Convergence Properties of the Heterogeneous Deffuant-Weisbuch Model [⋆]

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Abstract

The Deffuant-Weisbuch (DW) model is a bounded-confidence opinion dynamics model that has attracted much recent interest. Despite its simplicity and appeal, the DW model has proved technically hard to analyze and its most basic convergence properties, easy to observe numerically, are only conjectures.

This paper solves the convergence problem for the heterogeneous DW model with the weighting factor not less than 1/2. We establish that, for any positive confidence bounds and initial values, the opinion of each agent will converge to a limit value almost surely, and the convergence rate is exponential in mean square. Moreover, we show that the limiting opinions of any two agents either are the same or have a distance larger than the confidence bounds of the two agents. Finally, we provide some sufficient conditions for the heterogeneous DW model to reach consensus.

Key words: Opinion dynamics, consensus, Deffuant model, gossip model, bounded confidence model

1 Introduction

The field of opinion dynamics studies the dynamical processes regarding the formation, diffusion, and evolution of public opinion about certain events and object of interest in social systems. The study of opinion dynamics can be traced back to the two-step communication flow model in (Katz and Lazarsfeld, 1955) and the social power and averaging model in (French Jr., 1956). The model by French Jr. (1956) was then elaborated by Harary (1959) and rediscovered by DeGroot (1974). Other notable developments include the model by Friedkin and Johnsen (1990) with attachment to initial opinions, a general influence network theory (Friedkin, 1998), social impact theory (Latané, 1981), and dynamic social impact theory (Latané, 1996). A comprehensive review of opinion dynamics models is given in the two tutorials Proskurnikov and Tempo (2017, 2018) and the textbook Bullo (2019).

In recent years, significant attention has focused on so-called *bounded confidence* (BC) models of opinion dynamics. In these models one individual is willing to accord influence to another only if their pair-wise opinion difference is below a threshold (i.e., the confidence bound). (Deffuant et al., 2000) propose their now well-known BC model called the Deffuant-Weisbuch (DW) model or Deffuant model. In this model a pair of individuals is selected randomly at each discrete time step and each individual updates its opinion if the other individ-

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ual's opinion lies within its confidence bound. A second well-known BC model is the Hegselmann-Krause (HK) model (Hegselmann and Krause, 2002), where all individuals update their opinions synchronously by averaging the opinions of individuals within their confidence bounds.

As reported in (Lorenz, 2007, 2010), simulation results for the DW model have revealed numerous interesting phenomena such as consensus, polarization and fragmentation. However, the DW model is in general hard to analyze due to the nonlinear state-dependent inter-agent topology. Current analysis results focus on the homogeneous case in which all the agents have the same confidence bound. The convergence of the homogeneous DW model has been proved in (Lorenz, 2005) and its convergence rate is established in (Zhang and Chen, 2015). Some research has considered also modified DW models. For example, (Como and Fagnani, 2011) consider a generalized DW model with an interaction kernel and investigate its scaling limits when the number of agents grows to infinity; (Zhang and Hong, 2013) generalize the DW model by assuming that each agent can choose multiple neighbors to exchange opinion at each time step. Despite all this progress, the analysis of the heterogeneous DW model is still incomplete in that its convergence properties are yet to be established.

It is worth remarking that the analysis of the HK model is also similarly restricted to the homogeneous case; the convergence of the heterogeneous HK model is only conjectured in our previous work (MirTabatabaei and Bullo, 2012) and has since been established in (Chazelle and Wang, 2017) only for the special case that the confidence bound of each agent is either 0 or 1. In general, numerous conjectures remain open for heterogeneous bounded-confidence models.

This paper establishes the convergence properties of the heterogeneous DW model with the weighting factor is not less than 1/2. We show that, for any positive confidence bounds and initial opinions, the opinion of each agent converges almost surely to a limiting value, and the convergence rate is exponential in mean square. Additionally we prove that the limiting values of any two agents' opinions are either identical or have a distance larger than the confidence bounds of the two agents. Moreover, we show that a sufficient, and in some cases also necessary, condition for almost sure consensus; the intuitive condition is expressed as a function of the largest confidence bound in the group.

The paper is organized as follows. Section 2 introduces the heterogeneous DW model and our main results. Section 3 contains the proofs of our results. Finally, Section 4 concludes the paper.

2 The heterogeneous DW model and our main convergence results

This paper considers the following DW model proposed in (Deffuant et al., 2000). In a group of $n \geq 3$ agents, we assume each agent $i \in \{1, \dots, n\}$ has a real-valued opinion $x_i(t) \in \mathbb{R}$ at each discrete time $t \in \mathbb{Z}_{\geq 0}$. We let $x(t) := (x_1(t), \dots, x_n(t))^\top$ assume, without loss of generality, that $x(0) \in [0,1]^n$. We let $r_i > 0$ denote the *confidence bound* of the agent i and we assume, without loss of generality,

$$r_1 \geq r_2 \geq \cdots \geq r_n > 0.$$

We let the constant $\mu \in (0,1)$ denote the weighting factor. We let $\mathbb{1}_{\{\cdot\}}$ denote the indicator function, i.e., we let $\mathbb{1}_{\{\omega\}} = 1$ if the property ω holds true and $\mathbb{1}_{\{\omega\}} = 0$ otherwise. At each time $t \in \mathbb{Z}_{\geq 0}$, a pair $\{i_t, j_t\}$ is independently and uniformly selected from the set of all pairs $\mathcal{N} = \{\{i,j\} \mid i,j \in \{1,\dots,n\}, i<j\}$. Subsequently, the opinions of the agents i_t and j_t are updated according to

$$\begin{cases} x_{i_t}(t+1) = x_{i_t}(t) \\ \qquad + \mu \mathbb{1}_{\{|x_{j_t}(t) - x_{i_t}(t)| \leq r_{i_t}\}} (x_{j_t}(t) - x_{i_t}(t)), \\ x_{j_t}(t+1) = x_{j_t}(t) \\ \qquad + \mu \mathbb{1}_{\{|x_{j_t}(t) - x_{i_t}(t)| \leq r_{j_t}\}} (x_{i_t}(t) - x_{j_t}(t)), \end{cases} \tag{1}$$

whereas the other agents' opinions remain unchanged:

$$x_k(t+1) = x_k(t), \ \text{ for } k \in \{1,\dots,n\} \setminus \{i_t, j_t\}. \tag{2}$$

If $r_1 = \cdots = r_n$, the DW model is called homogeneous, otherwise heterogeneous.

Previous works (Lorenz, 2005) show that the homogeneous DW model (1)-(2) always converges to a limit opinion profile. Simulations reported in (Lorenz, 2007) show that this property holds also for the heterogeneous case; but a proof for this statement is lacking. Simulations in (Deffuant et al., 2000; Weisbuch et al., 2002) show that the parameter μ mainly affects the convergence time and so previous works (Lorenz, 2007, 2010) simplified the model by setting $\mu = 1/2$. This paper considers the case when $\mu \in [1/2, 1)$.

Before stating our convergence results, we need to define the probability space of the DW model. If the initial state $x(0)$ is a deterministic vector, we let $\Omega = \mathcal{N}^\infty$ be the sample space, $\mathcal{F}$ be the Borel σ-algebra of Ω, and $\mathbb{P}$ be the probability measure on $\mathcal{F}$. Then the probability space of the DW model is written as $(\Omega, \mathcal{F}, \mathbb{P})$. It is worth mentioning that $\omega \in \Omega$ refers to a particular path of agent pairs for opinion update. If the initial state is a random vector, we let $\Omega = [0,1]^n \times \mathcal{N}^\infty$ be the sample space and, similarly to the case of deterministic initial state, the probability space is defined by $(\Omega, \mathcal{F}, \mathbb{P})$.

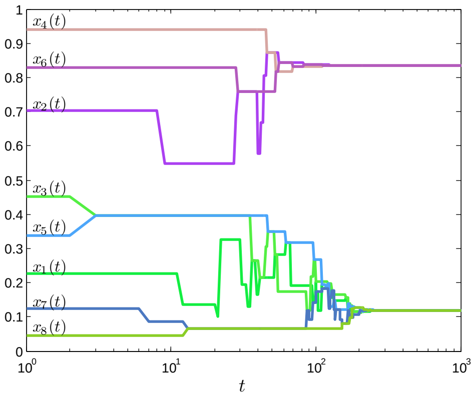

Fig. 1. A convergent simulation of the heterogeneous DW model

Let $\|\cdot\|$ denote the ℓ_2-norm (Euclidean norm). The main results of this paper can be formulated as follows.

Theorem 1 (Convergence and convergence rate of heterogeneous DW model) *Consider the heterogeneous DW model (1)-(2) with* $\mu \in [1/2, 1)$*. For any initial state* $x(0) \in [0,1]^n$*,*

(i) there exists a random vector $x^* \in [0,1]^n$ *satisfying* $x_i^* = x_j^*$ *or* $|x_i^* - x_j^*| > \max\{r_i, r_j\}$ *for all* $i \neq j$*, such that* $x(t)$ *converges to* x^* *almost surely (a.s.) as* $t \to \infty$*, and*

(ii) $\mathbb{E}\|x(t) - x^*\|^2 \leq nc^{\lfloor \frac{t}{2(T+1)} \rfloor} + \frac{n}{4}\left(1 - \frac{8\mu(1-\mu)}{n(n-1)}\right)^{\lfloor \frac{t}{2} \rfloor}$ *with* $T := (n-1)^2(1 + \lceil \log_{1-\mu} \frac{r_n}{r_1} \rceil)\lceil \frac{1-r_n}{(1-\mu)^2 r_n} \rceil$ *and* $c := 1 - \frac{2^T}{n^T(n-1)^T}$*.*

The proof of Theorem 1 is postponed to Section 3. Fig. 1 displays the simulation results for a heterogeneous DW model (1)-(2) with $\mu = 1/2$ and $n = 8$, while the agents' confidence bounds equal to $0.5, 0.41, 0.35, 0.24, 0.175, 0.165, 0.12, 0.047$ respectively. Consistently with Theorem 1, Fig. 1 shows that the individual opinions converge to two distinct limit values and that the distance between the two values is larger than $r_1 = 0.5$.

Theorem 1 leads to two corollaries on convergence to consensus. By consensus we mean that all agents' opinions converge to the same value.

Corollary 2 (Almost sure consensus for large confidence bound) *Consider the heterogeneous DW model (1)-(2) with* $\mu \in [1/2, 1)$*. If the largest confidence bound* r_1 *is not less than 1, then for any initial state* $x(0) \in [0,1]^n$ *the system reaches consensus a.s.*

Corollary 3 (Almost sure consensus if and only if large confidence bound) *Consider the heterogeneous DW model (1)-(2) with* $\mu \in [1/2, 1)$*. Assume that the initial state* $x(0)$ *is randomly distributed in* $[0,1]^n$ *and that its joint probability density has a lower bound* $\rho_{\min} > 0$*, that is, for any real numbers* a_i, b_i*,* $i \in \{1, \ldots, n\}$*, with* $0 \leq a_i < b_i \leq 1$*,*

$$\mathbb{P}\Big(\bigcap_{i=1}^{n} \{x_i(0) \in [a_i, b_i]\} \Big) \geq \rho_{\min} \prod_{i=1}^{n} (b_i - a_i). \quad (3)$$

Then the heterogeneous DW model reaches consensus almost surely if and only if the largest confidence bound $r_1 \geq 1$*.*

Remark 4 *Condition (3) can be satisfied if* $\{x_i(0)\}_{i=1}^n$ *are mutually independent and have positive probability densities over* $[0,1]$*. Examples include independent uniform, or independent truncated Gauss distributions. On the other hand, condition (3) cannot be satisfied if there exists* $x_i(0)$ *which has zero probability density in a subinterval of* $[0,1]$ *with positive measure, e.g., if* $x_i(0)$ *is a discrete random variable.*

Corollary 3 provides a sufficient and necessary condition for almost sure consensus when the initial opinions are randomly distributed. However, for settings when almost sure consensus is not guaranteed, the probability of achieving consensus is unknown. In the remainder of this section, we provide simulation results for the consensus probability of the heterogeneous DW model. Let $\mu = 1/2$ and $n = 10$. Suppose that agent 1 has a maximal confidence bound $r_{\max}$ whose value is chosen over the set $\{\frac{i}{20} : i = 1, 2, \ldots, 20\}$. We approximate the consensus probability via the Monte Carlo method. We run 1000 samples for each value of $r_{\max}$. In each sample, we assume the initial opinions are independently and uniformly distributed on $[0,1]$, while the confidence bounds of agents $2, 3, \ldots, 10$ are independently and uniformly distributed on $[0, r_{\max}]$. Fig. 2 shows the estimated consensus probability of the heterogeneous DW model (1)-(2) as a function of the maximal confidence bound $r_{\max}$.

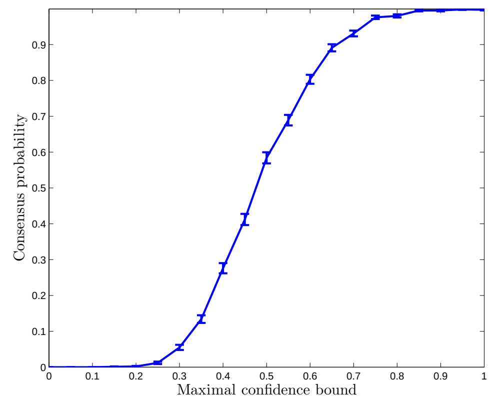

Fig. 2. The estimated consensus probability with respect to the maximal confidence bound $r_{\max}$, where the error bars denote the standard deviations of the estimated probability of reaching consensus at the points of $r_{\max}$.

3 Proof of convergence results

The proof of Theorem 1 requires multiple steps. We adopt the method of "transforming the analysis of a stochastic system into the design of control algorithms" first proposed by (Chen, 2017). This method requires the construction of a new system called as *DW-control system* to help with the analysis of the DW model. The following Lemma 5 gives the connection between the DW model and DW-control system. Also, we introduce a new concept called *maximal-confidence cluster* whose basic properties will be provided in the following Lemmas 6-7. In the following Lemmas 8 and 10, we design control algorithms on DW-control system around the maximal-confidence cluster. The proof of Theorem 1 follows from these lemmas.

3.1 DW-control system and connection to DW model

Consider the DW protocol (1)-(2) where, at each time t, the pair $\{i_t, j_t\}$ is not selected randomly but instead treated as a control input. In other words, assume that $\{i_t, j_t\}$ is chosen from the set $\mathcal{N}$ arbitrarily as a control signal. We call such a control system the *DW-control system.*

Given $S \subseteq \mathbb{R}^n$, we say S *is reached at time* t if $x(t) \in S$ and *is reached in the time interval* $[t_1, t_2]$ if there exists $t \in [t_1, t_2]$ such that $x(t) \in S$.

The following lemma builds a connection between the DW model and DW-control system.

Lemma 5 (Connection between DW model and DW-control system) *Let* $S \subseteq [0,1]^n$ *be a set of states. Assume there exists a duration* $t^* > 0$ *such that for any* $x(0) \in [0,1]^n$*, we can find a sequence of pairs* $\{i'_0, j'_0\}, \{i'_1, j'_1\}, \ldots, \{i'_{t^*-1}, j'_{t^*-1}\}$ *for opinion update which guarantees* S *is reached in the time interval* $[0, t^*]$*. Let* $a := 1 - \frac{2^{t^*}}{n^{t^*}(n-1)^{t^*}}$*. Then, under the DW protocol, for any initial state* $x(0) \in [0,1]^n$ *we have*

$$\mathbb{P}\left(\tau \geq t\right) \leq a^{\lfloor t/(t^*+1) \rfloor}, \quad \forall t \geq 1,$$

where $\tau := \min\{t' : x(t') \in S\}$ *is the time when* S *is firstly reached.*

PROOF. First according to the rule of the DW protocol (1)-(2) we get $x(t) \in [0,1]^n$ for all $t \geq 0$. Also, for any $x(t) \in [0,1]^n$, by the assumption of this lemma we can find a sequence of pairs $\{i'_t, j'_t\}, \{i'_{t+1}, j'_{t+1}\}, \ldots, \{i'_{t+t^*-1}, j'_{t+t^*-1}\}$ for opinion update such that S is reached in $[t, t+t^*]$ under the DW-control system. Thus, under the DW protocol, for any $t \geq 0$ and $x(t) \in [0,1]^n$ we have

$$\begin{aligned}
&\mathbb{P}\left(\{S \text{ is reached in } [t, t+t^*]\} \,|x(t)\right) \\
&\quad \geq \mathbb{P}\Big(\bigcap_{s=t}^{t+t^*-1} \{\{i_s, j_s\} = \{i'_s, j'_s\}\} |x(t)\Big) \\
&\quad = \prod_{s=t}^{t+t^*-1} \mathbb{P}\big(\{i_s, j_s\} = \{i'_s, j'_s\}\big) \\
&\quad = \frac{1}{|\mathcal{N}|^{t^*}} = \frac{2^{t^*}}{n^{t^*}(n-1)^{t^*}},
\end{aligned} \tag{4}$$

where the first and second equalities use the fact that $\{i_t, j_t\}$ is uniformly and independently selected from the set $\mathcal{N}$, and $|\mathcal{N}|$ denotes the cardinality of the set $\mathcal{N}$.

Set E_t to be the event that S is reached in $[t, t+t^*]$, and let E_t^c be the complement set of E_t. For any integer $M > 0$ and $x(0) \in [0,1]^n$, Bayes' Theorem and equation (4) imply

$$\begin{aligned}
&\mathbb{P}\big(\{S \text{ is not reached in } [0, (t^*+1)M-1]\} \,|x(0)\big) \\
&= \mathbb{P}\Big(\bigcap_{m=0}^{M-1} E^c_{m(t^*+1)} |x(0)\Big) \\
&= \mathbb{P}\left(E^c_0|x(0)\right) \prod_{m=1}^{M-1} \mathbb{P}\Big(E^c_{m(t^*+1)} \Big| x(0), \bigcap_{0 \leq m' < m} E^c_{m'(t^*+1)}\Big) \\
&\leq \Big(1 - \frac{2^{t^*}}{n^{t^*}(n-1)^{t^*}}\Big)^M = a^M.
\end{aligned} \tag{5}$$

For any integer $M > 0$ and $x(0) \in [0,1]^n$, by (5) we have

$$\begin{aligned}
&\mathbb{P}\big(\tau \geq (t^*+1)M|x(0)\big) \\
&\quad = \mathbb{P}\big(\{S \text{ is not reached in } [0, (t^*+1)M-1]\} \,|x(0)\big) \\
&\quad \leq a^M,
\end{aligned}$$

and, in turn,

$$\mathbb{P}\left(\tau \geq t|x(0)\right) \leq \mathbb{P}\left(\tau \geq \left\lfloor \frac{t}{T} \right\rfloor T|x(0)\right) \leq a^{\lfloor t/(t^*+1) \rfloor}.$$

□

According to Lemma 5, to prove the convergence of the DW model, we only need to design control algorithms for DW-control system such that a convergence set is reached. Before the design of such control algorithms we introduce some useful notions.

3.2 Maximal-confidence clusters and properties

Recall that we assume $r_1 \geq r_2 \geq \cdots \geq r_n > 0$. For any opinion state $x = (x_1, \ldots, x_n) \in [0,1]^n$, let $C_1(x) \subseteq$

$\{1,\ldots,n\}$ be the set of the agents that can connect to agent 1 directly or indirectly with the confidence bound r_1, i.e., $i \in C_1(x)$ if and only if $|x_i - x_1| \leq r_1$ or there exists some agents $1', 2', \ldots, k' \in \{1,\ldots,n\}$ such that $|x_i - x_{1'}| \leq r_1, |x_{1'} - x_{2'}| \leq r_1, \ldots, |x_{k'} - x_1| \leq r_1$. From this definition we have $1 \in C_1(x)$.

Set $\widetilde{C}_1(x) := \{1,\ldots,n\} \setminus C_1(x)$. If $\widetilde{C}_1(x)$ is not empty, we let $i_2 := \min_{i \in \widetilde{C}_1(x)} i$ and define $C_2(x) \subseteq \widetilde{C}_1(x)$ to be the set of the agents that can connect to agent i_2 directly or indirectly with the confidence bound r_{i_2}. Set $\widetilde{C}_2(x) := \{1,\ldots,n\} \setminus (C_1(x) \cup C_2(x))$. If $\widetilde{C}_2(x)$ is not empty, we let $i_3 := \min_{i \in \widetilde{C}_2(x)} i$ and define $C_3(x) \subseteq \widetilde{C}_2(x)$ to be the set of the agents that can connect to agent i_3 directly or indirectly with the confidence bound r_{i_3}. Repeat this process until there exists an integer K such that $\widetilde{C}_K(x) = \emptyset$. We call the sets $C_1(x), C_2(x), \ldots, C_K(x)$ *maximal-confidence (MC) clusters*. Note that MC clusters are quite different from connected components in graph theory.

To illustrate the definition of MC clusters we give an example, visualized Fig. 3: Assume that $n = 7$ and that the agents are labeled by $1, 2, \ldots, 7$. We suppose $r_1 \geq r_2 \geq \cdots \geq r_7$. With the confidence bound r_1 the agent 1 can connect to agents 5 and 7, and the agent 7 can connect to agent 3; however agent 3 cannot connect to agent 2. Thus, the first MC cluster $C_1(x)$ is $\{1, 3, 5, 7\}$. The remaining agents are $2, 4$, and 6. With the confidence r_2 the agent 2 can connect to agent 4, and the agent 4 can connect to agent 6, so the second MC cluster $C_2(x)$ is $\{2, 4, 6\}$.

The following lemma can be derived immediately from the definition of MC cluster.

Lemma 6 (Distance between maximal-confidence clusters) *For any opinion state $x \in [0,1]^n$ and two different MC clusters $C_i(x)$ and $C_j(x)$, let $r_{\max}^{ij} := \max_{k \in C_i(x) \cup C_j(x)} r_k$ be the maximal confidence bound of all agents in $C_i(x)$ and $C_j(x)$. Then, the opinion values of agents in $C_i(x)$ are all $r_{\max}^{ij}$ bigger or smaller than those in $C_j(x)$, i.e.,*

$$
\begin{aligned}
x_k - x_l &> r_{\max}^{ij} \quad \forall k \in C_i(x), l \in C_j(x), \qquad \textit{or} \\
x_l - x_k &> r_{\max}^{ij} \quad \forall k \in C_i(x), l \in C_j(x).
\end{aligned}
$$

Under the DW protocol (1)-(2), the MC clusters have the convex property as follows.

Lemma 7 (Convexity of maximal-confidence clusters) *Consider the DW protocol (1)-(2) with arbitrary initial state and update pairs $\{\{i_t, j_t\}\}_{t \geq 0}$. For any $t \geq 0$ and any MC cluster $C_i(x(t))$, the opinion values of all agents in $C_i(x(t))$ will always stay in the interval $[x_{\min}^i(t), x_{\max}^i(t)]$ at the time $s \geq t$, i.e.,*

$$x_{\min}^i(t) \leq x_j(s) \leq x_{\max}^i(t), \quad \forall j \in C_i(x(t)), s \geq t,$$

where $x_{\min}^i(t) := \min_{k \in C_i(x(t))} x_k(t)$ and $x_{\max}^i(t) := \max_{k \in C_i(x(t))} x_k(t)$ denote the minimal and maximal opinion values of all agents in $C_i(x(t))$ respectively.

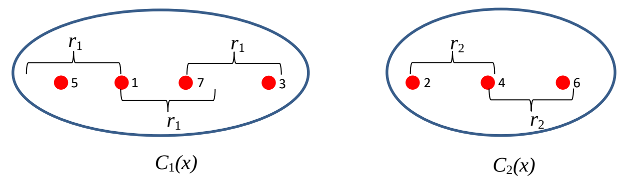

Fig. 3. Two MC clusters $C_1(x)$ and $C_2(x)$. The distance between two adjacent nodes in $C_1(x)$ (or $C_2(x)$) is not bigger than the r_1 (or r_2), but the distance between the agents 3 and 2 is bigger than r_1.

PROOF. Assume that at time t all MC clusters are $C_1 = C_1(x(t)), C_2 = C_2(x(t)), \ldots, C_K = C_K(x(t))$. By Lemma 6 we can order these clusters as

$$C_{j_1} \prec C_{j_2} \prec \cdots \prec C_{j_K},$$

and get, for $1 \leq k \leq K-1$,

$$\min_{l \in C_{j_{k+1}}} x_l(t) - \max_{l \in C_{j_k}} x_l(t) > r^{k,k+1}, \tag{6}$$

where $C_i \prec C_j$ means that at time t the opinion values of the agents in C_i are all less than those in C_j, and $r^{k,k+1} := \max_{l \in C_{j_k} \cup C_{j_{k+1}}} r_l$.

By the DW protocol (1)-(2), if the update pair $\{i_t, j_t\}$ belongs to different MC clusters then from (6) we have $x_{i_t}(t+1) = x_{i_t}(t)$ and $x_{j_t}(t+1) = x_{j_t}(t)$; if $\{i_t, j_t\}$ belongs to a same MC cluster C_{j_k} then $x_{i_t}(t+1)$ and $x_{j_t}(t+1)$ will stay in the interval $[x_{\min}^{j_k}(t), x_{\max}^{j_k}(t)]$. Thus, for $1 \leq k \leq K-1$,

$$\min_{l \in C_{j_{k+1}}} x_l(t+1) - \max_{l \in C_{j_k}} x_l(t+1) > r^{k,k+1}.$$

Repeating this process yields our result. $\square$

With the definition and properties of MC clusters we can design control algorithms and complete final proof of our results in the following subsection.

3.3 Design of control algorithms

Throughout this subsection we assume $\mu \in [1/2, 1)$ and $r_1 \geq r_2 \geq \cdots \geq r_n > 0$. We first design control algorithms to split a MC cluster into different MC clusters, or reduce its diameter by a certain value in finite time.

Lemma 8 *Let $t \geq 0$ and $x(t) \in [0,1]^n$ be arbitrarily given. Let $C_i(x(t))$ be an arbitrary MC cluster, in which the agents' maximal and minimal confidence bounds are $r^i_{\max}$ and $r^i_{\min}$ respectively. Assume*

$$\max_{M,m\in C_i(x(t))}[x_M(t)-x_m(t)] > r^i_{\min}. \tag{7}$$

Then, under the DW-control system, there is a sequence of agent pairs $\{i'_t, j'_t\}, \{i'_{t+1}, j'_{t+1}\}, \dots, \{i'_{t+t^*-1}, j'_{t+t^*-1}\}$ with

$$t^* \leq (|C_i(x(t))|-1)^2 \left(1+\lceil \log_{1-\mu} r^i_{\min}/r^i_{\max}\rceil\right)$$

for opinion update, such that one of the following two results holds:

(i) the agents in $C_i(x(t))$ split into different MC clusters at time $t+t^*$; and
(ii) we have

$$\begin{aligned}&\max_{M,m\in C_i(x(t))}[x_M(t+t^*)-x_m(t+t^*)]\\ &\leq \max_{M,m\in C_i(x(t))}[x_M(t)-x_m(t)] - (1-\mu)^2 r^i_{\min}.\end{aligned}$$

The proof of Lemma 8 is quite complicated. We put it in Appendix A.

Remark 9 *The result in Lemma 8 cannot be extended to the case when $\mu < 1/2$. For example, assume $n = 3$, $(r_1, r_2, r_3) = (0.4, 0.3, 0.2)$, and $(x_1(0), x_2(0), x_3(0)) = (0.1-\varepsilon, 0.1+\varepsilon, 0.5)$, where $\varepsilon \in (0, 0.2(1-\mu)^2)$ is a small constant. Then, the three agents form a MC cluster, however the interaction exists only between agents 1 and 2. Because*

$$\lim_{t\to\infty}\begin{bmatrix}1-\mu & \mu\\ \mu & 1-\mu\end{bmatrix}^t = \begin{bmatrix}0.5 & 0.5\\ 0.5 & 0.5\end{bmatrix},$$

we know $x_1(t)\uparrow 0.1$, $x_2(t)\downarrow 0.1$ as $t\to\infty$ if we always choose $\{1,2\}$ as the opinion update pair. In fact, we cannot find a finite sequence of opinion update pairs such that either result (i) or result (ii) in Lemma 8 holds.

For any opinion state $x \in [0,1]^n$ and any MC cluster $C_i(x)$, we say that $C_i(x)$ is a *complete cluster* if any agent in $C_i(x)$ can interact with others with the minimal confidence bound of $C_i(x)$, i.e.,

$$\max_{j,k\in C_i(x)}|x_j - x_k| \leq \min_{j\in C_i(x)} r_j.$$

Lemma 8 leads to control algorithms such that all MC clusters become complete clusters in finite time.

Lemma 10 *Consider the DW-control system. Then for any initial state, there exists a sequence of agent pairs $\{i'_0, j'_0\}, \{i'_1, j'_1\}, \dots, \{i'_{T-1}, j'_{T-1}\}$ with*

$$T \leq (n-1)^2\left(1+\left\lceil \log_{1-\mu}\frac{r_n}{r_1}\right\rceil\right)\left\lceil\frac{1-r_n}{(1-\mu)^2 r_n}\right\rceil$$

for opinion update such that all MC clusters are complete clusters at time T.

PROOF. Assume that at time t all agents are divided into K_t MC clusters labeled as $C_1(x(t)),\dots, C_{K_t}(x(t))$. Define

$$f_i(t) := \begin{cases}0, \text{ if } C_i(x(t)) \text{ is a complete cluster,}\\ \max\limits_{M,m\in C_i(x(t))}[x_M(t)-x_m(t)], \text{ otherwise,}\end{cases}$$

and $F(t) := \sum_{i=1}^{K_t} f_i(t)$. By Lemma 6 we have $F(t) \in \{0\}\cup(r_n, 1]$, and all MC clusters are complete clusters at time t if and only if $F(t) = 0$. If $F(t) > 0$, by Lemmas 6 and 8 there is a sequence of agent pairs $\{i'_t, j'_t\}, \{i'_{t+1}, j'_{t+1}\}, \dots, \{i'_{t+t^*-1}, j'_{t+t^*-1}\}$ with

$$t^* \leq (n-1)^2\left(1+\lceil\log_{1-\mu} r_n/r_1\rceil\right)$$

for opinion update, such that

$$F(t+t^*) \leq F(t) - (1-\mu)^2 r_n.$$

With this process repeated, we can find a sequence of agent pairs $\{i'_0, j'_0\}, \{i'_1, j'_1\}, \dots, \{i'_{T-1}, j'_{T-1}\}$ with

$$T \leq (n-1)^2\left(1+\left\lceil \log_{1-\mu}\frac{r_n}{r_1}\right\rceil\right)\left\lceil\frac{1-r_n}{(1-\mu)^2 r_n}\right\rceil$$

for opinion update, such that $F(T) = 0$. □

3.4 Final proofs

Proof of Theorem 1 The proof of convergence rate partly uses the idea appearing in Section II.B of (Boyd et al., 2006). Let τ be the first time when all MC clusters are complete clusters under the DW protocol (1)-(2). By Lemmas 10 and 5,

$$\mathbb{P}(\tau \geq t) \leq c^{\lfloor t/(T+1)\rfloor}, \quad \forall t \geq 1. \tag{8}$$

Then $\mathbb{P}(\tau < \infty) = 1$. Label the MC clusters as $C_1, \dots, C_K$ at time τ. By Lemmas 6 and 7, for $t \geq \tau$ all MC clusters $C_1, \dots, C_K$ remain unchanged, i.e., if node i belongs to a cluster C_j at time τ then it will always belong to C_j for $t > \tau$.

Next, we consider the dynamics when $t \geq \tau$. Define the matrix $P(t) \in [0,1]^{n\times n}$ by

$$
\begin{aligned}
&(P_{ii}(t), P_{jj}(t), P_{ij}(t), P_{ji}(t)) := \\
&\begin{cases}
(1-\mu, 1-\mu, \mu, \mu), & \text{if } \{i,j\} \text{ is the opinion update} \\
& \text{pair at time } t \text{ and belongs the} \\
& \text{same MC cluster,} \\
(1,1,0,0), & \text{otherwise.}
\end{cases}
\end{aligned} \tag{9}
$$

for all $i < j$. Then $P(t)$ is a symmetric stochastic matrix. By the protocol (1)-(2), and the facts that all MC clusters are complete clusters, and two agents in different MC clusters have no interaction, we can get

$$x(t+1) = P(t)x(t), \quad \forall t \geq \tau.$$

Also, because $C_1, \dots, C_K$ remain unchanged, there exists a permutation matrix $Q \in \{0,1\}^{n\times n}$ such that

$$Q^\top P(t) Q = \operatorname{diag}(W_1(t), \dots, W_K(t)) := W(t), \tag{10}$$

where $W_k(t)$ is a $|C_k| \times |C_k|$ matrix corresponding to the MC cluster C_k. For $z(t) := Q^\top x(t)$, we have

$$
\begin{aligned}
z(t+1) &= Q^\top x(t+1) = Q^\top P(t)x(t) \\
&= Q^\top P(t) Q z(t) = W(t)z(t) \\
&= W(t)\cdots W(\tau)z(\tau) \\
&= \operatorname{diag}\big(W_1(t)\cdots W_1(\tau), \dots, \\
&\qquad W_K(t)\cdots W_K(\tau)\big)z(\tau).
\end{aligned} \tag{11}
$$

Set $J_0 := 0$ and $J_k := |C_1| + |C_2| + \cdots + |C_k|$ for $1 \leq k \leq K$. Let

$$\vec{z}_k(t) := (z_{J_{k-1}+1}(t), z_{J_{k-1}+2}(t) \dots, z_{J_k}(t))^\top.$$

By (11), for $1 \leq k \leq K$ we have

$$\vec{z}_k(t+1) = W_k(t)\vec{z}_k(t) = W_k(t)\cdots W_k(\tau)\vec{z}_k(\tau). \tag{12}$$

Let $\overline{z_k}(t) := \frac{\mathbf{1}_{|C_k|}^\top \vec{z}_k(t)}{|C_k|}\mathbf{1}_{|C_k|}$ be the average vector of $\vec{z}_k(t)$, where $\mathbf{1}_{|C_k|} = (1,\dots,1)^\top$ is a $|C_k|$−dimensional column vector. By (9) and (10) we know $W_k(t)$ is a symmetric stochastic matrix so that

$$
\begin{aligned}
\overline{z_k}(t+1) &= \frac{[\mathbf{1}_{|C_k|}^\top W_k(t)]\vec{z}_k(t)}{|C_k|}\mathbf{1}_{|C_k|} = \frac{\mathbf{1}_{|C_k|}^\top \vec{z}_k(t)}{|C_k|}\mathbf{1}_{|C_k|} \\
&= \overline{z_k}(t) = \cdots = \overline{z_k}(\tau).
\end{aligned} \tag{13}
$$

Set $\vec{y}_k(t) := \vec{z}_k(t) - \overline{z_k}(t)$. We note that if $|C_k| = 1$ then $\vec{y}_k(t) = 0$. Thus, we only need to consider the case when $|C_k| \geq 2$. By (12) and (13) we have

$$
\begin{aligned}
\vec{y}_k(t+1) &= \vec{z}_k(t+1) - \overline{z_k}(t+1) \\
&= W_k(t)(\vec{z}_k(t) - \overline{z_k}(t)) = W_k(t)\vec{y}_k(t).
\end{aligned}
$$

Therefore, we can write

$$\mathbb{E}\left[\|\vec{y}_k(t+1)\|^2 | \vec{y}_k(t)\right] = \vec{y}_k^\top(t)\mathbb{E}\left[W_k^2(t)\right]\vec{y}_k(t). \tag{14}$$

Because an agent pair for opinion update is selected uniformly and independently from $\mathcal{N}$ at each time, by (9) we have

$$
\begin{aligned}
\mathbb{E}\left[W_k^2(t)\right]_{ij} &= \sum_{l=1}^{|C_k|}\mathbb{E}[W_k(t)]_{il}[W_k(t)]_{lj} \\
&= \mathbb{E}\left([W_k(t)]_{ii}[W_k(t)]_{ij} + [W_k(t)]_{ij}[W_k(t)]_{jj}\right) \\
&= \mathbb{P}\left([W_k(t)]_{ij} > 0\right)[(1-\mu)\mu + \mu(1-\mu)] \\
&= \frac{4\mu(1-\mu)}{n(n-1)}, \qquad \forall i \neq j,
\end{aligned}
$$

and then $\mathbb{E}\left[W_k^2(t)\right]_{ii} = 1 - \dfrac{4\mu(1-\mu)(|C_k|-1)}{n(n-1)}$. Let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|C_k|}$ be the eigenvalues of $\mathbb{E}W_k^2(t)$, while $\xi_1, \dots, \xi_{|C_k|}$ be the corresponding unit right eigenvectors. It can be computed that

$$\lambda_2 = \cdots = \lambda_{|C_k|} = 1 - \frac{|C_k| 4\mu(1-\mu)}{n(n-1)}.$$

Also, $\xi_1 = \frac{\mathbf{1}_{|C_k|}}{\sqrt{|C_k|}} \perp \vec{y}_k(t)$, and $\xi_i \perp \xi_j$ for $i \neq j$, we have

$$
\begin{aligned}
&\vec{y}_k^\top(t)\mathbb{E}\left[W_k^2(t)\right]\vec{y}_k(t) \\
&\quad = \left(\sum_{l=2}^{n}(\xi_l^\top \vec{y}_k(t))\xi_l\right)^\top \left(\sum_{l=2}^{n}(\xi_l^\top \vec{y}_k(t))\lambda_l \xi_l\right) \\
&\quad = \left(1 - \frac{|C_k| 4\mu(1-\mu)}{n(n-1)}\right)\vec{y}_k^\top(t)\vec{y}_k(t).
\end{aligned}
$$

Combining this with (14) we have

$$\mathbb{E}\|\vec{y}_k(t+1)\|^2 = \left(1 - \frac{|C_k| 4\mu(1-\mu)}{n(n-1)}\right)\mathbb{E}\|\vec{y}_k(t)\|^2. \tag{15}$$

Using (15) repeatedly we can get

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k=1}^{K}\|\vec{y}_k(t)\|^2 | t \geq \tau, C_1, \dots, C_K\right] \\
&\quad = \mathbb{E}\Big[\sum_{1\leq k\leq K, |C_k|\geq 2}\left(1 - \frac{|C_k| 4\mu(1-\mu)}{n(n-1)}\right)^{t-\tau} \\
&\qquad\qquad \times \|\vec{y}_k(\tau)\|^2 \Big| \, t \geq \tau, C_1, \dots, C_K\Big] \\
&\quad \leq \frac{n}{4}\left(1 - \frac{8\mu(1-\mu)}{n(n-1)}\right)^{t-\tau},
\end{aligned} \tag{16}
$$

where the inequality uses the Popoviciu inequality (Popoviciu, 1935) which says for any real numbers $b_1, \ldots, b_m$ we have

$$\frac{1}{m}\sum_{l=1}^{m}\left(b_l - \frac{b_1+\cdots+b_m}{m}\right)^2 \le \frac{1}{4}\left(\max_l b_l - \min_l b_l\right)^2.$$

Let $x^* := Q(\overline{z_1}^\top(\tau), \ldots, \overline{z_K}^\top(\tau))^\top$. Since $z(t)$ is a rearrangement of entries of $x(t)$, by (8), (16) and the total probability formula we have

$$\begin{aligned}
&\mathbb{E}\|x(t)-x^*\|^2 \\
&= \mathbb{P}\left(\tau > \frac{t}{2}\right)\mathbb{E}\left[\|x(t)-x^*\|^2 \middle| \tau > \frac{t}{2}\right] \\
&\quad + \mathbb{P}\left(\tau \le \frac{t}{2}\right)\mathbb{E}\left[\|x(t)-x^*\|^2 \middle| \tau \le \frac{t}{2}\right] \\
&\le nc^{\lfloor \frac{t}{2(T+1)} \rfloor} + \frac{n}{4}\left(1 - \frac{8\mu(1-\mu)}{n(n-1)}\right)^{\lfloor \frac{t}{2} \rfloor}. \qquad (17)
\end{aligned}$$

For any constant $\varepsilon > 0$, by (17) and the Markov's inequality we can get

$$\sum_{t=1}^{\infty}\mathbb{P}(\|x(t)-x^*\| > \varepsilon) \le \sum_{t=1}^{\infty}\frac{\mathbb{E}\|x(t)-x^*\|^2}{\varepsilon^2} < \infty,$$

then by the Borel-Cantelli lemma we have a.s. $x(t) \to x^*$ as $t \to \infty$. By Lemma 6 and the definition of x^* we obtain $x_i^* = x_j^*$ or $|x_i^* - x_j^*| > \max\{r_i, r_j\}$ for any $i \ne j$. $\square$

Proof of Corollary 2 By Theorem 1 we have $x(t)$ a.s. converges to a limit point $x^* \in [0,1]^n$ which satisfies either $|x_1^* - x_i^*| = 0$ or $|x_1^* - x_i^*| > r_1$ for all $2 \le i \le n$. Because $r_1 \ge 1$, we have $|x_1^* - x_i^*| = 0$ for all $2 \le i \le n$, which indicates x^* is a consensus state. $\square$

Proof of Corollary 3 If $r_1 \ge 1$, then Corollary 2 implies that the system reaches consensus a.s.

If $r_1 < 1$, then equation (3) implies

$$\begin{aligned}
&\mathbb{P}\Big(x_1(0) \in \Big[0, \frac{1-r_1}{3}\Big], \bigcap_{i=2}^{n}\Big\{x_i(0) \in \Big[\frac{2+r_1}{3}, 1\Big]\Big\}\Big) \\
&\qquad\qquad \ge \rho_{\min}\left(\frac{1-r_1}{3}\right)^n.
\end{aligned}$$

Also, if $x_1(0) \in [0, \frac{1-r_1}{3}]$ and the event $\bigcap_{i=2}^{n}\{x_i(0) \in [\frac{2+r_1}{3}, 1]\}$ takes place, then $|x_1(0) - x_i(0)| = \frac{1+2r_1}{3} > r_1$ for $2 \le i \le n$. In turn, this implies that the system cannot reach consensus because the agent 1 can never interact with the agents $2, \ldots, n$. $\square$

4 Conclusions

Bounded confidence (BC) models of opinion dynamics adopt a mechanism whereby individuals are not willing to accept other opinions if these other opinions are beyond a certain confidence bound. These models have attracted significant mathematical and sociological attention in recent years. One well-known BC model is the Deffuant-Weisbuch (DW) model, in which a pair of agents is selected randomly at each time step, and each agent in the pair updates its opinion if the other agent's opinion in the pair is within its confidence bound. Because the inter-agent topology of the DW model is coupled with the agents' states, the heterogeneous DW model is hard to analyze. This paper proves the convergence of a heterogeneous DW model and shows the mean-square error is bounded by a negative exponential function of time.

As directions for future research, it remains to prove the convergence of the heterogeneous DW model with the weighting factor $\mu \in (0, 1/2)$. From Remark 9, the convergence for the case $\mu \in (0, 1/2)$ cannot be deduced directly by the current method. A more ingenious control design may be required to establish that the DW-control system converges to a set with invariant topology in finite time.

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A The proof of Lemma 8

The proof of this lemma is identical for all cases $t = 0, 1, 2, \ldots$. To simplify the exposition we consider only the case when $t = 0$.

Assume the agents j and k have the minimal and maximal opinions among $C_i(x(0))$ at time 0 respectively, i.e.,

$$x_j(0) = \min_{m \in C_i(x(0))} x_m(0), \quad x_k(0) = \max_{m \in C_i(x(0))} x_m(0).$$

Also, assume that the agent l has the maximal confidence bound $r^i_{\max}$ in $C_i(x(0))$.

We first consider the case when $x_l(0) \geq \frac{x_k(0)+x_j(0)}{2}$. From (7) we have

$$x_l(0) \geq x_j(0) + r^i_{\min}/2. \tag{A.1}$$

Let

$$\underline{A}(s) := \{m \in C_i(x(0)) : x_m(s) < x_j(0) + (1-\mu)^2 r^i_{\min}\}.$$

We aim to control the agent l to episodically come and pull out one more agent from $\underline{A}(s)$, or otherwise we have a split of clusters. The control strategy can be divided into the following steps:

Step 1: Control the agent pairs for opinion update until one of the following two events happens:
(E1) The agents in $C_i(x(0))$ split into different MC clusters;
(E2) $|\underline{A}(s)| = |\underline{A}(0)| - 1$, where $|\cdot|$ denote the cardinality of a set.

Let i_0' be the agent in $C_i(x(0))$ which has the smallest opinion within the confidence bound of agent l at time 0, i.e.,

$$i_0' = \underset{m \in C_i(x(0))}{\arg\min} \{x_m(0) : |x_l(0) - x_m(0)| \leq r_l\}.$$

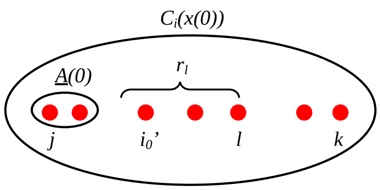

Fig. A.1. An example for the relation of $C_i(x(0))$, $\underline{A}(0)$, and agents j, k, l, and i_0'.

An example for the relation of $C_i(x(0))$, $\underline{A}(0)$, and agents j, k, l, and i_0' is shown in Fig. A.1. Set

$$T_1 := \max\left\{\left\lceil \log_{1-\mu} \frac{r_{i_0'}}{x_l(0) - x_{i_0'}(0)} \right\rceil, 0\right\}.$$

We can get $T_1 \le \lceil \log_{1-\mu} r_n \rceil$ is uniformly bounded. Choose $\{i_0', l\}$ as the agent pair for opinion update at times $0, 1, \dots, T_1$. If $T_1 = 0$, we have $x_l(0) - x_{i_0'}(0) \le r_{i_0'}$, then by the protocol (1)-(2) and the fact of $\mu \in [1/2, 1)$ we get

$$\begin{aligned} x_l(1) &= (1-\mu)x_l(0) + \mu x_{i_0'}(0) \\ &\le (1-\mu)x_{i_0'}(0) + \mu x_l(0) = x_{i_0'}(1) \end{aligned} \quad (A.2)$$

If $T_1 \ge 1$, by the definition of T_1 we have

$$\begin{aligned} &T_1 - 1 < \log_{1-\mu} \frac{r_{i_0'}}{x_l(0) - x_{i_0'}(0)} \le T_1 \iff \\ &(1-\mu)^{-T_1+1} r_{i_0'} < x_l(0) - x_{i_0'}(0) \le (1-\mu)^{-T_1} r_{i_0'}. \end{aligned} \quad (A.3)$$

Using (A.3) and the protocol (1)-(2) repeatedly we obtain

$$\begin{cases} x_{i_0'}(s) = x_{i_0'}(0) \\ x_l(s) = x_{i_0'}(0) + (1-\mu)^s (x_l(0) - x_{i_0'}(0)) \end{cases}$$

for $s = 1, \dots, T_1$, and

$$\begin{cases} x_{i_0'}(T_1+1) = x_{i_0'}(0) + \mu(1-\mu)^{T_1}(x_l(0) - x_{i_0'}(0)) \\ x_l(T_1+1) = x_{i_0'}(0) + (1-\mu)^{T_1+1}(x_l(0) - x_{i_0'}(0)) \end{cases}. \quad (A.4)$$

We continue our discussion by considering the following two cases:

Case I: $i_0' \in \underline{A}(0)$. By (A.2), (A.3), and (A.4) we get

$$\begin{aligned} x_{i_0'}(T_1+1) &\ge x_l(T_1+1) \\ &= x_{i_0'}(0) + (1-\mu)^{T_1+1}(x_l(0) - x_{i_0'}(0)) \\ &> x_{i_0'}(0) + (1-\mu)^2 r_{i_0'} \\ &\ge x_j(0) + (1-\mu)^2 r_{\min}^i. \end{aligned} \quad (A.5)$$

Because all agents except l and i_0' keep their opinions invariant during the time $[0, T_1+1]$, by (A.5) we have

$$|\underline{A}(T_1+1)| = |\underline{A}(0)| - 1. \quad (A.6)$$

Case II: $i_0' \notin \underline{A}(0)$. By (A.2) and (A.4) we get

$$x_l(T_1+1) \le x_{i_0'}(T_1+1) < x_l(0). \quad (A.7)$$

Let $\mathcal{L}_l(s)$ denote the set of the agents in $C_i(x(0))$ whose opinions at time s are less than $x_l(s)$, i.e.,

$$\mathcal{L}_l(s) := \{m \in C_i(x(0)) : x_m(s) < x_l(s)\}.$$

By (A.7) we have

$$|\mathcal{L}_l(T_1+1)| \le |\mathcal{L}_l(0)| - 1. \quad (A.8)$$

Let i_1' be the agent in $C_i(x(0))$ which has the smallest opinion within the confidence bound of agent l at time $T_1 + 1$, i.e.,

$$\begin{aligned} i_1' = \operatorname*{arg\,min}_{m \in C_i(x(0))} \{&x_m(T_1+1) : \\ &|x_l(T_1+1) - x_m(T_1+1)| \le r_l\}. \end{aligned}$$

If $x_{i_1'}(T_1+1) = x_l(T_1+1)$, the agents in $C_i(x(0))$ split into different MC clusters; otherwise, let

$$\begin{aligned} T_2 &:= T_1 + 1 \\ &+ \max\left\{\left\lceil \log_{1-\mu} \frac{r_{i_1'}}{x_l(T_1+1) - x_{i_1'}(T_1+1)} \right\rceil, 0\right\}, \end{aligned}$$

and choose $\{i_1', l\}$ as the agent pair for opinion update at times $T_1+1, T_1+2, \dots, T_2$.

If $i_1' \in \underline{A}(0)$, similar to case I we get $|\underline{A}(T_2+1)| = |\underline{A}(0)| - 1$.

If $i_1' \notin \underline{A}(0)$, similar to (A.8) we have

$$|\mathcal{L}_l(T_2+1)| \le |\mathcal{L}_l(T_1+1)| - 1. \quad (A.9)$$

Repeat the above process until the agents in $C_i(x(0))$ split into different MC clusters, or $|\underline{A}(T_p+1)| = |\underline{A}(0)| - 1$ for some positive integer p. By (A.8)-(A.9) we get that

$$p \le |\mathcal{L}_l(0)| - |\underline{A}(0)| + 1 \le |C_i(x(0))| - |\underline{A}(0)|.$$

From this inequality and the definition of $T_1, T_2, \dots$ we have

$$T_p + 1 \le \left(|C_i(x(0))| - |\underline{A}(0)|\right)\left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right). \quad (A.10)$$

Let t_1 be the minimal time such that E1 or E2 happens. By (A.6) and (A.10) we have

$$t_1 \le (|C_i(x(0))| - |\underline{A}(0)|)\left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right). \tag{A.11}$$

If E1 happens at time t_1, our result i) holds; otherwise, we need to carry out the following Step 2.

Step 2: For $s \ge t_1$ we control the agent l moves toward the right until E1 or one of the following two events happens:
(**E3**) $x_l(s) \ge x_j(0) + r_{\min}^i/2$;
(**E4**) $\max_{m\in C_i(x(0))} x_m(s) \le x_k(0) - (1-\mu)^2 r_{\min}^i$;

For $s \ge t_1$, let i'_s be the agent in $C_i(x(0))$ which has the biggest opinion within the confidence bound of agent l at time s, i.e.,

$$i'_s = \underset{m\in C_i(x(0))}{\arg\max}\, \{x_m(s) : |x_l(s) - x_m(s)| \le r_l\}.$$

Choose $\{i'_s, l\}$ as the agent pair for opinion update, until at least one of the events E1, E3, and E4 happens. Let t_2 be the minimal time that E1, E3, or E4 happens. For $s \in [t_1, t_2)$, since E1 and E4 do not happen at time s,

$$x_l(s+1) = (1-\mu)x_l(s) + \mu x_{i'_s}(s) > x_l(s).$$

By the similar method as Step 1, each agent in $C_i(x(0))\backslash(\underline{A}(t_1) \cup \{l\})$ can be chosen at most $1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil$ times for opinion update during $[t_1, t_2)$. Then,

$$\begin{aligned} & t_2 - t_1 \\ & \le (|C_i(x(0))| - |\underline{A}(t_1)| - 1)\left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right) \\ & = (|C_i(x(0))| - |\underline{A}(0)|)\left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right). \end{aligned} \tag{A.12}$$

If E4 happens, Lemma 7 implies

$$\begin{aligned} & \max_{M,m\in C_i(x(0))} [x_M(t_2) - x_m(t_2)] \\ & \le \max_{M\in C_i(x(0))} x_M(t_2) - x_j(0) \\ & \le x_k(0) - x_j(0) - (1-\mu)^2 r_{\min}^i, \end{aligned}$$

which indicates our result ii) holds; if E1 happens, our result i) holds at time t_2; otherwise, we need to carry out next Step.
... ...

Step $2m+1$: For $s \ge t_{2m}$, we use the similar control method as Step 1. Let t_{2m+1} be the minimal time such that E1 happens or $|\underline{A}(t_{2m+1})| = |\underline{A}(t_{2m-1})| - 1$. Similar to (A.11) we have

$$\begin{aligned} & t_{2m+1} - t_{2m} \\ & \le (|C_i(x(0))| - |\underline{A}(t_{2m-1})|)\left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right) \\ & = (|C_i(x(0))| - |\underline{A}(0)| + m)\left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right). \end{aligned} \tag{A.13}$$

Step $2m+2$: For $s \ge t_{2m+1}$, we use the similar control method as Step 2. Let t_{2m+2} be the minimal time such that E1, E3, or E4 happens. Similar to (A.12) we have

$$\begin{aligned} & t_{2m+2} - t_{2m+1} \\ & \le (|C_i(x(0))| - |\underline{A}(t_{2m+1})| - 1)\left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right) \\ & = (|C_i(x(0))| - |\underline{A}(0)| + m)\left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right). \end{aligned} \tag{A.14}$$

The above process will end at Step $2|\underline{A}(0)| - 1$ because $\underline{A}(t_{2|\underline{A}(0)|-1}) = \emptyset$. By Lemma 7 and the definition of $\underline{A}(s)$ we have

$$\begin{aligned} & \max_{M,m\in C_i(x(0))} [x_M(t_{2|\underline{A}(0)|-1}) - x_m(t_{2|\underline{A}(0)|-1})] \\ & \le x_k(0) - \min_{m\in C_i(x(0))} x_m(t_{2|\underline{A}(0)|-1}) \\ & \le x_k(0) - x_j(0) - (1-\mu)^2 r_{\min}^i, \end{aligned} \tag{A.15}$$

which indicates our result ii) holds when $t^* = t_{2|\underline{A}(0)|-1}$. Set $t_0 := 0$. By (A.14) and (A.15) we have

$$\begin{aligned} & t_{2|\underline{A}(0)|-1} \\ & = \sum_{m=0}^{|\underline{A}(0)|-2} (t_{2m+2} - t_{2m}) + t_{2|\underline{A}(0)|-1} - t_{2|\underline{A}(0)|-2} \\ & \le \Big(\sum_{m=0}^{|\underline{A}(0)|-2} 2\,(|C_i(x(0))| - |\underline{A}(0)| + m) \\ & \quad + |C_i(x(0))| - 1 \Big) \left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right) \\ & = \left[(2|\underline{A}(0)| - 1)|C_i(x(0))| + (-|\underline{A}(0)| - 1)|\underline{A}(0)| + 1\right] \\ & \quad \times \left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right) \\ & \le (|C_i(x(0))| - 1)^2 \left(1 + \lceil \log_{1-\mu} r_{\min}^i / r_{\max}^i \rceil\right), \end{aligned}$$

where the last inequality uses the fact that $|\underline{A}(0)| \le |C_i(x(0))| - 1$.

For the case when $x_l(0) < [x_k(0) + x_j(0)]/2$, we can set

$$\bar{A}(s) := \{m \in C_i(x(0)) : x_m(s) > x_k(0) - (1-\mu)^2 r_{\min}^i\},$$

and use the similar method as the case $x_l(0) \ge [x_k(0) + x_j(0)]/2$ to control $\bar{A}(s)$ becomes empty. $\square$