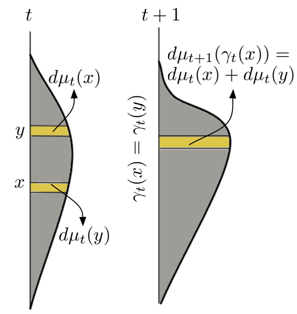



**Figure 4.1:** A schematic illustration of push forward of a measure $\mu_t$ via the flow map $\gamma_t$.

## 4.2 System Properties

In this section, we analyze fundamental properties of Eulerian HK systems which lead to the convergence of the system. Let us start with few definitions and notations. We denote the absolute continuity of any measure $\mu$ with respect to Lebesgue measure $\mathcal{L}^1$ with $\mu \ll \mathcal{L}^1$. We denote the smallest and largest opinions along supp $\mu_t$ by $x_{\min}(t)$ and $x_{\max}(t)$, respectively, and the length of interval $x_{\max}(t) - x_{\min}(t)$ by $|\text{supp } \mu_t|$. Flow map $\gamma_t(x)$ is called bi-Lipschitz, if for any $x, y \in \text{supp } \mu_t$ there exists $L_t \geq 1$ such that

$$|y - x|/L_t \leq |\gamma_t(y) - \gamma_t(x)| \leq L_t|y - x|. \tag{4.4}$$

For any finite mass distribution $\mu_t \ll \mathcal{L}^1$, we define the *opinion average* over any interval $[a,b] \in \mathbb{R}$ where $\int_{[a,b]} d\mu_t(z)$ is nonzero by

$$y_t([a,b]) = \frac{\int_{[a,b]} z d\mu_t(z)}{\int_{[a,b]} d\mu_t(z)}. \tag{4.5}$$

Moreover, consider $a < b \in \mathbb{R}$ and $x, y \in \mathbb{R}_{>0}$, then

$$\max_{x \in [x_1, x_2], y \in [y_1, y_2]} \frac{xa + yb}{x + y} = \frac{x_1 a + y_2 b}{x_1 + y_2}. \tag{4.6}$$

**Lemma 4.2.1** (Bounds on opinion average)**.** *Consider a finite mass distribution $\mu \ll \mathcal{L}^1$, whose support is a closed bounded interval of $R$. Assume that its density function $\rho(x) \geq 0$ satisfies $\rho(x) \in [\rho_{\min}, \rho_{\max}]$ for all $x \in$ supp $\mu$ with $0 < \rho_{\min} \leq \rho_{\max} < \infty$. Then, for all $a, b \in$ supp $\mu$, the opinion average over $[a,b]$, denoted by $y([a,b])$, can be bounded as follows:*

$$a \; < \; \frac{b + a\sqrt{\rho_{\max}/\rho_{\min}}}{1 + \sqrt{\rho_{\max}/\rho_{\min}}} \; \leq \; y([a,b]) \; \leq \; \frac{a + b\sqrt{\rho_{\max}/\rho_{\min}}}{1 + \sqrt{\rho_{\max}/\rho_{\min}}} \; < \; b. \tag{4.7}$$

*Proof.* Since $\mu$ is assumed to be a finite absolutely continuous measure, there exists a Lebesgue integrable density function $\rho$ such that $\mu(E) = \int_E \rho(z) dz$ for all Borel subsets $E \in \mathbb{R}$. Here, we prove the upper bound of the average, and proof to the lower bound is similar. We maximize $y([a,b])$ for the following *step density function* over the variable $c \in [a,b]$:

$$\rho(x) = \begin{cases} \rho_{\min}, & \text{if } x \in [a,c), \\ \\ \rho_{\max}, & \text{if } x \in [c,b], \end{cases} \tag{4.8}$$

According to the first mean value theorem for integrals, one can show that for any bounded density function $\rho(x) \in [\rho_{\min}, \rho_{\max}]$ with $x \in [a, b]$, there exists a real value $c \in [a, b]$ such that $\rho$'s average over $[a, b]$ is equal to the average of step density (4.8) over $[a, b]$, which is equal to

$$y([a,b]) = \frac{\int_a^b z\rho(z)dz}{\int_a^b \rho(z)dz} = \frac{(c^2 - a^2)\rho_{\min}/2 + (b^2 - c^2)\rho_{\max}/2}{(c-a)\rho_{\min} + (b-c)\rho_{\max}} =: \frac{f}{g}.$$

Owing to the differentiability of $y([a, b])$ with respect to $c$, the maximum of $y([a, b])$ over $c$ can be computed by letting $\partial y([a, b])/\partial c$ equal to zero.

$$\frac{\partial y([a,b])}{\partial c} = \frac{c(\rho_{\min} - \rho_{\max})g - (\rho_{\min} - \rho_{\max})f}{g^2} = 0.$$

Hence, the critical point $c = f/g = y([a, b])$ gives maximum $y([a, b])$,

$$2(c^2 - ac)\rho_{\min} + 2(bc - c^2)\rho_{\max} = (c^2 - a^2)\rho_{\min} + (b^2 - c^2)\rho_{\max},$$
$$\Rightarrow \quad c = \frac{a + b\sqrt{\rho_{\max}/\rho_{\min}}}{1 + \sqrt{\rho_{\max}/\rho_{\min}}}.$$

$\square$

**Lemma 4.2.2** (Convergence of opinion average)**.** *Consider a time-varying finite mass distribution $\mu_t \ll \mathcal{L}^1$ such that for all $t \geq 0$, supp $\mu_t$ is some closed bounded interval of $R$ and strictly contains $a, b \in \mathbb{R}$ where $a < b$. If the opinion average over $[a, b]$, denoted by $y_t([a, b])$, satisfies $\lim_{t\to\infty} y_t([a, b]) = a$, then $\mu_t([a, b])$ converges to a scaled Dirac Delta distribution centered at $a$.*

*Proof.* By contradiction, assume that there exists an interval $[c, d]$, where $a < c < d < b$ such that $\mu_t([c, d]) > m$ for some $m \in \mathbb{R}_{>0}$ and all $t \geq 0$. Therefore, if we denote the density function of $\mu_t$ by $\rho_t$, then

$$y_t([a,b]) = \frac{\int_{[a,b]} z d\mu_t(z)}{\int_{[a,b]} d\mu_t(z)} = \frac{\int_a^b z\rho_t(z)dz}{\int_a^b \rho_t(z)dz} = \frac{(\int_a^c z\rho_t(z)dz + \int_d^b z\rho_t(z)dz) + \int_c^d z\rho_t(z)dz}{(\int_a^c \rho_t(z)dz + \int_d^b \rho_t(z)dz) + \int_c^d \rho_t(z)dz}.$$

Since $\mu_t$ is finite, there exits $M \in \mathbb{R}_{>0}$ such that $\mu_t([a, b]) \leq M$ for all $t \geq 0$. According to equation (4.6) and Lemma 4.2.1,

$$y_t([a,b]) > \frac{aM + cm}{M + m} = a + \frac{(c-a)m}{M+m},$$

which contradicts the convergence of $y_t([a, b])$ to $a$. □

The following theorem on Eulerian HK systems is equivalent to Theorem 5.2.1 on Eulerian HK systems with input whose input is zero. Consequently, the proof to the latter theorem can be employed to the former by setting the input equal to a zero distribution, and for the interest of brevity, we omit this proof from this chapter.

**Theorem 4.2.3** (Properties of $\mu_t$ and $\gamma_t$)**.** *Consider an Eulerian HK system whose initial distribution $\mu_0 \ll \mathcal{L}^1$ is finite and supp $\mu_0$ is a closed bounded interval, then for all $t \geq 0$ such that $|supp\ \mu_t| > 2r$,*

*(i) $\mu_t \ll \mathcal{L}^1$ is finite and supp $\mu_t$ is a closed bounded interval;*

*(ii) for any $x, y \in supp\ \mu_t$, if $x < y$, then $\gamma_t(x) < \gamma_t(y)$;*

*(iii)* $\gamma_t(x)$ *is bi-Lipschitz with respect to* $x$*;*

*(iv) supp* $\mu_t$ *strictly contains supp* $\mu_{t+1}$*; and*

*(v)* $x_{\min}(t+1) = \gamma_t(x_{\min}(t))$ *and* $x_{\max}(t+1) = \gamma_t(x_{\max}(t))$.

The following lemma establishes the idea of "rich gets richer and poor gets poorer". Roughly speaking, the lemma states that for any population $E$ with close enough opinions, if the density of the population in $E$'s confidence range is higher than the density of the population just outside of $E$'s confidence range, then the opinions of the $E$ population get closer to each other in one iteration.

**Lemma 4.2.4.** *Assume that in an Eulerian HK system, the mass distribution* $\mu_t$ *is finite and absolutely continuous, and denote its density function by* $\rho_t : \text{supp } \mu_t \to \mathbb{R}_{\geq 0}$*. Consider any* $x \in \text{supp } \mu_t$*, such that* $\rho_t(x) > 0$.

*(i)* $\rho_{t+1}\big(\gamma_t(x)\big) > 0$*;*

*(ii)* $\partial\Delta_t(x)/\partial x = \rho_t(x)/\rho_{t+1}\big(\gamma_t(x)\big) - 1$*; and*

*(iii) if*

$$\begin{aligned} \max\{\rho_t(x+r), \rho_t(x-r)\} &< \frac{\mu_t([x-r, x+r])}{2r} \quad \textit{or} \\ \min\{\rho_t(x+r), \rho_t(x-r)\} &> \frac{\mu_t([x-r, x+r])}{2r}, \end{aligned} \tag{4.9}$$

*then* $\rho_{t+1}\big(\gamma_t(x)\big) > \rho_t(x)$ *or* $\rho_{t+1}\big(\gamma_t(x)\big) < \rho_t(x)$*, respectively.*

*Proof.* Regarding part (i), absolute continuity of $\mu_t$ and $\mu_{t+1}$ together with equation 4.2 results in

$$\lim_{\epsilon \to 0} \rho_t(x)\epsilon = \lim_{\epsilon \to 0} \rho_{t+1}\big(\gamma_t(x)\big)\Big(\gamma_t(x+\epsilon) - \gamma_t(x)\Big). \tag{4.10}$$

According to Theorem 4.2.3 part (ii), $\gamma_t(x+\epsilon) - \gamma_t(x) \neq 0$, which proves our claim.

Regarding part (ii), we have

$$\frac{\partial \Delta_t(x)}{\partial x} = \lim_{\epsilon \to 0} \frac{\Delta_t(x+\epsilon) - \Delta_t(x)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\gamma_t(x+\epsilon) - \gamma_t(x)}{\epsilon} - 1.$$

Employing equation 4.10 in the right hand side results in the claimed statement.

Regarding part (iii), we show that if the inequality (4.9) holds true, then $\rho_{t+1}\big(\gamma_t(x)\big) > \rho_t(x)$, and the proof to the second is similar. First, consider a ball of infinitesimal radius $\epsilon \in \mathbb{R}_{>0}$ centered at $x$ denoted by $\mathcal{B}_\epsilon(x)$,

$$\begin{aligned}\rho_t(x) = \lim_{\epsilon \to 0} \frac{\mu_t(\mathcal{B}_\epsilon(x))}{2\epsilon} &= \lim_{\epsilon \to 0} \frac{\mu_{t+1}\big(\gamma_t(\mathcal{B}_\epsilon(x))\big)}{2\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\rho_{t+1}\big(\gamma_t(x)\big)\big(\gamma_t(x+\epsilon) - \gamma_t(x-\epsilon)\big)}{2\epsilon}. \end{aligned} \tag{4.11}$$

Regarding right hand side,

$$\begin{aligned}\lim_{\epsilon \to 0} \gamma_t(x+\epsilon) &= \lim_{\epsilon \to 0} \frac{(x+r)\hat{\mu}_1 + y\hat{\mu}_2}{\hat{\mu}_1 + \hat{\mu}_2}, \\ \lim_{\epsilon \to 0} \gamma_t(x-\epsilon) &= \lim_{\epsilon \to 0} \frac{(x-r)\hat{\mu}_3 + y\hat{\mu}_2}{\hat{\mu}_3 + \hat{\mu}_2},\end{aligned}$$

where $\hat{\mu}_{1,2,3}$ denote $\mu_t(\mathcal{B}_\epsilon(x+r))$, $\mu_t([x-r+\epsilon, x+r-\epsilon])$, and $\mu_t(\mathcal{B}_\epsilon(x-r))$, respectively, and $y$ denotes the opinion average over interval $[x-r+\epsilon, x+r-\epsilon]$. Here, we prove that the factor of $\rho_{t+1}\big(\gamma_t(x)\big)$ in the right hand side of equation (4.11) is strictly less than one. Knowing that $\lim_{\epsilon\to 0}\mu_t(\mathcal{B}_\epsilon(x\pm r))/2\epsilon = \rho_t(x\pm r)$

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\gamma_t(x+\epsilon)-\gamma_t(x-\epsilon)}{2\epsilon} &= \lim_{\epsilon\to 0}\frac{\gamma_t(x+\epsilon)-x-\gamma_t(x-\epsilon)+x}{2\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{2r\hat{\mu}_1\hat{\mu}_3/\epsilon+\rho_t(x+r)\hat{\mu}_2(x-y+r)+\rho_t(x-r)\hat{\mu}_2(y-x+r)}{(\hat{\mu}_1+\hat{\mu}_2)(\hat{\mu}_3+\hat{\mu}_2)}.\end{aligned}$$

Since $|y-x|<r$, both $x-y+r$ and $y-x+r$ are positive, and according to the inequality (4.9), $\max\{\rho_t(x+r),\rho_t(x-r)\}<\lim_{\epsilon\to 0}\hat{\mu}_2/2r$. Therefore,

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{\gamma_t(x+\epsilon)-\gamma_t(x-\epsilon)}{2\epsilon}\\ &\quad<\lim_{\epsilon\to 0}\frac{r\rho_t(x+r)\hat{\mu}_3+r\rho_t(x-r)\hat{\mu}_1+2r\max\{\rho_t(x+r),\rho_t(x-r)\}\hat{\mu}_2}{(\hat{\mu}_1+\hat{\mu}_2)(\hat{\mu}_3+\hat{\mu}_2)}\\ &\quad<\lim_{\epsilon\to 0}\frac{\hat{\mu}_2(\hat{\mu}_1+\hat{\mu}_3)+\hat{\mu}_2^2}{(\hat{\mu}_1+\hat{\mu}_2)(\hat{\mu}_3+\hat{\mu}_2)}<1.\end{aligned}$$

□

## 4.3 Convergence Behavior

The main result of this chapter is presented in this section, where we prove that in Eulerian HK models, under mild assumptions, the mass distribution converges to a sum of Dirac Delta functions. In the following context, we call a single point $x\in\mathbb{R}$ an atom with respect to a measure $\mu$ if $x\in\operatorname{supp}\mu$ and $\mu(x)>0$. Moreover, if every $\mu$-measurable set of positive measure contains an atom, then $\mu$ is purely atomic or atomic in short.

**Lemma 4.3.1.** *If in an Eulerian HK system* $\ll \mathcal{L}^1$ *is finite and supp* $\mu_0$ *is a closed bounded interval, then for all* $t \geq 0$ *such that* $|\text{supp } \mu_t| > 2r$,

*(i) supp* $\mu_t$ *strictly contains supp* $\mu_{t+1}$, *and*

*(ii)* $x_{\min}(t+1) = \gamma_t(x_{\min}(t))$ *and* $x_{\max}(t+1) = \gamma_t(x_{\max}(t))$.

In the following context, we call a single point $x \in \mathbb{R}$ an atom with respect to a measure $\mu$ if $x \in \text{supp } \mu$ and $\mu(x) > 0$. Moreover, if every $\mu$-measurable set of positive measure contains an atom, then $\mu$ is called purely atomic or atomic in short.

**Theorem 4.3.2.** *Consider an Eulerian HK system, with confidence bound* $r$, *and with initial condition such that* $\mu_0 \ll \mathcal{L}^1$ *is finite and supp* $\mu_0$ *is a closed bounded interval. If* $|\text{supp } \mu_t| > 2r$ *for all* $t \geq 0$, *then* $\mu_t$ *converges in the weak-star topology to an atomic measure, whose atoms are separated by a distance greater than* $r$.

*Proof.* This system satisfies the conditions of Theorem 4.2.3 and Lemma 4.3.1. Therefore, $\mu_t \ll \mathcal{L}^1$, supp $\mu_t$ is a closed bounded interval, and supp $\mu_t \subset$ supp $\mu_{t-1}$. Since $x_{\min}(t)$ is a strictly increasing function of time and supp $\mu_t$ is a subset of supp $\mu_0$, there exists an opinion $x_1$ in the interior of supp $\mu_t$ such that $x_{\min}(t)$ converges to $x_1$. Thus, there exists $\tau$ after which $x_1 - x_{\min}(t) < r$, and in the remainder of this proof $t$ is assumed to be larger than $\tau$.

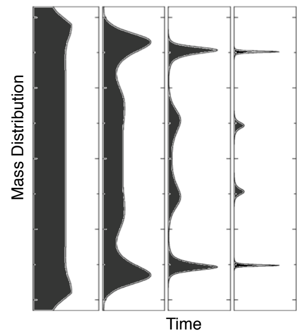



**Figure 4.2:** A schematic illustration of convergence of an Eulerian HK system with no input in the weak-star topology to an atomic measure, whose atoms are separated by a distance greater than $r$.

First, we prove that the mass distribution over interval $(x_1, x_1 + r)$ converges to zero. Let us denote the intervals $[x_{\min}(t), x_1)$ and $[x_1, x_{\min}(t) + r]$ by $I_t$ and $\hat{I}_t$, respectively, and the density function of $\mu_t$ by $\rho_t$. Then, we define the following opinion average

$$y_t(\hat{I}_t) := \frac{\int_{x_1}^{x_{\min}(t)+r} z\rho_t(z)dz}{\int_{x_1}^{x_{\min}(t)+r} \rho_t(z)dz}.$$

By Lemma 4.3.1, $x_{\min}(t+1) = \gamma_t(x_{\min}(t))$, hence

$$\lim_{t\to\infty} x_{\min}(t+1) = \lim_{t\to\infty} \frac{\int_{x_{\min}(t)}^{x_1} z\rho_t(z)dz + y_t(\hat{I}_t)\mu_t(\hat{I}_t)}{\int_{x_{\min}(t)}^{x_1} \rho_t(z)dz + \mu_t(\hat{I}_t)}.$$

On the other hand, since $x_{\min}(t)$ converges to $x_1$ as time goes to infinity,

$$\lim_{t\to\infty} \int_{x_{\min}(t)}^{x_1} z\rho_t(z)dz = x_1 \lim_{t\to\infty} \mu_t(I_t).$$

Hence,

$$\lim_{t\to\infty} x_{\min}(t+1) - x_1 = \lim_{t\to\infty} \frac{x_1\mu_t(I_t) + y_t(\hat{I}_t)\mu_t(\hat{I}_t)}{\mu_t(I_t) + \mu_t(\hat{I}_t)} - x_1$$
$$= \lim_{t\to\infty} \frac{(y_t(\hat{I}_t) - x_1)\mu_t(\hat{I}_t)}{\mu_t(I_t) + \mu_t(\hat{I}_t)} = 0.$$

Consequently, the following two cases are possible:

1) $\mu_t(\hat{I}_t)$ converges to zero. Since $\rho_t$ is Lebesgue-integrable over supp $\mu_t$,

$$\lim_{x_{\min}(t)\to x_1} \mu_t(\hat{I}_t) = \lim_{x_{\min}(t)\to x_1} \int_{x_1}^{x_{\min}(t)+r} \rho_t(z)dz = \int_{x_1}^{x_1+r} \rho_t(z)dz = \mu_t((x_1, x_1+r)). \tag{4.12}$$

It follows that if $\mu_t(\hat{I}_t)$ converges to zero, then $\mu_t((x_1, x_1+r))$ converges to zero.

2) $y_t(\hat{I}_t)$ converges to $x_1$. Then, according to Lemma 4.2.2, the mass distribution over $\hat{I}_t$ converges to a Dirac Delta distribution centered at $x_1$, and hence $\mu_t((x_1, x_{\min}(t)+r])$ converges to zero. Therefore, equation (4.12) implies that $\mu_t((x_1, x_1+r))$ converges to zero.

Second, owing to a lower bound on $\mu_t(\text{supp }\mu_t)$ two cases are possible: i) $\mu_t$ converges to a single atom at $x_1$, which proves our theorem. ii) There exists opinion $\hat{x}_2 \in [x_1+r, x_{\max}(t)]$ such that $\rho(\hat{x}_2) > \rho_{\min}$ for some $\rho_{\min} \in \mathbb{R}_{>0}$ and all $t \geq 0$. Denote the minimum opinion over all $\hat{x}_2$'s by $x_2$. Therefore, for any $x \in [x_1+r, x_2)$, $\rho(x)$ converges to zero, and thus $\mu_t([x_1+r, x_2))$ converges to zero. According to the first part of this proof, $\mu_t((x_1, x_1+r))$ also converges to zero, hence owing to $x_2 \geq x_1+r$, $\mu_t([x_2-r, x_2))$ converges to zero as time goes to infinity. Let us denote the opinion intervals $[x_2 - r, x_2)$ and $[x_2, x_2 + r]$ by $J_1$ and $J_2$, and their opinion averages by $y_t(J_1)$ and $y_t(J_2)$, respectively. Then,

$$\gamma_t(x_2) = \frac{y_t(J_1)\mu_t(J_1) + y_t(J_2)\mu_t(J_2)}{\mu_t(J_1) + \mu_t(J_2)}.$$

Next, we prove that either $y_t(J_2)$ converges to $x_2$ or $\mu_t(J_2)$ converges to zero. By contradiction assume that there exist $\Delta \in \mathbb{R}_{>0}$ and $\mu_{\min} \in \mathbb{R}_{>0}$ such that $y_t(J_2) - x_2 > \Delta$ and $\mu_t(J_2) > \mu_{\min}$. As stated above, for any $\epsilon \in \mathbb{R}_{>0}$, there exists $T \geq 0$ such that $\mu_t(J_1) < \epsilon$ for all $t \geq T$. Knowing that $y_t(J_1) - x_2 > -r$, equation (4.6) tells us that for all $t \geq T$

$$\gamma_t(x_2) - x_2 = \frac{(y_t(J_1) - x_2)\mu_t(J_1) + (y_t(J_2) - x_2)\mu_t(J_2)}{\mu_t(J_1) + \mu_t(J_2)} > \frac{-r\epsilon + \Delta\mu_{\min}}{\epsilon + \mu_{\min}}$$

Consider $\epsilon_1, \delta \in \mathbb{R}_{>0}$ such that $\epsilon_1 < \Delta\mu_{\min}/r$ and

$$\delta = \frac{-r\epsilon_1 + \Delta\mu_{\min}}{\epsilon_1 + \mu_{\min}}.$$

On the other hand, it follows from $\rho(x_2) > \rho_{\min}$ and absolute continuity of $\mu_t$ that for any $\delta \in \mathbb{R}_{>0}$, there exists $\epsilon_2 \in \mathbb{R}_{>0}$ such that $\mu_t(\mathcal{B}_\delta(x_2)) > \epsilon_2$ for all $t \geq 0$, where $\mathcal{B}_\delta(x)$ is an open ball centered at $x$ with radius $\delta$.

Now, let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, then again by equation (4.6) for all $t \geq T$,

$$\delta = \frac{-r\epsilon_1 + \Delta\mu_{\min}}{\epsilon_1 + \mu_{\min}} \leq \frac{-r\epsilon + \Delta\mu_{\min}}{\epsilon + \mu_{\min}} < \gamma_t(x_2) - x_2,$$

and $\mu_t(\mathcal{B}_\delta(x_2)) > \epsilon$. Therefore, for any $x \in \mathcal{B}_\delta(x_2)$, $x < \gamma_t(x_2)$, and thus according to Theorem 4.2.3 part (ii), $\gamma_t^{-1}(\mathcal{B}_\delta(x_2)) \in J_1$. Based on equation (4.2),

$$\mu_{t+1}(\mathcal{B}_\delta(x_2)) = \mu_t(\gamma_t^{-1}(\mathcal{B}_\delta(x_2))) < \mu_t(J_1) < \epsilon,$$

which contradicts the assumption that $\mu_t(\mathcal{B}_\delta(x_2)) > \epsilon_2 \geq \epsilon$ for all $t \geq 0$. Therefore, it is true that either $y_t(J_2)$ converges to $x_2$ or $\mu_t(J_2)$ converges to zero. In former case, Lemma 4.2.2 tells us that $\mu_t([x_2, x_2+r])$ converges to a Dirac Delta function centered at $x_2$, and thus it can be concluded from both cases that $\mu_t((x_2, x_2+r])$ converges to zero.

Third, we repeat the second part of this proof for the opinion interval $[x_2 + r, x_{\max}(t)]$ and so on.

Finally, for every bounded and continuous test function $\eta$

$$\lim_{t\to\infty} \int_{\mathbb{R}} \eta(z)\mu_t(dz) = \eta(x_1)\mu_1 + \eta(x_2)\mu_2 + \eta(x_3)\mu_3 + \cdots =: \int_{\mathbb{R}} \eta(z)\mu_\infty(dz),$$

where, the measures $\mu_\infty([x_{\min}(\infty), x_1+r))$, $\mu_\infty([x_1+r, x_2+r))$, $\mu_\infty([x_2+r, x_3+r))$, $\ldots$ are denoted by $\mu_1$, $\mu_2$, $\mu_3$, $\ldots$, respectively. Hence, $\mu_t$ converges in the weak-star topology to an atomic measure $\mu_\infty$, whose atoms, $\{\mu_i : i = 1, 2, 3, \ldots\}$, are far apart with at least distance $r$, see Figure 4.2. $\square$

## 4.4 Summary

This chapter studied the behavior of an Eulerian bounded confidence model of opinion dynamics. In this model, a population is distributed over an opinion set and updates its opinion via the opinion of the population inside the confidence range. We proved some fundamental properties of this system's dynamics, and we derived a simple sufficient condition for opinion consensus. Employing these results, we proved the convergence of population's distribution to a sum of Dirac Delta functions.

# Chapter 5

# Opinion Manipulation

Decision-making in a society is a complex process, which is led to the final state by *endogenous* and *exogenous* factors. One of the most influential exogenous factors is the mainstream media that acts as a real-time input owing to its easy access to the public. Owing to the media's easy access to public, they can quickly get out their message and hence act as a real-time input in the opinion evolution of decision makers. Media influence decisions by employing some well known techniques such as repeated exposure to experts' messages. In the 21st century, the direct influence of media on public has been replaced by a two-way relationship, with the increase in popularity of new technologies such as blogging [56]. In this sense, the message sent by media will be restated by public blogs, while each blog's report is biased by the owner's opinion. In order to accommodate the input of our opinion dynamics model to these properties of media influence, we envision this exogenous factor as a background Gaussian signal centered at the opinion of an expert. The variance of this input depends on many factors such as message repetition, expert's importance, public's different interpretation, and blogs' rebroadcasting. On the other hand, the influence of media on the public depends on the public's attitude towards it [30]. This concept for a voter decision-maker who ignores the message from an opposite political predisposition is called "partisan resistance". However, the voter receives a biased version of such message through his own party's reporters or blogs. This feature is reflected in our model by assuming that each agent associated with one opinion receives exogenous input information within its opinion confident ranges. The effect of media on opinion formation with pairwise pairwise gossip interactions is numerically analyzed in [9].

The contributions of this chapter can be summarized as follows. We propose a reasonable model for exogenous inputs in the Eulerian HK opinion-dynamics model. We derive a simple sufficient condition for the system to reach opinion consensus. We establish some important properties of the Eulerian HK model with a time-varying input. We represent the exogenous input by a background Gaussian distribution centered at the advertised opinion. We introduce the *attraction range* of an input, which is the largest range of opinions that the input can attract to its center. We conjecture a linear relation between attraction range, input's variance, and confidence bound. Accordingly, we compare two different manipulation strategies that aim to increase the population who vote positively in finite time. Finally, we present a real world example of decision making in a committee of experts, whose interconnection network is constructed via their meetings transcripts. In [10], the medical device advisory panel in the US Food and Drug Administration is analyzed, and a novel method in construction of experts' interconnection network via employing the *Author-Topic* model to their meetings documents is presented. However, the dynamics of the process of decision making by the committee is not analyzed in [10]. Here, we introduce a Lagrangian mutlidimensional HK model whose interconnection network is strongly correlated with the real example's network. We highlight common features between the dynamics of the introduced model and the presented real world example. We approximate the Lagrangian model with an Eulerian HK model, and then we discuss manipulation strategies that can alter the final state of opinion evolution process.

This chapter is organized as follows. In Section 5.1 we introduce the mathematical model. In Section 5.2 we present the main results and establish properties of the dynamical system. In Section 5.3 we discuss manipulation strategies. In Section 5.4 we present a real world example. Finally, Section 5.5 contains conclusion and future directions.

## 5.1 Eulerian HK Model with Input

Here, we propose a reasonable model for exogenous inputs in Eulerian HK model of opinion dynamics introduced in Section 4.1. The opinion set belongs to a continuous state space in $\mathbb{R}$, each opinion value is denoted by $x$, and the mass distribution of agents over the opinion set is represented by $\mu_t : \mathbb{R} \to \mathbb{R}_{\geq 0}$ at discrete time steps $t$. At $t+1$, this population updates its opinion to $\gamma_t(x)$, which is called the flow map of mass distribution $\gamma_t : \text{supp } \mu_t \subseteq \mathbb{R} \to \mathbb{R}$ and is defined as follows:

$$\gamma_t(x) = \frac{\int_{[x-r,x+r]} z d\mu_t(z) + \int_{[x-r,x+r]} z du_t(z)}{\int_{[x-r,x+r]} d\mu_t(z) + \int_{[x-r,x+r]} du_t(z)}. \tag{5.1}$$

In above equation, $r$ is the confidence bound of agents, and $u_t$ represents the distribution of exogenous background input at time $t$, which is also assumed to be a Radon probability measure for simplicity of analysis. Again, since the state space is continuous by definition, the mass distribution $\mu_t(x)$, whose sum over opinion space is preserved over time, is assumed to be a finite Borel probability measure on $R$. The mass distribution is tracked by

$$\mu_{t+1} = \gamma_t \# \mu_t, \tag{5.2}$$

and for every Borel set $E \in \mathbb{R}$,

$$\mu_{t+1}(E) = \mu_t(\gamma_t^{-1}(E)). \tag{5.3}$$

**Definition 5.1.1** (Eulerian HK System with Input). *We call the dynamical system in which a mass distribution $\mu_t$ defined over a continuous state space is being pushed forward with flow map (5.1) under the influence of input $u_t$ an* Eulerian HK system with Input.

Based on Lemma 4.2.1, some properties of mass distribution $\mu$ determine how close $\mu$'s opinion average over an interval can be to the interval's boundary. On the other hand, the following lemma demonstrates that if the opinion average of a time varying mass distribution $\mu_t$ over an interval converges to the boundary of that interval, then $\mu_t$ converges to a Dirac Delta distribution centered at that boundary.

**Lemma 5.1.2** (Opinion average limit). *Assume that in an Eulerian HK system with input, the mass distribution $\mu_t \ll \mathcal{L}^1$ is finite and supp $\mu_t$ is close bounded and contains $[a, b] \in \mathbb{R}$ for all $t \geq 0$. The opinion average over $[a, b]$, denoted by $y_t([a, b])$, satisfies $\lim_{t\to\infty} y_t([a, b]) = a$ or $\lim_{t\to\infty} y_t([a, b]) = b$ if and only if $\mu_t([a, b])$ converges to a scaled Dirac Delta distribution centered at $a$ or $b$, respectively.*

*Proof.* We prove that $\lim_{t\to\infty} y_t([a, b]) = a$ is a sufficient condition for the convergence of $\mu_t([a, b])$ to a scaled Dirac Delta distribution centered at $a$, and the obvious proof to its necessity and convergence to the other bound $b$ is omitted. By contradiction, assume that there exists an interval $c \in (a, b)$ such that $\mu_t([c, b]) > m$ for some $m \in \mathbb{R}_{>0}$ and all $t \geq 0$. Therefore, if we denote the density function of $\mu_t$ by $\rho_t$, then

$$y_t([a,b]) = \frac{\int_{[a,b]} z d\mu_t(z)}{\int_{[a,b]} d\mu_t(z)} = \frac{\int_a^b z\rho_t(z)dz}{\int_a^b \rho_t(z)dz} = \frac{\int_a^c z\rho_t(z)dz + \int_c^b z\rho_t(z)dz}{\int_a^c \rho_t(z)dz + \int_c^b \rho_t(z)dz}.$$

Since $\mu_t$ is finite, there exits $M \in \mathbb{R}_{>0}$ such that $\mu_t([a, b]) \leq M$ for all $t \geq 0$. According to equation (4.6) and Lemma 4.2.1,

$$y_t([a,b]) > \frac{aM + cm}{M + m} = a + \frac{(c-a)m}{M+m},$$

which contradicts the convergence of $y_t([a, b])$ to $a$. □

Here, we introduce two assumptions on the initial states and inputs that are employed in parts of the context.

**Assumption 5.1.1.** *For an Eulerian HK system with input, $\mu_0 \ll \mathcal{L}^1$ is finite and supp $\mu_0$ is a closed bounded interval, and $u_t \ll \mathcal{L}^1$ for all $t \geq 0$.*

**Assumption 5.1.2.** *The set supp $u_t$ is contained in the set supp $\mu_t$ for all $t \geq 0$.*

The interpretation of Assumption 5.1.2 is that the manipulator can only advertise for opinions that have a non-zero population assigned to them. In other words, the manipulator disregards the opinions that nobody believes in, which is compatible with our claim that the logic behind a distributed influence is the public's different interpretation and rebroadcast of the message by blogs, who are part of the population.

## 5.2 Dynamic Properties of the Model

This section analyzes some fundamental properties of Eulerian HK systems with time-varying inputs, and gives a simple sufficient condition for opinion consensus.

**Theorem 5.2.1** (Properties of an Eulerian HK system with input)**.** *If an Eulerian HK system with input satisfies Assumption 5.1.1, then for any $t \geq 0$ such that $|supp\ \mu_\tau| > 2r$ for all $\tau \leq t$,*

*(i) $\mu_t \ll \mathcal{L}^1$ is finite and supp $\mu_t$ is a closed interval;*

*(ii) for any $x, y \in supp\ \mu_t$, if $x < y$, then $\gamma_t(x) < \gamma_t(y)$; and*

*(iii) $x \mapsto \gamma_t(x)$ is bi-Lipschitz.*

*Proof.* Here, we first prove that if statement (i) holds at any time $t$, then statements (ii) and (iii) will hold at $t$. Next, if the three statements hold at any $t$, then statement (i) holds at $t + 1$. Finally, since $\mu_0$ satisfies statement (i), the three statements hold for all $t$. For brevity, we denote the sum of the mass and input distributions with $\nu_t := \mu_t + u_t$. Since $u_t$ satisfies Assumption 5.1.1, if statement (i) holds at any $t$, then $\nu_t \ll \mathcal{L}^1$ is finite and supp $\nu_t$ is a closed bounded interval. Hence, $\nu_t$'s density function $\rho_t(x) \geq 0$ exists and satisfies $\rho_t(x) \in [\rho_{\min}(t), \rho_{\max}(t)]$ for all $x \in$ supp $\nu_t$ with $0 < \rho_{\min}(t) \leq \rho_{\max}(t) < \infty$.

Regarding part (ii), for any $x, y \in \text{supp } \mu_t$ and $x < y$, since $x \pm r$ or $y \pm r$ may not belong to supp $\nu_t$,

$$\gamma_t(x) = \frac{\int_a^b z\rho_t(z)dz}{\int_a^b \rho_t(z)dz}, \quad \gamma_t(y) = \frac{\int_p^q z\rho_t(z)dz}{\int_p^q \rho_t(z)dz}, \tag{5.4}$$

where $[a, b] = [x-r, x+r] \cap \text{supp } \nu_t$ and $[p, q] = [y-r, y+r] \cap \text{supp } \nu_t$. Equivalently,

$$\gamma_t(x) = \frac{\int_a^p z\rho_t(z)dz + \int_p^b z\rho_t(z)dz}{\int_a^p \rho_t(z)dz + \int_p^b \rho_t(z)dz} =: \frac{\hat{S}_1 + \hat{S}_2}{S_1 + S_2}, \tag{5.5}$$

$$\gamma_t(y) = \frac{\int_p^b z\rho_t(z)dz + \int_b^q z\rho_t(z)dz}{\int_p^b \rho_t(z)dz + \int_b^q \rho_t(z)dz} =: \frac{\hat{S}_2 + \hat{S}_3}{S_2 + S_3}. \tag{5.6}$$

It follows from properties of $\nu_t$ that Lemma 4.2.1 holds, and considering the integration intervals of $\hat{S}_i$'s and $S_i$'s, for nonzero $S_i$'s we have

$$\frac{\hat{S}_1}{S_1} < \frac{\hat{S}_2}{S_2} < \frac{\hat{S}_3}{S_3} \;\Rightarrow\; \hat{S}_1 S_2 < \hat{S}_2 S_1, \;\; \hat{S}_2 S_3 < \hat{S}_3 S_2, \;\; \text{and} \;\; \hat{S}_1 S_3 < \hat{S}_3 S_1.$$

Notice that based on assumption $|\text{supp } \mu_t| > 2r$, at least one of the $S_1$ or $S_3$ should be nonzero, moreover, since supp $\nu_t$ is a closed interval, the terms $S_1 + S_2$ and $S_1 + S_3$ are nonzero. Consequently, only one term out of the three terms $S_1$, $S_2$ and $S_3$ can be zero, and the following inequality always holds:

$$\begin{aligned} &\hat{S}_1 S_2 + \hat{S}_1 S_3 + \hat{S}_2 S_2 + \hat{S}_2 S_3 < \hat{S}_2 S_1 + \hat{S}_2 S_2 + \hat{S}_3 S_1 + \hat{S}_3 S_2, \\ &\qquad \Rightarrow \frac{\hat{S}_1 + \hat{S}_2}{S_1 + S_2} < \frac{\hat{S}_2 + \hat{S}_3}{S_2 + S_3} \;\Rightarrow\; \gamma_t(x) < \gamma_t(y). \end{aligned}$$

Regarding part (iii), the bi-Lipschitz property of the flow map $\gamma_t(x)$ asserts that for any $x, y \in \text{supp } \mu_t$ equation (4.4) holds for some $L_t \geq 1$. Assume that

$x < y$, and according to part (ii), $\gamma_t(x) < \gamma_t(y)$. Then, two different cases are possible:

1) $y - x \geq 2r$, hence,

$$\gamma_t(y) - \gamma_t(x) < y - x + 2r \leq 2(y - x),$$

and it follows from Lemma 4.2.1 that

$$\gamma_t(y) - \gamma_t(x) > \frac{(b-a) + (q-p)}{1 + \sqrt{\rho_{\max}(t)/\rho_{\min}(t)}},$$

where the flow maps are given by equations (5.4). Since $y - x \leq |\text{supp } \mu_t|$,

$$\begin{aligned}\frac{(b-a) + (q-p)}{1 + \sqrt{\rho_{\max}(t)/\rho_{\min}(t)}} &= \frac{(b-a+q-p)(y-x)}{(1 + \sqrt{\rho_{\max}(t)/\rho_{\min}(t)})(y-x)} \\ &\geq \frac{(b-a+q-p)(y-x)}{|\text{supp } \mu_t|(1 + \sqrt{\rho_{\max}(t)/\rho_{\min}(t)})}.\end{aligned}$$

Finally,

$$L_t = \min\{2, \frac{|\text{supp } \mu_t|(1 + \sqrt{\rho_{\max}(t)/\rho_{\min}(t)})}{b-a+q-p}\}.$$

Since $b-a+q-p \leq |\text{supp } \mu_t| + 2r$, $1+\sqrt{\rho_{\max}(t)/\rho_{\min}(t)} \geq 2$, and $|\text{supp } \mu_t| > 2r$,

$$\frac{|\text{supp } \mu_t|(1 + \sqrt{\rho_{\max}(t)/\rho_{\min}(t)})}{b-a+q-p} \geq \frac{2|\text{supp } \mu_t|}{|\text{supp } \mu_t| + 2r} \geq 1,$$

which confirms that $L_t \geq 1$.

2) $y - x < 2r$, hence following equations (5.5) and (5.6), we have

$$\gamma_t(y) - \gamma_t(x) = \frac{\hat{S}_2 + \hat{S}_3}{S_2 + S_3} - \frac{\hat{S}_1 + \hat{S}_2}{S_1 + S_2} = \frac{\hat{S}_2 S_1 + \hat{S}_3 S_1 + \hat{S}_3 S_2 - \hat{S}_1 S_2 - \hat{S}_1 S_3 - \hat{S}_2 S_3}{(S_1 + S_2)(S_2 + S_3)}.$$

Based on statement (i), the following inequalities can be derived:

$$S_1 < (p-a)\rho_{\max}(t) \le (y-x)\rho_{\max}(t), \qquad \hat{S}_1 < p(p-a)\rho_{\max}(t) \le q(y-x)\rho_{\max}(t)$$

$$S_3 < (q-b)\rho_{\max}(t) \le (y-x)\rho_{\max}(t), \qquad \hat{S}_3 < q(q-b)\rho_{\max}(t) \le q(y-x)\rho_{\max}(t),$$

$$S_2 < (b-p)\rho_{\max}(t) \le 2r\rho_{\max}(t), \qquad \hat{S}_2 < b(b-p)\rho_{\max}(t) \le 2rq\rho_{\max}(t),$$

$$(S_1+S_2)(S_2+S_3) > r^2\rho_{\min}(t)^2.$$

Consequently,

$$\gamma_t(y) - \gamma_t(x) < \frac{2r|q|\rho_{\max}(t)^2(y-x) + |q|\rho_{\max}(t)^2(y-x)^2 + 2r|q|\rho_{\max}(t)^2(y-x)}{r^2\rho_{\min}(t)^2}.$$

Again since $y-x<2r$ and $|q| \le \max\{|x_{\max}(t)|, |x_{\min}(t)|\}$,

$$\gamma_t(y) - \gamma_t(x) < \frac{6r\max\{|x_{\max}(t)|, |x_{\min}(t)|\}\rho_{\max}(t)^2}{r^2\rho_{\min}(t)^2}(y-x) =: L_1(y-x).$$

It follows from $|\mathrm{supp}\ \mu_t| > 2r$ that $\max\{|x_{\max}(t)|, |x_{\min}(t)|\} \ge r$, and thus $L_1 > 1$. As stated above, either $S_1$ or $S_3$ is nonzero. Without loss of generality, assume that $S_1$ is nonzero, and hence $p-a = y-x$. It follows from

$$\frac{\hat{S}_3+\hat{S}_2}{S_3+S_2} \ge \frac{\hat{S}_2}{S_2}$$

that

$$\gamma_t(y) - \gamma_t(x) \ge \frac{\hat{S}_2}{S_2} - \frac{\hat{S}_1+\hat{S}_2}{S_1+S_2} =: c(x_2-x_1), \tag{5.7}$$

where

$$x_1 = \frac{\hat{S}_1}{S_1}, \quad x_2 = \frac{\hat{S}_2}{S_2}, \quad \text{and} \quad c = \frac{S_1}{S_1 + S_2} \geq \frac{(p-a)\rho_{\min}(t)}{(b-p)\rho_{\max}(t) + (p-a)\rho_{\max}(t)}.$$

$$\Rightarrow \gamma_t(y) - \gamma_t(x) \geq \frac{(p-a)\rho_{\min}(t)}{(b-a)\rho_{\max}(t)}(x_2 - x_1) > \frac{(y-x)\rho_{\min}(t)}{2r\rho_{\max}(t)}(x_2 - x_1).$$

By Lemma 4.2.1,

$$x_2 - x_1 > \frac{b - p + p - a}{\sqrt{\rho_{\max}(t)/\rho_{\min}(t)} + 1} \geq \frac{r}{\sqrt{\rho_{\max}(t)/\rho_{\min}(t)} + 1}.$$

Therefore,

$$\gamma_t(y) - \gamma_t(x) > \frac{\rho_{\min}(t)}{2\rho_{\max}(t)(\sqrt{\rho_{\max}(t)/\rho_{\min}(t)} + 1)}(y - x) =: \frac{1}{L_2}(y - x).$$

Clearly $L_2 \geq 1$, therefore, $L_t = \min\{L_1, L_2\}$.

Regarding part (i), we now prove that if statements (i), (ii), and (iii) hold at time $t$, then statement (i) holds at $t+1$. First, we prove that the flow map

$$\gamma_t(x) = \frac{\int_{x-r}^{x+r} z\rho_t(z)dz}{\int_{x-r}^{x+r} \rho_t(z)dz} =: \frac{f(x)}{g(x)}$$

is continuous. Knowing that if two functions $f$ and $g$ are continuous and $g \neq 0$, then the quotient $f/g$ is also continuous, we show the continuity of the function $f(x)$ at all points $c \in \text{supp } \mu_t$, and the proof to the continuity of the denominator is similar. For all $x \in \text{supp}\ \mu_t$, $g(x) > 0$, and

$$\begin{aligned}\lim_{x\to c} f(x) = \lim_{x\to c}\int_{x-r}^{x+r} zd\rho(z) = \lim_{\pm\epsilon\to 0}\int_{c\pm\epsilon-r}^{c\pm\epsilon+r} zd\rho(z)\\ = \lim_{\pm\epsilon\to 0}\big(-\int_{c-r}^{c\pm\epsilon-r} zd\rho(z) + \int_{c+r}^{c\pm\epsilon+r} zd\rho(z)\big) + \int_{c-r}^{c+r} zd\rho(z)\\ = \lim_{\pm\epsilon\to 0}\big(\pm\epsilon(r-c)\rho(c-r) + \pm\epsilon(c+r)\rho(c+r)\big) + f(c).\end{aligned}$$

Due to the finiteness and absolute continuity of $\nu_t$, $f(c)$ exists for all $c \in \text{supp}\ \mu_t$ and above limit converges to zero, hence, $\lim_{x\to c} f(x) = f(c)$. We have shown that $\gamma_t(x)$ is strictly monotone and continuous with respect to $x$, therefore, this map is also invertible. Second, we prove absolute continuity of $\mu_{t+1}$. It is shown that $\gamma_t$ has the following properties: 1) Since any continuous function defined on Borel sets is a Borel measurable function, $\gamma_t^{-1}(E)$ is Borel measurable for any Borel set $E \in \mathbb{R}$. 2) The bi-Lipschitz map $\gamma_t$ satisfies

$$\mathcal{L}(\gamma_t^{-1}(E)) \leq C_t\mathcal{L}(E)$$

for some constant $C_t \in \mathbb{R}_{>0}$. According to Theorem 2 in [45], if the flow map $\gamma_t$ satisfies above two properties and $\mu_t \ll \mathcal{L}^1$, then $\mu_{t+1} \ll \mathcal{L}^1$. Third, a continuous function maps a compact set to another compact set, hence, $\gamma_t$ maps the closed bounded interval supp $\mu_t$ to another closed bounded interval supp $\mu_{t+1}$. Fourth, we establish bounds on $\mu_{t+1}$'s density function. For any $x, y \in \text{supp}\ \mu_{t+1}$ and

$x < y$, equation (4.2) gives

$$\int_x^y \hat{\rho}_{t+1}(z)dz = \int_{\gamma_t^{-1}(x)}^{\gamma_t^{-1}(y)} \hat{\rho}_t(z)dz.$$

where $\hat{\rho}_\tau(z)$ is $\mu_\tau$'s density function, and in view of condition (i),

$$0 < \hat{\rho}_{\min}(\tau) \leq \hat{\rho}_\tau(z) \leq \hat{\rho}_{\max}(\tau) < \infty$$

over supp $\mu_\tau$. Therefore,

$$(\gamma_t^{-1}(y) - \gamma_t^{-1}(x))\hat{\rho}_{\min}(t) \leq \int_{\gamma_t^{-1}(x)}^{\gamma_t^{-1}(y)} \hat{\rho}_t(z)dz \leq (\gamma_t^{-1}(y) - \gamma_t^{-1}(x))\hat{\rho}_{\max}(t).$$

Provided that $\gamma_t$ is bi-Lipschitz,

$$\frac{1}{L_t}(y-x)\hat{\rho}_{\min}(t) \leq \int_{\gamma_t^{-1}(x)}^{\gamma_t^{-1}(y)} \hat{\rho}_t(z)dz \leq L_t(y-x)\hat{\rho}_{\max}(t),$$
$$\Rightarrow \frac{1}{L_t}(y-x)\hat{\rho}_{\min}(t) \leq \int_x^y \hat{\rho}_{t+1}(z)dz \leq L_t(y-x)\hat{\rho}_{\max}(t).$$

The limit of above inequality as $y$ converges to $x$ gives

$$\frac{1}{L_t}(y-x)\hat{\rho}_{\min}(t) \leq (y-x)\hat{\rho}_{t+1}(x) \leq L_t(y-x)\hat{\rho}_{\max}(t)$$
$$\Rightarrow \frac{1}{L_t}\hat{\rho}_{\min}(t) \leq \hat{\rho}_{t+1}(x) \leq L_t\hat{\rho}_{\max}(t) \quad \forall x \in \text{supp } \mu_{t+1}.$$

Finally, since supp $\mu_t$ is bounded for all $t \geq 0$, we have

$$\mu_{t+1}(\text{supp } \mu_{t+1}) = \mu_t(\gamma_t^{-1}(\text{supp } \mu_{t+1})) = \mu_t(\text{supp } \mu_t).$$

Therefore, if $\mu_t$ is finite, then $\mu_{t+1}$ is finite. □

**Lemma 5.2.2** (Sufficient condition for consensus)**.** *Assume that an Eulerian HK system with input satisfies Assumption 5.1.2, $\mu_0, u_0 \ll \mathcal{L}^1$, and $\mu_0$ is a finite measure with closed bounded support. If $\mu_0$ and $u_t$ are distributed symmetrically around the center of supp $\mu_0$, and $|\text{supp } \mu_0| \leq 2r$, then the mass distribution reaches an opinion consensus in finite time.*

*Proof.* Let us denote $\mu_t + u_t$ by $\nu_t$. Owing to the absolute continuity of $\mu_0$ and $u_0$ and according to Lemma 4.2.1, $x_{\min}(1) > x_{\min}(0)$ and $x_{\max}(1) < x_{\max}(0)$. Since $\nu_0$ and $u_t$ are symmetrically distributed and the confidence bounds are homogeneous for all opinions, the distribution $\nu_t$ remains symmetric around the center of supp $\nu_t$ for all $t \geq 0$. Hence, defining $x_{\text{mid}}(t) := (x_{\min}(t) + x_{\max}(t))/2$, $x_{\text{mid}}(t) = x_{\text{mid}}$ is constant for all $t$. Now, we show that for any $\Delta \in \mathbb{R}_{>0}$, there exists $\delta \in \mathbb{R}_{>0}$ such that

$$\gamma_0(x_{\text{mid}} + \delta) - x_{\text{mid}} < \Delta, \tag{5.8}$$

where

$$\gamma_0(x_{\text{mid}} + \delta) = \frac{\int_{[x_{\min}(0)+\delta, x_{\max}(0)]} z d\nu_0(z) dz}{\int_{[x_{\min}(0)+\delta, x_{\max}(0)]} d\nu_0(z) dz}.$$

Owing to absolute continuity of $\nu_0$, $\gamma_0(x)$ is a continuous function of $x$, and thus,

$$\lim_{\delta \to 0} \gamma_0(x_{\text{mid}} + \delta) = \frac{\int_{[x_{\min}(0), x_{\max}(0)]} z d\nu_0(z) dz}{\int_{[x_{\min}(0), x_{\max}(0)]} d\nu_0(z) dz} = \gamma_0(x_{\text{mid}}) = x_{\text{mid}}.$$

If we let $\Delta = x_{\max}(0) - x_{\max}(1) = x_{\min}(1) - x_{\min}(0)$, then equation (5.8) gives $\mu_1([x_{\text{mid}} - \Delta, x_{\text{mid}} + \Delta]) \geq \mu_0([x_{\text{mid}} - \delta, x_{\text{mid}} + \delta]) > 0$. Moreover, at time $t = 1$, the population over $[x_{\text{mid}}-\Delta, x_{\text{mid}}+\Delta]$ considers the total population's opinion in its opinion update and thus reaches consensus at $\gamma_1(x) = \gamma_1(x_{\text{mid}}) = x_{\text{mid}}$ in the next iteration. Consequently, at $t = 2$, there exists an atomic measure centered at $x_{\text{mid}}$, whose atom weight is denoted by $\nu_{\text{mid}}$. Since $\gamma_t(x_{\text{mid}}) = x_{\text{mid}}$, the weight of the atomic measure centered at $x_{\text{mid}}$ is greater than or equal to $\nu_{\text{mid}}$ for all $t \geq 2$. Now, for all $t \geq 2$, we compute a strictly positive lower bound for $x_{\min}(t+1) - x_{\min}(t)$, which is equal to the lower bound on $x_{\max}(t)-x_{\max}(t+1)$, and proves that $|\text{supp } \nu_t|$ is strictly decreasing and converges to zero. Since for any $x$ such that $x < x_{\min}(t)$ or $x > x_{\max}(t)$, $d\nu_t(x) = 0$, $|\text{supp } \nu_t| \leq |\text{supp } \nu_0| \leq 2r$ for all $t \geq 0$, and it follows that the intervals $[x_{\min}(t), x_{\min}(t)+r]$ and $[x_{\max}(t)-r, x_{\max}(t)]$ contain the central point $x_{\text{mid}}$. For all $x \in [x_{\min}(t), x_{\text{mid}})$,

$$\gamma_t(x) \geq \frac{x_{\min}(t)\hat{\nu}_t + x_{\text{mid}}\nu_{\text{mid}}}{\hat{\nu}_t + \nu_{\text{mid}}},$$

where $\hat{\nu}_t := \int_{[x_{\min}(t), x_{\min}(t)+r]} d\nu_t(x) - \nu_{\text{mid}}$, whose upper bound we denote by $\hat{\nu}_{\max}$. Therefore,

$$x_{\min}(t+1) \geq x_{\min}(t) + \frac{(x_{\text{mid}} - x_{\min}(t))\nu_{\text{mid}}}{\hat{\nu}_{\max} + \nu_{\text{mid}}},$$

which tells us that the lower bound on $x_{\min}(t+1) - x_{\min}(t)$ is $(x_{\text{mid}} - x_{\min}(t))$ multiplied by a constant. Consequently, there exists time $\tau \geq 2$ such that $x_{\text{mid}} - x_{\min}(\tau) < r - (x_{\max}(\tau) - x_{\min}(\tau))/2$, and thus the mass distribution reaches a consensus at $\tau + 1$. $\square$

**Lemma 5.2.3.** *If an Eulerian HK system with input satisfies Assumptions 5.1.1 and 5.1.2, then for all $t \geq 0$ such that $|supp\ \mu_t| > 2r$,*

*(i) supp $\mu_t$ strictly contains supp $\mu_{t+1}$, and*

*(ii) $x_{\min}(t+1) = \gamma_t(x_{\min}(t))$ and $x_{\max}(t+1) = \gamma_t(x_{\max}(t))$.*

*Proof.* This system satisfies the conditions of Theorem 5.2.1, and part (i) of the theorem tells us that supp $\mu_t$ is equal to the closed bounded interval $[x_{\min}(t), x_{\max}(t)]$ for all $t \geq 0$. Hence, statement (i) asserts

$$x_{\min}(t) < x_{\min}(t+1) < x_{\max}(t+1) < x_{\max}(t).$$

Let us prove the lower bound's inequality, and the prove to the upper bound's is similar. For all $t$, based on Assumption 5.1.2, the support of measure $\nu_t = \mu_t + u_t$ is equal to supp $\mu_t$. Therefore, the density function of $\nu_t$ is equal to zero below $x_{\min}(t)$ and strictly greater than zero above $x_{\min}(t)$, and hence, $\gamma_t(x_{\min}(t)) > x_{\min}(t)$. According to Theorem 5.2.1 part (ii), for all $y \in$ supp $\mu_t$ and $y > x_{\min}(t)$, $\gamma_t(y) > \gamma_t(x_{\min}(t))$. Therefore, $\gamma_t(x_{\min}(t))$ is the smallest opinion in the set supp $\mu_{t+1}$, i.e., $\gamma_t(x_{\min}(t)) = x_{\min}(t+1)$, and thus $x_{\min}(t+1) > x_{\min}(t)$. $\square$

Notice that (1) if supp $u_t$ is not contained in supp $\mu_t$, then supp $\mu_{t+1}$ is not necessarily contained in supp $\mu_t$; (2) Lemma 5.2.3 also holds for an Eulerian HK system without input, that is, $u_t = 0$ for all $t \geq 0$.

## 5.3 Discussion on Exogenous Input

In this section, we consider a large population that votes positively or negatively in a series of opinion polls until final election. We aim to maximize the size of population with positive opinions in the final election. Moreover, we discuss two different manipulation strategies: *direct* and *distracting*. In the direct strategy, the manipulator broadcasts a positive opinion for all times. On the contrary, in the distracting strategy the manipulator first broadcasts a neutral or mildly negative opinion to attract the attention of people with strong negative opinions, and only later broadcasts the positive opinion. Loosely speaking, the distracting strategy implements a well-known subconscious persuasion method: in dealing with someone with different beliefs, a manipulator would start with a moderate opinion to win the trust of that person. Examples of indirect manipulation are observed in political election strategies and are referred to as "ideological shifts." In 2008, Romney positioned himself as the conservative alternative to McCain. However, in 2012, "he lost very conservative primary voters to Santorum by 14 percentage points (36 to 50 percent), but carried both moderate and liberal voters (39 to 33 percent)" [7]. Another technique in measuring ideological shifts, is the proportion of funds raised by a candidate against his/her ideology [8]. Finally, we show numerically that for broad ranges of parameters and initial conditions, the indirect strategy outperforms the direct strategy.

### 5.3.1 Bounded Attraction Range

Before analyzing the effect of ideological shift in the Eulerian HK system dynamics, we discuss how clustering of opinions prevents reaching a global opinion consensus under the influence of Guassian input. This phenomenon is due to the *3-sigma rule*, that is, about 99.7% of values drawn from a Gaussian distribution are within $3\sigma$ distance away from the mean. Consequently, the effect of input on our dynamical system out of 3-sigma region can be ignored.

**Definition 5.3.1** (Attraction range)**.** *Consider an an Eulerian HK system whose input is time invariant and is equal to* $u \sim \mathcal{N}(\hat{x}, \sigma^2)$*. We call the opinion interval* $[y, z] \in \mathbb{R}$*, denoted by* $\mathcal{R}(u)$*, the* attraction range *of input* $u$*, if* $[y, z]$ *is the maximal interval with the following property:*

$$\lim_{t\to\infty} \gamma_t \circ \cdots \circ \gamma_0(y) = \lim_{t\to\infty} \gamma_t \circ \cdots \circ \gamma_0(z) = \hat{x}.$$

If the system satisfies conditions of Theorem 5.2.1, then part (ii) tells us that $\mu_0(\mathcal{R}(u))$ represents the *attracted population*, the total population that reaches an opinion consensus at the center of input. Figure 5.1 illustrates a linear relation between $|\mathcal{R}(u)|$, $\sigma$, and $r$ for $|\mathcal{R}(u)| < 0.6|\text{supp}\ \mu_0|$ in evolutions of Eulerian HK

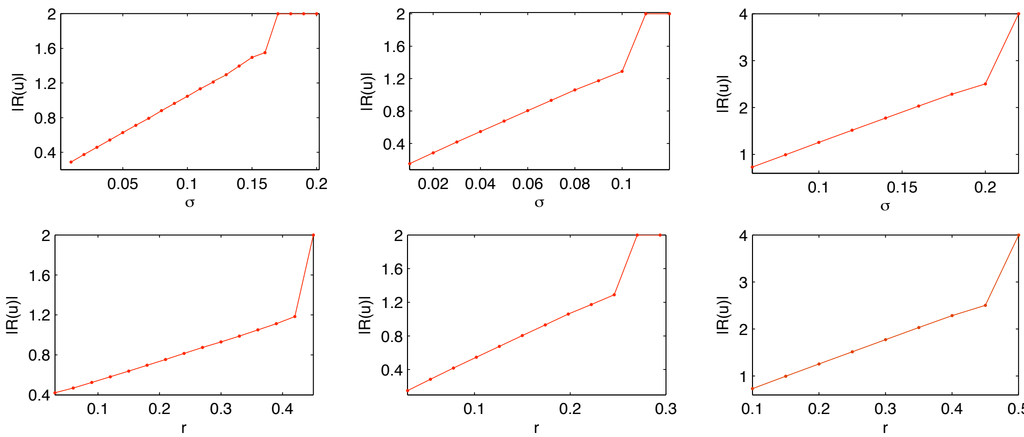



**Figure 5.1:** In evolutions of Eulerian HK systems with uniform initial distribution $\mu_0 \sim \mathcal{U}(-x_0, x_0)$ and input $u \sim \mathcal{N}(0, \sigma^2)$, the length of $\mathcal{R}(u)$ is found for different values of following parameters: $\sigma$, $x_0$, and confidence bound $r$. Top left: $x_0 = 1$, $r = 0.1$ and $\sigma \in \{0.01, 0.02, \ldots, 0.17\}$, bottom left: $x_0 = 1$, $\sigma = 0.04$ and $r \in \{0.03, 0.06, \ldots, 0.45\}$, middle: $x_0 = 1$ and $(\sigma, r) \in \{(0.01, 0.03), \ldots, (0.12, 0.3)\}$, right: $x_0 = 2$ and $(\sigma, r) \in \{(0.06, 0.1), \ldots, (0.22, 0.5)\}$.

systems with uniform initial distribution $\mu_0 \sim \mathcal{U}(-x_0, x_0)$ and input $u \sim \mathcal{N}(0, \sigma^2)$, which leads us to our conjecture.

**Conjecture 5.3.2** (Linear relation between attraction range and system parameters)**.** *Consider an Eulerian HK system with uniform initial mass distribution $\mu_0$ whose support is a closed interval and $u \sim \mathcal{N}(\hat{x}, \sigma^2)$, where $\hat{x} \in$ supp $\mu_0$. If $\sigma$ is sufficiently small, then $|\mathcal{R}(u)| = a\sigma + br + c$, with $a, b \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}$.*

The following conjecture defines an upper bound on $\mathcal{R}(u)$.

**Conjecture 5.3.3** (Upper bound on attraction range)**.** *Consider an Eulerian HK system with initially uniform mass distribution* $\mu_0$ *and* $u \sim \mathcal{N}(\hat{x}, \sigma^2)$*, where* $\hat{x} \in$ *supp* $\mu_0$*. Assume that there exists* $y, z \in$ *supp* $\mu_1$ *such that* $y < \hat{x} < z$*, and* $\gamma_1(y) = y$*,* $\gamma_1(z) = z$*. Then, the smallest interval* $[\gamma_0^{-1}(y), \gamma_0^{-1}(z)]$ *(since* $y$ *and* $z$ *may not be unique) is an upper bound on* $\mathcal{R}(u)$*.*

According to Conjecture 5.3.3 and Theorem 5.2.1, the following equality gives us the value of $y$, and owing to the symmetry of the input around its mean, one can write $\mathcal{R}(u) = 2|\hat{x} - y|$. Hence, $\mathcal{R}(u)$ can be computed as a function of $r$ and $\sigma$, whose computation is omitted for brevity:

$$\gamma_1(y) - y = 0 \Rightarrow \frac{\int_{\gamma_0^{-1}(y-r)}^{\gamma_0^{-1}(y+r)} \frac{\gamma_0(z)-y}{2} dz + \int_{y-r}^{y+r} (z-y) du(z)}{\int_{\gamma_0^{-1}(y-r)}^{\gamma_0^{-1}(y+r)} \frac{1}{2} dz + \int_{y-r}^{y+r} du(z)} = 0.$$

**Remark 5.3.4** (Intuition behind Conjecture 5.3.3)**.** *Under the conditions of Conjecture 5.3.3, the flow map of all opinions in supp* $\mu_0$ *at* $t = 0$ *is closer than the opinion's value to the input's center. In other words, for all* $x \in$ *supp* $\mu_0$*, either of the following holds true:* $x \leq \gamma_0(x) \leq \hat{x}$ *or* $x > \gamma_0(x) > \hat{x}$*. However, owing to the Gaussian distribution of input, the mass distribution* $\mu_1$ *has a fluctuation around the center of input. This fluctuation results in a decrease in the mass distribution in the intervals* $[y, y + r]$ *and* $[z - r, z]$ *at* $t = 1$*, and the input's attraction power at* $y$ *and* $z$ *is neutralized by peer pressure from above* $y$ *and below* $z$*, respectively. Consequently, in future iterations, the population with opinions* $y$ *and* $z$ *are at-*

*tracted to the clusters, respectively, below and above the cluster attracted to the input.*

### 5.3.2 Comparison on Manipulation Strategies

In an Eulerian HK systems with $\mu_0 \sim \mathcal{U}(-x_0, x_0)$ and Gaussian input, we aim to maximize the population with positive opinions in finite time $T$, or equivalently, maximize the objective function $\int_0^1 d\mu_T(z)$. More precisely, if we assume that the input's variance is a fixed parameter and the input is influential on less than half of entire population at $t = 0$, hence $\sigma < |\text{supp } \mu_0|/12$, then we can define input's mean based on our strategy.

- *Direct strategy:* Manipulator advertises for positive opinions, and thus $u_t \sim \mathcal{N}(x_I(t), \sigma^2)$ with $0 < x_I(t) < |\mathcal{R}(u)|/2$ for all $t \leq T$.

- *Distracting strategy:* First, for all $t \leq \alpha T$, where $\alpha \in (0, 1)$, the manipulator advertises for negative opinions, thus $u_t \sim \mathcal{N}(x_{II}(t), \sigma^2)$ with $x_{II}(t) < 0$. One can assume that $x_{II}(t) = x_{\min}(t) + |\mathcal{R}(u)|/2$. Then, for all $\alpha T \leq t \leq T$, the manipulator advertises for positive opinions, thus $u_t \sim \mathcal{N}(x_I(t), \sigma^2)$ with $0 < x_I(t) < |\mathcal{R}(u)|/2$.

It can be explained heuristically that the distracting strategy outperforms the direct strategy. It follows from the assumption $\sigma < |\text{supp } \mu_0|/12$ and bounded-

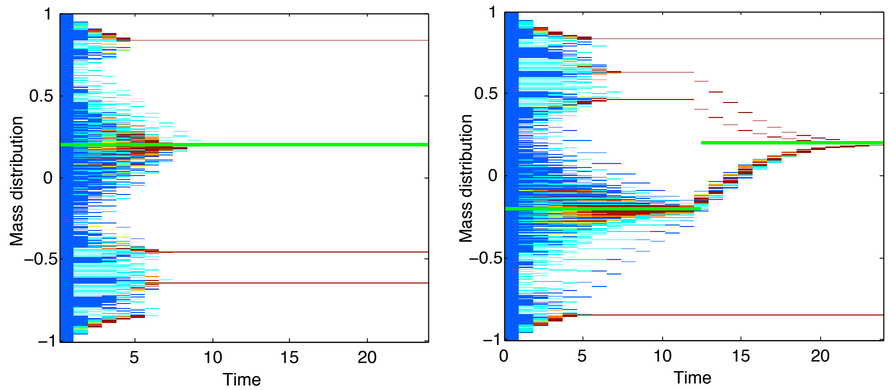



**Figure 5.2:** Two Eulerian HK systems with $\mu_0 \sim \mathcal{U}(-1, 1)$ under the influence of direct strategy with $u \sim \mathcal{N}(0.2, 0.1^2)$ (left) and distracting strategy $u \sim \mathcal{N}(-0.2, 0.1^2)$ for $t \leq 12$ and $u \sim \mathcal{N}(0.2, 0.1^2)$ for $12 < t \leq 25$ (right). In direct case 0.6525 portion of population is attracted to the input's center (green line), while in distracting case this portion is 0.8675.

ness of $|\mathcal{R}(u)|$ that the direct strategy prevents attraction of the population with opinions in the interval $[x_{\min}(0), -\mathcal{R}(u)/2]$. However, in distracting strategy, this population is in the attraction range of the first input before $\alpha T$, and hence there is a fluctuation of population centered at $x_{\mathit{II}}$ and closer to the input's mean after $\alpha T$. An example of this comparison is depicted in Figure 5.2.

## 5.4 A Real Example of Group Decision Making

A novel method in construction of experts' interconnection network via an analysis of the committee meeting transcripts is presented in [10]. In that article, the medical device advisory panel in the US Food and Drug Administration is analyzed, and it is claimed that panel members who use similar language tend to vote similarly. In [10], employing natural language processing tools, including *Author-Topic* model, they 1) examined topics of interest for each committee member, and 2) extracted the directed graph of the interconnection network between committee members. In the Author-Topic (AT) model, “words are viewed as discrete random variables, a document contains a fixed number of words, and each word takes one value from a predefined vocabulary. This model is a hierarchical generative model in which each word in a document is associated with two latent variables: an author, and a topic” [46]. Documents are generated by a two stage stochastic process. Authors and topics are associated with probability distributions over topics and words, respectively. In multi-author documents, the probability distribution over topics is a mixture of the distributions associated with the authors. In tests of AT model in [10] and [11], they held the number of topics constant for each meeting analyzed. Topics proportions are computed for each author, that is, the proportion of words spoken by that voting member in the corresponding topic. It is also investigated that panel members who speak often and focus on one topic, potentially display a depth of expertise and hence have higher influence. Multi-focus speakers are considered as mediators. Considering the definition of interconnection graphs in earlier versions of AT algorithm [53], two authors are said to be linked if they both spoke more that 20% of their time about at least one topic. Besides simplicity, this uniform 20% cutoff has the advantage of not counting for the total number of words a speaker contributes to a given topic. Therefore, a member who speaks rarely is not less likely to be linked. In a more sophisticated look to the analysis of the interconnection network of committee members, (Broniatowski, 2010) defines directed links based on temporal aspect of the data, that is, time lags in talking about a common topic. "A more influential speaker may change the subject, whereas a less influential speaker will remain on the subject introduced by the higher-status speaker." During a meeting, temporally ordered utterances defines time series for each speaker on one topic. In the plot of such time series for two speakers, if speaker $i$ speaks about the topic often before $j$ does, then we can say $i$ leads $j$. More specifically, those time series are used to generate the *topic-specific-cross-correlation* function for every two speakers on one topic, the signs of peaks of this function define the direction of the link between the two speakers. One future work mentioned in [11] is further developing their techniques using dynamic network analysis. Accordingly, we introduce a constrained two dimensional *synchronized bounded influence* model of opinion dynamics with exogenous inputs. Furthermore, we emphasize common features between construction of interconnection network in our model and above mentioned interaction network of a committee via their meetings transcript.

### 5.4.1 Multi-Topic Opinion Dynamics Model with Input

This section introduces a constrained two dimensional version of the synchronized bounded influence (SBI) model of opinion dynamics introduced and studied in Chapter 3. Consider $n$ agents discussing two topics, and after a finite number of meetings, they vote Yea or Nay on each topic. Let $x_i(t)$ denote agent $i$'s proportion of his time that is devoted to speak in favor of the first topic during the $t$th meeting. We assume that $x_i(t)$ is a negative number if agent $i$ speaks against the first topic, and thus $x_i(t)$ takes value in $[-1, 1]$. Similarly, we denote such proportion for the second topic by $y_i(t)$. Since we are considering only two topics of interest, the following constraint holds:

$$|x_i(t)| + |y_i(t)| = 1 \text{ for all } i \in \{1, \ldots, n\}. \tag{5.9}$$

In two dimensional bounded influence interaction, agent $i$'s opinion on one topic is affected by agent $j$'s opinion on that topic, if the difference in their proportions of talking about that topic is less than the *influence bound* of agent $j$ on that topic, denoted by $r_{x,j}, r_{y,j} \in \mathbb{R}_{>}0$. The *opinion vectors* $x(t), y(t) \in \mathbb{R}^n$ and the influence bounds vectors $r_x, r_y \in \mathbb{R}^n_{>0}$ are obtained by stacking all $x_i(t), y_i(t)$'s and $r_{x,j}, r_{y,j}$'s, respectively. In other words, the influence bound of each agent depends on the topic he is discussing. This assumption is based on the investigation and study by (Friedkin, 2011) on small group opinion dynamics concerned with changes of group members' positions on different issues. "For each issue in the sequence of issues that arise in a group, a process of interpersonal influence on the issue may unfold in a fixed structure of accorded influence" [21]. Moreover, influence bounds can be approximated from the real data in the following way. Having the conversational data from one set of meetings, (Broniatowski, 2010) computed the proportions of talking about different topics and defined a directed graph of interaction between members. If there is an edge from $i$ to $j$, we compute the average of their opinion differences in one topic over that set of meetings and let that value equal to the influence bound of $j$ on that topic.

As stated above, one of the most influential exogenous factors in committee decision making is the mainstream media. The techniques used by media in influencing opinions include using/misusing the experts or front runners and repeating a message. In our model of opinion dynamics: 1) we assume that the front runners are agents with larger influence bound; and 2) repeating the front runners' opinions is modeled by independent Gaussian variables centered around each front runner's opinion with an influence bound equal to that of front runner. Assume that after meeting $t$ the media advertises for the front runner $k$ in his opinion on the first topic. Therefore, we can say that the input is a set of Gaussian variables centered around $x_k(t)$, and the input's influence bound will be $r_{x,k}$. Hence, we enlarge the state space along to include the input variables.

According to the SBI model of opinion dynamics for each topic, considering $n$ agents with influence bounds vector $r$, we associate to each opinion vector $z \in \mathbb{R}^n$ and the input vector $u \in \mathbb{R}^m$ the *proximity digraph* $G(z, u)$ with nodes $\{1, \ldots, n+m\}$. Proximity digraph's edge set is defined as follows: the set of out-neighbors of node $i$ is $\mathcal{N}_i(z, u) = \{j \in \{1, \ldots, n\} : |v_i - v_j| \leq r_j\}$, where $v = [z, u]^T$. Clearly, all agents have self-loops in the proximity digraph, which represent the self-confidence of committee members, and we emphasize this confidence by considering self-weights $w_i$'s. The *adjacency matrix* corresponding to the mentioned proximity digraph is denoted by $A(z, u)$ and is an $n \times (n + m)$ matrix whose $i, j$ entry is defined by

$$a_{ij}(z, u) = \begin{cases} \dfrac{1}{|\mathcal{N}_i(z, u)| + w_i - 1}, & \text{if } j \neq i \in \mathcal{N}_i(z, u), \\[2ex] \dfrac{w_i}{|\mathcal{N}_i(z, u)| + w_i - 1}, & \text{if } j = i, \\[2ex] 0, & \text{if } j \notin \mathcal{N}_i(z, u), \end{cases}$$

where $|\mathcal{N}_i(z, u)|$ is the cardinality of $\mathcal{N}_i(z, u)$. In a two dimensional case of opinion dynamics model, owing to having topic-dependent influence bounds, the proximity digraphs associated to the two topic are independent from each other, while the two opinion vectors are related by constraint (5.9). We denote the opinion vectors on the two topics by $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, their corresponding input vectors by $u_x \in \mathbb{R}^m$ and $u_y \in \mathbb{R}^k$, the proximity digraphs by $G_x(x, u_x)$ and $G_y(y, u_y)$, and the adjacency matrices by $A_x(x, u_x)$ and $A_y(y, u_y)$, respectively. An example of a two dimensional opinion space and the associated proximity digraphs are illustrated in Figure 5.3. The dynamics of this updating rule is composed of two parts, first computation of the two intermediate vectors

$$\hat{x} = A_x(x(t), u_x(t)) \begin{bmatrix} x(t) \\ u_x(t) \end{bmatrix},$$
$$\hat{y} = A_y(y(t), u_y(t)) \begin{bmatrix} y(t) \\ u_y(t) \end{bmatrix}. \tag{5.10}$$

Second, computing the final opinion vectors

$$x(t+1) = \frac{\hat{x} + \mathrm{sign}(\hat{x})(\mathbf{1}_n - |\hat{y}|)}{2},$$
$$y(t+1) = \frac{\hat{y} + \mathrm{sign}(\hat{y})(\mathbf{1}_n - |\hat{x}|)}{2}, \tag{5.11}$$

where $\mathrm{sign} : \mathbb{R}^n \to \{-1, 0, 1\}^n$ is a map that associates entries of a vector to their signs. The logic behind above two step updating rule is to feed the change in one opinion vector into the other opinion vector, such that the constraint (5.9) is satisfied at the next iteration, that is, $|x(t+1)| + |y(t+1)| = 1$. The updating rule for one agent is schematically drawn in Figure 5.4.

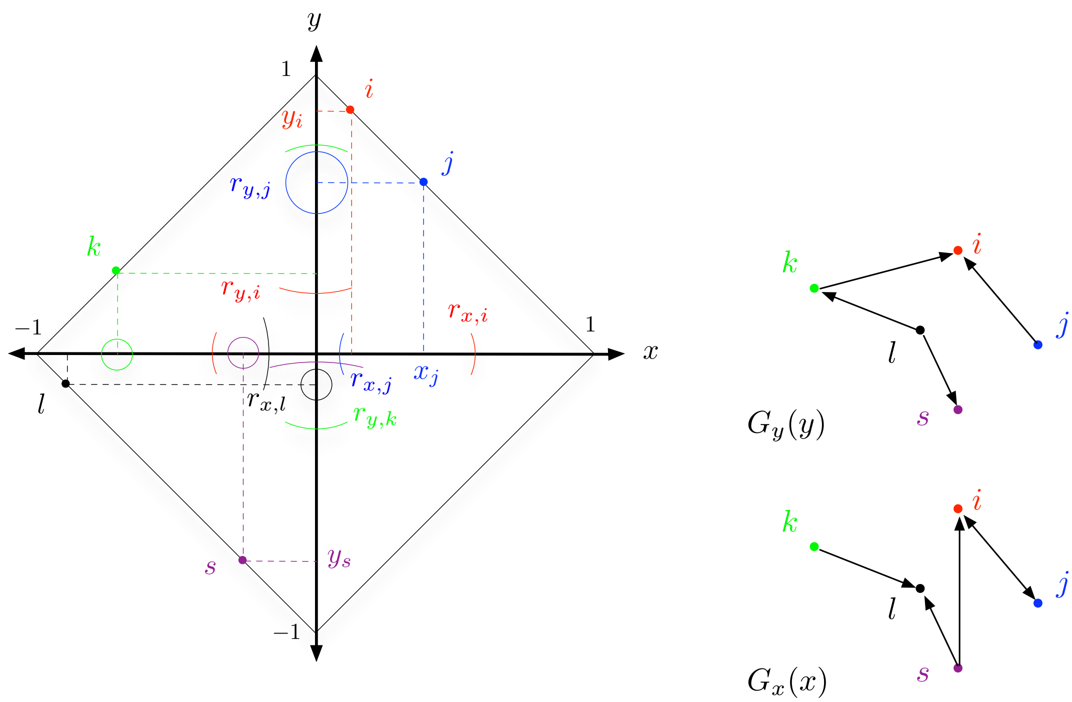



**Figure 5.3:** The two dimensional opinion space with topic dependent influence bounds for five agents (left) and the proximity digraphs associated to the two topics (right) are presented.

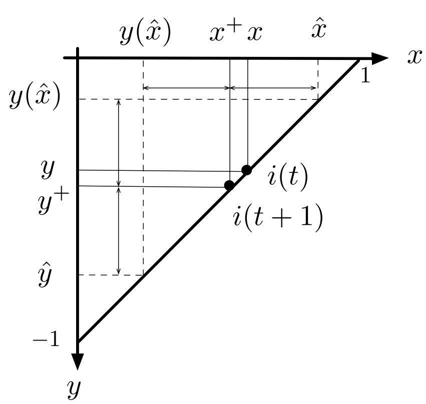



**Figure 5.4:** The schematic illustration of updating rule for one agent in constrained two dimensional SBI mdoel of opinion dynamics.

### 5.4.2 Problem Setup

In a committee of experts, there are two topics under discussion, and after a known number of meetings the committee votes for each topic. Here, we discuss strategies through which the media affects the results of voting on the two topics. We assume that the result of one topic is more important than the other, and hence the media might sacrifice the less important topic. Without loss of generality, assume that media values voting in favor of the first topic associated with opinion vector $x(t)$. In other words, the goal is having larger number of positive $x_i$'s after the final meeting, while the magnitude of these values are not decisive. We define similar strategies to those of Section 5.3.2:

- *Direct strategy:* After each meeting, the media advertises for one agent whose opinion lies in the positive $x$ and has influence on negative $x$. In other words, the input $u$ is a set of Gaussian variables centered around $x_i(t)$, where $i$ is such that $x_i(t) - r_i < 0$ and $x_i(t) > 0$.

- *Distracting strategy:* For the first half of the meetings, the media advertises for an opponent member who has a large negative $y$ with large bound of influence. During the second half of meetings, the media advertises for an agent with properties mentioned in direct strategy.

The distracting strategy is based on the fact that members of each political party tend to vote similarly on both topics, that is, each member tends to vote either in favor or against both topics. For instance, the positive $x$ and $y$ can represent the topic that members of one party are supporting. Advertising for a member with large negative $y$ will distract other members in his party with small negative $y$ but large negative $x$ from talking negatively about the first topic. This distraction eases the path for the second half of advertisements, where media tends to attract members with small negative $x$ toward voting in favor of the first topic. An example of the comparison of the two strategies on the two dimensional opinion evolution is illustrated in Figures 5.5 and 5.6. We aim to understand whether under uniformly randomly distributed initial opinion and bound vectors, the distracting strategy outperforms the direct strategy with higher probability. One approach to answer above question is to approximate the updating rule by a discrete-time continuous-space Markov Chain.

### 5.4.3 Eulerian Description by a Markovian Process

In Lagrangian models of opinion dynamics with finite population, discussed in Chapters 2 and 3, if the number of agents converges to infinity, then these models can be approximated by Eulerian models of opinion dynamics with infinite population, discussed in Chapter 4. For instance, consider a homogeneous HK

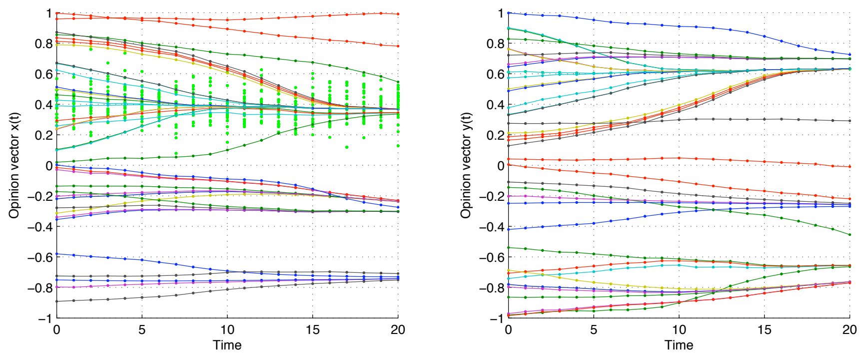



**Figure 5.5:** The trajectory of a constrained two dimensional SBI system, whose initial opinion vector and bounds vectors are uniformly randomly generated, under the influence of direct strategy is illustrated along $x$ (left) and $y$ (right) axes. The green dots represent the input, which is a set of Gaussian variable centered around one positive $x$. At final time the number of agents with positive $x$ is 22.

system with $n$ agents with opinions $x(t) = [x_1(t), \ldots, x_n(t)]$ and $m$ exogenous inputs $u(t) = [u_1(t), \ldots, u_m(t)]$. This system can be represented as an Eulerian HK system with the following mass or probability distribution and input at time $t$:

$$\mu_t(z) = \frac{1}{n}\sum_{i=1}^{n} \delta(z - x_i(t)),$$

$$u_t(z) = \frac{1}{m}\sum_{i=1}^{m} \delta(z - u_i(t)).$$

Hence, the flow map (5.1) gives us $\gamma_t = x(t+1)$. By updating the probability distribution via this flow map, the Delta Dirac function centered at any $x_i(t)$ will

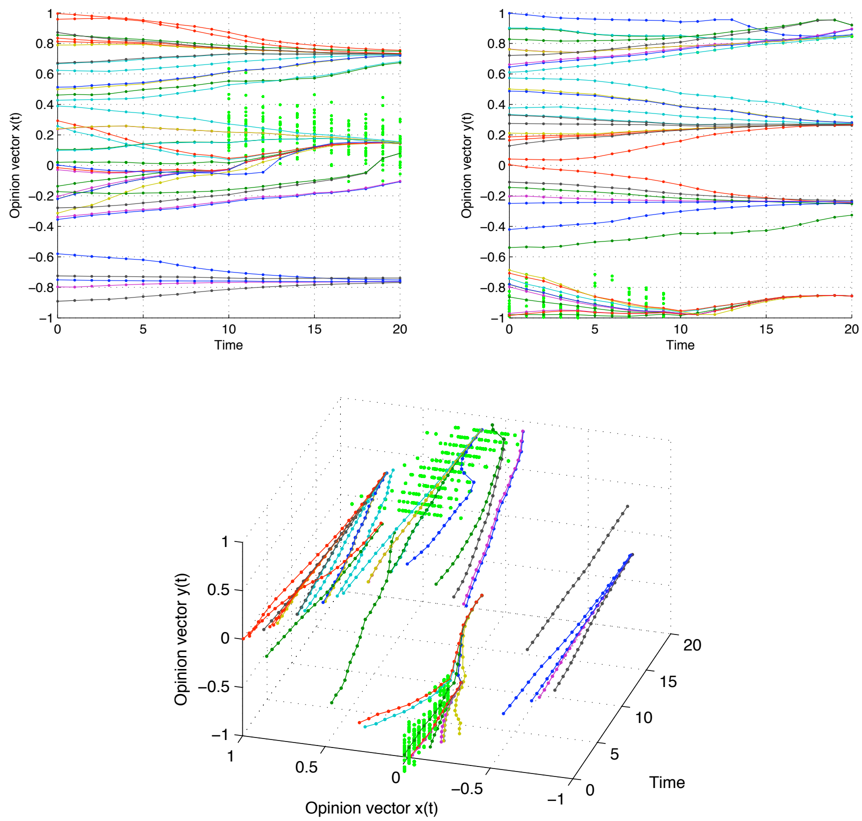



**Figure 5.6:** The trajectory of a constrained two dimensional SBI system with the same initial condition and bounds vector as the system of Figure 5.5 under the influence of distracting strategy is illustrated along $x$ (left) and $y$ (right) axes and in 3D (bottom). The green dots represent the input, which is a set of Gaussian variable centered around a negative $y$ for $t \in [0, 10]$ and a positive $x$ for $t \in [11, 20]$. At final time the number of members with positive $x$ is 32.

be transferred to $x_i(t+1)$. Therefore,

$$\mu_{t+1}(z) = \frac{1}{n}\sum_{i=1}^{n}\delta(z - x_i(t+1)),$$

which is consistent with the agent based HK system.

Accordingly, let $\mu_{x,t}$ and $\mu_{y,t}$ denote the probability distributions over the two opinion sets on the two topics at discrete times $t$. The initial probability distributions can be found by the real data on committee's opinions, here, we assume these distributions to be uniformly distributed over the given interval. Before proceeding to the dynamics of this system, we discuss the relation between the two probability distributions based on the constraint (5.9). As mentioned above, members of each political party tend to vote similarly on both topics, that is, each member tends to vote either in favor or against both topics. Therefore, agents with positive opinion $x$ tend to have a positive opinion $y$ with some probability $q > 0.5$ and vice versa. Based on constraint (5.9), if an agent devotes $x$ proportion of its time to the first topic, with probability $q$ he devotes $y^+(x) = \text{sign}(x) - x$ (similar sign to $x$) to the second topic and with probability $1-q$ he devotes $y^-(x) = x - \text{sign}(x)$ (opposite sign to $x$) to the second topic. Regarding initial distribution, assume that the probability distribution is uniformly distributed along the $x$ axis. Using the following strategy, one can prove that the probability distribution is also uniform along the $y$ axis. A uniform distribution tells us that $\mu_{0,x}(dx)$ centered at any $x_0 \in [-1,1]$ is equal to $dx/2$. For agents with opinion

$x_0$, $y_0$ is equal to $\text{sign}(x_0) - x_0$ with probability $qdx/2$. On the other hand, for agents with opinion $-x_0$, $y_0$ is equal to $x(0) - \text{sign}(x(0)) = \text{sign}(x_0) - x_0$ with probability $(1-q)dx/2$. Therefore, the probability distribution$\mu_{0,x}(dx)$ centered at $\text{sign}(x_0) - x_0$ is equal to $dx/2$.

(Broniatowski, 2010) claims that panel members who speak often and focus on one topic have higher influence. The empirical evidence confirms that influence bounds are both agent-dependent and opinion-dependent. In other words, if the same highly influential committee member changes his strategy and devotes less time to that topic, it is highly likely that he loses his influence. For brevity, in our Lagrangian description of the two dimensional SBI, we assume that the influence bounds are only agent-dependent. While, our Eulerian description of the model considers both agent-dependence and opinion dependence properties of the influence bounds. In this Eulerian model, we divide the opinion space into two regions *low influential (LI)* and *high influential (HI)* whose opinions are lower or higher than some value $\Delta \in \mathbb{R}_{>0}$, respectively. For any agent in the low or high influential region, the influence bound is assumed to be uniformly randomly distributed over $[\epsilon_1, \epsilon_2]$ or $[\zeta_1, \zeta_2]$, respectively, where $\epsilon_{1,2}, \zeta_{1,2} \in \mathbb{R}_{\geq 0}$. Moreover, $\epsilon_1 < \zeta_1$ and $\epsilon_2 < \zeta_2$. Therefore, the flow maps are computed by first considering the intermediate vectors in updating equations (5.10):

$$\hat{\gamma}_x(t,x) = \frac{\int_{x-\sigma}^{x+\sigma}\big(\int_{\zeta_1}^{\zeta_2}\mathbf{I}_{z\in HI} zP_x(t,z,\sigma)d\sigma + \int_{\epsilon_1}^{\epsilon_2}\mathbf{I}_{z\in LI} zP_x(t,z,\sigma)d\sigma\big)dz + \int_{x-\eta}^{x+\eta} uP_{x,u}(t,u)du}{\int_{x-\sigma}^{x+\sigma}\big(\int_{\zeta_1}^{\zeta_2}\mathbf{I}_{z\in HI} P_x(t,z,\sigma)d\sigma + \int_{\epsilon_1}^{\epsilon_2}\mathbf{I}_{z\in LI} P_x(t,z,\sigma)d\sigma\big)dz + \int_{x-\eta}^{x+\eta} P_{x,u}(t,u)du}.$$

Similarly for $y$ component we have:

$$\hat{\gamma}_y(t,y) = \frac{\int_{y-\sigma}^{y+\sigma}\big(\int_{\zeta_1}^{\zeta_2}\mathbf{I}_{z\in HI} zP_y(t,z,\sigma)d\sigma + \int_{\epsilon_1}^{\epsilon_2}\mathbf{I}_{z\in LI} zP_y(t,z,\sigma)d\sigma\big)dz + \int_{y-\eta}^{y+\eta} uP_{y,u}(t,u)du}{\int_{y-\sigma}^{y+\sigma}\big(\int_{\zeta_1}^{\zeta_2}\mathbf{I}_{z\in HI} P_y(t,z,\sigma)d\sigma + \int_{\epsilon_1}^{\epsilon_2}\mathbf{I}_{z\in LI} P_y(t,z,\sigma)d\sigma\big)dz + \int_{y-\eta}^{y+\eta} P_{y,u}(t,u)du}.$$

Finally, the updating equations (5.11) gives us the final flow maps:

$$\begin{aligned}\gamma_x(t,x) &= \frac{1}{2}\hat{\gamma}_x(t,x) \\ &\quad + \frac{1}{2}\mathrm{sign}(\hat{\gamma}_x(t,x))\big(q(1-|\hat{\gamma}_y(t,y^+(x))|) + (1-q)(1-|\hat{\gamma}_y(t,y^-(x))|)\big),\end{aligned}$$

$$\begin{aligned}\gamma_y(t,y) &= \frac{1}{2}\hat{\gamma}_y(t,y) \\ &\quad + \frac{1}{2}\mathrm{sign}(\hat{\gamma}_y(t,y))\big(q(1-|\hat{\gamma}_x(t,x^+(y))|) + (1-q)(1-|\hat{\gamma}_x(t,x^-(y))|)\big).\end{aligned}$$

An example of two dimensional constrained Eulerian HK model under the influence of input with direct and distracting strategy is illustrated in Figure 5.7, where the latter strategy outperforms the former. We believe that it is possible to prove that there is a strong correlation between two dimensional and one dimensional distracting strategies in an Eulerian point of view. Consequently, as we numerically established in Section 5.3, it can be shown that the distracting strategy in two dimensional HK model has a higher probability that the direct strategy in increasing the objective function in finite time.

## 5.5 Summary and Extensions

The formation of opinions in a large population is governed by endogenous (e.g., human interactions) and exogenous (e.g., media influence) factors. Considering a large population allows approximation of the decision making rules with Non-Bayesian "rule of thumb" methods. This chapter analyzed the behavior of an Eulerian bounded confidence model of opinion dynamics with time-varying input. In this model, a population is distributed over an opinion set and updates its opinion via 1) the opinion of the population inside the confidence range, and 2) the information from an exogenous input in that range. First, we proved some fundamental properties of this system's dynamics with time-varying input. We derived a simple sufficient condition for opinion consensus, and proved the convergence of population's distribution under time-invariant input to a sum of Dirac Delta functions. We computed an empirical upper bound on the largest range of opinions that a fixed Gaussian distributed input can attract to its center. We defined the *attraction range* of an input, and for a normally distributed input

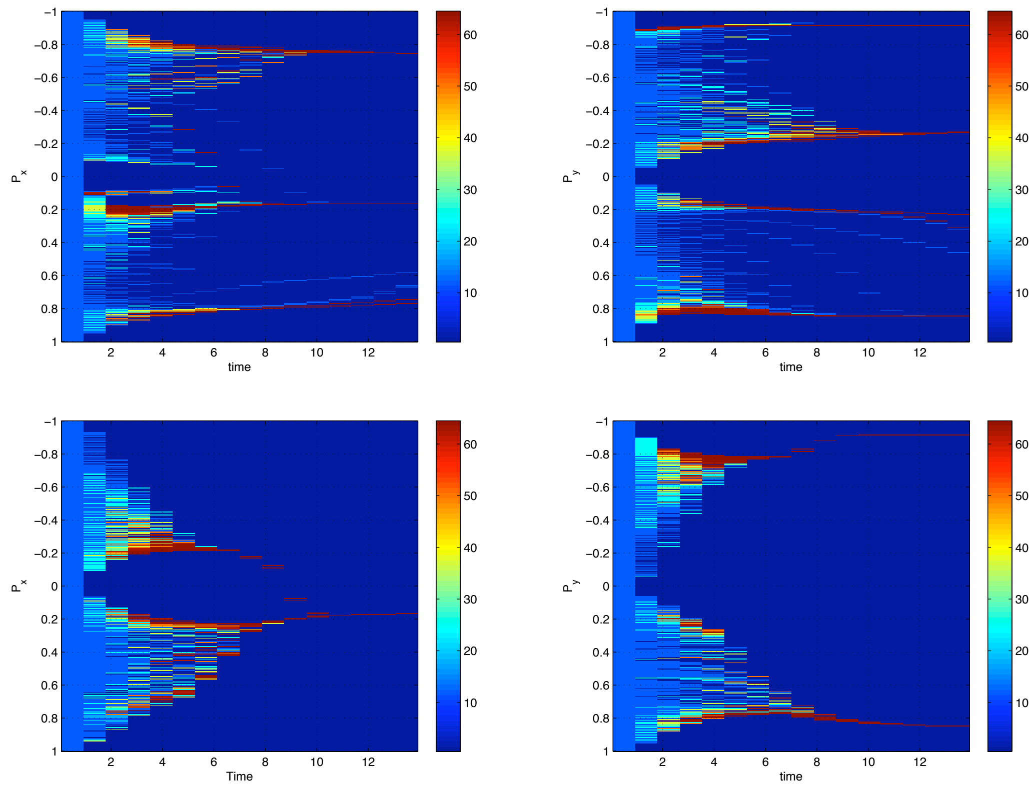



**Figure 5.7:** The evolution of probability or mass distribution of opinions along $x$ (left) and $y$ (right) axes under the influence of an input in direct (top) and distracting (bottom) strategies. The initial probability distributions are uniform on both axes. For opinions in interval $[-0.6, 0.6]$, the influence bound range is $[0, 0.2]$, and for the rest of opinions this value is $[0.1, 0.4]$. It can be seen that the distracting strategy attracts a higher population to the positive $x$ than the direct strategy.

and uniformly distributed initial population, respectively, we conjectured a linear relation between this range's length, population's confidence bound, and input's variance. Accordingly, we compared two manipulation strategies, where based on limited attraction range of manipulator, we discussed how one outperforms the other. Finally, we presented a real world example and discussed the effect of introduced strategies on evolution of opinions.

There is a great potential for further investigation of the effect of exogenous inputs on opinion evolution in a social network. (1) Under the effect of an input with Normal distribution on a large population, there exists a bounded opinion range in which the population gets attracted to the input's advertising value. One future challenge is computing the exact boundary of mentioned opinion range as a function of the system's parameters. Such a boundary acts as a tipping point in the dynamics of the systems, out of which agents cluster away from the input and inside which agents reach consensus at input's center. (2) We believe that it may be possible to find the opinion value $y$ that is most influenced by the input in a finite number of time steps $T$. More specifically, define $y(T)$ and $y_u(T)$ to be the final opinions of population with initial opinion $y$ under and without influence of input for $T$ iterations, respectively. Then, for any distribution of input, we plan to find the opinion $y$ whose $|y_u(T) - y(T)|$ is maximum. (3) As stated above, we consider a large population that votes positively or negatively in a series of opinion polls until final election. We aim to maximize the population with positive opinions in the final election at $T$. In other words, we aim to increase the Lyapunov function

$$V(T) = \int_0^1 d\mu_T(x).$$

However, owing to the discontinuity of this function over opinion space, optimizing this Lyapunov function is a mathematically hard problem. Consequently, two possible breakouts could be: i) finding a smooth Lyapunov function; or ii) employing an appropriate greedy algorithm. (4) We plan to obtain a greedy optimal manipulation strategy for any initial distribution of population on opinion space. An optimum input can be defined as an input that would attract the largest population to its center or that would increase the population with positive opinion the most.

# Chapter 6

# Conclusions

Decision-making in a society is a complex process, which is led to the final state by *endogenous* and *exogenous* factors. The interaction of individuals via in person meetings or online social networks is an endogenous factor. One of the most influential exogenous factors is the mainstream media that acts as a real-time input owing to its easy access to the public. Considering a large population allows approximation of the decision making rules with Non-Bayesian "rule of thumb" methods without relying on detailed social psychological findings. This thesis mainly addresses complex problems in the analysis of opinion evolution in (a) heterogeneous societies and (b) societies with large population under the influence of exogenous events. In the study of opinion dynamics in heterogeneous social networks we classified the agents and proved the existence of a leader group, and we derived sufficient conditions for the convergence of agents opinions to a final set of decisions. In the study of opinion evolution in a large population driven by peer-to-peer interactions and exogenous events: we (1) proved the convergence of populations distribution to clusters concentrated around separate opinions; (2) computed an empirical upper bound on the largest population that a fixed input can attract to its center, and proved numerically that this largest attracted population is a linear function of the system's parameters such as agents bound of trust; and (3) compared different manipulation strategies and evaluated their performance based on the limited attraction range of the manipulator agent.

## 6.1 Summary

In Chapter 2, we studies the properties of a homogeneous model of opinion dynamics for a finite population, and we proved that the time complexity of the model is of the order $O(n^4)$.

In Chapter 3, we analyzed two heterogeneous models of opinion dynamics for a finite population. We proved the existence of a leader group for each group of agents and determined the controllable factors in the behavior of followers. We derived novel, sufficient conditions for the convergence of agents opinions to a final set of decisions. We designed an appropriate classification of agents, based on which we explained the behavior of a different group of agents in the long run. We studied the properties of possible final decisions of groups of agents.

In Chapter 4, we analyzed the evolution of opinions in a large population driven by peer-to-peer interactions. We described fundamental properties of this systems dynamics. We derived a sufficient condition for opinion consensus, and proved the convergence of populations distribution to clusters concentrated around separate opinions.

In Chapter 5, we analyzed the model of opinion dynamics for large populations, introduced in the previous chapter, under the influence of exogenous inputs. We proposed a novel model describing the influence of inputs and derived a sufficient condition for opinion consensus. We established some important properties of the model with a time-varying input. We computed an empirical upper bound on the largest population that a fixed input can attract to its center, and proved numerically that this largest attracted population is a linear function of the systems parameters such as agents bound of trust. We compared two manipulation strategies and evaluated their performance based on the limited attraction range of the manipulator agent. We presented a real world example and discussed the effect of introduced strategies on evolution of opinions.

## 6.2 Future Directions

Regarding modeling the opinion dynamics in a finite population, we considered an opinion dependent interconnection network whose nodes or agents are heterogeneous. This heterogeneous state-dependent interconnection topology leads to a poorly-understood complex dynamic behavior. The convergence of this discrete time state-dependent linear system with switching topology has been an open problem since 2002. Hence, the main future challenge is to prove that all trajectories of this system converge to steady states. One approach is to employ the results of [33], which establishes that the product of an infinite number of row-stochastic matrices with positive diagonals converges to a partly fixed matrix, with complete consensus matrices on the diagonal. This result leaves us with a finite number of limiting opinion vectors for our system, and hence a finite number of invariant neighborhoods for each limiting vector to which the trajectory can be asymptotically confined. Another future challenge is a probability analysis on the topology of the proximity digraph of the system, which leads us to understanding the reason behind common final outcomes of decision making processes.

In bounded confidence (BC) models of opinion dynamics, agents interacts if and only if their opinions differ less than a given threshold. Such discontinuity makes the model non-robust to small variation in system parameters and initial opinions; this lack of robustness is a weakness of the BC models. We plan to transition from this extreme discontinuous model to a model based on (1) smooth interactions and (2) interactions aware of spatial geographic proximity. With regards to (1), we envision modeling interactions as a process whose strength decays smoothly and possibly exponentially fast in opinion difference. Moreover, we plan to introduce some stochasticity in the interaction model. With regards to (2), we plan to study model in which the opinion of an agent evolves as a result of interactions with agents, which are simultaneously similar-minded and geographically close. This implies that multiple graphs (or a graph with multiple attributes) come into play.

We have studied heterogeneity in the graph structure, characterizing equilibria, convergence criteria and convergence rates. However, so far we have assumed homogeneity in the averaging coefficients defining the opinion dynamics and we have not yet exploited the fact that the confidence bounds are not uniformly distributed in a real society. We next intend to study heterogeneity in the averaging coefficients and power-law distributions of confidence radii and coefficients. Such heterogeneity in a graph leads to interesting convergence behaviors, which can be studied by a probability analysis on different possible structures of the graph. Additionally, under power-law distributions, some graph components are more likely to exist and convergence properties will depend upon it. Finally, we aim to study the effect of heterogeneity in the averaging coefficients on the total convergence rate of the system.

There is a great potential for further investigation of the effect of exogenous inputs on opinion evolution in a social network:

First, the effect of information assimilation dynamics on an opinion-dependent social network is largely unknown. This is due to two main reasons. On one hand, due to the bounded confidence constraint, it may happen that each agent has access to a subset of the events, so that inference has to be performed at the level of the social network. On the other hand, the limit behavior of opinion-dependent networks is dominated by the phenomena of clustering and loss of connectivity, which are intrinsically against the possibility to reach a global agreement. It is thus an open issue whether and when social learning through observation and communication is effective. For instance, under the effect of an input with bounded domain on a distribution of population, there exists a bounded opinion range in which the population gets attracted to the input's advertising value. One future challenge is computing the exact boundary of mentioned opinion range as a function of the system's parameters. Such a boundary acts as a tipping point in the dynamics of the systems, out of which agents cluster away from the input and inside which agents reach consensus at input's center.

Second, we plan to consider general models of opinion dynamics with strategic manipulation. Our investigations start with the notion of an influential agent, as a misbehaving agent which we assume has (1) some goal about changing the others' opinions, (2) partial or complete knowledge of the current opinion of the other agents, and (3) the capability of setting her own opinion as a function of time. In the presence of influential agents and under a bounded confidence interaction model, a preliminary control-theoretic analysis shows that even one single influential agent may drive all the network to agreement at an arbitrary value of choice  although this process may possibly take a long time. Regarding this future challenge, we believe that it may be possible to find the population whose opinion is the most influenced by any strategic manipulator in a finite number of time steps. In considering a large population that votes positively or negatively in a series of opinion polls until final election, we aim to maximize the size of population with positive opinions in the final election. One approach is finding a greedy optimal manipulation strategy for any initial distribution of population on opinion space. An optimum input can be defined as an input that would attract the largest population to its center or that would increase the population with positive opinion the most.

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