Perspectives on Network Systems and Mathematical Sociology

Francesco Bullo and Noah E. Friedkin

1 Introduction

Recent years have witnessed the emergence of a discipline of study focused on modeling, analyzing, and designing dynamic phenomena over networks. We refer to such systems as network systems; they are also equivalently referred to as multi-agent or distributed systems. This emerging discipline, rooted in graph theory, control theory, and matrix analysis, is increasingly relevant because of its broad set of application domains. Network systems appear naturally in (i) social networks and mathematical sociology, (ii) electric, mechanical and physical networks, and (iii) animal behavior, population dynamics, and ecosystems. Network systems are designed in the context of networked control systems, robotic networks, power grids, parallel and scientific computation, and transmission and traffic networks, to name a few.

Within this broad context, the disciplines of social networks and mathematical sociology have themselves received growing attention. Building on a classic history of work starting in the 50s, the study of influence systems and opinion dynamics has become a modern topic of interest to social scientists, engineers, computer scientists and physicists. The scientific trend towards quantitate analysis in the social sciences is motivated by the availability of insightful datasets and sharper statistical and mathematical analysis tools.

A recent outstanding survey on social networks is given in [Proskurnikov and Tempo, 2017]. Recent excellent treatments of network systems and their applica-

Francesco Bullo
Department of Mechanical Engineering and Center for Control, Dynamical-Systems and Computation, 2325 Engineering Bldg II, University of California at Santa Barbara, Santa Barbara, CA 93106-5070, USA e-mail: bullo@engineering.ucsb.edu
Noah E. Friedkin
Department of Sociology and Center for Control, Dynamical-Systems and Computation, University of California at Santa Barbara, Santa Barbara, CA 93106-5070, USA e-mail: friedkin@soc.ucsb.edu

tions are given in the recent books [Ren and Beard, 2008, Bullo et al., 2009, Mesbahi and Egerstedt, 2010, Bai et al., 2011, Cristiani et al., 2014, Francis and Maggiore, 2016, Arcak et al., 2016, Bullo, 2018] and recent related surveys include [Martínez et al., 2007, Ren et al., 2007, Garin and Schenato, 2010, Cao et al., 2013, Oh et al., 2015]. The books and articles [Newman, 2003, Boccaletti et al., 2006, Castellano et al., 2009, Easley and Kleinberg, 2010, Jackson, 2010, Newman, 2010] are instead excellent references on network science.

Against this background, this chapter is a review document intended for scientists interested in network systems and cooperative control as well as social networks and mathematical sociology. This chapter has a dual focus. First, we review classic results in the theory of linear network systems and place them in an algebraic framework based on Perron-Frobenius and algebraic graph theory. For example we characterize the set of non-negative matrices in terms of irreducibility and primitivity. Second, we focus on mathematical sociology and describe models of opinion dynamics in social influence systems, including the classic French-Harary-DeGroot and the Friedkin-Johnsen models. Motivated by recent empirical evidence on opinion dynamics along single issues and sequences of issues, we then describe some mathematical models for the evolution of social power and influence systems via the reflected appraisal mechanism.

Paper organization and related literature

Sections 2 and 3 review Perron Frobenius and algebraic graph theory. Classic references on this material include [Gantmacher, 1959, Seneta, 1981, Horn and Johnson, 1985]. This content may be regarded as a highly-abbreviated version of the first part of the recent textbook [Bullo, 2018].

Section 4 describes models of opinion dynamics. This classic field initiated with the seminal papers by French [1956], Harary [1959], Abelson [1964], and DeGroot [1974]. The classic discrete-time linear averaging model is well known as the DeGroot model, but a more accurate historic name would be the French-Harary-DeGroot model since modeling concepts were contained in [French, 1956] and analysis results in [Harary, 1959]. It is worth remarking how the 15 years before DeGroot the mathematical analysis in [Harary, 1959] was rather sophisticated already and included the concept of average consensus. The second model we review is the Friedkin-Johnsen model, which is an elaboration of the French-Harary-DeGroot. Documented in [Friedkin and Johnsen, 1999, 2011], this model is still based upon linear averaging but it includes also an attachment to initial opinions. Recent results on variations of this model are given in [Ravazzi et al., 2015, Frasca et al., 2015, Parsegov et al., 2015, Friedkin et al., 2016b, Parsegov et al., 2017].

Section 5 reviews the empirical findings on influence system evolution in small deliberative groups that are documented and analyzed in Friedkin et al. [2016a] and Friedkin and Bullo [2017]. The human subject experiments focused on both the opinion formation process on a single issue as well as on the influence network evolution that takes place along a sequence of opinion dynamic issues. Via multilevel linear regression analysis, we provide statistical evidence that the observed human subjects behavior is consistent with (1) the Friedkin-Johnsen model for single-issue opinion formation and (2) a reflected appraisal mechanism for the network evolution along issues. We remark that the papers Friedkin et al. [2016a] and Friedkin and Bullo [2017] report a rich collection of opinion dynamics phenomena and issue-sequence effects on influence network structure, only some of which are reviewed here.

Section 6 reviews the mathematical model of social power and influence network evolution proposed by Jia et al. [2015]. The key idea is to combine the French-Harary-DeGroot model of opinion dynamics with the Friedkin formalization of the reflected appraisal mechanism. Recent results on this model and its variations include the following. [Jia et al., 2017b] completes the analysis in [Jia et al., 2015] by treating the case of reducible interaction matrices. For single-time scale models, [Chen et al., 2017] proposes a continuous-time distributed model and [Jia et al., 2017a] proposes a dynamical flow model of interpersonal appraisals. Only preliminary results in [MirTabatabaei et al., 2014] are known at this time for the case of stubborn individuals. [Ye et al., 2018] obtains results on exponential convergence and the setting of time-varying interaction networks. [Chen et al., 2018] treats the case of switching and stochastic interaction matrices.

2 Perron–Frobenius theory

Here we review the widely-established Perron–Frobenius theory for non-negative matrices. We start by classifying non-negative matrices in terms of their zero/nonzero pattern and of the asymptotic behavior of their powers.

Definition 1 (Irreducible and primitive matrices). A square $n \times n$ non-negative matrix A, for $n \geq 2$, is

(i) *irreducible* if $\sum_{k=0}^{n-1} A^k$ is positive,
(ii) *primitive* if there exists $k \in \mathbb{N}$ such that A^k is positive.

A matrix that is not irreducible is said to be *reducible*.

In equivalent words, the matrix A is irreducible if, for any pair of indices (i, j) there exists an exponent $k = k(i, j) \leq (n-1)$ such that $(A^k)_{ij} > 0$. It is not hard to show that, if a non-negative matrix is primitive, then it is also irreducible.

We now state the main results in Perron-Frobenius theory and characterize the properties of the spectral radius of a non-negative matrix.

Theorem 1 (Perron-Frobenius Theorem). *Consider a square $n \times n$ non-negative matrix A, for $n \geq 2$. If A is irreducible, then*

(i) there exists a simple positive eigenvalue λ satisfying $\lambda \geq |\mu| \geq 0$ for all other eigenvalues μ,

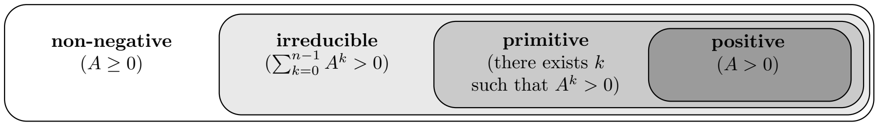


Fig. 1: The set of non-negative square matrices and its increasingly smaller subsets of irreducible, primitive and positive matrices.

(ii) the right and left eigenvectors v_{right} and v_{left} of λ are unique and positive, up to rescaling.

If additionally A is primitive, then

(iii) the eigenvalue λ satisfies $\lambda > |\mu|$ for all other eigenvalues μ.

The real non-negative eigenvalue λ is the spectral radius $\rho(A)$ of A and it is usually referred to as the *dominant or Perron eigenvalue* of A. The right and left eigenvectors v_{right} and v_{left} (unique up to rescaling and selected non-negative) of the dominant eigenvalue λ are called the *right and left dominant eigenvectors*, respectively.

Finally, the Perron–Frobenius Theorem for primitive matrices has immediate consequences for the asymptotic behavior of the discrete time dynamical system $x(k+1) = Ax(k)$, that is, for the powers A^k as $k \to \infty$.

Proposition 1 (Powers of primitive matrices). *Consider a square $n \times n$ non-negative matrix A, for $n \geq 2$. Let λ be the dominant eigenvalue and let v_{right} and v_{left} be the right and left dominant eigenvectors of A normalized so that they are both positive and satisfy $v_{\text{right}}^\top v_{\text{left}} = 1$. Then*

$$\lim_{k\to\infty} \frac{A^k}{\lambda^k} = v_{\text{right}} v_{\text{left}}^\top.$$

3 Algebraic graph theory

In this section we review some basic and prototypical results that involve correspondences between graphs and adjacency matrices. We let G denote a weighted digraph and A its weighted adjacency matrix or, equivalently, we let A be a non-negative matrix and we let G be its *associated weighted digraph* (i.e., the digraph with nodes $\{1,\dots,n\}$ and with weighted adjacency matrix A).

We start with some basic definitions about a directed graph G. A node i is *globally reachable* if, for every other node j, there exists a directed walk in G from node j to node i. A directed graph is *strongly connected* if each node is globally reachable. A *subgraph* of G is a subset of nodes and edges of G. A subgraph H is a *strongly connected component* of G if H is strongly connected and any other subgraph of

G containing H is not strongly connected. A directed graph G is *aperiodic* if there exists no integer that divides the length of each cycle of G.

We will also need the notion of condensation of a digraph. Given a directed graph G, the *condensation digraph* of G is formed by contracting each strongly connected component into a single node and letting an arc exist from one component to another if and only if at least one arc exists from a member of one component to a member of the other in G. The condensation digraph is acyclic and, therefore, contains at least one sink.

The first result we present relate the powers of the adjacency matrix with directed walks on the graph.

Lemma 1. *Let G be an unweighted digraph with unweighted adjacency matrix $A_{0,1} \in \{0,1\}^{n\times n}$. For all $i,j \in \{1,\dots,n\}$ and $k \in \mathbb{N}$, the (i,j) entry of $A_{0,1}^k$ equals the number of directed walks of length k (including walks with self-loops) from node i to node j.*

Moreover, if G is a weighted digraph with weighted adjacency matrix A, then the (i,j) entry of A^k is positive if and only if there exists a directed walk of length k (including walks with self-loops) from node i to node j.

Theorem 2 (Connectivity properties of the digraph and positive powers of the adjacency matrix). *Let G be a weighted digraph with $n \geq 2$ nodes and weighted adjacency matrix A. The following statements are equivalent:*

(i) A is irreducible, that is, $\sum_{k=0}^{n-1} A^k > 0$;
(ii) there exists no permutation matrix P such that $P^\top AP$ is block triangular;
(iii) G is strongly connected;
(iv) for all partitions $\{\mathscr{I},\mathscr{J}\}$ of the index set $\{1,\dots,n\}$, there exists $i \in \mathscr{I}$ and $j \in \mathscr{J}$ such that $\{i,j\}$ is an edge in G.

Let us remark that, instead of the order in which we presented matters here, most references define an irreducible matrix through property (ii) or, possibly, through property (iv).

Theorem 3 (Strongly connected and aperiodic digraph and primitive adjacency matrix). *Let G be a weighted digraph with weighted adjacency matrix A. Then G is strongly connected and aperiodic if and only if A is primitive.*

4 Mathematical models for the evolution of opinions

This section reviews some classic models for opinion dynamics. We focus on basic linear and affine models, whose relevance is established empirically.

We start by presenting some convergence results for systems of the form

$$x(k+1) = Ax(k), \qquad \text{where } A \text{ is row-stochastic.} \tag{1}$$

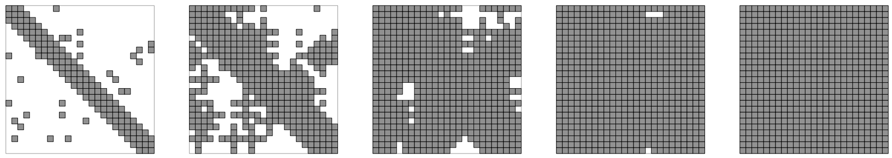


Fig. 2: These five images depict increasing powers of a non-negative matrix $A \in \mathbb{R}^{25\times 25}$. The digraph associated to A is strongly connected and has self-loops at each node so that, by Theorem 3, there exists k (in this case $k = 5$) such that $A^k > 0$.

Recall that the non-negative square matrix A is said to be row-stochastic if all its row-sums are equal to one, that is, if $A\mathbb{1}_n = \mathbb{1}_n$. Therefore, the right eigenvector of the eigenvalue 1 can be selected as $\mathbb{1}_n$.

The discrete-time averaging model (1) is well known as the DeGroot model, but a more accurate historic name would be the French-Harary-DeGroot model, as discussed in the introduction. The matrix A describes an *interpersonal influence network*.

Theorem 4 (Consensus for row-stochastic matrices with a globally-reachable aperiodic strongly-connected component). *Let A be a row-stochastic matrix and let G be its associated digraph. The following statements are equivalent:*

(A1) *the eigenvalue* 1 *is simple,* $\rho(A) = 1$*, and all other eigenvalues have magnitude strictly smaller than* 1*,*
(A2) *A is semi-convergent (i.e.,* $\lim_{k\to\infty} A^k$ *exists and is finite) and* $\lim_{k\to\infty} A^k = \mathbb{1}_n v_{\text{left}}^\top$*, for some* $v_{\text{left}} \in \mathbb{R}^n$*,* $v_{\text{left}} \geq 0$*, and* $\mathbb{1}_n^\top v_{\text{left}} = 1$*,*
(A3) *the digraph associated to A contains a globally reachable node and the subgraph of globally reachable nodes is aperiodic.*

If any, and therefore all, of the previous conditions are satisfied, then the matrix A is said to be indecomposable *and the following properties hold:*

(i) $v_{\text{left}} \geq 0$ *is the left dominant eigenvector of A and* $(v_{\text{left}})_i > 0$ *if and only if node i is globally reachable;*
(ii) the solution to the averaging model $x(k+1) = Ax(k)$ *in equation (1) satisfies*

$$\lim_{k\to\infty} x(k) = \big(v_{\text{left}}^\top x(0)\big)\mathbb{1}_n;$$

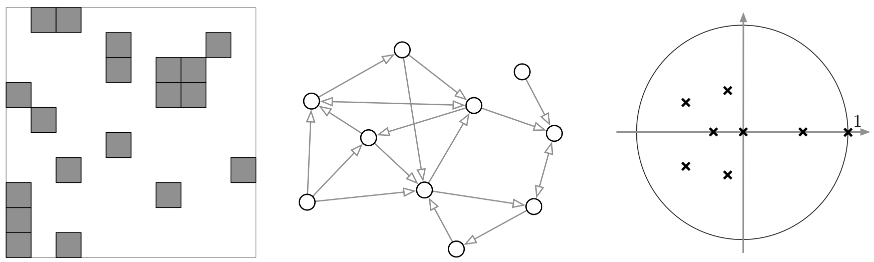
In this case we say that the dynamical system achieves consensus*;*
(iii) if additionally A is doubly-stochastic, then $v_{\text{left}} = \frac{1}{n}\mathbb{1}_n$ *(because* $A^\top \mathbb{1}_n = \mathbb{1}_n$ *and* $\frac{1}{n}\mathbb{1}_n^\top \mathbb{1}_n = 1$*) so that*

$$\lim_{k\to\infty} x(k) = \frac{\mathbb{1}_n^\top x(0)}{n}\mathbb{1}_n = \text{average}\big(x(0)\big)\mathbb{1}_n.$$

In this case we say that the dynamical system achieves average consensus.

The limiting vector is therefore a weighted average of the initial conditions. The relative weights of the initial conditions are the convex combination coefficients $(v_{\text{left}})_1, \dots, (v_{\text{left}})_n$. In a social influence network, the coefficient $(v_{\text{left}})_i$ is regarded as the "social influence" of agent i.

In Figure 3 we show a nonnegative matrix that is indecomposable, together with its directed graph and its spectrum.



(a) A rows-stochastic matrix; in each row, nonzero entries are equal and sum to 1.

(b) The corresponding digraph has an aperiodic subgraph of globally reachable nodes.

(c) The spectrum of the adjacency matrix includes a dominant eigenvalue.

Fig. 3: An example indecomposible row-stochastic matrix, its associated digraph consistent with Theorem 4(A2), and its spectrum consistent with Theorem 4(A1)

The implication (A3) $\Longrightarrow$ (ii) amounts to a result in which the structure of the network determines its function, i.e., the asymptotic behavior of the averaging system.

Next, we consider the general case of digraphs that do not contain globally reachable nodes, that is, digraphs whose condensation digraph has multiple sinks. In what follows, we say that a node is *connected* with a sink of a digraph if there exists a directed walk from the node to any node in the sink.

Theorem 5 (Convergence for row-stochastic matrices with multiple aperiodic sinks). *Let A be a row-stochastic matrix, let G be its associated digraph, and let $M \geq 2$ be the number of sinks in the condensation digraph $C(G)$. If each of the M sinks is aperiodic, then*

(i) the semi-simple eigenvalue $\rho(A) = 1$ has multiplicity equal M and is strictly larger than the magnitude of all other eigenvalues, hence A is semi-convergent,
(ii) there exist M left eigenvectors of A, denoted by $v^m_{\text{left}} \in \mathbb{R}^n$, for $m \in \{1, \dots, M\}$, with the properties that: $v^m_{\text{left}} \geq 0$, $\mathbb{1}_n^\top v^m_{\text{left}} = 1$ and $(v^m_{\text{left}})_i$ is positive if and only if node i belongs to the m-th sink,
(iii) the solution to the averaging model $x(k+1) = Ax(k)$ with initial condition $x(0)$ satisfies

$$\lim_{k\to\infty} x_i(k) = \begin{cases} (v_{\text{left}}^m)^\top x(0), & \textit{if node } i \textit{ belongs to the } m\textit{-th sink}, \\ (v_{\text{left}}^m)^\top x(0), & \textit{if node } i \textit{ is connected only with the } m\textit{-th sink}, \\ \sum_{m=1}^{M} z_{i,m}\big((v_{\text{left}}^m)^\top x(0)\big), & \textit{if node } i \textit{ is connected to more than one sink}, \end{cases}$$

where, for each node i connected to more than one sink, the coefficients $z_{i,m}$, $m \in \{1,\dots,S\}$, are convex combination coefficients and are strictly positive if and only if there exists a directed walk from node i to the sink m.

Note that convergence does not occur to consensus (not all components of the state are equal) and the final value of all nodes is independent of the initial values at nodes which are not in the sinks of the condensation digraph. We summarize the discussion in this section with a figure summarizing the asymptotic behavior of the French-Harary-DeGroot discrete-time averaging systems; see Figure 4.

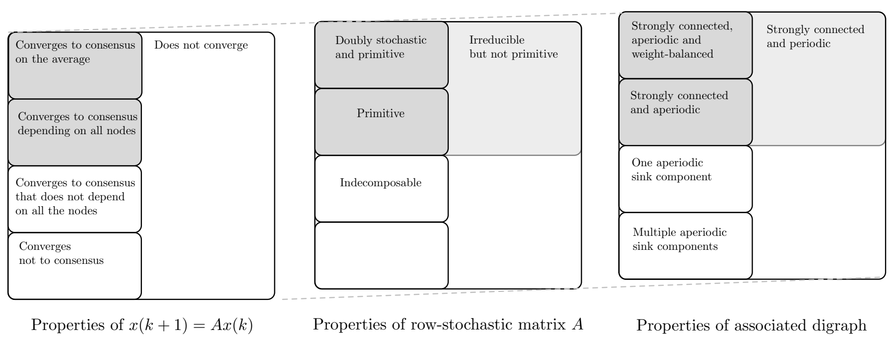


Fig. 4: Corresponding properties for the discrete-time averaging dynamical system $x(k+1) = Ax(k)$, the row-stochastic matrix A and the associated weighted digraph.

We next consider the opinion dynamics model by Friedkin and Johnsen [1999] which, for generic parameter values, features persistent disagreement and lack of consensus. As before we let A be a row-stochastic matrix whose associated digraph describes an interpersonal influence network. We assume that every individual is naturally given an *openness level* $\lambda_i \in [0,1]$, $i \in \{1,\dots,n\}$, describing how open the individual is to interpersonal influence and, therefore, to changing her initial opinion about a subject. We then define $\Lambda = \text{diag}(\lambda_1,\dots,\lambda_n)$, where diag is the standard operator that maps an array to a diagonal matrix.

The *Friedkin-Johnsen model* of opinion dynamics is defined by

$$x(k+1) = \Lambda A x(k) + (I_n - \Lambda)x(0), \tag{2}$$

where, for individual i, $x_i(k)$ represents the current opinion and $x_i(0)$ represents the initial opinion or prejudice. The *Friedkin-Johnsen model* is again an averaging model with stubborn individuals in the sense that here every individual i exhibits an attachment $(1-\lambda_i)$ to its initial opinion $x_i(0)$.

Theorem 6 (Persistent disagreement in the Friedkin-Johnsen model). *Consider a square $n \times n$ non-negative matrix A, for $n \geq 2$, and a diagonal matrix Λ with entries in $[0,1]$. Assume that*

(A1) *at least one individual has a strictly positive attachment to its initial opinion, that is, $\lambda_i < 1$ for at least one individual i; and*
(A2) *the interpersonal influence network contains directed walks from each individual with openness level equal to 1 to an individual j with openness level $\lambda_j < 1$.*

Then the following statements hold:

(i) the matrix ΛA is convergent, that is, $\rho(\Lambda A) < 1$,
(ii) the total influence matrix *$V = (I_n - \Lambda A)^{-1}(I_n - \Lambda)$ is well defined and row-stochastic, and*
(iii) the limiting opinions satisfy $\lim_{k\to+\infty} x(k) = Vx(0)$.

We conclude with some remarks. As predicted in the model formulation, consensus is not achieved asymptotically because of the attachment to initial opinions. If Assumption (A1) is not satisfied and therefore $\Lambda = I_n$, then we recover the French-Harary-DeGroot opinion dynamics model.

Finally, it is worth noting that the original work [Friedkin and Johnsen, 1999], see also [Friedkin and Johnsen, 2011], make the additional assumption that $\lambda_i = 1 - a_{ii}$, for $i \in \{1,\dots,n\}$. This additional assumption is justified by sociological reasons and introduces coupling between the openness level and the interpersonal influence values. Other properties of this model are studied in [Bindel et al., 2015, Friedkin et al., 2016a, Ravazzi et al., 2015].

5 Empirical findings on the evolution of opinions and influence networks

We here review the empirical findings on influence system evolution in small deliberative groups that are documented and analyzed in [Friedkin et al., 2016a, Friedkin and Bullo, 2017]. The human subject experiments focus on both the opinion formation process on a single issue as well as on the influence network evolution that takes place along a sequence of issues.

5.1 The Friedkin-Johnsen model on judgmental issues

We collected data in experiments on 30 groups of 4 individuals assembled to discuss a sequence of 15 risk/reward choice-dilemma issues. Choice-dilemma issues are judgmental issues, in which no absolute truth exists. In risk/reward dilemmas individuals develop opinions about the minimum level of confidence (measured as a scalar value in the $[0,1]$ interval) required to accept a risky option with a high payoff over a less risky option with a low payoff. In other words, individuals are asked to answer questions of the following type:

> What is your minimum level of confidence (scored 0-100) required to accept a risky option with a high payoff rather than a less risky option with a low payoff?

Questions are selected from a variety of domains, including medical, financial, and professional. The groups are assembled for face-to-face discussion and put under pressure to reach consensus via instructions similar to "reaching consensus is desirable, but not required." Each human subject recorded privately and chronologically on each issue:

(i) an initial opinion about the issue prior to the-group discussion,
(ii) a final opinion after the group-discussion (which lasted anywhere between 3-27 minutes), and
(iii) an allocation of "100 influence units" to the 4 components of the group. These influence units are described as follows: "these allocations represent your appraisal of the relative influence of each group member's opinion on your own final opinion."

The 15 issues were presented in random order and subjects were assigned to groups randomly to eliminate bias in group composition. We refer to [Friedkin et al., 2016a] for details about the maximum-likelihood multilevel random-intercept linear regression and its software implementation.

In summary this regression analysis, presented in Table 1 below, confirms that the Friedkin-Johnsen model has predictive value for the final opinion achieved by a group discussing risk/reward choice issues.

Table 1: Prediction of an individual's final opinion on an issue. Opinions are scaled $0-100$. Notes: F-J stands for Friedkin-Johnsen. Standard errors are in parentheses; * $p \leq 0.05$ ** $p \leq 0.01$ *** $p \leq 0.001$; balanced random-intercept multilevel longitudinal design; maximum likelihood estimation with robust standard errors; $n = 1,800$.

	(a)	(b)	(c)
F-J prediction		0.897^{***}	1.157^{***}
		(0.018)	(0.032)
initial opinion			-0.282^{***}
			(0.031)
constant	58.975^{***}	5.534	6.752^{***}
	(1.550)	(1.176)	(1.124)
log likelihood	-8579.835	-7329.003	-7241.097

5.2 *The Friedkin-Johnsen model on intellective and multidimensional issues*

We next briefly describe how opinion averaging models are predictive also in the setting of intellective and resource allocation issues. In other words, we extend our analysis from the risk/reward choice dilemmas to two other types of issues: analytical reliability problems with exact answers and multi-dimensional constrained resource allocation issues.

First, the empirical findings in [Friedkin and Bullo, 2017] deal with analytical reliability problems based on Bayesian reasoning. These problems have an exact answer and, therefore are referred to as intellective issues. It is known that in such problems often, but not always, "truth wins" in the sense that the correct answer propagates from a correct individual to the others in the group. Here is an example drawn from the medical field.

> Two medical teams are working independently to achieve a cure for a disease. Team A succeeds if it can solve two scientific problems A_1 and A_2 with independent success probabilities $P[A_1] = 0.60$ and $P[A_2] = 0.45$. Team B succeeds if it can solve three scientific problems B_1, B_2, and B_3, with independent success probability $P[B_1] = 0.80$, $P[B_2] = 0.85$, $P[B_3] = 0.95$. What is your estimate of the probability that the disease will be cured?

While we refer the reader to [Friedkin and Bullo, 2017] for the detailed findings, we summarize the work here by stating that the Friedkin-Johnsen model (i) has predictive value for the final opinion expressed by the group member and (ii) substantially clarifies how truth wins in groups engaged in sequences of intellective issues based on an evolving centrality of the truth in the groups.

Second, in forthcoming publications, we will report empirical findings on group decision-making on resource allocation distributions under conditions of uncertainty. Here is an example drawn from the political field.

> If you were a State Legislator, what would be your opinion on the percentage of state tax revenues that should be allocated to each the following categories: (i) Spending on Education, (ii) Spending on State Employee Wages, Health Care, and Pensions, (iii) Spending on State Physical Infrastructure Improvements, and (iv) All Other Categories (Welfare, Other Costs of Government, Etc.)? These percentages must sum to 100%.

Preliminary results indicate how multidimensional opinions are constrained to evolve in certain polytopic spaces and how a single Friedkin-Johnsen model is predictive of the final group decision. The findings establish a natural meshing of automatic polytopic decision spaces, weighted averaging models, and group decision making on uncertain resource allocation problems. These findings provide a mechanistic explanation for the bounded-rationality phenomenon of satisficing, that is, the achievement of satisfactory consensus distribution as described by the Nobel award winning work by Simon [1947].

5.3 A reflected appraisal mechanism explaining influence network evolution

We next consider network evolution phenomena along sequences and, specifically, we postulate a mechanism for network evolution. As documented in [Cooley, 1902, Gecas and Schwalbe, 1983, Friedkin, 2011], the reflected appraisal mechanism is a psychological process that affects the levels of closure-openness levels of individuals in response to an individual's perception of how others see and evaluate him or her. In this mechanism it is postulated that individuals react to their perception of their social influence, or social power, in the group decision making.

If an individual is perceived to have had a large role in influencing a group outcome, then that individual tends to elevate his or her own self-weight or, equivalently, his or her own closure level to interpersonal influence. Conversely, if an individual is perceived to have (or really does have) limited and diminishing influence on a group outcome, then the self-weight will tend to diminish.

It is a consequence of this postulated mechanism of reflected appraisal that individuals come to think of themselves in ways that are affected by what other individuals think of them. In other words, levels of stubbornness and closure-openness to interpersonal influence are ultimately social constructions and not personality characteristics.

To mathematize this group psychological mechanism, we start by describing loosely a simplified and crude model for it:

> Each individual dampens/elevates her self-weight according to her prior influence centrality in prior issues.
>
> Specifically, along the issue sequence $s = 1, 2, \dots$, the self-weight of each individual at issue $s+1$ is set equal to the relative control of that individual on the prior issue s.

Here, relative control over an issue outcome is tantamount to social power of an individual in the group. Here also note how we have simplified the mechanism (influence centrality) to assume that individuals are capable of perceiving from their peers their actual level of relative control.

With the notation introduced in Section 4 for the Friedkin-Johnsen model in equation (2), we define the following issue-dependent concepts:

$$
\begin{aligned}
A(s) &= \text{influence matrix at issue } s,\\
a_{ii}(s) &= \text{self-weight (level of closure to influence) of individual } i \text{ at issue } s,\\
V(s) &= \text{total influence matrix at issue } s,\\
c_i(s) = V(s)^\top \mathbb{1}_n/n &= \text{social power of individual } i \text{ at issue } s,\\
\bar{c}_i(s) = \frac{1}{s}\sum_{t=1}^{s} c_i(t) &= \text{issue-averaged social power of individual } i \text{ up until issue } s.
\end{aligned}
$$

We next perform a regression analysis of the empirical data collected in Friedkin et al. [2016a] to determine whether or not individuals' self-weights on issue $s+1$ adjust along the issue sequence $s = 1, 2\dots$ in correspondence with their social power at issue s or issue-averaged social power until issue s. As before we perform a maximum-likelihood multilevel random-intercept linear regression and we refer to [Friedkin et al., 2016a] for the corresponding technical details. The findings in Table 2 confirm that both social power and issue-averaged social power do indeed predict individuals' issue-specific self-weights on the following issues. The effect of social power $c_i(s)$ on self-weight $a_{ii}(s+1)$ is constant along the issue sequence. Remarkably, instead, the effect of issue-averaged social power $\bar{c}_i(s)$ on self-weight $a_{ii}(s+1)$ increases along the issue sequence.

Table 2: Prediction of an individual's level of closure to influence $a_{ii}(s+1)$ based on the individual's prior centrality $c_i(s)$ and time-averaged cumulative centrality $\bar{c}_i(s) = \frac{1}{s}\sum_{t=1}^{s} c_i(t)$. Standard errors are in parentheses. Notes: $^{*}\ p \leq 0.05$ $^{**}\ p \leq 0.01$ $^{***}\ p \leq 0.001$; balanced random-intercept multilevel longitudinal design; maximum likelihood estimation with robust standard errors; $n = 1,680$.

	(a)	(b)	(c)
$c_i(s)$		0.336^{***} (0.104)	
$\bar{c}_i(s)$			0.404^{**} (0.159)
s		0.002 (0.004)	-0.018^{***} (0.005)
$s \times c_i(s)$		0.171 (0.012)	
$s \times \bar{c}_i(s)$			0.095^{***} (0.018)
constant	0.643^{***} (0.016)	0.515^{***} (0.030)	0.498^{***} (0.039)
log likelihood	-367.331	-327.051	-293.656

6 Mathematical models for the evolution of influence networks

Motivated by the empirical findings in the previous section we now propose a basic dynamical model for the evolution of self-weight, social power, and influence networks through the process of reflected appraisal. The key references for this section are [Friedkin, 2011] where a first model is proposed and [Jia et al., 2015] where a comprehensive modeling and analysis framework is developed.

6.1 Models of reflected appraisal = Dynamics of the influence network

We start by revisiting the French-Harary-DeGroot model in equation (1) parametrized by a single row-stochastic matrix A. We start with the fundamental observation that the entries of A do not all have the same interpretation. From an applied psychological viewpoint, the diagonal entries are self-weight values, that is, measures of self-appraisal, levels of closure to interpersonal influence and stubbornness. The off-diagonal terms are instead interpersonal accorded weights, that is, they represent what influence an individual is willing to accord to another. Under mild connectivity assumptions, it is possible to re-parametrize the matrix A in the following way. First, we define the self-weights

$$A_{ii} =: x_i \in [0,1]. \tag{3}$$

Second, we assume the existence of a zero-diagonal row-stochastic matrix W, whose off-diagonal entries W_{ij} are *relative interpersonal accorded weights* satisfying the equality $A_{ij} =: (1-x_i)W_{ij}$. In short, we can now write

$$A(x) = \operatorname{diag}(x) + \operatorname{diag}(\mathbb{1}_n - x)W. \tag{4}$$

Before proceeding, we define the left dominant eigenvector for W to be $w = (w_1,\dots,w_n) = v_{\text{left}}(W)$. We recall that the right dominant eigenvector of W is $\mathbb{1}_n$ and that, under irreducibility assumptions, Theorem 1 implies that the left dominant eigenvector is positive and unique with the scaling $\mathbb{1}_n^\top w = 1$.

One can show that, after some manipulation and almost everywhere, the following equation relates the dominant eigenvector of $A(x)$ with that of W:

$$v_{\text{left}}(A(x)) = \Big(\frac{w_1}{1-x_1},\dots,\frac{w_n}{1-x_n}\Big) \Big/ \sum_{i=1}^{n} \frac{w_i}{1-x_i}.$$

We are now ready to implement in simple, even crude, mathematical form the reflected appraisal mechanism described in the previous section: "along issues $s = 1,2,\dots,$ individual dampens/elevates self-weight according to prior influence centrality." We turn this into the following equation:

$$x(s+1) = v_{\text{left}}(A(x(s))), \tag{5}$$

that is, the self-weights are set equal to the relative control of the individuals on prior issues, i.e., their social power. Note that, after at most one iteration, the state of this system takes value in the simplex $\Delta_n = \{y \in \mathbb{R}^n \mid y \geq 0, \mathbb{1}_n^\top y = 1\}$. The definition of this dynamical system is illustrated in Figure 5. We refer to the dynamical system (5) as to the DeGroot-Friedkin model, as introduced in [Jia et al., 2015] and motivated by the foundational works in [DeGroot, 1974, Friedkin, 2011].

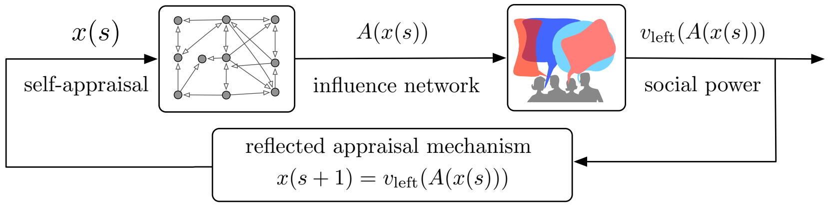


Fig. 5: An illustration of the reflected appraisal mechanism as a feedback mechanism, leading to the definition of a closed-loop dynamical system. Here $A(x)$ is given as in equation (4).

6.2 Equilibrium and asymptotic convergence analysis

Now that we have defined a dynamical system for the evolution of self-weights and social power, we can investigate what long-term predictions are consistent this model. It is of interest to characterize the existence and stability of equilibria, the role of network structure and parameters, and whether the influence system has a tendency towards the emergence of *autocracy* (social power concentrated in one individual) and *democracy* (social power equitably distributed among all individuals).

Theorem 7 (Equilibria and convergence). *Let W be the zero-diagonal row-stochastic matrix of relative interpersonal accorded weights and consider the resulting DeGroot-Friedkin model in equation (5), for $n \geq 3$. Assume that W is irreducible, that w is its dominant left eigenvector, and that its associated digraph does not have star topology. Then*

(i) in the interior of the simplex there exists a unique fixed point $x^* = x^*(w_1, \dots, w_n)$,
(ii) from almost all initial conditions the following convergence result holds:

$$\lim_{s\to\infty} x(s) = \lim_{s\to\infty} v_{\text{left}}(A(x(s))) = x^*,$$

so that, in other words, individuals forget their initial conditions, and
(iii) the fixed point is characterized by a phenomenon of accumulation of social power and self-appraisal at the top in the following sense:

- *the fixed point x^* has same ordering of $(w_1, \dots, w_n)$, i.e., if $w_i \geq w_j$ then also $x_i^* \geq x_j^*$, and*
- *x^* is an extreme version of $(w_1, \dots, w_n)$ in the sense that there exists a social power threshold p such that, each individual i satisfies either $x_i^* < w_i < p$ or $p < w_i < x_i^*$.*

A special case of this result is the emergence of democracy for matrices W of relative interpersonal accorded weights that are doubly-stochastic. In this case, one can easily verify that the theorem above implies:

(i) the unique non-trivial fixed point is $\dfrac{\mathbb{1}_n}{n}$, and

(ii) $\lim_{s\to\infty} x(s) = \lim_{s\to\infty} v_{\text{left}}(A(x(s))) = \frac{\mathbb{1}_n}{n}$.

In other words, such networks are characterized by uniform social power and no power accumulation at the top. In simple words, one may say that the influence system is functioning as a democracy.

The other relevant special case is that of a star topology associated to W; this setting is not a direct consequence of Theorem 7 and required an ad-hoc analysis. In this case, the DeGroot-Friedkin dynamics leads to the emergence of autocracy in the following sense. If W has star topology with center j:

(i) there are no fixed points, other than the vertices of the simplex, and
(ii) $\lim_{s\to\infty} x(s) = \lim_{s\to\infty} v_{\text{left}}(A(x(s))) = e_j$,

where e_j is the jth vector of the canonical basis. In other words, individual j, the center node of the star topology, comes to be the autocrat of the influence system. In this case, the topology of the interpersonal accorded weights leads to extreme power accumulation, in the sense that the autocrat j has full power.

Naturally we refer to the original paper for a much more detailed treatment and for the detailed proofs. It is worth, however, to review the method of proof for the statements in the main Theorem 7. We first establish the existence of the equilibrium point x^* via the Brouwer Fixed Point Theorem. Uniqueness is proved by contradiction through an elementary calculation. We next establish the following monotonicity property. Let $i_{\max}$ denote the individual with maximum $\frac{x_j(0)}{x_j^*}$, for simplicity let us here assume that it is unique. Then it turns out that $i_{\max}$ remains the index corresponding to the largest $\frac{x_j(s)}{x_j^*}$ for all subsequent issues s. (A similar result holds for $i_{\min}$.) In turn this monotonicity allows us to prove convergence via a variation on classic "max-min" Lyapunov function:

$$V(x) = \max_j \left(\ln \frac{x_j}{x_j^*}\right) - \min_j \left(\ln \frac{x_j}{x_j^*}\right).$$

It is historically interesting to mention that, to the best of our knowledge, the earliest work introducing a max-min Lyapunov function is the work [Tsitsiklis et al., 1986] on distributed optimization. This work is however related to the classic work by Birkhoff [1957]. We also refer to [Sepulchre et al., 2010] for a review of this history and for a study of consensus in non-commutative spaces.

7 Conclusions

This chapter has reviewed a large literature on the mathematics of network systems and its application to the study of dynamical models for the evolution of opinions and influence systems. We have presented both mathematical results and empirical findings.

Overall our recent works provide a new perspective on influence networks and social power, grounded in multiple human-subject experiments and based on both multi-level regression and control theoretical analysis. We have designed, executed, and analyzed experiments on group discussions for judgmental and intellective issues. We have proposed, analyzed, and validated a novel dynamical model with feedback. In turn this model provides a novel mechanism that may explain the phenomenon of power accumulation and emergence of autocracy in certain influence networks.

Ongoing and future research will focus on (1) studying the mathematical robustness of our findings to modeling assumptions, (2) studying and modeling the evolution of the matrix of interpersonal accorded weights, and (3) performing larger-scale controlled experiments perhaps via online software. We will also endeavor to design and validate intervention strategies to influence group discussions.

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