UNIVERSITY OF CALIFORNIA
Santa Barbara

On Orientation Localization for Relative Sensing Networks

A Thesis submitted in partial satisfaction
of the requirements for the degree of

Master of Science

in

Mechanical Engineering

by

Giulia Piovan

Committee in Charge:

Professor Francesco Bullo, Chair

Professor Jeffrey Moehlis

Professor Bradley Paden

September 2010

The Thesis of
Giulia Piovan is approved:

Professor Jeffrey Moehlis

Professor Bradley Paden

Professor Francesco Bullo, Committee Chairperson

September 2010

On Orientation Localization for Relative Sensing Networks

Acknowledgements

This Thesis is based on my work with my advisor, Prof. Francesco Bullo, in collaboration with Iman Shames and Prof. Brian D. O. Anderson from the Australian National University and National ICT Australia, and Prof. Baris Fidan from the Department of Mechanical and Mechatronics Engineering, University of Waterloo, Canada.

Part of this material has been published in [7].

Abstract

On Orientation Localization for Relative Sensing Networks

Giulia Piovan

This thesis develops a novel localization theory for networks of nodes that measure each other's relative position, i.e., we assume that nodes do not have the ability to perform measurements expressed in a common reference frame. The thesis begins with some basic definitions of frame localizability and orientation localizability. Based on some key kinematic relationships, we characterize orientation localizability for planar networks with angle-of-arrival sensing. We then address the orientation localization problem in the presence of noisy measurements. Our first algorithm computes a least-squares estimate of the unknown node orientations in a ring network given angle-of-arrival sensing. For arbitrary connected graphs, our second algorithm exploits kinematic relationships among the orientations of nodes in loops in order to reduce the effect of noise. We establish the convergence of the algorithm, and through some simulations we show that the algorithm reduces the mean-square error due to the noisy measurements. We then consider networks in 3-D space and we explore necessary and sufficient conditions for orientation localizability in the noiseless case.

Contents

List of Figures

Chapter 1

Introduction

1.1 Introduction

Sensor networks are used in a large number of applications which cover a wide range of fields, such as, surveillance, controls, communications, monitoring areas, intrusion detection, vehicle tracking and mapping. One of the key problems in sensor networks is localization, i.e., determining the location of each sensor in the network.

We address the aforementioned problem in a distributed manner, by assuming that any node in the network has its own reference frame, and does not have any knowledge about its physical position in the environment or the position of the other nodes. Each node, through a sensor, can detect the relative position of any node inside a given sensor footprint. The measurements are affected by noise, so we extend our analysis to the noisy case. We call *frame localization* the problem of computing the relative location and orientation of each node of the network with respect to each other. We aim to solve the problem through a distributed algorithm, which computes the estimate of the angle associated to every edge of the graph by distributing the error of every cycle on its edges.

Network localization has been the center of extensive research work, and the various approaches are due to different assumptions on the deployment of the nodes and the way sensors work. In some cases, there is the use of some special nodes, whose positions are known, called beacons or anchors. Particular interest arises from the works of Roumeliotis and coworkers, [12] and [9], in which the problem of determining the relative position for a pair of robots moving in 2D or 3D is studied using only distance measurements between the robots. A distributed method for 3-D sensor network orientation and translation localization was proposed in [10]. Also notable is the beautiful treatment in [1], where a theory of localization emerges.

This thesis contains several contributions. First, we present a novel formulation of the frame localizability and frame computational localization problem for networks with relative sensing. Second, we define a characterization of frame localizability for planar networks, focusing on consistency for the orientation localization problem. Third, we compute a least-squares estimate of the unknown node orientations in a ring network. Fourth, we consider arbitrary connected graphs and provide a distributed algorithm for planar orientation localization which exploits kinematic relationships among the orientation of nodes in loops in order to reduce the effect of noise. Fifth, we provide some simulations in order to validate our algorithm results. Finally, we consider networks in the three-dimensional space and we explore necessary or sufficient conditions for orientation localizability in the noiseless case.

The document is organized as follows. In Chapter II, we review some kinematic conventions and elements of graph theory that are used throughout the paper. Chapter III is dedicated to the description of the network model and the problem with some preliminary relationships on relative positions. Chapter IV studies the orientation localizability of the network considering measurement noise, and some simulation results are shown. Chapter V explores the orientation localizability problem for noiseless networks in the three-dimensional space. Finally, Chapter VI gives a short review of what has been done, with conclusions and aims for future work.

Chapter 2

Preliminaries

2.1 Elements of kinematics

We let $\mathbb{R}$ and $\mathbb{C}$ denote real and complex numbers, respectively. We let $\|\boldsymbol{v}\|$ denote the Euclidean norm of the vector $\boldsymbol{v} \in \mathbb{R}^d$. We define the versor operator $\text{vers}\colon \mathbb{R}^d \to \mathbb{R}^d$ by $\text{vers}(\boldsymbol{0}) = \boldsymbol{0}$ and $\text{vers}(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$ for $\boldsymbol{v} \neq \boldsymbol{0}$. Given a scalar θ, we let $\text{proj}(\theta)$ take value in $[-\pi, \pi[$, where the map $\text{proj}\colon \mathbb{R} \to [-\pi, \pi[$ is defined by

$$\text{proj}(\theta) = (\theta + \pi)\text{mod}2\pi - \pi. \tag{2.1}$$

We let $\angle z$ denote the phase of $z \in \mathbb{C}$. We will be interested in measurements expressed in different reference frames. Accordingly, it is useful to review some basic kinematic conventions. We let $\Sigma_1 = \{p_1, \boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}_1\}$ be a fixed reference frame in $\mathbb{R}^3$. A point q and a vector $\boldsymbol{v}$ expressed with respect to frame Σ_1 are denoted by q^1 and $\boldsymbol{v}^1$, respectively. Next, let $\Sigma_2 = \{p_2, \boldsymbol{x}_2, \boldsymbol{y}_2, \boldsymbol{z}_2\}$ be a reference

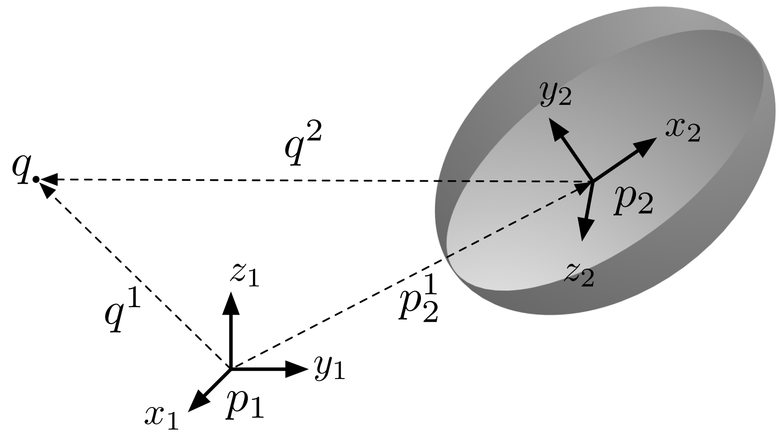


Figure 2.1: Two reference frames in $\mathbb{R}^3$

frame fixed with a moving body. The origin of Σ_2 is the point p_2, denoted by p_2^1 when expressed with respect to Σ_1. The orientation of Σ_2 is characterized by the 3-dimensional rotation matrix $\boldsymbol{R}_2^1$, whose columns are the frame vectors $\{\boldsymbol{x}_2, \boldsymbol{y}_2, \boldsymbol{z}_2\}$ of Σ_2 expressed with respect to Σ_1. We recall here the definition of the set of rotation matrices in d-dimensions, for $d \in \{2, 3\}$: $SO(d) = \{\boldsymbol{R} \in \mathbb{R}^{d\times d} |\ \boldsymbol{R}\boldsymbol{R}^T = I_d,\ \det(\boldsymbol{R}) = +1\}$. With these notations, reference frame transformations are described by

$$q^1 = \boldsymbol{R}_2^1 q^2 + p_2^1, \qquad \text{and} \qquad \boldsymbol{v}^1 = \boldsymbol{R}_2^1 \boldsymbol{v}^2. \tag{2.2}$$

Recall also $\boldsymbol{R}_2^1 = (\boldsymbol{R}_1^2)^T$. Analogously, we let S^i denote the point set S as expressed in the reference frame Σ_i. Finally, if three reference frames Σ_i, $i \in \{1, 2, 3\}$, are considered, then simple bookkeeping arguments lead to

$$\boldsymbol{R}_2^1\boldsymbol{R}_3^2\boldsymbol{R}_1^3 = I_3, \quad \text{and} \quad \boldsymbol{R}_2^1 = \boldsymbol{R}_3^1\boldsymbol{R}_2^3. \tag{2.3}$$

Next, it is convenient to present a planar case version of these notions. In the planar case, p_1 and p_2 take values in $\mathbb{R}^2$, the reference frames consist of only two orthonormal vectors, and the rotation matrices take values in $SO(2)$. It is convenient to identify $\mathbb{R}^2$ with the set of complex numbers $\mathbb{C}$ and to denote the unit imaginary number by $\sqrt{-1} \in \mathbb{C}$. If we describe the planar rotation matrix $\boldsymbol{R}_2^1 \in SO(2)$ by its unit-length complex number representation $\exp(\theta_2^1\sqrt{-1})$, with angle $\theta_2^1 \in [-\pi, \pi[$, then the second part of eq. (2.2) reads $\boldsymbol{v}^1 = \exp(\theta_2^1\sqrt{-1})\boldsymbol{v}^2$.

Finally, we review the exponential representation of rotations. For the unit vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$, we use Rodrigues' rotation formula [6] to define the rotation matrix about axis $\boldsymbol{\omega}$ of an angle γ as

$$\exp\left(\gamma\widehat{\boldsymbol{\omega}}\right) = I_3 + \sin\gamma\widehat{\boldsymbol{\omega}} + (1-\cos\gamma)\widehat{\boldsymbol{\omega}}^2, \tag{2.4}$$

where, as usual, $\widehat{\boldsymbol{\omega}} \in \mathbb{R}^{3\times 3}$ is defined by

$$\widehat{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{2.5}$$

We also recall that for any $\boldsymbol{R} \in SO(d)$,

$$\boldsymbol{R}\exp(\widehat{\boldsymbol{v}}) = \exp(\widehat{\boldsymbol{R}\boldsymbol{v}})\boldsymbol{R}. \tag{2.6}$$

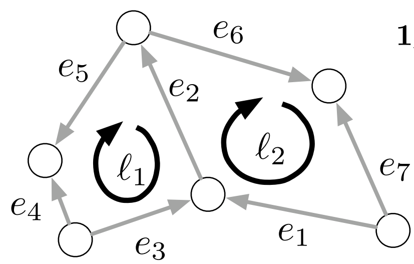
2.2 Elements of graph theory

We review a few useful notions from graph theory [2, 4]. We let $G = (V, E)$ represent an undirected graph G, with vertex set $V \triangleq \{v_i\}_{i=1}^{n}$ and edge set E with cardinality m. $G_d = (V, E_d)$ defines a directed graph associated to G, where E_d is an orientation of E. We denote a directed edge from vertex v_i to v_j by $e_{ij} = (i, j)$. If the graph is undirected, (i, j) is equivalent to (j, i).

Definition 1 (Path and cycle) *Let G be either a directed or undirected non-empty graph. A* path *is a non-empty graph $P = (V_P, E_P) \subseteq G$ of the form $V_P \triangleq \{v_i\}_{i=1}^{k}$ and $E_p \triangleq \{(j_i, j_{i+1})\}_{i=1}^{k-1}$, where $\{j_1, \cdots, j_k\}$ is a permutation of $v_1, \cdots, v_k$. Furthermore, every sequence of edges that form a closed path in G and do not visit the same node twice, except the start/end node, is called a* cycle *and it is denoted by ℓ.*

The direction of a cycle is the order in which the nodes are visited. We let $\mathcal{L}(G)$ denote the set of all cycles ℓ of G, and $|\ell|$ the number of edges in ℓ. It should be noted that, in a digraph G_d, the cycle directions are independent of the direction of the individual edges composing the cycles.

Definition 2 (Cycle vector) *For $\ell \in \mathcal{L}(G_d)$, the* cycle vector *is the vector $\mathbf{1}_\ell \in \{-1, 0, +1\}^m \subset \mathbb{R}^m$ whose i^{th} entry is $+1$ if the i^{th} edge belongs to ℓ and its* *orientation is consistent with the orientation of* ℓ*,* -1 *if the* i^{th} *edge belongs to* ℓ *and its orientation is opposite to the orientation of* ℓ*, and is* 0 *otherwise.*



$\mathbf{1}_{\ell_1} = [0, -1, -1, 1, -1, 0, 0]$

$\mathbf{1}_{\ell_2} = [1, 1, 0, 0, 0, 1, -1]$

Figure 2.2: Example of cycle vector

Definition 3 (Set of cycle and fundamental cycle vectors) *The* set of cycle vectors *is* $L = \{\mathbf{1}_\ell |\ \ell \in \mathcal{L}(G_d)\}$*. A* set of fundamental cycle vectors $L_f \subseteq L$ *is a subset of* L *that constitute a base for* L*. The elements of* L_f *are called fundamental cycle vectors.*

Given a set of fundamental cycle vectors L_f, we let $\mathcal{L}_f(G_d)$ denote the associated fundamental cycles $\mathcal{L}_f(G) = \{\ell \in \mathcal{L}(G_d)|\ \mathbf{1}_\ell \in L_f\}$.

Definition 4 (Cycle and fundamental cycle matrix) *The* cycle matrix C *of a directed graph* G_d *is the* $k \times m$ *matrix* $C = [\mathbf{1}_{\ell_1}, \ldots, \mathbf{1}_{\ell_k}]^T$ *where* k *is the cardinality of* L*, and* m *is the number of edges of* G_d*. The* $r \times m$ *matrix* $C_f \subseteq C$*, with* $r = \dim(L_f)$*, such that each row represents a fundamental cycle vector in* L_f*, is called the* fundamental cycle matrix*:*

$$C_f = [\mathbf{1}_{\ell_1}, \ldots, \mathbf{1}_{\ell_r}]^T, \quad \textit{for all} \quad \mathbf{1}_{\ell_i} \in L_f. \tag{2.7}$$

Note that C_f is not unique since it depends on the choice of the fundamental cycles vectors, and it is a full rank matrix.

Theorem 5 (Number of independent cycles [2]) *If G_d has n vertices and m edges, than the dimension of the fundamental cycle space L_f is $m-n+1$, i.e., there are $m-n+1$ independent cycles.*

Chapter 3

Network model and localization problems

In what follows we describe our notion of *network equipped with relative sensors*. We consider a group of n nodes in $\mathbb{R}^d$, for $d \in \{2, 3\}$, and we assume that a reference frame Σ_i with origin p_i, for $i \in \{1, \dots, n\}$, is attached to each node. Also, we assume $p_i \neq p_j$ for all $i \neq j$, and we label the 1^{st} node the *reference node*.

3.1 Relative sensing model

Each node i activates a sensor that detects the presence and returns a measurement about the relative position of any node inside a given sensor footprint. We let $S_i \subset \mathbb{R}^d$ be the *sensor footprint* of node i and S_i^i be its expression in the Σ_i frame; we shall assume that all node sensors are equal, so that we write $S_i^i = S^i$. We assume that there exists a map $\text{sns} \colon \mathbb{R}^d \to \mathbb{R}^k$, for some k, called the *sensing function*, such that node i acquires the symbol $\mathrm{sns}(p^i_j)$ for each node $j \in \{1, \dots, n\} \setminus \{i\}$ that satisfies $p^i_j \in S^i$. There are different kind of sensors:

Range sensing: Node i measures $\|p^i_j\|$, i.e., $\mathrm{sns}(p^i_j) = \|p^i_j\| \in \mathbb{R}_{\geq 0}$, for all nodes j within a fixed sensing range r from i, that is, the footprint S^i is a disk of radius r and the function sns returns the norm of its argument.

Angle-of-arrival sensing: Node i measures $\mathrm{vers}(p^i_j)$, i.e., $\mathrm{sns}(p^i_j) = \mathrm{vers}(p^i_j) \in \mathbb{R}^d$, for all nodes j within a fixed sensing range r from i, that is, the footprint S^i is a disk of radius r and the function sns returns the spherical coordinates of its argument.

Complete sensing (range and angle-of-arrival): Node i measures p^i_j, i.e., $\mathrm{sns}(p^i_j) = p^i_j$ in $\mathbb{R}^d$, for all nodes j within a fixed sensing range r from i, that is, the footprint S^i is a disk of radius r and the function sns returns its argument.

Given the nodes $p_1, \dots, p_n$, the directed *sensing graph*, $G_d = (V_S, E_d)$ is the directed graph where vertex v_i corresponds to node i and the directed edge $(i, j) \in E_d$ if $p^i_j \in S^i$, that is, if node j is inside the sensor footprint of node i. In what follows, we assume that the sensor footprint S^i is a unit-radius disk with p_i as the center, so that the sensing graph is the so-called unit-disk geometric graph illustrated in Figure 3.1. With this assumption, if node i senses node j, then node j senses node i as well. Therefore, if $(i, j) \in E_d$, then $(j, i) \in E_d$ as well. To simplify notation we use an undirected graph $G_S = (V_S, E_S)$ with vertex set V_S and undirected edge set E_S satisfying $(i, j) \in E_S \iff (i, j) \in E_d \iff (j, i) \in E_d$. We call G_S the undirected sensing graph or simply the sensing graph. We further assume that a pair of nodes i and j communicate with each other if and only if they can sense each other, i.e., $(i, j) \in E_S$. In summary, the physical components

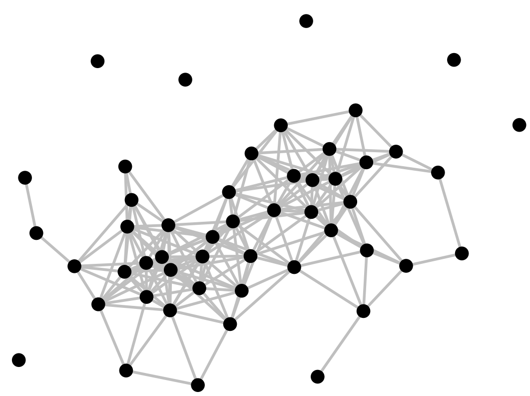


Figure 3.1: The disk graph in $\mathbb{R}^2$

of a *relative sensing network* consist of n nodes with identifiers in $\{1, \ldots, n\}$, with configurations in $\mathbb{R}^d \times SO(d)$, and with relative sensors described by the sensor footprint S^i and sensing function sns.

Note that the sensor we will use throughout the thesis is the *angle-of-arrival*.

3.2 The frame localization problem

Loosely speaking, we call *frame localization* the problem of computing the location and orientation of each node of a relative sensing network. Additionally, we call *orientation localization* the problem of computing the orientation of each node of a relative sensing network. We begin with questions about the uniqueness of these localization problems.

Problem 6 ***(Frame and orientation localizability)*** *Given a relative sensing network with reference node* 1*, provide graph theoretical conditions under which:*

(frame localizability:) *the reference frame transformations* $\{\boldsymbol{R}_i^1, p_i^1\}$*, for all* $i \in \{2, \dots, n\}$*, are uniquely determined by the relative measurements;*

(orientation localizability:) *the orientations* $\boldsymbol{R}_i^1$*, for all* $i \in \{2, \dots, n\}$*, are uniquely determined by the relative measurements.* □

Problem 7 ***(Centralized and distributed localization)*** *Given a frame (respectively, orientation) localizable network, give a centralized or distributed algorithm to compute the reference frames transformation* $\{\boldsymbol{R}_i^1, p_i^1\}$ *(respectively, the orientations* $\boldsymbol{R}_i^1$*), for all* $i \in \{2, \dots, n\}$*. Give algorithms for both noise-less and noisy sensor measurements.* □

Finally, for the above questions, we are interested in complexity in arbitrary networks and expected computational complexity in random geometric networks.

Remark 8 (Data referencing motivation) *It is worth remarking that the frame localization problem needs to be solved in relative sensing networks if measurements taken by arbitrary sensors in their respective reference frames need to be expressed (and possibly fused) in a common unique reference frame. Measurements might include positions of targets, environment boundaries, etc.* □

3.3 Preliminary relationships

In three dimensions, for any sensing and communication undirected edge (i, j), the basic relationship between the relative positions p_i^j and p_j^i and the change of frame rotation matrix $\boldsymbol{R}_j^i$ can be computed from (2.2) to be $p_j^i = -\boldsymbol{R}_j^i p_i^j$. It is possible to write a normalized version of this equation that applies to angle-of-arrival measurements:

$$\operatorname{vers}(p_j^i) = -\boldsymbol{R}_j^i \operatorname{vers}(p_i^j), \tag{3.1}$$

And its planar version, where relative positions are complex numbers and rotations matrices are unit-length complex numbers, is:

$$\theta_j^i = \operatorname{proj}(\angle p_j^i - \angle p_i^j + \pi). \tag{3.2}$$

Remark 9 ***(Measurements and variables)*** *Recall that the two nodes i and j measure each other's relative positions p_j^i and p_i^j, respectively. The unknown variable in eq. (3.1) is the rotation matrix $\boldsymbol{R}_j^i$ with d degrees of freedom.* □

Lemma 10 (Feasible orientations) *Given unit-length measurements $u_j^i = \mathrm{vers}(p_j^i)$ and $u_i^j = \mathrm{vers}(p_i^j)$, let $\boldsymbol{H}_j^i \in SO(3)$ be defined by $\boldsymbol{H}_j^i = \exp\left(\alpha_j^i\, \widehat{\boldsymbol{e}_j^i}\right)$, where $\boldsymbol{e}_j^i \in \mathbb{R}^3$, $\alpha_j^i \in [0, \pi]$ are defined[1] by*

$$\boldsymbol{e}_j^i = \begin{cases} \mathrm{vers}(u_j^i \times u_i^j), & \text{if } u_j^i \times u_i^j \neq \mathbf{0}, \\ \text{any unit-length vector} \perp u_j^i, & \text{otherwise}, \end{cases}$$

$$\alpha_j^i = \mathrm{atan}_2(\|u_j^i \times u_i^j\|, -u_j^i \cdot u_i^j).$$

Then, every solution to eq. (3.1) may be written as $\boldsymbol{R}_j^i = \exp\left(\beta\, \widehat{u_j^i}\right)\boldsymbol{H}_j^i$, for an appropriate angle $\beta \in [-\pi, \pi[$.

Proof: First, let us show that $\boldsymbol{H}_j^i$ is solution of (3.1):

$$\begin{aligned} \boldsymbol{H}_j^i u_i^j &= \exp\left(\alpha_j^i\, \widehat{\boldsymbol{e}_j^i}\right) u_i^j, \\ &= u_i^j \cos\alpha_j^i + (\boldsymbol{e_j^i} \times u_i^j)\sin\alpha_j^i + (1-\cos\alpha_j^i)(\boldsymbol{e_j^i}\cdot u_i^j)u_i^j. \end{aligned}$$

Because $\boldsymbol{e_j^i}$ and u_i^j are mutually orthogonal unit vectors, we have that

$$\boldsymbol{H}_j^i u_i^j = u_i^j \cos\alpha_j^i + \boldsymbol{n}\sin\alpha_j^i, \tag{3.3}$$

[1] *For any point (x, y) in the plane except for the origin, let $\mathrm{atan}_2(y, x)$ be the angle between the horizontal positive axis and the point (x, y) measured counterclockwise.*

where $\boldsymbol{n}$ is a unit vector perpendicular to the plane containing $\boldsymbol{e_j^i}$ and u_i^j whose direction is given by their cross product. Let us consider the orthonormal base $\{u_i^j, \boldsymbol{n}, \boldsymbol{e_j^i}\}$. Then, eq. (3.3) represents the rotation of axis u_i^j around axis e_j^i of an angle α_j^i, where α_j^i is, by definition, the angle between u_i^j and $-u_j^i$. Therefore $\boldsymbol{H}_j^i u_i^j = -u_j^i$, that is, $\boldsymbol{H}_j^i$ is solution of (3.1). Now, for an arbitrary angle $\gamma \in [-\pi, \pi[$, we compute

$$\begin{aligned}
&\exp\left(\gamma\, \widehat{u_j^i}\right) u_j^i = \\
&= u_j^i \cos\gamma + (u_j^i \times u_j^i)\sin\gamma + (1-\cos\gamma)(u_j^i \cdot u_j^i)u_j^i \\
&= u_j^i \cos\gamma + u_j^i - u_j^i \cos\gamma = u_j^i.
\end{aligned} \tag{3.4}$$

Then $u_j^i = -\exp\left(\beta\, \widehat{u_j^i}\right)\boldsymbol{H}_j^i u_i^j$, for $\beta = -\gamma \in [-\pi, \pi[$, i.e., $\exp\left(\beta\, \widehat{u_j^i}\right)\boldsymbol{H}_j^i$ is solution of (3.1) for all $\beta \in]-\pi, \pi]$.

Now, we want to show that any solution of (3.1) takes such a form. Suppose the matrix $\tilde{\boldsymbol{R}} \in SO(3)$ is solution of (3.1). We obtain $\tilde{\boldsymbol{R}} u_i^j = \exp\left(\beta\, \widehat{u_j^i}\right)\boldsymbol{H}_j^i u_i^j$, which can be easily written as $\exp(-\alpha_j^i\, \widehat{\boldsymbol{e}_j^i})\exp(-\beta\, \widehat{u_j^i})\tilde{\boldsymbol{R}} u_i^j = u_i^j$. It is known that any rotation of a fixed vector that yields the same vector is equivalent to a rotation of the vector about itself by any angle. Then

$$\exp(-\alpha_j^i\, \widehat{\boldsymbol{e}_j^i})\exp(-\beta\, \widehat{u_j^i})\tilde{\boldsymbol{R}} = \exp(-\mu\, \widehat{u_i^j}),$$

for any $\mu \in [-\pi, \pi[$. From (3.1) and (2.6) we obtain

$$\exp(-\mu\, \widehat{u_i^j}) = \exp(-\mu\, \widehat{\tilde{\boldsymbol{R}}^{-1} u_j^i}) = \tilde{\boldsymbol{R}}^{-1}\exp(-\mu\, \widehat{u_i^j})\tilde{\boldsymbol{R}},$$

and $\tilde{\boldsymbol{R}} \exp(-\alpha_j^i \ \widehat{\boldsymbol{e}_j^i}) = \exp((-\mu + \beta) \ \widehat{u_j^i})$. Therefore, any solution of (3.1) can be written as $\tilde{\boldsymbol{R}} = \exp\left(\varphi \ \widehat{u_j^i}\right) \boldsymbol{H}_j^i$, for any $\varphi \in [-\pi, \pi[$. ■

Chapter 4

Two-dimensional frame localization

4.1 Orientation localizability with angle-of- arrival sensors

Our first localizability result follows.

Theorem 11 (Orientation localizability for two- dimensional networks with angle-of-arrival sensing) *Consider a planar relative sensing network ($d = 2$) and with noiseless angle-of-arrival sensing. The following statements are equivalent:*

(i) the sensing graph is connected, and

(ii) the network is orientation localizable.

Proof: For every undirected edge (i, j) of the sensing graph, the angles $\angle p_j^i$ and $\angle p_i^j$ are measured. Therefore, eq. (3.2) implies that the relative angle θ_j^i is uniquely determined from the measurements. Now, let us prove (i) $\implies$ (ii). If the network is connected, then there exists a path from the reference node 1 from each $i \neq 1$. From equation (2.3), the angle θ_i^1 is uniquely determined as the sum of the relative angles along the path connecting i to the reference node. Let us now prove (ii) $\implies$ (i). Assume that there exists no path from node i to the reference node 1. Therefore, i and 1 belong to distinct connected components with the network. No measurement is available about the relative orientation of each node in the component containing i with respect to any node in the component containing 1. Therefore, it is not possible that only a single orientation θ_i^1 is compatible with the measurements. ■

Proposition 12 *A network with only range measurements is not orientation localizable.*

Proof: The range measurement is independent of the reference frame, i.e., given a fixed geometry of a network, each node may have an infinite number of orientations. ■

Proposition 13 (Sufficient conditions for localizability) *A network with n nodes capable of angle-of-arrival measurement is both frame localizable and orien-*

tation localizable if the sensing graph is rigid and at least one of the edge lengths is known.

In order to prove Proposition 13, we introduce the following notions and lemma from [8]. Consider a reference frame with configuration $p\colon V \to \mathbb{R}^2$, i.e., the position of every point of the network.

Definition 14 (Constraint) *The length constraints, L, are pairs of points whose lengths are fixed, and the direction constraints, D, are pairs of points whose directions are fixed.*

It is then possible to speak of the direction graph $F = (V, D)$ and the length graph $G = (V, L)$ and consider the double graph $FG = (V; D, L)$. We can measure the distance between every two points of the network through the *rigidity function*, defined as $\zeta\colon \mathbb{R}^2 \to \mathbb{R}$, and $\zeta(p)_{i,j} = \|p_i - p_j\|^2$ for $j \in]i, |V|]$. Note that ζ is continuously differentiable with respect to p.

Definition 15 (Rigidity matrix) *The* rigidity matrix *for p is the matrix $D(p)$ defined by $\zeta'(p) = 2D(p)$, where $\zeta'(p) = \frac{d\zeta}{dp}$.*

We call $D(FG, p)$ the *constraint matrix* of the graph, which consists in the rows of $D(p)$ that correspond to the edges in L and D.

Definition 16 (Independent constraints) *A set of constraints is* independent *if the corresponding rows of the constraint matrix are independent.*

Lemma 17 (Number of independent constraints) *A graph with n nodes and $2n-3$ independent bearing constraints plus any single length constraint, has $2n-2$ independent constraints and a 2-dimensional space of translations in the plane.*

Proof: [Proof of Proposition 13] Since the sensing graph is rigid, it is connected and it has at least $2n-3$ independent edges. Hence, according to Theorem 11, it is orientation localizable, and according to Lemma 17, the corresponding framework has a 2-dimensional space of translation in the plane. But, fixing the origin as any node $i \in \{1, \dots, n\}$, $\{R^i_j, p^i_j\}$ for any $j \in \{1, \dots, n\}$ are uniquely determined. Hence the network is frame localizable. ■

4.2 Orientation localization with noisy angle-of-arrival sensors

Now we follow Theorem 11, and we consider a network with nodes in the plane and with angle-of-arrival sensing. We assume that, for each undirected edge (i, j) of the sensing graph, nodes i and j measure, respectively, the angles $\angle p^i_j + n^i_j$ and $\angle p^j_i + n^j_i$, where we suppose the noises n^j_i and n^i_j to be independent, Gaussian random variables with zero mean and variance σ. Therefore, for each undirected edge (i, j), we can measure only

$$y_j^i = \text{proj}((\angle p_j^i + n_j^i) - (\angle p_i^j + n_i^j) + \pi), \tag{4.1}$$

and not the true relative orientation θ_j^i as in eq. (3.2).

Remark 18 (Redundant measurements in cycles) *If the sensing graph is a tree, then there is no redundant measurement and we cannot reduce the effect of measurement noise on our angle estimates. However, for every cycle in the network, we can enforce a cycle constraint (see eq.* (2.3)*). We formalize this statement as follows.* □

Let $G_S = (V_S, E_S)$ be the undirected sensing graph with n nodes and m edges. We assign a direction to each edge in E_S in the following way: the direction is from j to i if $i > j$. Noting that this direction assignment is different from/independent of the sensing/communication relations, let us denote the directed graph obtained, by $G_d = (V_S, E_d)$. Consider the oriented edge $e = (j, i) \in E_d$, with $i > j$. Let ψ_e denote the estimate of the true relative angle associated to e, $\theta_e = \theta_i^j$. Let $\psi \in \mathbb{R}^m$ denote the vector of angle estimates for all the edges of the graph. Analogously, we let y denote the measurement vector with components $y_e = y_i^j$, for $i > j$. For $\ell \in \mathcal{L}(G_d)$, the *cycle error* ϵ_ℓ at ψ is

$$\boldsymbol{\epsilon}_\ell = \text{proj}(\mathbf{1}_\ell \cdot \psi), \tag{4.2}$$

where the map proj: $\mathbb{R} \to [-\pi, \pi[$ is defined in (2.1) and the map proj: $\mathbb{R}^n \to [-\pi, \pi[^n$ is defined by

$$\text{proj}([x_1, \ldots, x_n]^T) = [\text{proj}(x_1), \ldots, \text{proj}(x_n)]^T. \tag{4.3}$$

Note that $\text{proj}(\mathbf{1}_\ell \cdot \psi) = \text{proj}\left(\sum_{f \in \ell} \pm\psi_f\right)$, where $\pm$ indicates whether or not the direction of the edge f is concordant with the direction of the cycle ℓ which f belongs to.

It is evident that for a set of angle estimates to be consistent, the cycle error must be zero. Accordingly, in what follows, we aim to solve the least-squares estimation problem:

$$\begin{aligned} \min_{\psi} \quad & \|\psi - y\|^2 \\ \text{subj. to} \quad & \text{proj}(\mathbf{1}_\ell \cdot \psi) = 0, \text{ for all } \ell \in \mathcal{L}(G_d). \end{aligned} \tag{4.4}$$

Note that the optimal ψ lives in a set of countable affine subspaces; once the optimal affine subspace is determined, the optimal estimate is computed via a linear projection.

Remark 19 (Analogy with the Kirchhoff Voltage Law) *The constraint in eq.* (4.4) *can be regarded as a generalization of the Kirchhoff Voltage Law (KVL) for electrical networks, which states that the sum of voltage drops around a closed loop is zero. We can look at the vector of angle estimates ψ as the vector of potential drops on every edge of the graph. The sum of the potential drops on any* *cycle ℓ (projected on $[-\pi, \pi[$ since we are using angles instead of voltages), is then given by* $\text{proj}(\mathbf{1}_\ell \cdot \psi)$*. Imposing the constraint that this quantity is equal to zero is equivalent to imposing the KVL on the network.* □

4.3 Optimal estimation in a ring

Now, suppose the sensing graph G_S is a ring with nodes $\{1, \dots, n\}$ and with undirected edges $(i, (i+1) \bmod n)$, for $i \in \{1, \dots, n\}$. In what follows, we write $(i+1)$ to denote $(i+1) \bmod n$. Compute a set of angle estimates ψ^i_{i+1} by

$$\psi^i_{i+1} = y^i_{i+1} - \frac{1}{n}\Big(\text{proj}\,\Big(\sum_{i=1}^{n-1} y^i_{i+1} - y^1_n\Big)\Big), \tag{4.5}$$

and $\psi^1_n = -\psi^n_1$. In vectorial form eq. (4.5) reads

$$\psi = y - \mathbf{1}_\ell \frac{1}{n}\,\text{proj}(\mathbf{1}_\ell \cdot y), \tag{4.6}$$

where $y = [y^1_2, y^2_3, \cdots, y^1_n]^T$ and $\psi = [\psi^1_2, \psi^2_3, \cdots, \psi^1_n]^T$. This affine map is a projection onto one of the affine subspaces that describe the constraint in the optimization problem (4.4). It is interesting to note that eq. (4.6) is a particular case of the Kaczmarz's projection method [5] for solving a system of linear equations through iterative projections.

Theorem 20 (Solution to the least squares) *The angle estimates computed in eq.* (4.5) *are the solution to the least squares estimation problem* (4.4)*.*

Proof: We start by noting that we can rewrite the constraint set of (4.4) as $(\mathbf{1}_\ell \cdot \psi) = 2k\pi, \quad k \in \mathbb{Z}$. It is clear that there must exist $k' \in \mathbb{Z}$ and $|\delta| \leq \pi$ such that $\sum_{i=1}^{n-1} y_{i+1}^i - y_n^1 = 2k'\pi + \delta$. Then, we know that the orthogonal projection of y onto $(\mathbf{1}_\ell \cdot \psi) = 2k'\pi$ minimizes (4.4). Now we want to show that the vector ψ calculated by (4.5) is the orthogonal projection of y onto the appropriate affine subspace. The normal vectors of the hyperplanes in the constraint subspaces defined by (4.4) are equal and are exactly $\mathbf{1}_\ell$. The orthogonal projection of y can be calculated by computing the point of intersection of the line $L = \{y + t\mathbf{1}_\ell | \ t \in \mathbb{R}\}$ with the hyperplane $(\mathbf{1}_\ell \cdot \psi) = 2k'\pi$, i.e., finding the point $\psi = [\psi_2^1, \psi_3^2, \cdots, -\psi_n^1]^T$ satisfying

$$\sum_{i=1}^{n-1} \psi_{i+1}^i - \psi_n^1 = 2k'\pi, \qquad \psi_{i+1}^i - y_{i+1}^i = -\psi_n^1 + y_n^1,$$

for $i = 1, \ldots, n-1$. Therefore, from the second equation,

$$n\psi_{i+1}^i = ny_{i+1}^i + \Big(\sum_{j=1}^{n-1} \psi_{j+1}^j - \psi_n^1\Big) - \Big(\sum_{j=1}^{n-1} y_{j+1}^j - y_n^1\Big).$$

Then, for all $i \in \{1, \ldots, n-1\}$,

$$\psi_{i+1}^i = y_{i+1}^i - \frac{1}{n}\delta = y_{i+1}^i - \frac{1}{n}\Big(\operatorname{proj}\Big(\sum_{j=1}^{n-1} y_{j+1}^j - y_n^1\Big)\Big),$$

$$\psi_n^1 = y_n^1 - \Big(-\frac{1}{n}\Big(\operatorname{proj}\Big(\sum_{j=1}^{n-1} y_{j+1}^j - y_n^1\Big)\Big)\Big),$$

which is indeed (4.5). ■

4.4 An iterative estimation algorithm for arbitrary graphs

We now consider an arbitrary network G_d with set of cycles $\mathcal{L}(G_d)$ and we propose a natural generalization of the optimal estimation algorithm (4.6). Let $\hat{\mathcal{L}} \subseteq \mathcal{L}(G_d)$ be a subset of the cycle set, and let ψ_e denote the estimate of the angle associated to the edge e. For $0 < \kappa \ll 1$, consider the following "cycle-distributed" discrete-time system:

$$\begin{aligned} \psi_e(0) &= y_e, \\ \psi_e(t+1) &= \psi_e(t) - \kappa \sum_{\ell \in \hat{\mathcal{L}}:\, e \in \ell} (\mathbf{1}_\ell \cdot \boldsymbol{e}_e) \operatorname{proj}(\mathbf{1}_\ell \cdot \psi(t)), \end{aligned} \tag{4.7}$$

where $\boldsymbol{e}_i$ is the m-dimensional vector whose i-th entry is 1, and all the other entries are equal to zero. We will often focus our attention to the case where the set of cycles $\hat{\mathcal{L}}$ is a set of fundamental cycles L_f. Such a protocol is "cycle-distributed" in the sense that it requires communications with neighboring cycles only.

Theorem 21 (Exponential convergence) *Consider a planar relative sensing network $\mathcal{N}$ with noisy angle-of-arrival sensing and sensing graph $G_S = (V_S, E_S)$ with n vertices and m edges, and its associated directed graph $G_d = (V_S, E_d)$. Let L_f be a fundamental cycle set for the digraph with associated fundamental cycle matrix C_f. The solution of the discrete-time system* (4.7) *with $\hat{\mathcal{L}} = \mathcal{L}_f$ converges* *exponentially fast with exponential convergence factor* $\rho = (1-\kappa)^2$, *to the set of angles with zero cycle error for* $\kappa < 2/(1+\lambda_{\max}(F))$, *where* $F = C_f C_f^T$, *and* $\lambda_{\max}(F)$ *is the maximum eigenvalue of* F.

Proof: Let $\dim(L_f) = r$. Given the fundamental cycles $\ell_1, \ldots, \ell_r$, define the *cycle error vector* $\boldsymbol{\epsilon}$ at ψ by $\boldsymbol{\epsilon} = [\epsilon_{\ell_1}, \ldots, \epsilon_{\ell_r}]^T$, where ϵ_{ℓ_i} is defined by (4.2), for all $i \in \{1, \ldots, r\}$. With this notation we have

$$\psi(t+1) = \psi(t) - \kappa \sum_{\ell \in \mathcal{L}(G_d)} \mathbf{1}_\ell \epsilon_\ell(t).$$

Then for every loop $\alpha \in \mathcal{L}(G_d)$,

$$\hat{\epsilon}_\alpha(t+1) = \hat{\epsilon}_\alpha(t) - \kappa \sum_{\ell \in \mathcal{L}(G_d)} (\mathbf{1}_\alpha \cdot \mathbf{1}_\ell)\epsilon_\ell(t),$$

where $\hat{\epsilon}_\alpha(t) = (\mathbf{1}_\alpha \cdot \psi(t))$. By choosing a base of independent loops ℓ_i, $i \in \{1, \ldots, r\}$, and an associated fundamental cycle matrix C_f as defined in (2.7), we can write this for all the loops as vector $\hat{\boldsymbol{\epsilon}}$, whose evolution is given by

$$\hat{\boldsymbol{\epsilon}}(t+1) = \hat{\boldsymbol{\epsilon}}(t) - \kappa C_f C_f^T \boldsymbol{\epsilon}(t),$$

and

$$\boldsymbol{\epsilon}(t+1) = \mathrm{proj}((I_r - \kappa F)\boldsymbol{\epsilon}(t)), \tag{4.8}$$

where $F = C_f C_f^T$. Note that F is symmetric positive definite. Consider now the associated linear system

$$\boldsymbol{x}(t+1) = (I_r - \kappa F)\boldsymbol{x}(t),$$

and the Lyapunov function candidate $V(\boldsymbol{x}) = \boldsymbol{x}^T P \boldsymbol{x}$, with $P = I_r$. Next, for $\kappa \in \,]0, 2[$, we define $Q = (2\kappa - \kappa^2) I_r > 0$. Noting that $A = I_r - \kappa F$, we find the values of κ such that the discrete-time Lyapunov inequality $A^T P A - P \leq -Q$ holds. Because F is symmetric positive definite, it can be diagonalized as $F = U \Lambda U^T$, with an orthogonal matrix U and a positive definite diagonal matrix $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_r\}$. Accordingly, the above discrete-time Lyapunov inequality reads

$$U(I_r - \kappa\Lambda)^T (I_r - \kappa\Lambda) U^T - I_r \leq -U(2\kappa - \kappa^2) U^T,$$

which is satisfied if and only if $(1 - \kappa\lambda_i)^2 - 1 + 2\kappa - \kappa^2 < 0$, for $i \in \{1, \ldots, r\}$. In turn, this is satisfied for $\kappa < 2/(1 + \lambda_{\max}(F)) < 2$. Additionally, one can show that $P - Q = \rho I_r$ where $\rho = (1 - \kappa)^2$.

We are now ready to study the nonlinear system (4.8). It is straightforward to verify that the inequality $V(\text{proj}(\boldsymbol{x})) \leq V(\boldsymbol{x})$ holds for all $\boldsymbol{x} \in \mathbb{R}^r$. Therefore,

$$V(\boldsymbol{\epsilon}(t+1)) = V(\text{proj}(A\boldsymbol{\epsilon}(\boldsymbol{t}))) \leq V(A\boldsymbol{\epsilon}(\boldsymbol{t})) = \boldsymbol{\epsilon}(t)^T A^T A \boldsymbol{\epsilon}(t).$$

From the discrete-time Lyapunov inequality and from $P - Q = \rho I_r$, we compute $\boldsymbol{\epsilon}(t)^T A^T A \boldsymbol{\epsilon}(t) \leq \rho V(\boldsymbol{\epsilon}(t))$, so that

$$V(\boldsymbol{\epsilon}(t)) \leq \rho^t V(\boldsymbol{\epsilon}(0)).$$

Given κ in $]0, 2/(1 + \lambda_{\max}(F))[$, we know that $\rho < 1$ and, therefore, the cycle error converges to zero exponentially fast. ■

At this time, it is not known whether the proposed algorithm computes the optimal least-square estimate of the unknown angles. Numerical experiments in Section 4.6 illustrate however its compelling performance in this regard.

4.5 Some remarks on complexity

in order to speed up the exponential convergence factor ρ of algorithm (4.7), it is desirable to maximize κ. To compute the largest possible κ that guarantees convergence, it is natural to ask how to choose C_f, i.e., how to choose the fundamental cycle set in order to minimize the maximum eigenvalue of the matrix $F = C_f C_f^T$. At this time, we only provide the following conservative analysis. One can see that $\text{trace}(F) = \sum_{i\in\{1,\dots,r\}} |\ell_i|$, and, since $\lambda_{\max}(F) < \text{trace}(F)$, exponential convergence of algorithm (4.7) is guaranteed if

$$\kappa < \frac{2}{1 + \sum_{i\in\{1,\dots,r\}} |\ell_i|}.$$

From Theorem 5, we know the fundamental cycle space has rank $m - n + 1$ in a digraph with n nodes and m edges. In the worst-case, it is possible for a digraph to have order n^2 edges and it is certainly true that each cycle has at most order n edges. Therefore, in the worst-case, we can only choose

$$\kappa \in O\Big(\frac{1}{n^3}\Big).$$

Suppose now instead that (i) the graph is planar, so that it has at most $3n-6$ edges, and that (ii) we consider only cycles with bounded length (e.g., in a planar graph that is a triangulation, one can choose a fundamental cycle set with all cycles of length 3). Then we can choose

$$\kappa \in O\left(\frac{1}{n}\right).$$

More generally, how to choose a fundamental cycle set to minimize the sum of cycle lengths is an optimization problem known as the *minimum cycle basis problem*. In the beautiful work of Elkin and coworkers [3], the authors construct a fundamental cycle basis for an unweighted undirected graph of length $O(n^2)$.

4.6 Simulations

We provide some simulations to illustrate the performance of the proposed distributed algorithm considering $\hat{\mathcal{L}}$ as a set of independent cycles. We consider arbitrary network configurations with fixed node positions and varying sensing footprints. Convergence of (4.7) is shown in Figure 4.1, which refers to a complete graph with 10 nodes, using (4.7) and noise variance $\sigma^2 = 0.01$. It is easy to see that the angle estimate for each node converges to a fixed solution. Figure 4.2 shows the cycle error (4.2) for every loop for the same graph. As expected, the cycle error for each loop converges to zero.

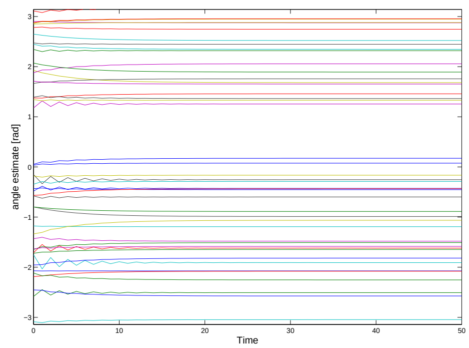


Figure 4.1: Angle estimate.

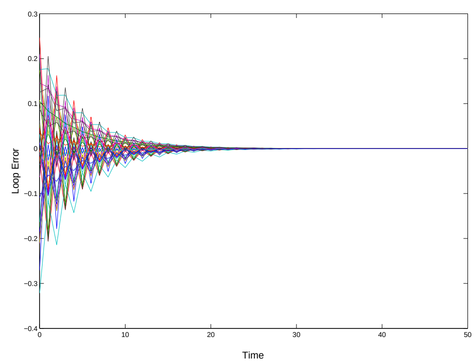


Figure 4.2: Loop error.

Different number of edges in the network lead to different number of independent loops. The empty dots in in Figure 4.3 represent the mean square error of the measurements $\|y-\theta\|^2$, whereas the full dots represent the mean square error of the estimate $\|\psi-\theta\|^2$. As expected, $\|y-\theta\|^2$ stays about constant with the number of independent cycles, whereas $\|\psi-\theta\|^2$ and decreases as the number of independent cycles increases.

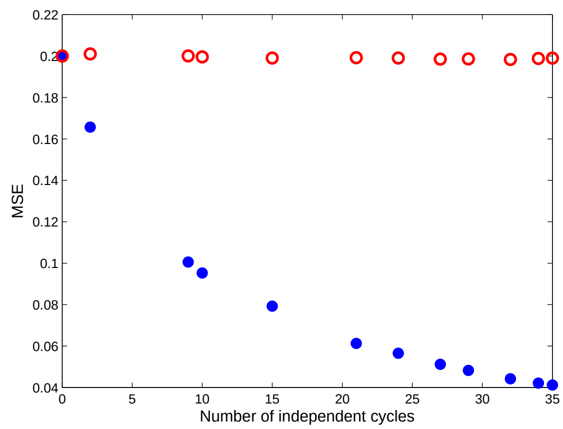


Figure 4.3: Evolution of mean square error (MSE).

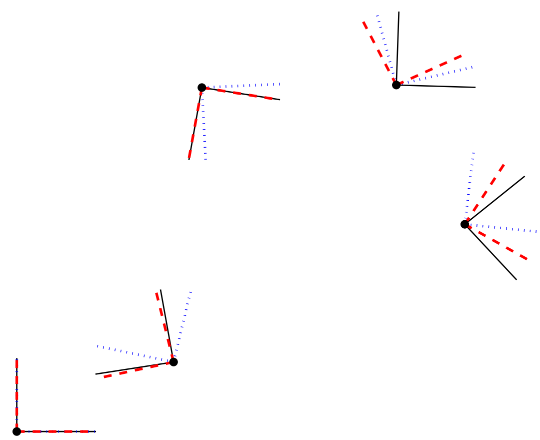


Figure 4.4: Complete network performing self-localization.

Figure 4.4 shows a complete network performing self-localization considering a set of independent cycles. The solid lines are the true frames. The dotted lines are the measured frames (with respect to the pre-specified frame in the bottom left corner). The dashed line are the estimated frame computed by algorithm (4.7). From the figure, it is possible to see that, even thought not every single angle estimate is better than the measurement, the overall estimate MSE is smaller than the measurement MSE.

Chapter 5

Three-dimensional frame localization

Here, we consider a network composed by three nodes in 3-dimensional space with a complete sensing graph, i.e., a triangle. The setup is illustrated in Figure 5.1.

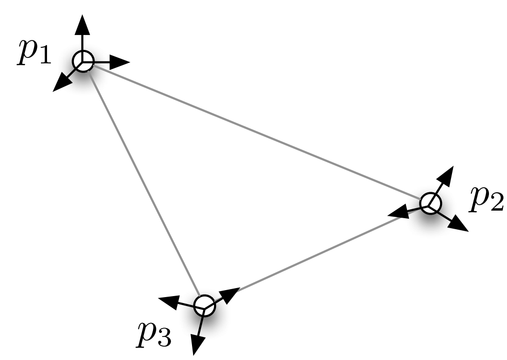


Figure 5.1: Three nodes in $\mathbb{R}^3$ with complete sensing

Before introducing the main result we consider the following proposition, presented in [11]):

Proposition 22 (Inverse non-orthogonal Euler angles) *Given unit-length vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$, with $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ non-coplanar, and a rotation matrix $R \in SO(3)$, consider the system of equations for the unknowns $\{\beta_1, \beta_2, \beta_3\} \in [-\pi, \pi[^3$:*

$$\exp\left(\beta_1\, \widehat{\boldsymbol{r}_1}\right) \exp\left(\beta_2\, \widehat{\boldsymbol{r}_2}\right) \exp\left(\beta_3\, \widehat{\boldsymbol{r}_3}\right) = R. \tag{5.1}$$

Define $a, b, c \in \mathbb{R}$ by

$$\begin{aligned} a = -\boldsymbol{r}_1^T \widehat{\boldsymbol{r}}_2{}^2 \boldsymbol{r}_3, \qquad b = \boldsymbol{r}_1^T \widehat{\boldsymbol{r}}_2 \boldsymbol{r}_3, \\ c = \boldsymbol{r}_1^T R \boldsymbol{r}_3 - \boldsymbol{r}_1^T \boldsymbol{r}_3 - \boldsymbol{r}_1^T \widehat{\boldsymbol{r}}_2^2 \boldsymbol{r}_3. \end{aligned} \tag{5.2}$$

If $c^2 > a^2 + b^2$, then the system of equations (5.1) *has no solutions. Vice versa, if $c^2 \leq a^2 + b^2$, then the system of equations* (5.1) *has two (possibly identical) solutions with β_2 determined by*

$$(\beta_2)_{1,2} = \operatorname{atan}_2(b, a) \pm \operatorname{atan}_2(\sqrt{a^2 + b^2 - c^2}, c), \tag{5.3}$$

and β_1 and β_3 determined as follows. After choosing a particular $i \in \{1, 2\}$ and corresponding $(\beta_2)_i$, β_1 is the unique common solution to

$$\beta_1 = \operatorname{atan}_2(b_{11}, a_{11}) \pm \operatorname{atan}_2(\sqrt{a_{11}^2 + b_{11}^2 - c_{11}^2}, c_{11}),$$

$$\beta_1 = \operatorname{atan}_2(b_{12}, a_{12}) \pm \operatorname{atan}_2(\sqrt{a_{12}^2 + b_{12}^2 - c_{12,i}^2}, c_{12,i}),$$

and, analogously, β_3 is the unique common solution to

$$\beta_3 = \operatorname{atan}_2(b_{31}, a_{31}) \pm \operatorname{atan}_2(\sqrt{a_{31}^2 + b_{31}^2 - c_{31}^2}, c_{31}),$$

$$\beta_3 = \operatorname{atan}_2(b_{32}, a_{32}) \pm \operatorname{atan}_2(\sqrt{a_{32}^2 + b_{32}^2 - c_{32,i}^2}, c_{32,i}),$$

where, for $(j,k) = (1,3)$ *or* $(j,k) = (3,1)$,

$$a_{11} = \boldsymbol{r}_2^T \widehat{\boldsymbol{r}}_1^2 R \boldsymbol{r}_3,$$

$$b_{11} = \boldsymbol{r}_2^T \widehat{\boldsymbol{r}}_1 R \boldsymbol{r}_3,$$

$$c_{11} = \boldsymbol{r}_2^T \boldsymbol{r}_1 \boldsymbol{r}_1^T R \boldsymbol{r}_3 - \boldsymbol{r}_2^T \boldsymbol{r}_3,$$

$$a_{12} = \boldsymbol{r}_3^T \widehat{\boldsymbol{r}}_1^2 R \boldsymbol{r}_3,$$

$$b_{12} = \boldsymbol{r}_3^T \widehat{\boldsymbol{r}}_1 R \boldsymbol{r}_3,$$

$$c_{12,i} = \boldsymbol{r}_3^T \boldsymbol{r}_1 \boldsymbol{r}_1^T R \boldsymbol{r}_3 - \boldsymbol{r}_3^T \exp\left((\beta_2)_i \, \widehat{\boldsymbol{r}_2}\right) \boldsymbol{r}_3,$$

$$a_{31} = \boldsymbol{r}_1^T R \widehat{\boldsymbol{r}}_3^2 \boldsymbol{r}_2,$$

$$b_{31} = \boldsymbol{r}_1^T R \widehat{\boldsymbol{r}}_3 \boldsymbol{r}_2,$$

$$c_{31} = -\boldsymbol{r}_1^T \boldsymbol{r}_2 + \boldsymbol{r}_1^T R \boldsymbol{r}_3 \boldsymbol{r}_3^T \boldsymbol{r}_2,$$

$$a_{32} = \boldsymbol{r}_1^T R \widehat{\boldsymbol{r}}_3^2 \boldsymbol{r}_1,$$

$$b_{32} = \boldsymbol{r}_1^T R \widehat{\boldsymbol{r}}_3 \boldsymbol{r}_1,$$

$$c_{32,i} = -\boldsymbol{r}_1^T \exp\left((\beta_2)_i \, \widehat{\boldsymbol{r}_2}\right) \boldsymbol{r}_1 + \boldsymbol{r}_1 R \boldsymbol{r}_3 \boldsymbol{r}_3^T \boldsymbol{r}_1.$$

We can now consider a network with setup as illustrated in Figure 5.1.

Lemma 23 *Consider a network composed by three nodes in 3-dimensional space with angle of arrival sensing. Pick any one of the three nodes as reference. If the sensing graph is the complete graph and the nodes are in generic positions with generic orientations, then there are precisely two feasible configurations for the three nodes and, therefore, the network is not orientation localizable.*

Proof: The frame localizability problem is described as follows. First, the unknown variables are the three matrices $\boldsymbol{R}^i_{i+1}$, for $i \in \{1, 2, 3\}$, where we write $(i+1)$ to denote $(i+1)$ mod 3. These matrices have each three degrees of freedom, for a total of 9 degrees of freedom. Second, assuming unit-length angle of arrival measurements u^j_i, for $i \neq j \in \{1, \dots, 3\}$, the constraint equations arising from the measurements and from the closed kinematics chain relationships (2.3) are:

$$
\begin{aligned}
u^1_2 &= -\boldsymbol{R}^1_2 u^2_1, \quad u^2_3 = -\boldsymbol{R}^2_3 u^2_3, \\
u^3_1 &= -\boldsymbol{R}^3_1 u^1_3, \quad I_3 = \boldsymbol{R}^1_2 \boldsymbol{R}^2_3 \boldsymbol{R}^3_1.
\end{aligned}
\tag{5.4}
$$

Given these measurements and according to Lemma 10, we compute the three rotation matrices $\boldsymbol{H}^i_{i+1}$ and we know that there exist three angles $\beta^i_{i+1} \in [-\pi, \pi[$ such that

$$
\boldsymbol{R}^i_{i+1} = \exp\left(\beta^i_{i+1}\, \widehat{u^i_{i+1}}\right) \boldsymbol{H}^i_{i+1}.
$$

Thus, equations (5.4) admit a unique solution $\{\boldsymbol{R}^1_2, \boldsymbol{R}^2_3, \boldsymbol{R}^3_1\}$ precisely when there exist unique angles $\beta^i_{i+1} \in [-\pi, \pi[$ such that

$$
I_3 = \exp\left(\beta^1_2\, \widehat{u^1_2}\right) \boldsymbol{H}^1_2 \exp\left(\beta^2_3\, \widehat{u^2_3}\right) \boldsymbol{H}^2_3 \exp\left(\beta^3_1\, \widehat{u^3_1}\right) \boldsymbol{H}^3_1.
$$

Applying eq. (2.6) repeatedly, we compute

$$
\begin{aligned}
&(\boldsymbol{H}^3_1)^T (\boldsymbol{H}^2_3)^T (\boldsymbol{H}^1_2)^T \\
&\qquad = \exp\left(\beta^1_2\, \widehat{u^1_2}\right) \exp\left(\beta^2_3\, \widehat{\boldsymbol{H}^1_2 u^2_3}\right) \exp\left(\beta^3_1\, \widehat{\boldsymbol{H}^1_2 \boldsymbol{H}^2_3 u^3_1}\right).
\end{aligned}
$$

We now rely on the assumption of generic positions and orientations to infer that $u_2^1 \nparallel \boldsymbol{H}_2^1 u_3^2$ and that $u_3^2 \nparallel \boldsymbol{H}_3^2 u_1^3$. Also the left hand side term $(\boldsymbol{H}_1^3)^T(\boldsymbol{H}_3^2)^T(\boldsymbol{H}_2^1)^T$ is generic. From the formulation of the problem, we know that at least a real solution for (5.4) exists. In particular, Proposition 22 tells us that such equations admit two solutions. Therefore, the network is not orientation localizable. ■

Now, let us consider a network composed by four nodes, whose connected sensing graph consists of two 3-nodes loops, with an edge in common. For example, consider the setup in Figure 5.2 (b).

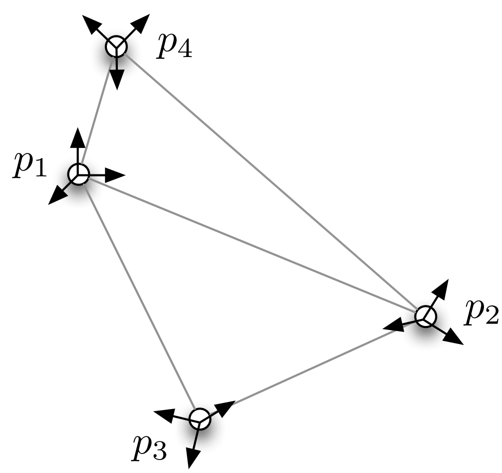


Figure 5.2: Four nodes in $\mathbb{R}^3$

Lemma 24 *Consider a network composed by four nodes in the 3-dimensional space with angle of arrival sensing. If the sensing graph is connected and there are at least two independent loops, then the network is orientation localizable.*

Proof: As in the three-nodes case, the set of equations (5.4) is extended by

$$\begin{aligned} u_2^1 &= -\boldsymbol{R}_2^1 u_1^2, \quad u_4^2 = -\boldsymbol{R}_4^2 u_2^4, \\ u_1^4 &= -\boldsymbol{R}_1^4 u_4^1, \quad I_3 = \boldsymbol{R}_2^1 \boldsymbol{R}_4^2 \boldsymbol{R}_1^4. \end{aligned} \tag{5.5}$$

As equations (5.4), also equations (5.5) admit two solutions, i.e. two different sets of values for β_i^j. It is straightforward to show that only one of the two solutions for β_2^1 of (5.5) matches with one of the solutions for β_2^1 in (5.4). Therefore, all angles β_i^j are uniquely determined, and the network is orientation localizable. ■

Lemma 25 *Necessary condition for a network in the* 3*-dimensional space with angle of arrival sensing to be orientation localizable is to have at least* 4 *nodes.*

Proof: If the network has less than 3 nodes, there are no loops, so it is not orientation localizable. Assume now the network has 3 nodes. If the sensing graph is not complete, the network has no loops and therefore is not orientation localizable. If the sensing graph is complete, according to Lemma 23, the network is not orientation localizable. ■

Lemma 26 *Any network in the* 3*-dimensional space with complete sensing graph is orientation localizable if it has at least* 4 *nodes.*

Proof: In a complete network every loop belongs to a three-edges loop. Therefore, from what has been shown before, the network is orientation localizable. ■

Definition 27 (3-dimensional triangulation) *Consider a connected network composed by nodes in 3-dimensional space. We call such network a* 3-dimensional triangulation *if there exists a basis for the cycle space such that each cycle in the basis has* 3 *nodes and it shares at least one edge with another cycle of the basis.*

Lemma 28 *Consider a network with* $n \geq 4$ *nodes in* 3*-dimensional space, and assume its angle of arrival sensing is a 3-dimensional triangulation. Then, the network is orientation localizable.*

Proof: From the definition of 3-dimensional triangulation, we know that the network can be divided into a basis for the cycle space such that each cycle in the basis has 3 nodes and it shares at least one edge with another cycle of the basis. Then, for each couple of loops we can use the result from Lemma 24. Therefore, the network is orientation localizable. ■

Chapter 6

Conclusions

This thesis introduces the frame localization problem in a connected network. For the planar orientation localization problem with angle-of-arrival (bearing) sensors, we developed an algorithm that reduces the effect of noise. Our algorithm computes the correct least-square estimate for ring networks in one step. Our algorithm is proved to converge exponentially fast and is validated through some simulations. For the three-dimensional case, we explore necessary and sufficient conditions for a network to be orientation localizable when no noise is present. We are currently extending the work in several directions. First, we want to improve the efficiency of the orientation localization algorithm and either show its least-square optimality or modify it to achieve least-square optimality. Second, we plan to address the problem of position localization, defined earlier in this paper. Third and finally, we aim to formulate conditions for frame localization in three dimensions in the noisy case.

Bibliography

[1] J. Aspnes, T. Eren, D. K. Goldenberg, A. S. Morse, W. Whiteley, Y. R. Yang, B. D. O. Anderson, and P. Belhumeur. A theory of network localization. *IEEE Transactions on Mobile Computing*, 5(12):1663–1678, 2006.

[2] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, New York, 2 edition, 2005.

[3] M. Elkin, C. Liebchen, and R. Rizzi. New length bounds for cycle bases. *Information Processing Letters*, 104(5):186–193, 2007.

[4] L. R. Foulds. *Graph Theory Applications*. Universitext. Springer, New York, 1995.

[5] S. Kaczmarz. Approximate solution of systems of linear equations. *International Journal of Control*, 57(6):1269–1271, 1993. (english translation).

[6] R. M. Murray, Z. X. Li, and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, Boca Raton, FL, 1994.

[7] G. Piovan, I. Shames, B. Fidan, F. Bullo, and B. D. O. Anderson. On frame and orientation localization for relative sensing networks. *IEEE Conf. on Decision and Control*, pages 2326–2331, 2008.

[8] B. Servatius and W. Whiteley. Constraining plane configurations in computer-aided design: Combinatorics of directions and lengths. *SIAM Journal on Discrete Mathematics*, 12:136–153, 1999.

[9] N. Trawny, X. S. Zhou, K. X. Zhou, and S. I. Roumeliotis. 3D relative pose estimation from distance-only measurements. In *IEEE/RSJ Int. Conf. on Intelligent Robots & Systems*, pages 1071–1078, San Diego, CA, Oct. 2007.

[10] R. Tron and R.Vidal. Distributed image-based 3-d localization of camera sensor networks. *IEEE Conf. on Decision and Control*, 48, 2009.

[11] J. Wittenburg and L. Lilov. Decomposition of a finite rotation into three rotations about given axes. *Multibody System Dynamics*, 9(4):353–375, 2003.

[12] X. S. Zhou and S. I. Roumeliotis. Robot-to-robot relative pose estimation from range measurements. *IEEE Transactions on Robotics*, 24(6):1379–1393, 2008.