Graph-Theoretic Small-Gain Theorems for Metzler Matrices and Monotone Systems

Xiaoming Duan, Saber Jafarpour, *Member, IEEE*, Francesco Bullo, *Fellow, IEEE*

Abstract—This paper studies the graph-theoretic conditions for stability of positive monotone systems. Using concepts from the input-to-state stability and network small-gain theory, we first establish necessary and sufficient conditions for the stability of linear positive systems described by Metzler matrices. Specifically, we define and compute two forms of input-to-state stability gains for Metzler systems, namely max-interconnection gains and sum-interconnection gains. Then, based on the max-interconnection gains, we show that the cyclic small-gain theorem becomes necessary and sufficient for the stability of Metzler systems; based on the sum-interconnection gains, we obtain novel graph-theoretic conditions for the stability of Metzler systems. All these conditions highlight the role of cycles in the interconnection graph and unveil how the structural properties of the graph affect stability. Finally, we extend our results to the nonlinear monotone system and obtain similar sufficient conditions for global asymptotic stability.

I. INTRODUCTION

Problem description and motivation: Much attention in recent years has been focused on multi-agent systems, but the majority of efforts has been devoted to averaging dynamics and consensus behavior. Much less attention has been drawn to dynamical flow systems, modeled as monotone or cooperative systems [16], [25]. Notable exceptions are a collection of recent papers motivated by applications to traffic and biological systems [2], [7] as well as the long-standing interest in positive systems [15], [22]. Despite these remarkable recent works, many open questions remain.

This paper focuses on a key foundational question for linear monotone systems, i.e., positive systems modeled by Metzler matrices, and on its application to the study of nonlinear monotone systems: what are graph-theoretical conditions for the Hurwitzness of a Metzler matrix? While a graph theoretical treatment is available for a subclass of Meztler matrices known as "compartmental matrices" [31], a general treatment is lacking. This is in stark contrast with the comprehensive understanding of the graph theoretical conditions guaranteeing convergence to consensus for row-stochastic matrices in averaging systems. Related to this open question is the work in [3]. The graph-theoretic conditions are particularly useful because they allow us to analyze stability based on the structural properties of the interconnection network given the existence of perturbations or uncertainties on the parameters.

For nonlinear monotone systems, much recent progress is documented in [8], [9], where a basic fundamental connection is built between monotone systems and contractive systems. A notable gap, however, remains, in explaining the relationship between the treatment of monotone contractive systems and the stability theory of network small gain developed in [12], [18].

In summary, we aim to develop an algebraic graph theory for monotone dynamical systems, starting with the linear case of Metzler matrices and continuing with the nonlinear setting and its connections with network small-gain theorems.

Literature review: Monotone dynamical systems appear naturally in numerous applications and have many appealing properties. The mathematical theory of nonlinear monotone systems has been vastly studied in dynamical system literature [16], [24], [25]. In control community, the notion of monotonicity has been extended to systems with inputs and outputs, and properties of the interconnected monotone systems have been studied [2]. It is well known that linear monotone systems (also referred to as linear positive systems) are described by Metzler matrices. Conditions for stability of Metzler matrices have been studied extensively in the literature. Narendra and Shorten, *et al.* established an iterative method based on the Schur complement to check the Hurwitzness of Metzler matrices in [21], [30]. A graph-theoretic characterization for diagonal stability of matrices whose underlying digraph is a cactus graph was proposed in [3]. Briat studied the sign stability of Metzler matrices and block Metzler matrices in [5]. Blanchini *et al.* studied switched Metzler systems and Hurwitz convex combinations in [4]. Stability of switched Metzler systems has also been studied in [20], where the authors provided guarantees for robustness with respect to delays. In [22], scalable methods for analysis and control of large-scale linear monotone systems have been studied. The admissibility, stability, and persistence of interconnected positive heterogenous systems have been studied in [14]. For nonlinear monotone systems, using novel connections to the contraction theory, Coogan established sufficient conditions for global stability of monotone systems [8], [9]. We refer the interested readers to [15] for a detailed study of linear positive systems and to the survey paper [28] for theoretical results and applications of interconnected monotone systems.

Small-gain theorems are arguably one of the fundamental results for stability of interconnected systems. Started with

the works by Zames [32], the early classical studies on small-gain theorems mostly focused on stability analysis using linear gains [23]. Introduction of the notion of input-to-state stability (ISS) in the seminal paper [27] triggered a paradigm shift in the study of small-gain theorems. More recent works on small-gain theorems focused on the input-to-state framework and they provided results in terms of nonlinear notions of input-to-state gains [12], [17].

Contributions: In this paper, we study the graph-theoretic stability conditions for Metzler matrices. By using concepts from the small-gain theorems for interconnected systems, we obtain necessary and sufficient conditions for Hurwitzness of Metzler matrices in terms of the input-to-state gains, and we also extend our results to the nonlinear monotone systems.

(i) We compute and characterize two types of input-to-state stability gains for linear Metzler systems, namely max-interconnection gains and sum-interconnection.

(ii) Using the max-interconnection and the sum-interconnection gains, we obtain two graph-theoretic characterizations for Hurwitzness of Metzler matrices. Our conditions highlight the role of cycles and cycle gains and provide valuable insights for connections between the network structure and network functions. In particular, our characterizations of Hurwitzness of Metzler matrices using the max-interconnection gains coincide with the well-known cyclic small gain theorem [18, Theorem 3.1]; based on the sum-interconnection gains, in addition to necessary and sufficient cycle gain conditions that depend the cycle structure of the interconnection graph, we also show that all cycle gains being less than 1 is a necessary condition and the sum of cycle gains being less than 1 is a sufficient condition.

(iii) As an independent contribution, we obtain graph-theoretic interpretations of Schur complements for Metzler matrices.

(iv) We extend our stability analysis using max-interconnection and sum-interconnection gains to nonlinear monotone systems. As a result, we provide two equivalent sufficient conditions for global stability of monotone nonlinear systems.

Paper organization: We review the known stability results for Metzler matrices in Section II. The input-to-state stability and two forms of ISS gains are introduced in Section III. We characterize different ISS gains for Metzler systems in Section IV. The graph-theoretic conditions for Hurwitzness of Metzler matrices are presented in Section V. We extend the conditions to nonlinear monotone systems in Section VI. We collect new results on Kron reduction of asymmetric graphs in VII. A few additional concepts and proofs are included in Section VIII. We conclude the paper in Section IX.

II. REVIEW OF METZLER MATRICES

A. Notation and preliminaries

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_{\geq 0}$ be the set of nonnegative real numbers. For a vector $v \in \mathbb{R}^n$, its Euclidean norm is denoted by $|v|$. Particularly, if $v \in \mathbb{R}$, then $|v|$ is the absolute value of v. For a finite set S, $|S|$ is the cardinality. For $t \geq 0$ and a time-varying vector signal $x : [0, t] \mapsto \mathbb{R}^n$, we define the norm

$$\|x\|_{[0,t]} = \sup_{s \in [0,t]} |x(s)|.$$

Moreover, for $x : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$, $\|x\|_\infty = \sup_{s \geq 0} |x(s)|$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$; it is a class $\mathcal{K}_\infty$ function if it is a class $\mathcal{K}$ function and $\lim_{s \to \infty} \alpha(s) = \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function if $\beta(s, t)$ is a class $\mathcal{K}$ function of s for fixed t, and a decreasing function of t with $\lim_{t \to \infty} \beta(s, t) = 0$ for fixed s.

For a matrix $A \in \mathbb{R}^{n \times n}$, its associated graph $\mathcal{G}(A) = (V, \mathcal{E}, A)$ is a weighted digraph defined as follows: $V = \{1, \ldots, n\}$ is the set of nodes, and $\mathcal{E} = \{(j, i) \mid i, j \in V, a_{ij} \neq 0\}$ is the set of edges. For $i \in V$, the neighbor set of node i is defined by $\mathcal{N}_i = \{j \in V \mid (j, i) \in \mathcal{E}\}$. A matrix $A \in \mathbb{R}^{n \times n}$ is irreducible if its associated digraph $\mathcal{G}(A)$ is strongly connected.

In a weighted digraph $\mathcal{G} = (V, \mathcal{E}, W)$, a simple cycle c in $\mathcal{G}$ is a directed path that starts and ends at the same node and has no repetitions other than the starting and ending nodes. For two simple cycles c_1 and c_2 in $\mathcal{G}$, c_1 and c_2 intersect if they share at least one common node, i.e., $c_1 \cap c_2 \neq \emptyset$; c_1 is a subset of c_2 if all the nodes on c_1 are also on c_2. Self loops are not considered as simple cycles in this paper.

For a matrix $A \in \mathbb{R}^{n \times n}$, the leading principal submatrices of A are given by A_I, where $I = \{1, \ldots, i\}$ is the set of indices for all $i \in \{1, \ldots, n\}$. In particular, when $I = \{1, \ldots, n\}$, we have $A_I = A$. A matrix $M \in \mathbb{R}^{n \times n}$ is Metzler if all its off-diagonal elements are nonnegative. A matrix $C \in \mathbb{R}^{n \times n}$ is compartmental if it is Metzler and has nonpositive column sums.

The following lemma will be used later in the paper.

Lemma 1 (Bounding sum by maximum). *Let $\{x_1, \ldots, x_n\}$ and $\{\alpha_1, \ldots, \alpha_n\}$ be a set of real and positive real numbers respectively. If $\sum_{i=1}^n \frac{1}{\alpha_i} \leq 1$, then*

$$\sum_{i=1}^n x_i \leq \max_{i \in \{1, \ldots, n\}} \{\alpha_i x_i\}.$$

Proof. Let $s \in \{1, \ldots, n\}$ satisfy $\alpha_i x_i \leq \alpha_s x_s$ for all $i \in \{1, \ldots, n\}$. Then

$$\sum_{i=1}^n x_i \leq \sum_{i=1}^n \frac{\alpha_s x_s}{\alpha_i} \leq \alpha_s x_s = \max_{i \in \{1, \ldots, n\}} \{\alpha_i x_i\}.$$

$\square$

B. Algebraic conditions for Hurwitzness of Metzler matrices

We collect a few well-known equivalent conditions for the Hurwitzness of Metzler matrices in the following Theorem.

Theorem 2 (Properties of Hurwitz Metzler matrices [6, Theorem 14.17] [15, Theorem 13])**.** *Let $M \in \mathbb{R}^{n \times n}$ be a Metzler matrix, then the following statements are equivalent:*

(i) *M is Hurwitz;*
(ii) *M is invertible and $-M^{-1} \geq 0$;*
(iii) *all leading principal minors of $-M$ are positive;*
(iv) *there exists $\xi \in \mathbb{R}^n$ such that $\xi > \mathbb{0}_n$ and $M\xi < \mathbb{0}_n$;*
(v) *there exists $\eta \in \mathbb{R}^n$ such that $\eta > \mathbb{0}_n$ and $\eta^\top M < \mathbb{0}_n$;*
(vi) *there exists a diagonal matrix $P \succ 0$ such that $M^\top P + PM \prec 0$.*

Remark 3. *(i) To the best of our knowledge, the equivalence of parts (i) and (iii) in Theorem 2 has not been fully exploited in the literature, and we build one of our main results based on this condition.*

(ii) If the Metzler matrices are symmetric, then the necessary and sufficient condition in Theorem 2(iii) is exactly the Sylvester's criterion for negative definiteness of general symmetric matrices.

(iii) The equivalence of parts (i) and (vi) in Theorem 2 implies that for Metzler matrices, the Hurwitzness and diagonal stability are equivalent.

Based on the Schur complement, Narendra *et al.* propose an iterative method to verify the Hurwitzness of a Metzler matrix [21]. Partition a Metzler matrix $M \in \mathbb{R}^{n \times n}$ as follows

$$M = \begin{bmatrix} M_{n-1} & b_{n-1} \\ c_{n-1}^\top & d_{n-1} \end{bmatrix}$$

where d_{n-1} is a scalar. The Schur complement of M with respect to d_{n-1} is given by $M[n-1] = M_{n-1} - \frac{b_{n-1}c_{n-1}^\top}{d_{n-1}}$. For $k \in \{1, \ldots, n-1\}$, define $M[k]$ iteratively as the Schur complement of $M[k+1]$ with respect to d_k, where $M[n] = M$, then the following statement holds.

Theorem 4 (Necessary and sufficient condition based on the Schur complement [21])**.** *A Metzler matrix $M \in \mathbb{R}^{n \times n}$ is Hurwitz if and only if for all $k \in \{1, \ldots, n\}$, all the diagonal elements of $M[k]$ are negative.*

By Theorem 4, we have the following necessary condition.

Corollary 5 (Negativity of diagonal elements)**.** *If a Metzler matrix $M \in \mathbb{R}^{n \times n}$ is Hurwitz, then all the diagonal elements of M are negative.*

III. ISS AND INTERCONNECTED SYSTEMS

We review the concepts of input-to-state stability and introduce the gain functions in two different forms for interconnected input-to-state stable systems [12], [18].

A. Input-to-state stability

Consider the system

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in \mathbb{R}^N$ is the state, $u \in \mathbb{R}^m$ is the input, and $f : \mathbb{R}^N \times \mathbb{R}^m \mapsto \mathbb{R}^N$ is a locally Lipschitz function and satisfies $f(\mathbb{0}_n, \mathbb{0}_m) = \mathbb{0}_n$. Then, we have the following definition for input-to-state stability.

Definition 6 (Input-to-state stability [27, Definition 2.1])**.** *System (1) is input-to-state stable if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(0) = x_0$ and any measurable and locally essentially bounded input u, the solution $x(t)$ satisfies, for all $t \geq 0$,*

$$|x(t)| \leq \max\{\beta(|x_0|, t), \gamma(\|u\|_\infty)\}. \tag{2}$$

The class $\mathcal{K}$ function γ in (2) is the *ISS gain* of the system.

Remark 7 (ISS Lyapunov function)**.** *In order to check ISS using Definition 6, we need to find an estimate for the trajectory of the system, which is computationally hard in general, if not impossible. However, one can show that ISS is equivalent to the existence of an ISS Lyapunov function. We refer the interested readers to [29, Theorem 1].*

B. Interconnection, ISS gains, and cyclic small-gain theorem

In this section, we study input-to-state stability for networked interconnected systems. Suppose the interaction between subsystems is described by a directed graph $G = (V, \mathcal{E})$, where $V = \{1, \ldots, n\}$ is the set of nodes and for all $i, j \in V$ and $i \neq j$, $(j, i) \in \mathcal{E}$ if x_j is an input to subsystem i. We consider a network of n interconnected dynamical systems with the interconnection graph G:

$$\dot{x}_i = f_i(x_i, x_{\mathcal{N}_i}, u_i), \quad \text{for all } i \in \{1, \ldots, n\}, \tag{3}$$

where $x_i \in \mathbb{R}^{n_i}$ and $x_{\mathcal{N}_i} = \begin{bmatrix} x_{i_1}, \ldots, x_{i_{k_i}} \end{bmatrix}^\top \in \mathbb{R}^{n_{\mathcal{N}_i}}$ with $\mathcal{N}_i = \{i_1, \ldots, i_{k_i}\}$ and $n_{\mathcal{N}_i} = \sum_{j=1}^{k_i} n_{i_j}$. For every $i \in V$, the function $f_i : \mathbb{R}^{n_i + n_{\mathcal{N}_i} + m_i} \to \mathbb{R}^{n_i}$ is a locally Lipschitz function satisfying $f_i(\mathbb{0}_{n_i}, \mathbb{0}_{n_{\mathcal{N}_i}}, \mathbb{0}_{m_i}) = \mathbb{0}_{n_i}$. For the interconnected system (3), it is desirable to study ISS of the interconnection using the ISS of each subsystem. We first introduce componentwise ISS for network systems.

Definition 8 (Componentwise ISS)**.** *An interconnected system (3) is componentwise ISS if every subsystem i is ISS for the input $\begin{bmatrix} x_{\mathcal{N}_i} & u_i \end{bmatrix}^\top \in \mathbb{R}^{n_{\mathcal{N}_i} + m_i}$.*

In other words, an interconnected network system is componentwise ISS if each subsystem, separated from the whole system, is ISS. In general, componentwise ISS does not guarantee ISS of the whole interconnected system, and conditions on the composition of ISS gains of the subsystems is required to ensure ISS of the whole system. In the following, we introduce two notions of gains.

Definition 9 (Max-interconnection ISS gains)**.** *Consider the interconnected system (3). The family of functions $\{\Psi_{ij}\} \in \mathcal{K} \cup \{0\}$ is a max-interconnection gain if, for every $i \in$*

$\{1, \ldots, n\}$, there exists $\beta_i \in \mathcal{KL}$ and $\Psi_i \in \mathcal{K}$ such that for any initial state $x(0) = x_0$, and any measurable and locally essentially bounded inputs u_i, the solution $x_i(t)$ satisfies, for all $t \geq 0$,

$$|x_i(t)| \leq \max_{j \in \mathcal{N}_i}\{\beta_i(|x_i(0)|, t), \Psi_{ij}(\|x_j\|_{[0,t]}), \Psi_i(\|u_i\|_\infty)\}.$$

Definition 10 (Sum-interconnection ISS gains)**.** *Consider the interconnected system* (3)*. The family of functions* $\{\Gamma_{ij}\} \in \mathcal{K} \cup \{0\}$ *is a sum-interconnection gain if, for every* $i \in \{1, \ldots, n\}$*, there exists* $\beta_i \in \mathcal{KL}$ *and* $\Gamma_i \in \mathcal{K}$ *such that for any initial state* $x(0) = x_0$*, and any measurable and locally essentially bounded inputs* u_i*, the solution* $x_i(t)$ *satisfies, for all* $t \geq 0$*,*

$$|x_i(t)| \leq \beta_i(|x_i(0)|, t) + \sum_{j \in \mathcal{N}_i} \Gamma_{ij}(\|x_j\|_{[0,t]}) + \Gamma_i(\|u_i\|_\infty).$$

The following theorem provides conditions on a set of max-interconnection ISS gains which guarantee ISS of the interconnected system (3).

Theorem 11 (Cyclic small-gain theorem [18, Theorem 3.2])**.** *Consider an interconnected system* (3) *with each subsystem* i *being componentwise ISS and with a family of max-interconnected gains* $\{\Psi_{ij}\}$*. The interconnected system* (3) *is ISS with* x *as the state and* u *as the input if, for every simple cycle* $c = (i_1, i_2, \ldots, i_k, i_1)$ *in the interconnection graph* G *and every* $s > 0$*,*

$$\Psi_{i_2 i_1} \circ \Psi_{i_3 i_2} \circ \cdots \circ \Psi_{i_1 i_k}(s) < s, \quad (4)$$

where $\circ$ *is the function composition.*

IV. ISS FOR METZLER SYSTEMS

In this section, we characterize the ISS gains for Metzler systems. Consider the continuous-time linear system

$$\dot{x} = Mx + u, \quad (5)$$

where $M \in \mathbb{R}^{n \times n}$ is a Metzler matrix and $u \in \mathbb{R}^n_{\geq 0}$ is the control input. The Metzler system (5) can be viewed as a network of n interconnected scalar systems, where the interconnection is characterized by the digraph $\mathcal{G}(M)$. More specifically, one can write the Metzler system (5) in the interconnection form (3) as,

$$\dot{x}_i = m_{ii}x_i + \sum_{j \in \mathcal{N}_i} m_{ij}x_j + u_i, \quad \text{for all } i \in \{1, \ldots, n\}. \quad (6)$$

In the following, we derive the sum-interconnection and max-interconnection ISS gains for the Metzler system (5).

Theorem 12 (ISS Metzler systems)**.** *The Metzler system* (5) *with interconnection digraph* $\mathcal{G}(M) = (V, \mathcal{E})$

(i) is componentwise ISS if and only if

$$m_{ii} < 0, \quad \text{for all } i \in \{1, \ldots, n\}; \quad$$

(ii) has sum-interconnection gains $\{s \mapsto \Gamma_{ij}(s) = \gamma_{ij}s\}$*, if it is componentwise ISS and the set of scalars* $\{\gamma_{ij}\}$ *satisfies* $\gamma_{ij} = 0$ *for all* $j \notin \mathcal{N}_i$ *and*

$$\frac{m_{ij}}{-m_{ii}} \leq \gamma_{ij}, \quad \text{for all } i \in \{1, \ldots, n\}, j \in \mathcal{N}_i; \quad (7)$$

(iii) has max-interconnection gains $\{s \mapsto \Psi_{ij}(s) = \psi_{ij}s\}$*, if it is componentwise ISS and the set of scalars* $\{\psi_{ij}\}$ *satisfies* $\psi_{ij} = 0$ *for all* $j \notin \mathcal{N}_i$ *and*

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}}\right) \psi_{ij}^{-1} < 1, \quad \text{for all } i \in \{1, \ldots, n\}; \quad (8)$$

(iv) is ISS if and only if M *is Hurwitz.*

Proof. Regarding part (i), since the dynamics of the ith subsystem given by (6) is linear, it is ISS if and only if $m_{ii} < 0$ [18, Theorem 1.3]. Therefore, the Metzler system (5) is componentwise ISS if and only if, for every $i \in \{1, \ldots, n\}$, we have $m_{ii} < 0$.

Regarding part (ii), the state trajectory $x_i(t)$ satisfies

$$x_i(t) = e^{m_{ii}t}x_i(0) + \sum_{j \in \mathcal{N}_i} m_{ij} \int_0^t e^{m_{ii}(t-\tau)}x_j(\tau)d\tau + \int_0^t e^{m_{ii}(t-\tau)}u_i(\tau)d\tau,$$

which implies

$$|x_i(t)| \leq e^{m_{ii}t}|x_i(0)| + \sum_{j \in \mathcal{N}_i} m_{ij} \int_0^t |e^{m_{ii}(t-\tau)}x_j(\tau)|d\tau$$
$$+ \int_0^t |e^{m_{ii}(t-\tau)}u_i(\tau)|d\tau$$
$$\leq e^{m_{ii}t}|x_i(0)| + \sum_{j \in \mathcal{N}_i} m_{ij}\|x_j\|_{[0,t]} \int_0^t e^{m_{ii}(t-\tau)}d\tau$$
$$+ \|u_i\|_\infty \int_0^t e^{m_{ii}(t-\tau)}d\tau$$
$$\leq e^{m_{ii}t}|x_i(0)| + \sum_{j \in \mathcal{N}_i} \frac{m_{ij}}{-m_{ii}}\|x_j\|_{[0,t]} + \frac{1}{-m_{ii}}\|u_i\|_\infty. \quad (9)$$

Therefore, the Metzler system (5) has a sum-interconnection ISS gain $\{s \mapsto \Gamma_{ij}(s) = \gamma_{ij}(s)\}$ if we have $\frac{m_{ij}}{-m_{ii}} \leq \gamma_{ij}$.

Regarding part (iii), by Lemma 1 and (9), we have

$$|x_i(t)| \leq \max\{\alpha_1 e^{m_{ii}t}|x_i(0)|,$$
$$\alpha_2 \sum_{j \in \mathcal{N}_i} \frac{m_{ij}}{-m_{ii}}\|x_j\|_{[0,t]}, \alpha_3 \frac{1}{-m_{ii}}\|u_i\|_\infty\}, \quad (10)$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\sum_{i=1}^3 \frac{1}{\alpha_i} \leq 1$. If (8) holds, then by Lemma 1, we have

$$\sum_{j \in \mathcal{N}_i} \frac{m_{ij}}{-m_{ii}}\|x_j\|_{[0,t]} < \max_j\{\psi_{ij}\|x_j\|_{[0,t]}\}.$$

Therefore, we can pick α_2 properly such that

$$\sum_{j \in \mathcal{N}_i} \frac{m_{ij}}{-m_{ii}}\|x_j\|_{[0,t]} \leq \frac{1}{\alpha_2} \max_j\{\psi_{ij}\|x_j\|_{[0,t]}\},$$

which combined with (10) imply that $\{\psi_{ij}\}$ are max-interconnection gains.

Regarding part (iv), this is a straightforward application of [18, Theorem 1.3]. $\square$

V. Graph-theoretic conditions for Hurwitzness of Metzler matrices

In this section, we first show that we only need to consider irreducible Metzler matrices. Then, we show that different ISS gains result in different graph-theoretic conditions for the stability of Metzler systems. In particular, if we use the max-interconnection ISS gains, then the cycle condition (4) in Theorem 11 is a necessary and sufficient condition for the stability of Metzler systems. On the other hand, if we use the sum-interconnection ISS gains, then we can obtain new necessary and sufficient graph-theoretic conditions.

A. Metzler matrices with reducible graphs

The following theorem allows us to restrict our attention to irreducible Metzler matrices.

Theorem 13 (Hurwitzness and strongly connected components). *For a Metzler matrix $M \in \mathbb{R}^{n \times n}$, M is Hurwitz if and only if all the connected components in the condensation of $\mathcal{G}(M)$ are Hurwitz.*

Proof. If M is irreducible, then the statement holds trivially since there is only one strongly connected component in the condensation of $\mathcal{G}(M)$, which is $\mathcal{G}(M)$ itself.

If M is reducible, then there exists a permutation matrix such that M can be brought into block upper triangular form where each block on the diagonal represents a strongly connected component. Therefore, M is Hurwitz if and only if all its strongly connected components are Hurwitz. $\square$

If $\mathcal{G}(M)$ is acyclic, then we have the following corollary.

Corollary 14 (Necessary and sufficient condition for acyclic graphs [5, Theorem 3.4]). *For a Metzler matrix $M \in \mathbb{R}^{n \times n}$ whose associated digraph $\mathcal{G}(M)$ is acyclic, M is Hurwitz if and only if all the diagonal elements of M are negative.*

Hereafter, we focus on irreducible Metzler matrices with negative diagonal elements.

B. Cycle gains and the case of a simple cycle

In this subsection, we define the sum-cycle gains and max-cycle gains for Metzler matrices, and we emphasize the importance of cycles through the case of a simple cycle.

Definition 15 (Cycle gains for Metzler matrices). *Let $M \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix with negative diagonal elements and $c = (i_1, i_2, \ldots, i_k, i_1)$ be a simple cycle in $\mathcal{G}(M)$. Then*

(i) a max-cycle gain of c is

$$\psi_c = (\psi_{i_2 i_1})(\psi_{i_3 i_2}) \ldots (\psi_{i_1 i_k}), \quad (11)$$

where the scalars $\{\psi_{ij}\}$ satisfy (8); and

(ii) the sum-cycle gain of c is

$$\gamma_c = \left(\frac{m_{i_2 i_1}}{-m_{i_2 i_2}} \right) \left(\frac{m_{i_3 i_2}}{-m_{i_3 i_3}} \right) \ldots \left(\frac{m_{i_1 i_k}}{-m_{i_1 i_1}} \right). \quad (12)$$

Remark 16 (Uniqueness of cycle gains). *The sum-cycle gains in (12) are uniquely defined for simple cycles in $\mathcal{G}(M)$*

because we pick specific sum-interconnection gains in (7). However, the max-cycle gains in (11) are not unique in general. For every solution of (8), one can compute a set of max-cycle gains for simple cycles.

If the irreducible Metzler matrix $M \in \mathbb{R}^{n \times n}$ with negative diagonal elements has the associated digraph $\mathcal{G}(M)$ being a simple cycle, i,e, M has the following form,

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 & \cdots & 0 \\ 0 & m_{22} & m_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n1} & 0 & \cdots & 0 & m_{nn} \end{bmatrix},$$

then we have the following theorem.

Theorem 17 (Necessary and sufficient condition for simple cycles). *Let $M \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix with negative diagonal elements whose associated digraph $\mathcal{G}(M)$ is a simple cycle $c = (1, n, \ldots, 2, 1)$. Then the following statements are equivalent:*

(i) M is Hurwitz;

(ii) $\gamma_c < 1$;

(iii) there exists a solution to (8) such that $\psi_c < 1$.

Proof. Regarding the equivalence between (i) and (ii): by Theorem 2(iii), M is Hurwitz if and only if all the leading principal minors of $-M$ are positive. If $i < n$ and $I = \{1, \ldots, i\}$, then the leading principal submatrices $(-M)_I$ of $-M$ are triangular matrices with positive diagonal elements and thus $\det((-M)_I) > 0$. When $I = \{1, \ldots, n\}$, we have

$$\det(-M) = (-1)^n \det(M)$$
$$= (-1)^n \left(\prod_{i=1}^{n} m_{ii} + (-1)^{n+1} m_{n1} \prod_{i=1}^{n-1} m_{i,i+1} \right)$$
$$= \prod_{i=1}^{n} (-m_{ii}) - m_{n1} \prod_{i=1}^{n-1} m_{i,i+1}.$$

Then $\det(-M) > 0$ if and only if

$$\prod_{i=1}^{n} (-m_{ii}) > m_{n1} \prod_{i=1}^{n-1} m_{i,i+1},$$

which is equivalent to $\gamma_c < 1$.

Regarding the equivalence between (ii) and (iii): notice that if we pick $\psi_{ij} = \frac{m_{ij}}{-m_{ii}} + \epsilon$ for sufficiently small $\epsilon > 0$, then (8) is satisfied and $\psi_c < 1$ is equivalent to $\gamma_c < 1$. $\square$

It is worth mentioning that the necessary and sufficient condition in Theorem 17 is a special case of a more general result in [3, Proposition 2] regarding diagonal stability.

Example 18 (A two by two Metzler matrix describing a flow system [6, Exercise 9.8]). *We apply Theorem 17 to a simple two by two case where the Metzler matrix describes a symmetric flow system $\dot{x} = Mx$. Suppose the Metzler matrix M has the following form*

$$M = \begin{bmatrix} g - f & f \\ f & -d - f \end{bmatrix},$$

where $f > 0$ is the flow rate between two nodes, $g > 0$ is the growth rate at node 1 and $d > 0$ is the decay rate at node 2. By Theorem 17, the flow system $\dot{x} = Mx$ is asymptotically stable if and only if

$$g - f < 0, \quad -d - f < 0, \text{ and } \frac{f^2}{(f-g)(d+f)} < 1.$$

Equivalently, we have

$$d > g \text{ and } f > \frac{dg}{d-g}.$$

This condition has a clear physical interpretation that in order for the two-node flow system $\dot{x} = Mx$ to be asymptotically stable, i.e., the flow does not accumulate in the system, the decay rate at one node must be larger than the growth rate at the other node and the flow rate between the nodes should be sufficiently large.

Theorem 17 states that a Metzler matrix whose associated digraph is a simple cycle is Hurwitz if and only if the cycle gain is less than 1. It turns out that, for irreducible Metzler matrices with general digraphs, the gains of the simple cycles play a central role in determining the Hurwitzness. Moreover, cycle gains in different forms (sum or max) lead to different graph-theoretic conditions.

C. Max-interconnection gains and Hurwitz Metzler matrices

In this subsection, we use the max-interconnection ISS gains of the Metzler system (5) to provide a necessary and sufficient condition for Hurwitzness of a Metzler matrix.

Theorem 19 (Max-interconnection characterization). *Let $M \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix with negative diagonal elements, $\mathcal{G}(M) = (V, \mathcal{E})$ be the associated digraph, and Φ be the set of simple cycles of $\mathcal{G}(M)$. Then the following conditions are equivalent:*

(i) M is Hurwitz;

(ii) for every $i \in V$ and $j \in \mathcal{N}_i$, there exists $\psi_{ij} > 0$ such that

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}} \right) \psi_{ij}^{-1} < 1, \quad \text{for all } i \in \{1, \ldots, n\},$$
(13)

$$\psi_c < 1, \quad \text{for all } c \in \Phi. \quad (14)$$

Proof. (ii) $\implies$ (i): Since the diagonal entries of M are negative, the Metzler system (5) is componentwise ISS by Theorem 12(i). By Theorem 12(iii), there exist max-interconnection gains $\{\psi_{ij}\}$ such that

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}} \right) \psi_{ij}^{-1} < 1, \quad \text{for all } i \in \{1, \ldots, n\}.$$

Thus, the sufficient condition in Theorem 11 is equivalent to the existence of $\psi_{ij} > 0$, for $i \in V$ and $j \in \mathcal{N}_i$ such that

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}} \right) \psi_{ij}^{-1} < 1, \quad \text{for all } i \in \{1, \ldots, n\},$$

$$\psi_c < 1, \quad \text{for all } c \in \Phi.$$

Therefore, by Theorem 11, the Metzler system (5) is ISS and asymptotically stable, which implies that M is Hurwitz.

(i) $\implies$ (ii): Suppose that M is Hurwitz, then by Theorem 2(iv) there exists $\xi > \mathbb{0}_n$ such that $M\xi < \mathbb{0}_n$. Therefore, $\text{diag}(\xi^{-1})M\text{diag}(\xi)$ is a Metzler matrix with negative row sums, which implies

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}} \right) \frac{\xi_j}{\xi_i} < 1, \quad \text{for all } i \in \{1, \ldots, n\}.$$

Note that, for every $(i_1, \ldots, i_k, i_1) \in \Phi$, we have

$$\frac{\xi_{i_2}}{\xi_{i_1}} \cdots \frac{\xi_{i_1}}{\xi_{i_k}} = 1.$$

Thus, we have

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}} \right) \frac{\xi_j}{\xi_i} < 1, \quad \text{for all } i \in \{1, \ldots, n\},$$

$$\frac{\xi_{i_2}}{\xi_{i_1}} \cdots \frac{\xi_{i_1}}{\xi_{i_k}} = 1, \quad \text{for all } (i_1, \ldots, i_k, i_1) \in \Phi.$$

By a straightforward continuity argument, one can show that, for every $i \in V$ and $j \in \mathcal{N}_i$, there exists $\psi_{ij} > 0$ such that

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}} \right) \psi_{ij}^{-1} < 1, \quad \text{for all } i \in \{1, \ldots, n\},$$

$$\psi_c < 1, \quad \text{for all } c \in \Phi.$$

This completes the proof. $\square$

By Theorem 19, we can prove the following corollary.

Corollary 20 (Diagonal Stability and Hurwitzness of Metzler matrices). *Let $M \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix with negative diagonal elements, $\mathcal{G}(M) = (V, \mathcal{E})$ be the associated digraph, and Φ be the set of simple cycles of $\mathcal{G}(M)$. Assume that $\mathcal{G}(M)$ is cactus. Then the following conditions are equivalent:*

(i) M is Hurwitz;

(ii) for every $c \in \Phi$ and every $i \in c$, there exists positive constant $\theta_i^c > 0$ such that

$$\prod_{i \in c} \theta_i^c > \gamma_c, \quad \text{for all } c \in \Phi,$$
$$\sum_{c \in \Phi} \theta_i^c = 1, \quad \text{for all } i \in c,$$
(15)

where γ_c is defined in equation (12).

Proof. We postpone the proof to Appendix B. $\square$

Remark 21. *(i) The condition in Corollary 20(ii) for Metzler matrices is the same as conditions (11) and (12) in [3, Theorem 1] for the diagonal stability of arbitrary matrices with cactus graphs. Therefore, in the context of Metzler matrices, Theorem 19 is a generalization of [3, Theorem 1] to arbitrary topologies.*

(ii) One can compute the positive constants ψ_{ij} in Theorem 19(ii) by solving the following feasibility problem

$$\begin{aligned} \text{Find} \quad & \xi \\ \text{subject to} \quad & \xi > \mathbb{0}_n, \\ & M\xi < \mathbb{0}_n. \end{aligned}$$

Then, for $i \in V$ and $j \in \mathcal{N}_i$, we can compute ψ_{ij} as

$$\psi_{ij} = \delta \frac{\xi_i}{\xi_j},$$

where $0 < \delta < 1$ is given by

$$\delta = \max_i \left\{ \sum_{j \in \mathcal{N}_i} \frac{m_{ij}}{-m_{ii}} \frac{\xi_j}{\xi_i} \right\}.$$

In order to check conditions (13) and (14), we need to compute the max-interconnection ISS gains using the method in Remark 21(ii). This computation is essentially equivalent to the well-known linear program in Theorem 2(iv).

D. Sum-interconnection gains and Hurwitz Metzler matrices

We first introduce the *disjoint cycle sets*.

Definition 22 (Disjoint cycle sets). *Let $M \in \mathbb{R}^{n \times n}$ be a Metzler matrix with the associated digraph $\mathcal{G}(M)$ and $\Phi = \{c_1, \ldots, c_r\}$ be the set of simple cycles in $\mathcal{G}(M)$, the* disjoint cycle sets K_ℓ^M *for $\ell \in \{1, \ldots, r\}$ are defined by*

$$K_\ell^M = \{\{c_{i_1}, \ldots, c_{i_\ell}\} \subset \Phi \,|\, c_{i_k} \cap c_{i_{k'}} = \emptyset,$$
$$k \neq k' \text{ and } k, k' \in \{1, \ldots, \ell\}\}.$$

Intuitively, the disjoint cycle sets K_ℓ^M are sets where each element is a set of ℓ cycles that are mutually disjoint. We collect the graph-theoretic interpretations for the disjoint cycle sets in Section VIII-A. With the disjoint cycle sets, we are ready to define the notion of total cycle gain of a Metzler matrix and its leading principal submatrices.

Definition 23 (Total cycle gain). *Let $M \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix with negative diagonal elements. For $i = \{1, \ldots, n\}$ and $I = \{1, \ldots, i\}$, the leading principal submatrix M_I has the associated digraph $\mathcal{G}(M_I)$, set of simple cycles $\Phi_{M_I} = \{c_1, \ldots, c_{r_{M_I}}\}$ and disjoint cycle sets $K_\ell^{M_I}$, $\ell \in \{1, \ldots, r_{M_I}\}$, then the* total cycle gain *of M_I is defined by*

$$
\gamma_{M_I} \\
= \begin{cases} \sum_{\ell=1}^{r_{M_I}} \sum_{\{c_{i_1}, \ldots, c_{i_\ell}\} \in K_\ell^{M_I}} (-1)^{\ell-1} \gamma_{c_{i_1}} \ldots \gamma_{c_{i_\ell}}, & \text{if } \Phi_{M_I} \neq \emptyset, \\ 0, & \text{if } \Phi_{M_I} = \emptyset. \end{cases}
$$
(16)

Example 24 (Disjoint cycle sets and total cycle gain). *We illustrate the definitions of the disjoint cycle sets and the total cycle gain in this example. Let $M \in \mathbb{R}^{6 \times 6}$ be an irreducible Metzler matrix with negative diagonal elements as follows*

$$
M = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 & 0 & m_{16} \\ m_{21} & m_{22} & m_{23} & 0 & 0 & 0 \\ 0 & m_{32} & m_{33} & 0 & 0 & 0 \\ 0 & 0 & m_{43} & m_{44} & m_{45} & 0 \\ 0 & 0 & 0 & m_{54} & m_{55} & 0 \\ m_{61} & 0 & 0 & 0 & m_{65} & m_{66} \end{bmatrix}.
$$

The associated weighted digraph $\mathcal{G}(M)$ is shown in Fig. 1. There are five cycles in $\mathcal{G}(M)$, i.e., $c_1 = (1, 2, 1)$,
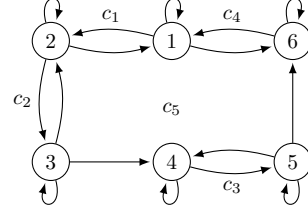

Fig. 1. The associated weighted digraph $\mathcal{G}(M)$

$c_2 = (2, 3, 2)$, $c_3 = (4, 5, 4)$, $c_4 = (6, 1, 6)$, $c_5 = (1, 2, 3, 4, 5, 6, 1)$, *and the disjoint cycle sets of M are:*

$$
\begin{aligned} K_1^M &= \{\{c_1\}, \{c_2\}, \{c_3\}, \{c_4\}, \{c_5\}\}, \\ K_2^M &= \{\{c_1, c_3\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_3, c_4\}\}, \\ K_3^M &= \{\{c_2, c_3, c_4\}\}, \\ K_4^M &= K_5^M = \emptyset. \end{aligned}
$$
(17)

According to (16), the total cycle gains of the leading principal submatrices are given by:

$$
\begin{aligned} \gamma_{M_{\{1\}}} &= 0, & \gamma_{M_{\{1,2\}}} &= \gamma_{c_1}, \\ \gamma_{M_{\{1,2,3\}}} &= \gamma_{c_1} + \gamma_{c_2}, & \gamma_{M_{\{1,2,3,4\}}} &= \gamma_{c_1} + \gamma_{c_2}, \\ \gamma_{M_{\{1,2,3,4,5\}}} &= \gamma_{c_1} + \gamma_{c_2} + \gamma_{c_3} - \gamma_{c_1}\gamma_{c_3} - \gamma_{c_2}\gamma_{c_3}, \\ \gamma_{M_{\{1,2,3,4,5,6\}}} &= \gamma_M = \gamma_{c_1} + \gamma_{c_2} + \gamma_{c_3} + \gamma_{c_4} + \gamma_{c_5} \\ &\quad - \gamma_{c_1}\gamma_{c_3} - \gamma_{c_2}\gamma_{c_3} - \gamma_{c_2}\gamma_{c_4} \\ &\quad - \gamma_{c_3}\gamma_{c_4} + \gamma_{c_2}\gamma_{c_3}\gamma_{c_4}. \end{aligned}
$$
(18)

With the above definitions, we now present a useful lemma.

Lemma 25 (Determinant and total cycle gain). *Let $M \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix with negative diagonal elements and let γ_{M_I} be the total cycle gain of M_I for $i \in \{1, \ldots, n\}$ and $I = \{1, \ldots, i\}$. Then*

$$\det(M_I) = (1 - \gamma_{M_I}) \prod_{j=1}^{i} m_{jj}.$$
(19)

Proof. We postpone the proof to Appendix A. $\qquad \square$

We are now ready to write the leading principal minor condition in Theorem 2(iii) in the graph-theoretic language.

Theorem 26 (Sum-interconnection characterization). *Let $M \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix with negative diagonal elements, $\mathcal{G}(M) = (V, \mathcal{E})$ be the associated digraph, and Φ be the set of simple cycles of $\mathcal{G}(M)$. Then the following statements hold:*

(i) (necessary condition) if M is Hurwitz then

$$\gamma_c < 1, \quad \text{for all } c \in \Phi;$$

(ii) (sufficient condition) if

$$\sum_{c \in \Phi} \gamma_c < 1,$$

then M is Hurwitz;

(iii) (necessary and sufficient condition) M is Hurwitz if and only if, for all $i \in \{1, \ldots, n\}$

$$\gamma_{M_I} < 1, \quad I = \{1, \ldots, i\}.$$

Proof. Regarding part (i), we postpone the proof to Section VIII-B, where an expansion algorithm for $\mathcal{G}(M)$ is given so that all the simple cycles can be identified by the leading principal submatrices and a simple proof is constructed.

Regarding part (ii), we prove the result by showing that Theorem 26(iii) holds. For all $i \in \{1, \ldots, n\}$ and $I = \{1, \ldots, i\}$, the leading submatrix M_I only involves a subset of Φ. If Φ_{M_I} is empty, then $\gamma_{M_I} = 0 < 1$. Otherwise, from (16), we know that γ_{M_I} has the following form:

$$\gamma_{M_I} = \sum_{\{c_{i_1}\} \in K_1^{M_I}} \gamma_{c_{i_1}} - \sum_{\{c_{i_1}, c_{i_2}\} \in K_2^{M_I}} \gamma_{c_{i_1}} \gamma_{c_{i_2}}$$
$$+ \sum_{\{c_{i_1}, c_{i_2}, c_{i_3}\} \in K_3^{M_I}} \gamma_{c_{i_1}} \gamma_{c_{i_2}} \gamma_{c_{i_3}}$$
$$+ \sum_{\ell=3}^{r_{M_I}} \sum_{\{c_{i_1}, \ldots, c_{i_\ell}\} \in K_\ell^{M_I}} (-1)^{\ell-1} \gamma_{c_{i_1}} \cdots \gamma_{c_{i_\ell}}.$$

Since for all $c \in \Phi$, we have $\gamma_c > 0$ and $\sum_{c \in \Phi} \gamma_c < 1$ by assumption, then we have that $\gamma_c < 1$ for all $c \in \Phi$ and $\sum_{\{c_{i_1}\} \in K_1^{M_I}} \gamma_{c_{i_1}} < 1$. Note that by the definition of $K_\ell^{M_I}$, for any $\{c_{i_1}, \ldots, c_{i_\ell}\} \in K_\ell^{M_I}$, we must have that all the subsets of $\{c_{i_1}, \ldots, c_{i_\ell}\}$ with $\ell - 1$ elements are contained in $K_{\ell-1}^{M_I}$. Thus, we have that, for all $k \geq 1$,

$$\sum_{\ell=2k}^{2k+1} \sum_{\{c_{i_1}, \ldots, c_{i_\ell}\} \in K_\ell^{M_I}} (-1)^{\ell-1} \gamma_{c_{i_1}} \cdots \gamma_{c_{i_\ell}} < 0.$$

Hence, we have for all $i \in \{1, \ldots, n\}$ and $I = \{1, \ldots, i\}$, $\gamma_{M_I} < 1$, and by Theorem 26(iii), M is Hurwitz.

Regarding part (iii), by Lemma 25, we have that for $i \in \{1, \ldots, n\}$ and $I = \{1, \ldots, i\}$,

$$\det((-M)_I) = \left(\prod_{j=1}^{i} (-m_{jj})\right)(1 - \gamma_{M_I}).$$

By Theorem 2(iii), M is Hurwitz if and only if for all $i \in \{1, \ldots, n\}$ and $I = \{1, \ldots, i\}$, $\det((-M)_I) > 0$, i.e.,

$$\left(\prod_{j=1}^{i} (-m_{jj})\right)(1 - \gamma_{M_I}) > 0,$$

which is equivalent to $\gamma_{M_I} < 1$. $\qquad\square$

Remark 27 (Necessary and sufficient condition in special graphs). *The sufficient condition for Hurwitzness in Theorem 26(ii) becomes necessary and sufficient when any two cycles share at least one common node in the digraph associated with the Metzler matrix.*

We give two simple examples illustrating that the condition in Theorem 26(i) is not sufficient and the condition in Theorem 26(ii) is not necessary.

Example 28 (Insufficiency of Theorem 26(i)). *Consider an irreducible Metzler matrix $M \in \mathbb{R}^{3 \times 3}$ as follows*

$$M = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

The associated weighted digraph $\mathcal{G}(M)$ is shown in Fig. 2. There are two cycles in $\mathcal{G}(M)$, i.e., $c_1 = (1, 2, 1)$ and $c_2 = (2, 3, 2)$, and the cycle gains are $\gamma_{c_1} = \gamma_{c_2} = \frac{1}{2}$. The cycle gains satisfy the condition in Theorem 26(i), but M is not Hurwitz since it has a zero eigenvalue.
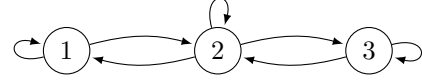

Fig. 2. The associated weighted digraph of M

Example 29 (Unnecessity of Theorem 26(ii)). *Consider an irreducible Metzler matrix $M \in \mathbb{R}^{4 \times 4}$ as follows*

$$M = \begin{bmatrix} -5 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The associated weighted digraph $\mathcal{G}(M)$ is shown in Fig. 3. There are three cycles in $\mathcal{G}(M)$, i.e., $c_1 = (1, 2, 1)$, $c_2 = (2, 3, 2)$ and $c_3 = (3, 4, 3)$, and the cycle gains are $\gamma_{c_1} = \frac{3}{5}$, $\gamma_{c_2} = \frac{1}{5}$ and $\gamma_{c_3} = \frac{1}{5}$. The cycle gains do not satisfy the sufficient condition in Theorem 26(ii), but one can check that M is Hurwitz.
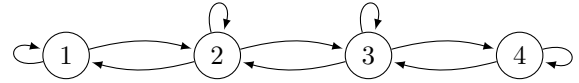

Fig. 3. The associated weighted digraph of M

We give the Hurwitzness conditions for Example 24.

Example 24 (Continued). *By Theorem 26(iii) and (18), the sufficient and necessary conditions for M to be Hurwitz are given by*

$$\gamma_{c_1} < 1, \qquad \gamma_{c_1} + \gamma_{c_2} < 1,$$
$$\gamma_{c_1} + \gamma_{c_2} + \gamma_{c_3} - \gamma_{c_1}\gamma_{c_3} - \gamma_{c_2}\gamma_{c_3} < 1, \qquad \gamma_M < 1,$$

which are equivalent to

$$\gamma_{c_1} + \gamma_{c_2} < 1, \tag{20}$$
$$\gamma_{c_1} + \gamma_{c_2} + \gamma_{c_3} - \gamma_{c_1}\gamma_{c_3} - \gamma_{c_2}\gamma_{c_3} < 1, \tag{21}$$
$$\gamma_M < 1. \tag{22}$$

It is not obvious whether the necessary conditions in Theorem 26(i) hold in this example. We show that (20)-(22) imply those necessary conditions in the following. From (20), since the cycle gains are positive, we know that $\gamma_{c_1} < 1$ and $\gamma_{c_2} < 1$. We can rewrite (21) as follows

$$\gamma_{c_3}(1 - \gamma_{c_1} - \gamma_{c_2}) < 1 - \gamma_{c_1} - \gamma_{c_2},$$

which along with (20) imply that $\gamma_{c_3} < 1$. *By using* (18)*, we can rearrange* (22) *as follows*

$$\gamma_{c_1}(1 - \gamma_{c_3}) + \gamma_{c_2} + \gamma_{c_3} - \gamma_{c_2}\gamma_{c_3} + \gamma_{c_5}$$
$$+ \gamma_{c_4}(1 - \gamma_{c_2})(1 - \gamma_{c_3}) < 1,$$

which is equivalent to

$$\gamma_{c_1}(1 - \gamma_{c_3}) + \gamma_{c_5} < (1 - \gamma_{c_4})(1 - \gamma_{c_2})(1 - \gamma_{c_3}). \quad (23)$$

Since all the terms on the left hand side of (23) *are positive, and on the right hand side we have* $\gamma_{c_2} < 1$ *and* $\gamma_{c_3} < 1$*, thus we must have that* $\gamma_{c_4} < 1$*. At the same time, since the term on the right hand side of* (23) *is less than* 1*, we must have that* $\gamma_{c_5} < 1$*.*

We conclude this section with a final result related to [11, Corollary 16]: the sufficient condition on the spectral radius of a so-called gain matrix becomes also necessary in the Metzler case.

Theorem 30 (Spectral radius condition on the gain matrix)**.** *Let* $M \in \mathbb{R}^{n \times n}$ *be a Metzler matrix with negative diagonal elements and define its associated* gain matrix $\Gamma \in \mathbb{R}^{n \times n}$ *by*

$$\Gamma_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{m_{ij}}{-m_{ii}}, & \text{if } i \neq j. \end{cases}$$

Then the following statements are equivalent:
 (i) M *is Hurwitz;*
 (ii) $\rho(\Gamma) < 1$*.*

Proof. (ii) $\implies$ (i): This follows from [11, Corollary 16].
 (i) $\implies$ (ii): Note that

$$\Gamma = I_n - \text{diag}(M)^{-1}M.$$

We prove the statement by contradiction. Suppose $\rho(\Gamma) = \lambda \geq 1$. Since Γ is a nonnegative matrix, the Perron-Frobenius Theorem implies the existence of a nonnegative vector $v \in \mathbb{R}^n$, $v \neq 0$ such that

$$\Gamma v = \lambda v.$$

In turn, this imples $(I_n - \text{diag}(M)^{-1}M)v = \lambda v$, or equivalently

$$v = (\lambda - 1)(-M^{-1})\text{diag}(M)v.$$

On the left hand side, we have that $v \geq \mathbb{0}_n$, and on the right hand side, we have that $(\lambda - 1)(-M^{-1})\text{diag}(M)v \leq \mathbb{0}_n$ because $-M^{-1} \geq 0$ by Theorem 2(ii). Then, we must have $v = \mathbb{0}_n$, which is a contradiction. Therefore, $\rho(\Gamma) < 1$. $\square$

VI. GRAPH-THEORETIC CONDITIONS FOR STABILITY OF NONLINEAR MONOTONE SYSTEMS

In this section, we extend our stability results to monotone nonlinear systems. Suppose the interaction between subsystems is described by a directed graph $G = (V, \mathcal{E})$, where $V = \{1, \ldots, n\}$ is the set of nodes and for all $i, j \in V$ and $i \neq j$, $(j, i) \in \mathcal{E}$ if x_j is an input to subsystem i. We consider a network of n interconnected dynamical systems with the interconnection graph G:

$$\dot{x}_i = f_i(x_i, x_{\mathcal{N}_i}), \quad \text{for all } i \in \{1, \ldots, n\}, \quad (24)$$

where $x_i \in \mathbb{R}$ and $x_{\mathcal{N}_i} = [x_{i_1}, \ldots, x_{i_{k_i}}]^\top \in \mathbb{R}^{|\mathcal{N}_i|}$ with $\mathcal{N}_i = \{i_1, \ldots, i_k\}$. For every $i \in \{1, \ldots, n\}$, the function $f_i : \mathbb{R}^{|\mathcal{N}_i|+1} \to \mathbb{R}$ is continuously differentiable. We assume that the interconnected system (24) is monotone, i.e., for every $x \in \mathbb{R}^n_{\geq 0}$, the Jacobian matrix $J(x)$ is Metzler. Moreover, we assume that $f(\mathbb{0}_n) = \mathbb{0}_n$. We show that our characterizations of stability for linear Metzler systems can be generalized to sufficient conditions for global stability of nonlinear monotone systems. In particular, we prove two global results for asymptotic stability of monotone interconnected networks based on the max-interconnection gains and the sum-interconnection gains.

Theorem 31 (Max-interconnection stability)**.** *Consider the interconnected nonlinear system* (24) *evolving on the positive orthant* $\mathbb{R}^n_{\geq 0}$ *with the interconnection graph* $G = (V, \mathcal{E})$*. Assume that* $f(\mathbb{0}_n) = \mathbb{0}_n$*, and for every* $x \in \mathbb{R}^n_{\geq 0}$*, the matrix* $J(x)$ *is Metzler with negative diagonal entries. Moreover, assume there exists a family of positive numbers* $\{\psi_{ij}\}$ *for* $i \in V$ *and* $j \in \mathcal{N}_i$ *such that:*

(i) for every $i \in \{1, \ldots, n\}$*,*

$$\sum_{j \in \mathcal{N}_i} \frac{J_{ij}(x)}{-J_{ii}(x)}\psi_{ij}^{-1} < 1, \quad \text{for all } x \in \mathbb{R}^n_{\geq 0}, \quad (25)$$

(ii) for every $c = (i_1, \ldots, i_k, i_1) \in \Phi$*,*

$$\psi_{i_2 i_1} \ldots \psi_{i_1 i_k} < 1.$$

Then $\mathbb{0}_n$ *is globally asymptotically stable for system* (24)*.*

Proof. Given $c > 0$, we define the set $B(c)$ and the real number $\delta(c)$ as follows:

$$B(c) = \{x \in \mathbb{R}^n_{\geq 0} \mid x \leq 2c\mathbb{1}_n\},$$
$$\delta(c) = \min_{x \in B(c)} \min_i \left(-J_{ii}(x) - \sum_{j \in \mathcal{N}_i} J_{ij}(x)\psi_{ij}^{-1} \right).$$

Since $B(c)$ is a compact set and (25) holds, we have that $\delta(c) > 0$. Let $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ be a class $\mathcal{KL}$ function given by $\beta(s, t) = se^{-\delta(s)t}$, where $\delta(s) > 0$ is nonincreasing function with respect to s. Consider the control system

$$\dot{x} = f(x) + \mathbb{0}_{n \times n}u, \quad (26)$$

where $u \in \mathbb{R}^n_{\geq 0}$. We first show that, for every $t \geq 0$ and every $i \in \{1, \ldots, n\}$,

$$x_i(t) \leq \max_j \{\beta(x_i(0), t), \psi_{ij}\|x_j\|_{[0,t]}, \|u_i\|_\infty\}. \quad (27)$$

Suppose that the statement (27) is not true. Therefore, there exist $i \in \{1, \ldots, n\}$, $t^* \geq 0$, and $\epsilon > 0$ such that

$$x_i(t^*) = \max_j \{\beta(x_i(0), t^*), \psi_{ij}\|x_j\|_{[0,t^*]}, \|u_i\|_\infty\}, \quad (28)$$

and for every $t \in (t^*, t^* + \epsilon)$,

$$x_i(t) > \max_j \{\beta(x_i(0), t), \psi_{ij}\|x_j\|_{[0,t]}, \|u_i\|_\infty\}. \quad (29)$$

Since $\mathbb{R}^n_{\geq 0}$ is convex, by the Mean Value Theorem [1, Proposition 2.4.7], there exists $\mathbb{0}_n \leq \xi_x \leq x$ such that

$$f_i(x) = J_{ii}(\xi_x)x_i + \sum_{j \in \mathcal{N}_i} J_{ij}(\xi_x)x_j. \tag{30}$$

By (28) and (29), we have that, for every j such that $(j,i) \in \mathcal{E}$ and every $t \in [t^*, t^* + \epsilon)$, we have $\|x_j\|_{[0,t]} \leq \psi_{ij}^{-1}x_i(t)$. Therefore, by (30), we have

$$\dot{x}_i(t) = f_i(x) \leq \left(J_{ii}(\xi_{x(t)}) + \sum_{j \in \mathcal{N}_i} J_{ij}(\xi_{x(t)})\psi_{ij}^{-1} \right) x_i(t). \tag{31}$$

We consider two cases in the following.

(i) $x_i(t^*) = \beta(x_i(0), t^*)$: In this case, for small enough ϵ and, for every $t \in [t^*, t^* + \epsilon)$, we have $\xi_{x(t)} \in B(x_i(0))$. Thus, by (31), we have

$$\dot{x}_i(t) \leq -\delta(x_i(0))x_i(t),$$

which implies that $x_i(t) \leq e^{-\delta(x_i(0))(t-t^*)}x_i(t^*)$. Thus, along with (28), we have, for every $t \in [t^*, t^* + \epsilon)$,

$$\begin{aligned} x_i(t) &\leq e^{-\delta(x_i(0))(t-t^*)}x_i(t^*) \\ &= e^{-\delta(x_i(0))t}x_i(0) \\ &\leq \max_j\{\beta(x_i(0), t), \|x_j\|_{[0,t]}, \|u_i\|_\infty\}, \end{aligned}$$

which is contradictory to (29).

(ii) $x_i(t^*) > \beta(x_i(0), t^*)$: In this case, we have $x_i(t^*) > x_i(0)e^{-\delta(x_i(0))t^*}$ and therefore

$$x_i(t^*) = \max_j\{\psi_{ij}\|x_j\|_{[0,t^*]}, \|u_i\|_\infty\}.$$

By (31), we have $\dot{x}_i(t) \leq 0$ for every $t \in [t^*, t^* + \epsilon)$. Since $\|x_j\|_{[0,t]}$ is nondecreasing with respect to t, for every $t \in [t^*, t^* + \epsilon)$,

$$\begin{aligned} x_i(t) &\leq \max_j\{\psi_{ij}\|x_j\|_{[0,t]}, \|u_i\|_\infty\} \\ &\leq \max_j\{\beta(x_i(0), t), \psi_{ij}\|x_j\|_{[0,t]}, \|u_i\|_\infty\}, \end{aligned}$$

which is contradictory to (29).

In both cases, we have a contradiction. Therefore, for every $t \geq 0$ and every $i \in \{1, \ldots, n\}$, $x_i(t)$ satisfies (27). Moreover, Theorem 31(ii) ensures that $\{\psi_{ij}\}_{(i,j) \in \mathcal{E}}$ satisfies $\psi_c < 1$, for every $c \in \Phi$. Therefore, by cyclic small-gain Theorem 11, the control system (26) is ISS, which implies that $\mathbb{0}_n$ is globally asymptotically stable for nonlinear dynamical system (24). $\qquad\square$

Theorem 32 (Sum-interconnection stability)**.** *Consider the interconnected nonlinear system* (24) *evolving on the positive orthant* $\mathbb{R}^n_{\geq 0}$ *with the interconnection graph* $G = (V, \mathcal{E})$. *Assume that* $f(\mathbb{0}_n) = \mathbb{0}_n$, *and for every* $x \in \mathbb{R}^n_{\geq 0}$, *the matrix* $J(x)$ *is Metzler with negative diagonal entries. Moreover, assume there exists a family of positive numbers* $\{\gamma_{ij}\}$ *for* $i \in V$ *and* $j \in \mathcal{N}_i$ *such that:*

(i) for every $i \in \{1, \ldots, n\}$,

$$\frac{J_{ij}(x)}{-J_{ii}(x)} \leq \gamma_{ij}, \quad \text{for all } x \in \mathbb{R}^n_{\geq 0},$$

(ii) for every $i \in \{1, \ldots, n\}$ *and* $I = \{1, \ldots, i\}$,

$$\gamma_{M_I} < 1,$$

where the Metzler matrix M *is defined as, for* $i', j' \in V$

$$m_{i'j'} = \begin{cases} \gamma_{i'j'}, & \text{if } (j', i') \in \mathcal{E}, \\ -1, & \text{if } i' = j', \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbb{0}_n$ *is globally asymptotically stable for system* (24).

Proof. By (ii) and Theorem 26(iii), M is Hurwitz. Thus, by Theorem 19, there exists a family of positive numbers $\{\psi_{ij}\}_{(i,j) \in \mathcal{E}}$ such that, for every $i \in \{1, 2, \ldots, n\}$,

$$\sum_{j \in \mathcal{N}_i} \frac{m_{ij}}{-m_{ii}}\psi_{ij}^{-1} \leq 1,$$

and $\psi_c < 1$ for every $c \in \Phi$. This implies that, for every $x \in \mathbb{R}^n_{\geq 0}$, we have

$$\sum_{j \neq i} \frac{J_{ij}(x)}{-J_{ii}(x)}\psi_{ij}^{-1} \leq \sum_{j \neq i} \gamma_{ij}\psi_{ij}^{-1} = \sum_{i \neq j} \frac{m_{ij}}{-m_{ii}}\psi_{ij}^{-1} \leq 1.$$

Therefore, for the family of positive numbers $\{\psi_{ij}\}_{(i,j) \in \mathcal{E}}$,

$$\sum_{j \neq i} \frac{J_{ij}(x)}{-J_{ii}(x)}\psi_{ij}^{-1} \leq 1, \quad \text{for all } i \in \{1, \ldots, n\},$$

and $\psi_c < 1$ for every $c \in \Phi$. Therefore, by Theorem 31, $\mathbb{0}_n$ is globally asymptotically stable for the dynamical system (24). $\qquad\square$

VII. KRON REDUCTION FOR METZLER MATRICES

In this section, we give graph-theoretic interpretations of Schur complements for irreducible Metzler matrices with negative diagonal elements, which is novel in its own and provides insights for the proof of Theorem 26(i). The Schur complement technique, known as Kron reduction in power engineering, has been developed for symmetric irreducible loopy Laplacian matrices in [13]. For an irreducible Metzler matrix $M \in \mathbb{R}^{n \times n}$ with negative diagonal elements, computing the Schur complement of M with respect to m_{nn} is graph-theoretically equivalent to removing node n and its associated edges in $\mathcal{G}(M)$ and modifying the connections and weights among the remaining nodes. Specifically, for all $i, j \in \{1, \ldots, n-1\}$,

(i) if both edge weights m_{in} and m_{ni} are nonzero, then the self weight of node i becomes $m_{ii} - \frac{m_{in}m_{ni}}{m_{nn}}$;
(ii) if both edge weights m_{in} and m_{nj} are nonzero, then edge weight on (j, i) becomes $m_{ij} - \frac{m_{in}m_{nj}}{m_{nn}}$; in particular, if $m_{ij} = 0$, then a new edge (j, i) establishes from node j to node i with edge weight $-\frac{m_{in}m_{nj}}{m_{nn}}$.

The topological changes are illustrated by an example in Fig. 4, where node five is removed by the Schur complement. The following observations regarding the Schur complement

can be made from Fig. 4: (i) the connectivity is maintained; (ii) the self weight of node i changes if and only if node i and node n form a simple cycle (i, n, i); (iii) the "path gain" $-\frac{m_{in}m_{nj}}{m_{nn}}$ of the path (j, n, i) is added to the weight on the edge (j, i), and in particular, if there exists no directed edge from node j to node i, i.e., $m_{ij} = 0$, then a directed edge appears in the reduced graph with the edge weight $-\frac{m_{in}m_{nj}}{m_{nn}}$.

(a) Original graph

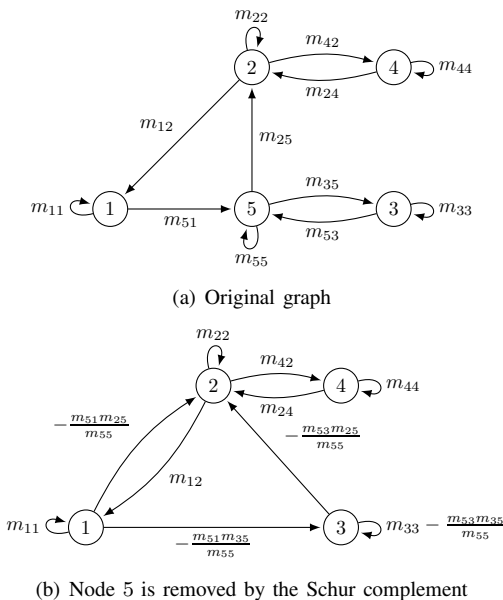(b) Node 5 is removed by the Schur complement

Fig. 4. Topological changes corresponding to the Schur complement

The one-node Kron reduction described above for irreducible Metzler matrices with negative diagonal elements may not be well defined when one tries to eliminate a subset of more than one nodes. However, the Schur complement is always well defined for the following sets of matrices.

Theorem 33 (Invariant sets under Schur complement). *The Schur complements are well-defined and leave the following sets of matrices invariant:*
(i) Hurwitz Metzler matrices;
(ii) irreducible compartmental matrices.

Proof. Regarding part (i), the result follows from Theorem 4 and the quotient identity of the Schur complement [10].

Regarding part (ii), since the compartmental matrix is irreducible, all its leading submatrices with order less than the dimension are invertible and therefore, the Schur complements are well defined. □

Theorem 34 (Spectral interlacing properties). *Let $M \in \mathbb{R}^{n \times n}$ be a symmetric Hurwitz Metzler matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n < 0$. The eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1} < 0$ of the Schur complement $M[n-1]$ are interlaced with those of M in the sense that*

$$\lambda_i \leq \mu_i \leq \lambda_{i+1}, \quad \text{for all } i \in \{1, \ldots, n-1\}.$$

Proof. By Theorem 33, the Schur complement is always well-defined for Hurwitz Metzler matrices. Then, the result follows from the interlacing properties for the schur complements of semidefinite matrices [26, Theorem 5]. □

VIII. ADDITIONAL CONCEPTS AND PROOFS

A. Cycle graphs, complementary cycle graphs and disjoint cycle sets

Let $M \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix with negative diagonal elements and $\Phi = \{c_1, \ldots, c_r\}$ be the set of simple cycles in $\mathcal{G}(M)$. Then the associated *cycle graph* of $\mathcal{G}(M)$ is the graph $\mathcal{G}_\Phi(M) = (V_\Phi, \mathcal{E}_\Phi)$ with the node set $V_\Phi = \{1, \ldots, r\}$ and the edge set $\mathcal{E}_\Phi$ given by

$$\mathcal{E}_\Phi = \{(i, j) \mid c_i \in \Phi, c_j \in \Phi, c_i \cap c_j \neq \emptyset\}.$$

We define the *complementary cycle graph* of $\mathcal{G}(M)$ by $\mathcal{G}_\Phi^c(M) = (V_\Phi, \mathcal{E}_\Phi^c)$. Note that while the graph $\mathcal{G}(M)$ is a weighted digraph, the graphs $\mathcal{G}_\Phi(M)$ and $\mathcal{G}_\Phi^c(M)$ are unweighted undirected graphs. Moreover, since M is irreducible, the cycle graph $\mathcal{G}_\Phi(M)$ is always connected. The *disjoint cycle set* K_ℓ^M is a set in which each element is a nonempty set of $\ell \geq 1$ cycles in Φ that form a complete graph in $\mathcal{G}_\Phi^c(M)$.

Example 35 (Cycle graphs, complementary cycle graphs and K_ℓ^M). *We illustrate the a few definitions using the Metzler matrix in Example 24, whose associated weighted digraph $\mathcal{G}(M)$ is shown in Fig. 1.*

The cycle graph $\mathcal{G}_\Phi(M)$ is given in Fig. 5(a) and the complementary cycle graph $\mathcal{G}_\Phi^c(M)$ is given in Fig. 5(b). From Fig. 5(b), one can check that the disjoint cycle sets are clearly given by (17).
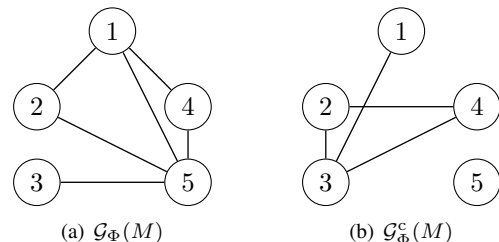

(a) $\mathcal{G}_\Phi(M)$ (b) $\mathcal{G}_\Phi^c(M)$

Fig. 5. Cycle graph and complementary cycle graph

B. Graph expansion and proof of Theorem 26(i)

Based on the observations from Section VII, we reverse the Schur complement process and propose a graph expansion algorithm for the associated graph of a Metzler matrix. The purpose of the expansion is to separate cycles so that no cycle is contained in another cycle.

For a Metzler matrix $M \in \mathbb{R}^{n \times n}$ associated with $\mathcal{G}(M) = (V, \mathcal{E}, M)$, we construct the expansion digraph $\mathcal{G}_{\exp}(M) = (V_{\exp}, \mathcal{E}_{\exp}, M_{\exp})$ and the expanded Metzler matrix $M_{\exp}$ using Algorithm 1.

In words, for a Metzler matrix $M \in \mathbb{R}^{n \times n}$, Algorithm 1 inserts a node on each directed edge in $\mathcal{G}(M)$ and assigns proper weights to the added nodes and edges.

Lemma 36. *For a Metzler matrix $M \in \mathbb{R}^{n \times n}$ and its expansion $M_{\exp}$, M is Hurwitz if and only if $M_{\exp}$ is Hurwitz.*

Proof. The Metzler matrix M can be recovered from $M_{\exp}$ by removing all the added nodes using the Schur complement,

---

Algorithm 1: Graph expansion for Metzler matrices

Input : $M \in \mathbb{R}^{n \times n}, \mathcal{G}(M) = (V, \mathcal{E}, M)$
Output: $\mathcal{G}_{\exp}(M) = (V_{\exp}, \mathcal{E}_{\exp}, M_{\exp})$

1 **Initialize:** $V_{\exp} = V$, $\mathcal{E}_{\exp} = \emptyset$, $M_{\exp} = M$, $k = 0$
2 **for** *every edge* $(i,j) \in \mathcal{E}$ **do**
3      $k = k + 1$
4      $V_{\exp} = V_{\exp} \cup \{n + k\}$
5      $\mathcal{E}_{\exp} = \mathcal{E}_{\exp} \cup \{(i, n+k), (n+k, j)\}$
6      $M_{\exp} = \begin{bmatrix} M_{\exp} & \mathbb{0}_{(n+k-1) \times 1} \\ \mathbb{0}_{1 \times (n+k-1)} & -1 \end{bmatrix}$
7      $M_{\exp}(n+k, i) = M_{\exp}(j, n+k) = \sqrt{m_{ji}}$
8 **end**

---

and the diagonal elements of the remaining nodes do not change during the elimination. Therefore, by Theorem 4, M is Hurwitz if and only if $M_{\exp}$ is Hurwitz. $\qquad \square$

Now we are ready to give a proof to Theorem 26(i).

Proof of Theorem 26(i). By construction, any cycle in $\mathcal{G}_{\exp}(M)$ can show up as a leading principal submatrix after a permutation on $M_{\exp}$. Since M is Hurwitz, $M_{\exp}$ is also Hurwitz and by Theorem 2(iii), the determinant of the negative leading principal submatrix must be positive, i.e., the cycle gain must be less than 1. $\qquad \square$

IX. CONCLUSION

In this paper, we obtained and characterized the graph-theoretic necessary and sufficient conditions for the Hurwitzness of Metzler matrices. By establishing connections with the well-known input-to-state stability theory and small-gain theorems, we were able to derive stability conditions for linear Metzler systems based on two different forms of ISS gains. These conditions give insights on how the cycles and cycle structures in the associated digraph of the Metzler matrices play a role in determining system stability. We also extended our results to the case of nonlinear monotone systems and obtained sufficient conditions for stability.

APPENDIX A
PROOF OF LEMMA 25

In order to prove Lemma 25, we need a few results regarding the graph-theoretic interpretations of determinants. For a weighted digraph $\mathcal{G} = (V, \mathcal{E}, W)$, a *factor* $F = \{c_1, \ldots, c_r\}$ of $\mathcal{G}$ satisfies

(i) each $c_i \in F$ is either a self loop or a simple cycle;
(ii) $c_i \cap c_j = \emptyset$, for all $i \neq j$;
(iii) $\cup_{i=1}^{r} c_i = V$.

Note that the set of factors may be empty and in this case the determinant of matrix corresponding to the digraph is 0.

For a matrix $A \in \mathbb{R}^{n \times n}$, the determinant of A can be computed based on the factors of $\mathcal{G}(A)$. For a simple cycle or a self loop c in $\mathcal{G}(A)$, we define $A(c)$ to be the product of the edge weights along the cycle or the self loop. Then, we have the following lemma.

Lemma 37 (Graph-theoretic interpretation of determinants [19, Theorem 1]). *Let $A \in \mathbb{R}^{n \times n}$ be a matrix with digraph $\mathcal{G}(A) = (V, \mathcal{E}, A)$. Suppose $\mathcal{G}(A)$ has factors $F_k = \{c_{k_1}, c_{k_2}, \ldots, c_{k_{r_k}}\}$, $k = 1, \ldots, q$, then*

$$\det(A) = (-1)^n \sum_{k=1}^{q} (-1)^{r_k} A(c_{k_1}) A(c_{k_2}) \cdots A(c_{k_{r_k}}). \tag{32}$$

In the case of irreducible Metzler matrices with negative diagonal elements, we can rewrite (32) in terms of the cycle gains. Let $M \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix with negative diagonal elements and $\Phi = \{c_1, \ldots, c_r\}$ be the set of simple cycles of $\mathcal{G}(M)$, then a *cycle factor* $F^c = \{c_1, \ldots, c_t\}$ of $\mathcal{G}(M)$ satisfies

(i) $F^c \subset \Phi$ and $F^c \neq \emptyset$;
(ii) $c_i \cap c_j = \emptyset$, for all $c_i, c_j \in F^c$ and $i \neq j$.

Suppose $\mathcal{G}(M)$ has cycle factors $F_k^c = \{c_{k_1}, c_{k_2}, \ldots, c_{k_{t_k}}\}$, $k = 1, \ldots, q$, then each cycle factor F_k^c can be expanded to a factor of $\mathcal{G}(M)$ by adding the self loops at the nodes that are not on any simple cycles in F_k^c and by doing this, all the factors except the one that consists of purely self loops can be recovered. Since the diagonal elements of M are negative, we can factor out $\prod_{i=1}^{n}(-m_{ii})$ in the general formula (32) and rewrite the equation for M as follows,

$$\det(M) = \prod_{i=1}^{n} m_{ii} + \prod_{i=1}^{n} m_{ii} \sum_{k=1}^{q} (-1)^{t_k} \gamma_{c_{k_1}} \gamma_{c_{k_2}} \cdots \gamma_{c_{k_{t_k}}}. \tag{33}$$

By definition, the disjoint cycle sets are related to the cycle factors as $K_\ell^M = \{F_k^c \,|\, t_k = \ell\}$, thus we can group the cycle factors with the same cardinality in (33) and obtain (19) for $I = \{1, \ldots, n\}$. For $i = \{1, \ldots, n-1\}$ and $I = \{1, \ldots, i\}$, the same procedure works for the leading principal submatrices M_I and (19) follows except for the case when Φ_{M_I} is empty. If Φ_{M_I} is empty, i.e., $\mathcal{G}(M_I)$ is acyclic, then the determinant $\det(M_I)$ is equal to the product of the diagonal elements. By (16), we have $\gamma_{M_I} = 0$ in this case and thus (19) holds.

APPENDIX B
PROOF OF COROLLARY 20

(i) $\implies$ (ii): Since M is Hurwitz, by Theorem 19, for every $(j, i) \in \mathcal{E}$, there exists $\psi_{ij} > 0$ such that

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}} \right) \psi_{ij}^{-1} < 1, \qquad \forall i \in \{1, \ldots, n\}, \tag{34}$$

$$\psi_c < 1, \qquad \forall c \in \Phi. \tag{35}$$

Let $c \in \Phi$ and assume that $c = (1, \ldots, k, 1)$. Then, for every $k' \in \{1, \ldots, k\}$, we define

$$\widehat{\theta}_{k'}^c = \begin{cases} \left(\frac{m_{k'+1, k'}}{-m_{k'+1, k'+1}} \right) \psi_{k'+1, k'}^{-1}, & k' \leq k-1, \\ \left(\frac{m_{1,k}}{-m_{11}} \right) \psi_{1,k}^{-1}, & k' = k. \end{cases}$$

First note that (35) can be written as

$$\prod_{i \in c} \widehat{\theta}_i^c > \gamma_c, \qquad \forall c \in \Phi.$$

Since $\mathcal{G}(M)$ is connected and cactus, no two simple cycles share an edge. Therefore, one can write (34) as follows:

$$\sum_{c \in \Phi} \widehat{\theta}_i^c < 1, \qquad \forall i \in c.$$

By a straightforward continuity argument, one can show that, for every $c \in \Phi$ and $i \in c$, there exists $\theta_i^c > 0$ such that

$$\prod_{i \in c} \theta_i^c > \gamma_c, \qquad \forall c \in \Phi,$$
$$\sum_{c \in \Phi} \theta_i^c = 1, \qquad \forall i \in c.$$

(ii) $\implies$ (i): Now suppose that, for every $c \in \Phi$ and every $i \in c$, there exists $\theta_i^c > 0$ which satisfies (15). Let $c = (1, \ldots, k, 1)$, and for every $k' \in \{1, \ldots, k-1\}$

$$\psi_{k'+1,k'} = \left(\frac{m_{k'+1,k'}}{-m_{k'+1,k'+1}} \right) (\theta_{k'}^c)^{-1},$$

and

$$\psi_{1,k} = \left(\frac{m_{1,k}}{-m_{11}} \right) (\theta_k^c)^{-1}.$$

By a continuity argument, (15) can be written as (34) and (35). Thus, by Theorem 19, the matrix M is Hurwitz. $\blacksquare$

C ACKNOWLEDGMENTS

REFERENCES

[1] R. Abraham, J. E. Marsden, and T. S. Ratiu. *Manifolds, Tensor Analysis, and Applications*, volume 75 of *Applied Mathematical Sciences*. Springer, 2 edition, 1988, ISBN 0387967907.

[2] D. Angeli and E. D. Sontag. Monotone control systems. *IEEE Transactions on Automatic Control*, 48(10):1684–1698, 2003. doi:10.1109/TAC.2003.817920.

[3] M. Arcak. Diagonal stability on cactus graphs and application to network stability analysis. *IEEE Transactions on Automatic Control*, 56(12):2766–2777, 2011. doi:10.1109/TAC.2011.2125130.

[4] F. Blanchini, P. Colaneri, and M. E. Valcher. Co-positive Lyapunov functions for the stabilization of positive switched systems. *IEEE Transactions on Automatic Control*, 57(12):3038–3050, 2012. doi:10.1109/TAC.2012.2199169.

[5] C. Briat. Sign properties of Metzler matrices with applications. *Linear Algebra and its Applications*, 515:53–86, 2017. doi:10.1016/j.laa.2016.11.011.

[6] F. Bullo. *Lectures on Network Systems*. CreateSpace, 1 edition, 2018, ISBN 978-1986425643. With contributions by J. Cortés, F. Dörfler, and S. Martínez. URL: http://motion.me.ucsb.edu/book-lns.

[7] G. Como. On resilient control of dynamical flow networks. *Annual Reviews in Control*, 43:80–90, 2017. doi:10.1016/j.arcontrol.2017.01.001.

[8] S. Coogan. Separability of Lyapunov functions for contractive monotone systems. In *IEEE Conf. on Decision and Control*, pages 2184–2189, Las Vegas, USA, December 2016. doi:10.1109/CDC.2016.7798587.

[9] S. Coogan. A contractive approach to separable Lyapunov functions for monotone systems, 2017. URL: https://arxiv.org/pdf/1704.04218.

[10] D. E. Crabtree and E. V. Haynsworth. An identity for the Schur complement of a matrix. *Proceedings of American Mathematical Society*, 22(2):364–366, 1969. doi:10.2307/2037057.

[11] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals and Systems*, 19(2):93–122, 2007. doi:10.1007/s00498-007-0014-8.

[12] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010. doi:10.1137/090746483.

[13] F. Dörfler and F. Bullo. Kron reduction of graphs with applications to electrical networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 60(1):150–163, 2013. doi:10.1109/TCSI.2012.2215780.

[14] Y. Ebihara, D. Peaucelle, and D. Arzelier. Analysis and synthesis of interconnected positive systems. *IEEE Transactions on Automatic Control*, 62(2):652–667, 2017. doi:10.1109/TAC.2016.2558287.

[15] L. Farina and S. Rinaldi. *Positive Linear Systems: Theory and Applications*. John Wiley & Sons, 2000, ISBN 0471384569.

[16] M. W. Hirsch. Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere. *SIAM Journal on Mathematical Analysis*, 16(3):423–439, 1985. doi:10.1137/0516030.

[17] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals and Systems*, 7:95–120, 1994. doi:10.1007/BF01211469.

[18] T. Liu, Z.-P. Jiang, and D. J. Hill. *Nonlinear Control of Dynamic Networks*. CRC Press, 2014, ISBN 9781138076617.

[19] J. S. Maybee, D. D. Olesky, P. Van den Driessche, and G. Wiener. Matrices, digraphs, and determinants. *SIAM Journal on Matrix Analysis and Applications*, 10(4):500–519, 1989. doi:10.1137/0610036.

[20] Z. Meng, W. Xia, K. H. Johansson, and S. Hirche. Stability of positive switched linear systems: Weak excitation and robustness to time-varying delay. *IEEE Transactions on Automatic Control*, 62(1):399–405, 2017. doi:10.1109/TAC.2016.2531044.

[21] K. S. Narendra and R. Shorten. Hurwitz stability of Metzler matrices. *IEEE Transactions on Automatic Control*, 55(6):1484–1487, 2010. doi:10.1109/TAC.2010.2045694.

[22] A. Rantzer. Scalable control of positive systems. *European Journal of Control*, 24:72–80, 2015. doi:10.1016/j.ejcon.2015.04.004.

[23] I. W. Sandberg. On the $\mathcal{L}_2$-boundedness of solutions of nonlinear functional equations. *The Bell System Technical Journal*, 43(4):1581–1599, 1964. doi:10.1002/j.1538-7305.1964.tb04098.x.

[24] S. Smale. On the differential equations of species in competition. *Journal of Mathematical Biology*, 3(1):5–7, 1976. doi:10.1007/BF00307854.

[25] H. L. Smith. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. American Mathematical Society, 1995, ISBN 082180393X.

[26] R. L. Smith. Some interlacing properties of the Schur complement of a Hermitian matrix. *Linear Algebra and its Applications*, 177:137–144, 1992. doi:10.1016/0024-3795(92)90321-Z.

[27] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989. doi:10.1109/9.28018.

[28] E. D. Sontag. Monotone and near-monotone biochemical networks. *Systems and Synthetic Biology*, 1(2):59–87, 2007. doi:10.1007/s11693-007-9005-9.

[29] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995. doi:10.1016/0167-6911(94)00050-6.

[30] M. Souza, F. Wirth, and R. Shorten. A note on recursive Schur complements, block Hurwitz stability of Metzler matrices, and related results. *IEEE Transactions on Automatic Control*, 62(8):4167–4172, 2017. doi:10.1109/TAC.2017.2682032.

[31] G. G. Walter and M. Contreras. *Compartmental Modeling with Networks*. Birkhäuser, 1999, ISBN 0817640193. doi:10.1007/978-1-4612-1590-5.

[32] G. Zames. On the input-output stability of time-varying nonlinear feedback systems – Part I: Conditions derived using concepts of loop gain, conicity, and positivity. *IEEE Transactions on Automatic Control*, 11(2):228–238, 1966. doi:10.1109/TAC.1966.1098316.