

Linear Stochastic Approximation Algorithms and Group Consensus over Random Signed Networks

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Abstract—This paper studies linear stochastic approximation (SA) algorithms and their application to multi-agent systems in engineering and sociology. As main contribution, we provide necessary and sufficient conditions for convergence of linear SA algorithms to a deterministic or random final vector. We also characterize the system convergence rate, when the system is convergent. Moreover, differing from non-negative gain functions in traditional SA algorithms, this paper considers also the case when the gain functions are allowed to take arbitrary real numbers. Using our general treatment, we provide necessary and sufficient conditions to reach consensus and group consensus for first-order discrete-time multi-agent system over random signed networks and with state-dependent noise. Finally, we extend our results to the setting of multi-dimensional linear SA algorithms and characterize the behavior of the multi-dimensional Friedkin-Johnsen model over random interaction networks.

Index Terms—stochastic approximation, linear systems, multi-agent systems, consensus, signed network

I. INTRODUCTION

Distributed coordination of multi-agent systems has drawn much attention from various fields over the past decades. For example, engineers control the formations of mobile robots, satellites, unmanned aircraft, and automated highway systems [12], [36]; physicists and computer scientists model the collective behavior of animals [37], [44]; sociologists investigate the evolution of opinion, belief and social power over social networks [10], [15], [22]. Many models for distributed coordination have been proposed and analyzed; a common thread in all these works is the study of a group of interacting agents trying to achieve a collective behavior by using neighborhood information allowed by the network topology.

Linear dynamical systems are a class of basic first-order dynamics with application to many practical problems in multi-agent systems, including distributed consensus of multi-agent systems, computation of PageRank, sensor localization

of wireless networks, opinion dynamics, and belief evolution on social networks [15], [32], [34]. If the operator in a linear dynamical system is time-invariant, then the study of this system is straightforward. However, practical systems are very often subject to random fluctuations, so that the operator in a linear dynamical system is time-variant and the system may not converge. To overcome this deficiency and eliminate the effects of fluctuation, a feasible approach is to adopt models based on the stochastic approximation (SA) algorithm [3], [6], [19], [20], [27], [28], [41].

The main idea of the SA algorithm is as follows: each agent has a memory of its current state. At each time step, each agent updates its state according to a convex combination of its current state and the information received from its neighbors. Critically, the weight accorded to its own state tends to 1 as time grows (as a way to model the accumulation of experience). The earliest SA algorithms were proposed by Robbins and Monro [38] who aimed to solve root finding problems. SA algorithms have then attracted much interest due to many applications such as the study of reinforcement learning [42], consensus protocols in multi-agent systems [6], and fictitious play in game theory [17]. A main tool in the study of SA algorithms (see [26, Chapter 5]) is the ordinary differential equations (ODE) method, which transforms the analysis of asymptotic properties of a discrete-time stochastic process into the analysis of a continuous-time deterministic process.

In this paper, we consider linear SA algorithms with random linear operators; these models are basic first-order protocols with numerous applications in engineering and sociology. Currently, there are two main threads on the theoretical research of linear SA algorithms. One thread is based on assumptions that guarantee the state of the system converges to a deterministic point [7], [8], [24], [25], [39]. Another thread is the research on consensus of multi-agent systems, where the system matrices are assumed to be row-stochastic [6], [19], [28]. These two threads only consider a part of linear operators, and the critical condition for convergence is still unknown. This paper develops appropriate analysis methods for linear SA algorithms and also provides some sufficient and necessary conditions for convergence which include critical conditions for convergence of linear operators. It is shown that under critical convergence conditions the state of the system will converge to random vectors, which is applied to consensus algorithms over signed networks. Moreover, an additional restriction of traditional SA algorithms is that only non-negative gain schedules are allowed. This paper relaxes this requirement and provides necessary and sufficient conditions for convergence of linear

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SA algorithms under arbitrary gains. In addition, we analyze the convergence rate of the system when it is convergent.

Our general theoretical results are directly applicable to certain multi-agent systems. The first application is to the study of consensus problems in multi-agent systems. As it is well known, numerous works provide sufficient conditions for consensus in time-varying multi-agent systems with row-stochastic interaction matrices; an incomplete list of references is [5], [6], [11], [28], [30], [40]; see also the classic works [4], [9], [43]. Recently, motivated by the study of antagonistic interactions in social networks, novel concepts of bipartite, group, and cluster consensus have been studied over signed networks (mainly focusing on continuous-time dynamical models); see [1], [29], [33], [45]. In this paper, we apply and extend our results on linear SA algorithms to the setting of first-order discrete-time multi-agent system over random signed networks and with state-dependent noise; for such models, we provide novel necessary and sufficient conditions to reach consensus and group consensus.

As the second application of our results, we study the Friedkin-Johnsen (FJ) model of opinion dynamics in social networks. The FJ model was first proposed in [14], where each agent is assumed to be susceptible to other agents' opinions but also to be anchored to his own initial opinion with a certain level of stubbornness. Ravazzi *et al.* proposed a gossip version of the FJ model in [34], whereby each link in the network is sampled uniformly and the agents associated with the link meet and update their opinions. The agents' opinions were proven to converge in mean square. Frasca *et al.* considered a symmetric pairwise randomization of FJ in [13], whereby a pair of agents are chosen to update their opinions. Our work, by exploiting stochastic approximation, largely relaxes the conditions for convergence when applied to FJ model over random interaction networks. The sociological meaning of stochastic approximated FJ model is that agents have cumulative memory about their previous opinions. The adoption of SA models in the study of human behavior is widely adopted in game theory and economics; e.g., see [17].

The main contributions of this paper are summarized as follows.

- 1) For linear SA systems, we provide some necessary and sufficient conditions to guarantee convergence by developing appropriate methods different from previous works. We derive some critical convergence conditions for linear operators for the first time. The convergence rate is also obtained when the system is convergent. Moreover, we consider the convergence of linear SA systems whose gain functions can take arbitrary real numbers.
- 2) Using our results, we get the necessary and sufficient conditions to reach consensus and group consensus of the first-order discrete-time multi-agent system over random signed networks and with state-dependent noise for the first time.
- 3) We extend our results to the multi-dimensional linear SA algorithms and provide applications to the multi-dimensional FJ model over random interaction networks.

Organization: The remainder of this paper is organized as follows. We briefly review the time-varying linear dynamical systems and propose a stochastic approximation version of it in Section II. The main results are presented in Section III. In particular, we introduce some preliminaries and assumptions in Subsection III-A. Sufficient conditions that guarantee the convergence of linear SA algorithms are obtained in Subsection III-B. We provide the results on convergence rate in the same subsection. In Subsection III-C, we prove that the sufficient condition is also necessary. The necessary and sufficient conditions for convergence are then summarized in Subsection III-D. We generalize the results to multi-dimensional models and discuss their application to group consensus and the FJ model in Section IV. Section V concludes the paper.

II. LINEAR DYNAMICAL SYSTEMS

A. Review of a time-varying linear dynamical system

In [16], [34] a time-varying linear dynamical system was considered as follows:

$$x(s+1) = P(s)x(s) + u(s), \quad s = 0, 1, \dots, \quad (1)$$

where $P(s) \in \mathbb{R}^{n \times n}$ is a matrix associated to the communication network between agents, and $u(s) \in \mathbb{R}^n$ is an input vector. Given a matrix $A \in \mathbb{R}^{n \times n}$, let $\rho(A)$ denote its spectral radius, i.e., $\rho(A) = \max_i |\lambda_i(A)|$, where $\lambda_i(A)$ is an eigenvalue of A . For system (1), if $P(s) \equiv P$, $u(s) \equiv u$, and $\rho(A) < 1$, then it is immediate to see that $x(s)$ converges to $(I_n - P)^{-1}u$.

In this paper we will consider the case when $\{P(s)\}$ and $\{u(s)\}$ are stochastic matrices and vectors respectively. We define the σ -algebra generated by $\{P(s)\}$ and $\{u(s)\}$ as $\mathcal{F}_t = \sigma(\{P(s), u(s)\}, 0 \leq s \leq t)$. The probability space is $(\Omega, \mathcal{F}_\infty, P)$.

Since the system (1) does not necessarily converge when $\{P(s)\}$ and $\{u(s)\}$ are stochastic, as an alternative, Ravazzi *et al.* [34] investigate the ergodicity of system (1) as follows.

Proposition 2.1 (Theorem 1 in [34]): Consider system (1) and assume $\{P(s)\}$ and $\{u(s)\}$ are sequences of independent identically distributed (i.i.d.) random matrices and vectors with finite first moments. Assume there exists a constant $\alpha \in (0, 1]$, a matrix $P \in \mathbb{R}^{n \times n}$ and a vector $u \in \mathbb{R}^n$ such that

$$\mathbb{E}[P(s)] = (1 - \alpha)I_n + \alpha P, \quad \mathbb{E}[u(s)] = \alpha u, \quad \forall s \geq 0.$$

If $\rho(P) < 1$, then $x(s)$ converges to a random variable in distribution, and $\frac{1}{s} \sum_{k=0}^{s-1} x(k)$ converges to $(I_n - P)^{-1}u$ almost surely.

In this paper we adopt the stochastic approximation method to average the effect of the stochastic $P(s)$ and $u(s)$ to the state $x(s)$. In this case we study the sufficient and necessary conditions for convergence of $x(s)$, and also obtain a convergence rate.

B. Linear SA algorithms over random networks

In this subsection we consider the stochastic-approximation version of system (1), formulated as:

$$x(s+1) = (1 - a(s))x(s) + a(s)[P(s)x(s) + u(s)], \quad s = 0, 1, \dots, \quad (2)$$

where $a(s) \in \mathbb{R}$ is the gain function. The system (1) is so called as linear SA algorithms [6]–[8], [19], [28], [39]. Compared to system (1), each agent in system (1) updates its state depending not only on the linear map $P(s)x(s) + u(s)$ but also on its own current state. If $a(s) = \frac{1}{s+1}$, then $x(s+1)$ equals the approximate average value of the previous s linear maps because $x(s)$ carries the information of the previous $s-1$ linear maps. Intuitively, in this case $x(s)$ approximately equals $\frac{1}{s} \sum_{k=0}^{s-1} x(k)$ in system (1), so that it should have the same limit as in Proposition 2.1. In fact, this result can be deduced by the following Proposition 3.1. Of course, this paper considers the more general case of $\{a(s)\}$ and $\{P(s)\}$.

The system (2) is a basic first-order discrete-time multi-agent system with much prior theoretical analysis. A main thread in the research of such a system is to study the setting in which $x(s)$ converges to a deterministic point. In [7], [8], convergence and convergence rates are studied for bounded linear operators with the assumption that there exists a matrix $P \in \mathbb{R}^{n \times n}$ whose eigenvalues' real parts are all less than 1 such that

$$\lim_{s \rightarrow \infty} \left(\sup_{s \leq t \leq m(s, T)} \left\| \sum_{i=s}^t a(i)(P(i) - P) \right\|_2 \right) = 0, \quad (3)$$

where $m(s, T) := \max\{k : a(s) + \dots + a(k) \leq T\}$ with T being an arbitrary positive constant, and $\|\cdot\|_2$ denotes the Euclidean norm. Later, Tadić relaxed the boundary condition of $P(s)$ and provided some convergence rates based on (3) and the assumption that the real parts of the eigenvalues of $P + \alpha I_n$ are all less than 1, where α is a positive constant [39]. Additionally, there are results on convergence rates by assuming that $\{I_n - P(s)\}_{s \geq 0}$ are a sequence of positive semi-definite matrices and $I_n - P$ is a positive definite matrix [24], [25]. Another thread in the theoretical research on system (2) is to consider its consensus behavior where $\{P(s)\}$ and $\{u(s)\}$ are assumed to be row-stochastic matrices and zero-mean noises respectively [6], [19], [28]. In addition, system (2) has many applications like computation of PageRank [46], sensor localization of wireless networks [23], distributed consensus of multi-agent systems, and belief evolution on social networks.

Despite all this prior theoretical research on system (2), a key problem remains unsolved: What is the necessary and sufficient condition for convergence regarding $\{P(s)\}$ and $u(s)$? Previous works focused on the case when the real parts of the eigenvalues of P are all assumed to be less than 1 [6]–[8], [19], [28], [39], but it is not known what happens when this condition is not satisfied. Also, traditional SA algorithms consider only non-negative gains, so another interesting problem is to investigate what happens if the gain function $a(s)$ can take arbitrary real numbers. This paper considers these two problems and studies the mean-square convergence of $x(s)$, whose definition is given as follows:

Definition 2.1: For an n -dimensional random vector x , we say $x(s)$ converges to x in mean square if

$$\mathbb{E}\|x\|_2^2 < \infty \text{ and } \lim_{s \rightarrow \infty} \mathbb{E}\|x(s) - x\|_2^2 = 0. \quad (4)$$

Also, we say $\{x(s)\}$ is mean-square convergent if there exists an n -dimensional random vector x such that (4) holds.

III. MAIN RESULTS

A. Informal statement of main results

We start with some notation. Given a matrix $A \in \mathbb{R}^{n \times n}$, define $\tilde{\rho}_{\max}(A) := \max_i \operatorname{Re}(\lambda_i(A))$ and $\tilde{\rho}_{\min}(A) := \min_i \operatorname{Re}(\lambda_i(A))$ to be the maximum and minimum values of the real parts of the eigenvalues of A respectively. It is easy to show that $|\tilde{\rho}_{\max}(A)| \leq \rho(A)$.

For $\{P(s)\}$ and $\{u(s)\}$, we relax the i.i.d. condition in [34] to the following assumption:

(A1) Suppose there exist a matrix $P \in \mathbb{R}^{n \times n}$ and a vector $u \in \mathbb{R}^n$ such that $\mathbb{E}[P(s) | x(s)] = P$ and $\mathbb{E}[u(s) | x(s)] = u$ for any $s \geq 0$ and $x(s) \in \mathbb{R}^n$. Also, assume $\mathbb{E}[\|P(s)\|_2^2 | x(s)]$ and $\mathbb{E}[\|u(s)\|_2^2 | x(s)]$ are uniformly bounded.

For $\{a(s)\}$, generally SA algorithms use the following assumption:

(A2) Assume $\{a(s)\}$ are non-negative real numbers independent with $\{x(s)\}$, and satisfying $\sum_{s=0}^{\infty} a(s) = \infty$ and $\sum_{s=0}^{\infty} a^2(s) < \infty$.

We will also consider the following alternative assumption.

(A2') Assume $\{a(s)\}$ are non-positive real numbers independent with $\{x(s)\}$, and satisfying $\sum_{s=0}^{\infty} a(s) = -\infty$ and $\sum_{s=0}^{\infty} a^2(s) < \infty$.

Under the assumptions (A1) and (A2), the previous works has investigated the cases when $\tilde{\rho}_{\max}(P) < 1$ and $P(s)x + u(s)$ is a bounded linear operator for all $s \geq 0$ [7], [8], or $\tilde{\rho}_{\max}(P + \alpha I_n) < 1$ [39], or $\{P(s)\}$ are row-stochastic matrices and $u = \mathbf{0}$ [6], [19], [28]. This paper will consider all the cases of P and u , and show the necessary and sufficient condition for the convergence of $x(s)$ in system (2) is $\tilde{\rho}_{\max}(P) < 1$, or $\tilde{\rho}_{\max}(P) = 1$ together with the following condition for P and u :

(A3) Assume any eigenvalue of P whose real part is 1 equals 1, and the eigenvalue 1 has the same algebraic and geometric multiplicities, and $\xi^T u = 0$ for any left eigenvector ξ^T of P corresponding to the eigenvalue 1.

Similarly, under (A1) and (A2') the necessary and sufficient condition for the convergence of $x(s)$ is $\tilde{\rho}_{\min}(P) > 1$, or $\tilde{\rho}_{\min}(P) = 1$ with (A3).

Also, we will study the convergence rates when $x(s)$ is convergent, and the convergence conditions when $\{a(s)\}$ are arbitrary real numbers.

B. Sufficient convergence conditions and convergence rates

Recall that P and u are the expectations of $P(s)$ and $u(s)$ respectively. Let

$$P = H^{-1} \operatorname{diag}(J_1, \dots, J_K) H := H^{-1} D H, \quad (5)$$

where $H \in \mathbb{C}^{n \times n}$ is an invertible matrix, and D is the Jordan normal form of P with

$$J_i = \begin{bmatrix} \lambda_{i'}(P) & 1 & & \\ & \lambda_{i'}(P) & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_{i'}(P) \end{bmatrix}_{m_i \times m_i}$$

for $1 \leq i \leq K$, where $\lambda_{i'}(P)$ is the eigenvalue of P corresponding to the Jordan block J_i .

Let r be the algebraic multiplicity of the eigenvalue 1 of P . We first consider the case $\tilde{\rho}_{\max}(P) = 1$ (or $\tilde{\rho}_{\min}(P) = 1$) with (A3), which implies that $r \geq 1$ and that the geometric multiplicity of the eigenvalue 1 is equal to r . We choose a suitable H such that $\lambda_1(P) = \dots = \lambda_r(P) = 1$. Then the Jordan normal form D can be written as

$$D = \begin{bmatrix} I_r & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(n-r) \times r} & \underline{D}_{(n-r) \times (n-r)} \end{bmatrix} \in \mathbb{C}^{n \times n}, \quad (6)$$

where $\underline{D} := \text{diag}(J_{r+1}, \dots, J_K) \in \mathbb{C}^{(n-r) \times (n-r)}$. For any vector $y \in \mathbb{C}^n$, throughout this subsection we set $\bar{y} := (y_1, \dots, y_r)^\top$ and $\underline{y} := (y_{r+1}, \dots, y_n)^\top$.

Theorem 3.1: (Convergence of linear SA algorithms at critical point) Consider the system (2) satisfying (A1), (A2), and (A3) with $\tilde{\rho}_{\max}(P) = 1$, or satisfying (A1), (A2'), and (A3) with $\tilde{\rho}_{\min}(P) = 1$. Let H be the matrix defined by (5) such that the Jordan normal form D has the form of (6). Then, for any initial state, $x(s)$ converges to $H^{-1}y$ in mean square, where \bar{y} is a random vector satisfying $\mathbb{E}\bar{y} = \overline{Hx(0)}$ and $\mathbb{E}\|\bar{y}\|_2^2 < \infty$, and $\underline{y} = (I_{n-r} - \underline{D})^{-1}Hu$.

From Theorem 3.1, $x(s)$ converges to a random vector under the critical condition $\tilde{\rho}_{\max}(P) = 1$ (or $\tilde{\rho}_{\min}(P) = 1$), which is different from the previous works where $x(s)$ converges to a deterministic vector under non critical conditions [6]–[8], [19], [28], [39]. Due to this difference, the traditional method cannot be used in the proof of Theorem 3.1. We propose a new method to prove this theorem as follows.

Proof of Theorem 3.1: Let $y(s) := Hx(s)$, $v(s) := Hu(s)$ and $D(s) := HP(s)H^{-1}$, then by (2) we have

$$\begin{aligned} H^{-1}y(s+1) \\ = (1 - a(s))H^{-1}y(s) + a(s)[P(s)H^{-1}y(s) + u(s)], \end{aligned} \quad (7)$$

which implies

$$y(s+1) = y(s) + a(s)[(D(s) - I_n)y(s) + v(s)]. \quad (8)$$

Let $v := \mathbb{E}[v(s)] = Hu$. From (5) we have $HP = DH$, which implies $H_iP = H_i$ for $1 \leq i \leq r$, where H_i is the i -th row of the matrix H . Thus, H_i , $1 \leq i \leq r$, is a left eigenvector corresponding to the eigenvalue 1. By (A3) we have

$$v_i = H_i u = 0, \quad \forall 1 \leq i \leq r. \quad (9)$$

Recall that $\underline{v} = (v_{r+1}, \dots, v_n)^\top$. Also, $I_{n-r} - \underline{D}$ is an invertible matrix, so we can set

$$z := \begin{bmatrix} \mathbf{0}_{r \times 1} \\ (I_{n-r} - \underline{D})^{-1}\underline{v} \end{bmatrix} \in \mathbb{C}^n.$$

From (6) and (9) we have

$$(D - I_n)z + v = \mathbf{0}_{n \times 1}. \quad (10)$$

Set $\theta(s) := y(s) - z$. From (8) we obtain

$$\theta(s+1) = \theta(s) + a(s)[(D(s) - I_n)(\theta(s) + z) + v(s)]. \quad (11)$$

We first consider the case when $\tilde{\rho}_{\max}(P) = 1$, which implies that $\underline{D} - I_{n-r}$ is a Hurwitz matrix. Thus, by the stability theory of continuous Lyapunov equation (see [18,

Corollary 2.2.4]), there exists a Hermitian positive definite matrix $A \in \mathbb{C}^{(n-r) \times (n-r)}$ such that

$$(\underline{D} - I_{n-r})^* A + A(\underline{D} - I_{n-r}) = -I_{n-r}, \quad (12)$$

where $(\cdot)^*$ denotes the conjugate transpose of the matrix or vector. Set

$$A_1 := \begin{bmatrix} I_r & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(n-r) \times r} & A_{(n-r) \times (n-r)} \end{bmatrix} \in \mathbb{C}^{n \times n},$$

then A_1 is still a Hermitian positive definite matrix. Define the Lyapunov function $V_1(\theta) := \theta^* A_1 \theta$. By (11), (A1) and (12), for any $\theta(s)$ we have

$$\begin{aligned} \mathbb{E}[V_1(\theta(s+1)) | \theta(s)] \\ \leq V_1(\theta(s)) + a(s)\theta^*(s) [(D - I_n)^* A_1 + A_1(D - I_n)] \theta(s) \\ + O(a^2(s)(\|\theta(s)\|_2^2 + 1)). \end{aligned} \quad (13)$$

From (6) and (12), we obtain

$$\begin{aligned} (D - I_n)^* A_1 + A_1(D - I_n) \\ = \begin{bmatrix} \mathbf{0}_{r \times r} & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(n-r) \times r} & (\underline{D} - I_{n-r})^* A + A(\underline{D} - I_{n-r}) \end{bmatrix} \\ = \begin{bmatrix} \mathbf{0}_{r \times r} & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(n-r) \times r} & -I_{n-r} \end{bmatrix}, \end{aligned} \quad (14)$$

so (13) implies

$$\mathbb{E}[V_1(\theta(s+1))] \leq [1 + c_1 a^2(s)] \mathbb{E}[V_1(\theta(s))] + c_2 a^2(s), \quad (15)$$

where c_1 and c_2 are two positive constants. Using (15) repeatedly we get

$$\begin{aligned} \mathbb{E}[V_1(\theta(s+1))] \\ \leq \prod_{i=0}^s [1 + c_1 a^2(i)] + \sum_{i=0}^s c_2 a^2(i) \prod_{j=i+1}^s [1 + c_1 a^2(j)] \\ < \infty \quad \text{as } s \rightarrow \infty, \end{aligned} \quad (16)$$

where the last inequality uses the condition that $\sum_{s=0}^{\infty} a^2(s) < \infty$. Also, because A_1 is a Hermitian positive definite matrix,

$$\frac{1}{\rho(A_1)} V_1(\theta(s)) \leq \|\theta(s)\|_2^2 \leq \frac{1}{\lambda_{\min}(A_1)} V_1(\theta(s)). \quad (17)$$

Combining (16) and (17) yields

$$\sup_s \mathbb{E}\|\theta(s)\|_2^2 \leq \sup_s \frac{\mathbb{E}[V_1(\theta(s))]}{\lambda_{\min}(A_1)} < \infty. \quad (18)$$

Inequality (18) shows that $\theta(s)$ will not diverge, however we need to prove its convergence. We first consider the convergence of $\underline{\theta}(s)$. Set

$$A_2 := \begin{bmatrix} \mathbf{0}_{r \times r} & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(n-r) \times r} & A_{(n-r) \times (n-r)} \end{bmatrix} \in \mathbb{C}^{n \times n}$$

¹Given two sequences of positive numbers $\{g_1(s)\}_{s=0}^{\infty}$ and $\{g_2(s)\}_{s=0}^{\infty}$, we say $g_1(s) = O(g_2(s))$ if there exist a constants $c > 0$ such that $g_1(s) \leq c g_2(s)$ for all $s \geq 0$.

and define $V_2(\theta) := \theta^* A_2 \theta = \underline{\theta}^* A \underline{\theta}$. Similar to (13), we have

$$\begin{aligned} & \mathbb{E}[V_2(\theta(s+1))] \\ & \leq \mathbb{E}\left[V_2(\theta(s)) + a(s)\theta^*(s)(D - I_n)^* A_2 \theta(s) \right. \\ & \quad \left. + a(s)\theta^*(s)A_2(D - I_n)\theta(s) + O(a^2(s)(\|\theta(s)\|_2^2 + 1))\right] \\ & = \mathbb{E}\left[V_2(\theta(s)) + a(s)\theta^*(s) \begin{bmatrix} \mathbf{0}_{r \times r} & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(n-r) \times r} & -I_{n-r} \end{bmatrix} \theta(s)\right] \\ & \quad + O(a^2(s)) \\ & \leq \left(1 - \frac{a(s)}{\rho(A)}\right) \mathbb{E}[V_2(\theta(s))] + O(a^2(s)), \end{aligned} \quad (19)$$

where the forth line uses (14) and (18), and the last inequality does a similar computation as (17). By (19) and Lemma A.1 in Appendix A, we obtain $\lim_{s \rightarrow \infty} \mathbb{E}[V_2(\theta(s))] = 0$, which implies

$$\lim_{s \rightarrow \infty} \mathbb{E}\|\underline{\theta}(s)\|_2^2 = 0. \quad (20)$$

It remains to consider the convergence of $\bar{\theta}(s)$. Set

$$A_3 := \begin{bmatrix} I_r & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(n-r) \times r} & \mathbf{0}_{(n-r) \times (n-r)} \end{bmatrix} \in \mathbb{R}^{n \times n},$$

and define $V_3(\theta) = \theta^* A_3 \theta = \bar{\theta}^* \bar{\theta}$. By (6) and (9) we get $A_3(D - I_n) = \mathbf{0}_{n \times n}$ and $A_3 v = \mathbf{0}_{n \times 1}$, thus by (A1) for any $i < j$ we have

$$\begin{aligned} & E\left[\left[(D(i) - I_n)(\theta(i) + z) + v(i)\right]^* A_3 \right. \\ & \quad \left. \times \left[(D(j) - I_n)(\theta(j) + z) + v(j)\right]\right] \\ & = E\left[\left[(D(i) - I_n)(\theta(i) + z) + v(i)\right]^* A_3 \right. \\ & \quad \left. \times \left[(D - I_n)(\theta(j) + z) + v\right]\right] = 0. \end{aligned} \quad (21)$$

Similarly, the equation (21) still holds for $i > j$. From these and (11) we get for any $s_2 > s \geq 0$,

$$\begin{aligned} & \mathbb{E}[V_3(\theta(s_2) - \theta(s))] \\ & = \mathbb{E}\left[V_3\left(\sum_{i=s}^{s_2-1} [\theta(i+1) - \theta(i)]\right)\right] \\ & = \mathbb{E}\left[\left(\sum_{i=s}^{s_2-1} a(i)\left[(D(i) - I_n)(\theta(i) + z) + v(i)\right]\right)^* A_3 \right. \\ & \quad \left. \times \left(\sum_{i=s}^{s_2-1} a(i)\left[(D(i) - I_n)(\theta(i) + z) + v(i)\right]\right)\right] \\ & = \sum_{i=s}^{s_2-1} a^2(i) \mathbb{E}\left[\left[(D(i) - I_n)(\theta(i) + z) + v(i)\right]^* \right. \\ & \quad \left. \times A_3 \left[(D(i) - I_n)(\theta(i) + z) + v(i)\right]\right] \\ & = O\left(\sum_{i=s}^{s_2-1} a^2(i)\right), \end{aligned} \quad (22)$$

where the last line uses (A1) and (18). Since $\sum_{i=0}^{\infty} a^2(i) < \infty$, from (22) we have

$$\begin{aligned} & \lim_{s \rightarrow \infty} \lim_{s_2 \rightarrow \infty} \mathbb{E}\|\bar{\theta}(s_2) - \bar{\theta}(s)\|_2^2 \\ & = \lim_{s \rightarrow \infty} \lim_{s_2 \rightarrow \infty} \mathbb{E}[V_3(\theta(s_2) - \theta(s))] = 0. \end{aligned} \quad (23)$$

By the Cauchy criterion (see [21, page 58]), $\bar{\theta}(s)$ has a mean square limit $\bar{\theta}(\infty)$. Also, from (11), (A1) and (9) we have

$$\begin{aligned} & \mathbb{E}[A_3 \theta(s+1)] \\ & = \mathbb{E}\left[\mathbb{E}[A_3 \theta(s+1) \mid \theta(s)]\right] \\ & = \mathbb{E}\left[A_3 \theta(s) + a(s)A_3[(D - I_n)(\theta(s) + z) + v]\right] \\ & = \mathbb{E}[A_3 \theta(s)] = \cdots = A_3 \theta(0), \end{aligned} \quad (24)$$

which is followed by

$$\mathbb{E}\bar{\theta}(\infty) = \bar{\theta}(0) = \bar{y}(0) = \overline{Hx}(0). \quad (25)$$

We remark that $x(s) = H^{-1}[\theta(s) + z]$. Let y be a vector satisfying $y = \underline{z} = (I_{n-r} - \underline{D})^{-1} \underline{v}$ and $\bar{y} = \bar{\theta}(\infty) + \bar{z} = \bar{\theta}(\infty)$. By (20) and (23) we have that $x(s)$ converges to $H^{-1}y$ in mean square. By (25) and (18) we get $\mathbb{E}\bar{y} = \overline{Hx}(0)$ and $\mathbb{E}\|\bar{y}\|_2^2 < \infty$.

For the case that $\tilde{\rho}_{\min}(P) = 1$, which implies $I_{n-r} - \underline{D}$ is a Hurwitz matrix. Set $b(s) = -a(s) \geq 0$ and substitute it to (11) we obtain

$$\theta(s+1) = \theta(s) + b(s)[(I_n - D(s))(\theta(s) + z) - v(s)].$$

Finally, a process similar to that from (12) to (25) yields our result. \square

For the case when $\tilde{\rho}_{\max}(P) < 1$ or $\tilde{\rho}_{\min}(P) > 1$, from the proof of Theorem 3.1 we have the following proposition:

Proposition 3.1: Consider the system (2) satisfying (A1), (A2) and $\tilde{\rho}_{\max}(P) < 1$, or satisfying (A1), (A2') and $\tilde{\rho}_{\min}(P) > 1$. Then, for any initial state, $x(s)$ converges to $(I_n - P)^{-1}u$ in mean square.

Proof: We can set $r = 0$ in the proof of Theorem 3.1, then we obtain that $x(s)$ converges to $H^{-1}(I_n - D)^{-1}Hu = (I_n - P)^{-1}u$ in mean square. \square

Next, we give the convergence rate when $x(s)$ is mean-square convergent.

Theorem 3.2: (Convergence rates of linear SA algorithms) Consider the system (2) satisfying (A1) and one of the following four cases: i) $\tilde{\rho}_{\max}(P) < 1$; ii) $\tilde{\rho}_{\min}(P) > 1$; iii) $\tilde{\rho}_{\max}(P) = 1$ with (A3); and iv) $\tilde{\rho}_{\min}(P) = 1$ with (A3). Let $\beta > 0, \gamma \in (\frac{1}{2}, 1]$, and α be a large positive number. Choose $a(s) = \frac{\alpha}{(s+\beta)^\gamma}$ if $\tilde{\rho}_{\max}(P) \leq 1$, and $a(s) = \frac{-\alpha}{(s+\beta)^\gamma}$ if $\tilde{\rho}_{\min}(P) \geq 1$. Then for any initial state,

$$\begin{aligned} & \mathbb{E}\|x(s) - x\|_2^2 \\ & = \begin{cases} O(s^{-\gamma}), & \text{if } \tilde{\rho}_{\max}(P) < 1 \text{ or } \tilde{\rho}_{\min}(P) > 1 \\ O(s^{1-2\gamma}), & \text{if } \tilde{\rho}_{\max}(P) = 1 \text{ or } \tilde{\rho}_{\min}(P) = 1 \end{cases} \end{aligned}$$

where x is a mean square limit of $x(s)$ whose expression is provided by Theorem 3.1 and Proposition 3.1.

The proof of this theorem is postponed to Appendix B.

Remark 1: For the case when $\tilde{\rho}_{\max}(P) < 1$, there exist results on the convergence and convergence rates of $x(s)$ provided some additional conditions hold, beside (A1)-(A2). For example, if $\lim_{s \rightarrow \infty} \sum_{k=0}^{s-1} \frac{\|P(s)\|_2}{s}$ a.s. exists and $\tilde{\rho}_{\max}(P + \alpha I_n) < 1$ with α being a positive constant, then Theorem 2 in [39] provides sufficient and necessary conditions

for the convergence rate of $x(s)$; if $\|x(s)\|_2$ is uniformly bounded a.s., then by the ODE method in SA theory (Theorem 5.2.1 in [26] or Theorem 2.2 in [2]) we have $x(s)$ converges to $(I_n - P)^{-1}u$ a.s. However, to the best of our knowledge, our results in Proposition 3.1 and Theorem 3.2 cannot be deduced from existing results without additional conditions.

C. Necessary conditions for convergence

We first consider necessary conditions of convergence under the assumptions (A1) and (A2) or (A2’):

Theorem 3.3: Consider the system (2) satisfying (A1). Then:

- i) If $\tilde{\rho}_{\max}(P) > 1$, or $\tilde{\rho}_{\max}(P) = 1$ but (A3) does not hold, there exist some initial states such that $x(s)$ is not mean-square convergent for any $\{a(s)\}$ satisfying (A2).
- ii) If $\tilde{\rho}_{\min}(P) < 1$, or $\tilde{\rho}_{\min}(P) = 1$ but (A3) does not hold, there exist some initial states such that $x(s)$ is not mean-square convergent for any $\{a(s)\}$ satisfying (A2’).

The proof of this theorem is postponed to Appendix C.

The necessary condition of convergence in Theorem 3.3 has a constraint that the gain function $\{a(s)\}$ must satisfy the assumption (A2) or (A2’). An interesting problem is to understand what happens if $\{a(s)\}$ are chosen as arbitrary real numbers. Obviously, from protocol (2) if $\{a(s)\}$ has only finite non-zero elements, then $x(s)$ will converge to a random variable. Thus, we only consider the setting whereby $x(s)$ does not converge to a deterministic vector for arbitrary gains.

Recall that

$$P = H^{-1} \text{diag}(J_1, \dots, J_K) H = H^{-1} D H,$$

where $H \in \mathbb{C}^{n \times n}$ is an invertible matrix, and D is the Jordan normal form of P . For $1 \leq i \leq K$, define

$$\tilde{I}_i = \text{diag}(0, \dots, I_{m_i}, \dots, 0) \in \mathbb{R}^{n \times n}, \quad (26)$$

which corresponds to the Jordan block J_i and then $D\tilde{I}_i = \text{diag}(0, \dots, J_i, \dots, 0)$. To study the necessary condition for convergence of system (2), we need the following two assumptions:

(A4) Assume there is a Jordan block J_j in D associated with the eigenvalue $\lambda_{j'}(P)$ such that $\text{Re}(\lambda_{j'}(P)) = 1$ and

$$\mathbb{E}[\|\tilde{I}_j H[(P(s) - P)x(s) + u(s) - u]\|_2^2 | x(s)] \geq c_1 \|x(s)\|_2^2 + c_2 \quad (27)$$

for any $s \geq 0$ and $x(s) \in \mathbb{R}^n$, where P , u , H , D and \tilde{I}_j are defined by (A1), (5), and (26), and c_1 and c_2 are constants satisfying $c_1 \geq 0$, $c_2 \geq 0$, and $c_1 + c_2 > 0$.

(A4’) Assume there are two Jordan blocks J_{j_1} and J_{j_2} associated with the eigenvalues $\lambda_{j_1'}(P)$ and $\lambda_{j_2'}(P)$ respectively such that $\text{Re}(\lambda_{j_1'}(P)) < 1 < \text{Re}(\lambda_{j_2'}(P))$ and (27) holds for $j = j_1, j_2$.

Theorem 3.4: Consider the system (2) satisfying (A1) and (A4) or (A4’). In addition, assume there exists a constant $c_3 > 0$ such that for any $s \geq 0$ and $x(s) \in \mathbb{R}^n$,

$$\mathbb{E}[\|(P(s) - P)x(s) + u(s) - u\|_2^2 | x(s)] \geq c_3. \quad (28)$$

Then for any deterministic vector $b \in \mathbb{R}^n$, any initial state $x(0) \neq b$, and any real number sequence $\{a(s)\}_{s \geq 0}$ independent with $\{x(s)\}_{s \geq 0}$, $x(s)$ cannot converge to b in mean square.

The proof of this theorem is postponed to Appendix D.

If $u(s)$ is a degenerate random vector which means that $\mathbb{E}\|u(s) - u\|_2^2 = 0$, then the condition (28) may not be satisfied.

Theorem 3.5: Consider the system (2) satisfying (A1), and $\mathbb{E}[\|u(s) - u\|_2^2 | x(s)] = 0$ for any $s \geq 0$ and $x(s) \in \mathbb{R}^n$. Assume (A4) or (A4’) holds but using

$$\mathbb{E}[\|\tilde{I}_j H(P(s) - P)x(s)\|_2^2 | x(s)] \geq c_1 \|x(s)\|_2^2 \quad (29)$$

instead of (27). For any deterministic vector $b \in \mathbb{R}^n$ and any initial state $x(0) \neq b$, if one of the following three conditions holds:

- i) $u \neq \mathbf{0}_{n \times 1}$ and $x(0) \neq \mathbf{0}_{n \times 1}$;
 - ii) $u \neq \mathbf{0}_{n \times 1}$, $x(0) = \mathbf{0}_{n \times 1}$, and $b \neq \alpha u$ for any $\alpha \in \mathbb{R}$; or
 - iii) $u = \mathbf{0}_{n \times 1}$, and the eigenvalues $\lambda_{j'}(P)$ in (A4), or $\lambda_{j_1'}(P)$ and $\lambda_{j_2'}(P)$ in (A4’) are not real numbers,
- then $x(s)$ cannot converge to b in mean square for any real number sequence $\{a(s)\}_{s \geq 0}$ independent with $\{x(s)\}_{s \geq 0}$.

The proof of this theorem is postponed to Appendix V.

D. Necessary and sufficient conditions for convergence

From Theorems 3.1 and 3.3 and Proposition 3.1, the following necessary and sufficient condition for convergence with non-negative gains is obtained immediately.

Theorem 3.6: (Necessary and sufficient condition for convergence of linear SA algorithms with non-negative gains) Consider the system (2) satisfying (A1) and (A2). Then $x(s)$ is mean-square convergent for any initial state if and only if $\tilde{\rho}_{\max}(P) < 1$, or $\tilde{\rho}_{\max}(P) = 1$ with (A3).

Remark 2: We remark that Theorem 3.6 is completely different from previous sufficient and necessary conditions of convergence in linear SA algorithms where only the case when $\tilde{\rho}_{\max}(P) < 1$ is considered and the assumptions are different from (A2) (Theorem 2 in [7]; Theorem 1 in [8]; Theorems 1 and 2 in [39]). In fact, the convergence of $x(s)$ at the critical point $\tilde{\rho}_{\max}(P) = 1$ has some applications such as the group consensus over random signed networks; see Subsection IV-A.

Similarly, from Theorems 3.1 and 3.3 and Proposition 3.1, the following necessary and sufficient condition for convergence with non-positive gain is obtained immediately.

Theorem 3.7: (Necessary and sufficient condition for convergence of linear SA algorithms with non-positive gains) Consider the system (2) satisfying (A1) and (A2’). Then $x(s)$ is mean-square convergent for any initial state if and only if $\tilde{\rho}_{\min}(P) > 1$, or $\tilde{\rho}_{\min}(P) = 1$ with (A3).

Remark 3: Compared to Theorem 1 in [34], Theorem 3.6 extends the convergence condition from $\rho(P) < 1$ to the sufficient and necessary condition. In fact, for the basic linear dynamical system $x(s+1) = Px(s) + u$, $x(s)$ converges if and only if $\rho(P) < 1$. However, if we consider the time-varying

linear dynamical system and adopt the SA method to eliminate the effect of fluctuation, then the convergence condition can be substantially weakened.

Theorems 3.6 and 3.7 have a constraint that the gain function $\{a(s)\}$ must satisfy the assumption (A2) or (A2'). Without this constraint we can get the following necessary and sufficient condition for convergence to a deterministic vector, but with some additional conditions on $\{u(s)\}$ or $\{P(s)\}$.

Theorem 3.8 (Necessary and sufficient condition for convergence of linear SA algorithms with arbitrary gains): Consider the system (2) which satisfies (A1). Suppose there exists a constant $c \in (0, 1)$ such that for any $s \geq 0$, $x(s) \in \mathbb{R}^n$, $\xi_1, \dots, \xi_m \in \{P_{ij}(s), 1 \leq i, j \leq n; u_i(s), 1 \leq i \leq n\}$ and $c_1, \dots, c_m \in \mathbb{C}$,

$$\mathbb{E} \left[\left| \sum_{i=1}^m c_i (\xi_i - \mathbb{E} \xi_i) \right|^2 \middle| x(s) \right] \geq c \sum_{i=1}^m |c_i|^2 \mathbb{E} [(\xi_i - \mathbb{E} \xi_i)^2 \middle| x(s)]. \quad (30)$$

In addition, assume one of the following two conditions holds:

- i) $\inf_{k,s} \mathbb{E} [(u_k(s) - u_k)^2 \middle| x(s)] > 0$.
- ii) $\mathbb{E} [\|u(s) - u\|^2 \middle| x(s)] = 0$, $u \neq \mathbf{0}_{n \times 1}$, $x(0) \neq \mathbf{0}_{n \times 1}$, and $\inf_{i,j,s} \mathbb{E} [(P_{ij}(s) - P_{ij})^2 \middle| x(s)] > 0$.

Then we can choose a real number sequence $\{a(s)\}_{s \geq 0}$ independent with $\{x(s)\}_{s \geq 0}$ such that $x(s)$ converges to a deterministic vector different from $x(0)$ in mean square if and only if $\tilde{\rho}_{\max}(P) < 1$ or $\tilde{\rho}_{\min}(P) > 1$.

Proof: If $\tilde{\rho}_{\max}(P) < 1$ or $\tilde{\rho}_{\min}(P) > 1$, by Proposition 3.1 we obtain that $x(s)$ converges to $(I_n - P)^{-1}u$ in mean square.

For $\tilde{\rho}_{\min}(P) \leq 1 \leq \tilde{\rho}_{\max}(P)$, we set $\tilde{P}(s) := P(s) - P$ and $\tilde{u}(s) := u(s) - u$. Define H and K by (5), and define \tilde{I}_i by (26). For any $j \in \{1, \dots, K\}$, since H is an invertible matrix, $\tilde{I}_j H$ contains at least one non-zero row $H_{j'}$. Thus, for any $x(s) \in \mathbb{R}^n$ we have

$$\begin{aligned} & \mathbb{E} [\|\tilde{I}_j H [\tilde{P}(s)x(s) + \tilde{u}(s)]\|_2^2 \middle| x(s)] \\ & \geq \mathbb{E} [|H_{j'} [\tilde{P}(s)x(s) + \tilde{u}(s)]|^2 \middle| x(s)] \\ & = \mathbb{E} \left[\left| \sum_{i,k} H_{j'i} \tilde{P}_{ik}(s) x_k(s) + \sum_i H_{j'i} \tilde{u}_i(s) \right|^2 \middle| x(s) \right] \\ & \geq c \sum_{i,k} |H_{j'i}|^2 \mathbb{E} [\tilde{P}_{ik}^2(s) \middle| x(s)] x_k^2(s) \\ & \quad + c \sum_i |H_{j'i}|^2 \mathbb{E} [\tilde{u}_i^2(s) \middle| x(s)], \end{aligned} \quad (31)$$

where the last inequality uses (30).

If Condition i) holds, we have there exists a constant $d_1 > 0$ such that $\mathbb{E} [\tilde{u}_i^2(s) \middle| x(s)] \geq d_1$ for $s \geq 0$ and $1 \leq i \leq n$. Combing this with (31) and the assumption $\tilde{\rho}_{\min}(P) \leq 1 \leq \tilde{\rho}_{\max}(P)$, we obtain that (28) and (A4) or (A4') hold. By Theorem 3.4, $x(s)$ cannot converge to a deterministic vector different from $x(0)$ in mean square.

If Condition ii) holds, we have $\mathbb{E} [\|\tilde{u}(s)\|_2^2 \middle| x(s)] = 0$ and there exists a constant $d_2 > 0$ such that $\mathbb{E} [\tilde{P}_{ik}^2(s) \middle| x(s)] \geq d_2$ for $s \geq 0$ and $1 \leq i, k \leq n$. By (31) we obtain

$$\begin{aligned} & \mathbb{E} [\|\tilde{I}_j H [\tilde{P}(s)x(s)]\|_2^2 \middle| x(s)] \\ & \geq cd_2 \sum_{i,k} |H_{j'i}|^2 x_k^2(s) = cd_2 \|x(s)\|_2^2 \sum_i |H_{j'i}|^2, \end{aligned}$$

which is followed by (29). By Theorem 3.5 i) $x(s)$ cannot converge to a deterministic vector different from $x(0)$ in mean square. \square

IV. SOME APPLICATIONS AND EXTENSION

A. Necessary and sufficient conditions for group consensus over random signed networks and with state-dependent noise

As we discuss in the Introduction, consensus problems in multi-agent systems have drawn a lot of attention from various fields including physics, biology, engineering and mathematics in the past two decades. Typically, a general assumption is adopted that the interaction matrix associated with the network is row-stochastic at every time. Recently, motivated by the possible antagonistic interaction in social networks, bipartite/group/cluster consensus problems have been studied over signed networks (focusing on continuous-time dynamic models), e.g., see [1], [29], [33], [45]. On the other hand, SA has become a effective tool for the distributed consensus to eliminate the effects of fluctuations [3], [6], [19], [20], [27], [28], [41]. Interestingly, if we consider the linear SA algorithms over random signed networks with state-dependent noise, from Theorems 3.1, 3.6 and 3.7 we can obtain some results for the consensus or group consensus.

Assume the system contains n agents. Each agent i has a state $x_i(s) \in \mathbb{R}$ at time s which can represent the opinion, social power or others, and is updated according to the current state and the interaction from the others. In detail, for $1 \leq i \leq n$ and $s \geq 0$, the state of agent i is updated by

$$\begin{aligned} x_i(s+1) &= (1 - a(s))x_i(s) \\ & \quad + a(s) \sum_{j \in \mathcal{N}_i(s)} P_{ij}(s) [x_j(s) + f_{ji}(x(s))w_{ji}(s)], \end{aligned} \quad (32)$$

where $a(s) \geq 0$ is the gain at time s , $\mathcal{N}_i(s)$ is the neighbors of node i at time s , $P_{ij}(s)$ is the weight of the edge (j, i) at time s , and $f_{ji}(x(s))w_{ji}(s)$ is the noise of agent i receiving information from agent j at time s . Here we consider the noise may be state-dependent which means that $f_{ji}(x(s))$ is a function of the state vector $x(s)$. Let $P_{ij}(s) = 0$ if $j \notin \mathcal{N}_i(s)$, and set

$$u_i(s) := \sum_{j \in \mathcal{N}_i(s)} P_{ij}(s) f_{ji}(x(s)) w_{ji}(s),$$

then system (32) can be rewritten as

$$x(s+1) = (1 - a(s))x(s) + a(s) [P(s)x(s) + u(s)].$$

If $P_{ij}(s)$ is a stationary stochastic process with uniformly bounded variance, and $w_{ji}(s)$ is a zero-mean noise with uniformly bounded variance for any $x(s)$, $P_{ij}(s)$, and $j \in \mathcal{N}_i(s)$, then (A1) is satisfied with $u = \mathbf{0}_{n \times 1}$.

We say the subsets $S_1, \dots, S_{r'} (r' \geq 1)$ is a *partition* of $\{1, \dots, n\}$ if $\emptyset \subset S_i \subseteq \{1, \dots, n\}$ for $1 \leq i \leq r'$, $S_i \cap S_j = \emptyset$ for $i \neq j$, and $\cup_{i=1}^{r'} S_i = \{1, \dots, n\}$. Following [45] with some modifications we introduce the definition for group consensus:

Definition 4.1: Let the subsets $S_1, \dots, S_{r'}$ be a partition of $\{1, \dots, n\}$. If $x(s)$ is mean-square convergent, and $\lim_{s \rightarrow \infty} \mathbb{E}|x_i(s) - x_j(s)| = 0$ when i and j belong to a same subset, then we say $x(s)$ asymptotically reaches $\{S_i\}_{i=1}^{r'}$ -group consensus in mean square.

The group consensus turns to cluster consensus if different groups have different limit values [16].

From Definition 4.1 we can know that consensus is a special case of the $\{S_i\}_{i=1}^{r'}$ -group consensus with $r' = 1$. Before the statement of our results, we need to introduce some notations and an assumption:

For a partition $S_1, \dots, S_{r'}$ of $\{1, \dots, n\}$, let $\mathbb{1}^i \in \mathbb{R}^n (1 \leq i \leq r')$ denote the column vector satisfying $\mathbb{1}_k^i = 1$ if $k \in S_i$ and $\mathbb{1}_k^i = 0$ otherwise. A linear combination of $\{\mathbb{1}^i\}_{i=1}^{r'}$ is $c_1 \mathbb{1}^1 + \dots + c_{r'} \mathbb{1}^{r'}$ with $c_1, \dots, c_{r'} \in \mathbb{C}$ being constants.

(A5) Assume any eigenvalue of P whose real part is 1 equals 1, and the algebraic and geometric multiplicities of the eigenvalue 1 equal $r \in [1, r']$, and any right eigenvector of P corresponding to the eigenvalue 1 can be written as a linear combination of $\{\mathbb{1}^i\}_{i=1}^{r'}$.

With Theorems 3.1, 3.6 and Proposition 3.1 we obtain the following result:

Theorem 4.1: (Necessary and sufficient condition for group consensus with non-negative gains) Consider the system (2) or (32) satisfying (A1) with $u = \mathbf{0}_{n \times 1}$ and (A2). Let $S_1, \dots, S_{r'}$ be a partition of $\{1, \dots, n\}$. Then $x(s)$ asymptotically reaches $\{S_i\}_{i=1}^{r'}$ -group consensus in mean square for any initial state if and only if $\tilde{\rho}_{\max}(P) < 1$, or (A5) holds with $\tilde{\rho}_{\max}(P) = 1$.

Proof: Before proving our result, we introduce some notes first. For any matrix $A \in \mathbb{C}^{n \times n}$, let A_i and A^i denote the i -th row and i -th column of A respectively. Set $A^{[i:j]} = (A^i, A^{i+1}, \dots, A^j) \in \mathbb{C}^{n \times (j-i+1)}$.

We first consider the sufficient part. If $\tilde{\rho}_{\max}(P) < 1$, by Proposition 3.1 and the fact $u = \mathbf{0}_{n \times 1}$ we obtain that $x(s)$ converges to $\mathbf{0}_{n \times 1}$ in mean square for all initial states. Hence, the $\{S_i\}$ -group consensus can be reached.

If (A5) holds with $\tilde{\rho}_{\max}(P) = 1$, which implies that (A3) holds together with the fact $u = \mathbf{0}_{n \times 1}$. Let $P = H^{-1}DH$, where H is an invertible matrix, and D is the Jordan normal form of P with the same expression as (6). Then, by Theorem 3.1, for any initial state there exist random variables y_1, \dots, y_r such that in mean square

$$x(s) \rightarrow y_1[H^{-1}]^1 + \dots + y_r[H^{-1}]^r \quad \text{as } s \rightarrow \infty. \quad (33)$$

Also, from $PH^{-1} = DH^{-1}$ and (6) we have

$$P[H^{-1}]^i = [H^{-1}]^i, \quad 1 \leq i \leq r. \quad (34)$$

Hence, by (33) and (A5), there exist random variables $z_1, \dots, z_{r'}$ such that in mean square

$$x(s) \rightarrow z_1 \mathbb{1}^1 + \dots + z_{r'} \mathbb{1}^{r'} \quad \text{as } s \rightarrow \infty,$$

which implies that $x(s)$ asymptotically reaches $\{S_i\}_{i=1}^{r'}$ -group consensus in mean square for any initial state.

Next we prove the necessary part. Since $x(s)$ asymptotically reaches $\{S_i\}_{i=1}^{r'}$ -group consensus in mean square for any initial state, then, by Definition 4.1, $x(s)$ is mean-square convergent for any initial state. Hence, by Theorem 3.6, we obtain that $\tilde{\rho}_{\max}(P) < 1$, or (A3) holds with $\tilde{\rho}_{\max}(P) = 1$.

It remains to show (A5) holds for the case when (A3) holds. For any complex right eigenvector $\mathbf{a} + \mathbf{b}i \in \mathbb{C}^n$ of P corresponding to eigenvalue 1, we have $P\mathbf{a} = \mathbf{a}$ and $P\mathbf{b} = \mathbf{b}$, which implies that \mathbf{a} and \mathbf{b} are real right eigenvectors of P corresponding to eigenvalue 1. Thus, any complex right eigenvector of P corresponding to the eigenvalue 1 can be written as a linear combination of real right eigenvectors corresponding to the eigenvalue 1. Also, from (6) we have $PH^{-1} = H^{-1}D$ if and only if (34) and $P[H^{-1}]^{[r+1,n]} = [H^{-1}]^{[r+1,n]}D$ hold. Thus, we can choose suitable H such that $P = H^{-1}DH$ and $[H^{-1}]^1, \dots, [H^{-1}]^r$ are real vectors. By Theorem 3.1, we have

$$\lim_{s \rightarrow \infty} \mathbb{E}x(s) = \sum_{i=1}^r H_i x(0) \cdot [H^{-1}]^i \quad (35)$$

Also, from $HH^{-1} = I_n$ we have $H_i[H^{-1}]^j$ equals 1 if $i = j$ and 0 otherwise. If we choose $x(0) = [H^{-1}]^i (1 \leq i \leq r)$, by (35) we have $\lim_{s \rightarrow \infty} \mathbb{E}x(s) = [H^{-1}]^i$. Because for any initial state, $x(s)$ asymptotically reaches $\{S_i\}_{i=1}^{r'}$ -group consensus in mean square, which implies $\mathbb{E}x(s)$ also asymptotically reaches $\{S_i\}_{i=1}^{r'}$ -group consensus, $[H^{-1}]^i (1 \leq i \leq r)$ can be written as a linear combination of $\{\mathbb{1}^j\}_{j=1}^{r'}$. From the linear independence of $[H^{-1}]^1, \dots, [H^{-1}]^r$ we have $r \leq r'$, and $[H^{-1}]^1, \dots, [H^{-1}]^r$ is a basis of the eigenspace \mathcal{R}^1 which consists of all the right eigenvectors of P corresponding to the eigenvalue 1 and together with the zero vector. Hence, any vector in \mathcal{R}^1 can be written as a linear combination of $[H^{-1}]^1, \dots, [H^{-1}]^r$, and thus a linear combination of $\{\mathbb{1}^j\}_{j=1}^{r'}$. \square

Similar to Theorem 4.1 we have the following theorem:

Theorem 4.2: (Necessary and sufficient condition for group consensus with non-positive gains) Consider the system (2) or (32) satisfying (A1) with $u = \mathbf{0}_{n \times 1}$ and (A2'). Let $S_1, \dots, S_{r'}$ be a partition of $\{1, \dots, n\}$. Then $x(s)$ asymptotically reaches $\{S_i\}_{i=1}^{r'}$ -group consensus in mean square for any initial state if and only if $\tilde{\rho}_{\min}(P) > 1$, or (A5) holds with $\tilde{\rho}_{\min}(P) = 1$.

By Theorems 4.1 and 4.2 with $r' = 1$, we immediately obtain the following two corollaries for consensus:

Corollary 4.1: Consider the system (2) or (32) satisfying (A1) with $u = \mathbf{0}_{n \times 1}$ and (A2). Then $x(s)$ asymptotically reaches consensus in mean square for any initial state if and only if one of the following condition holds:

- i) $\tilde{\rho}_{\max}(P) < 1$;
- ii) The sum of each row of P equals 1, and P has $n - 1$ eigenvalues whose real parts are all less than 1.

Corollary 4.2: Consider the system (2) or (32) satisfying (A1) with $u = \mathbf{0}_{n \times 1}$ and (A2'). Then $x(s)$ asymptotically reaches consensus in mean square for any initial state if and

only if one of the following condition holds:

- i) $\tilde{\rho}_{\min}(P) > 1$;
- ii) The sum of each row of P equals 1, and P has $n - 1$ eigenvalues whose real parts are all bigger than 1.

The communication topology is an important aspect in the research of multi-agent systems consensus. In fact, our result can also give some topology conditions of consensus for some special P . We first introduce some definitions concerning graphs. For a matrix $A \in \mathbb{R}^{n \times n}$ with $A_{ij} \geq 0$ for $j \neq i$. let $\mathcal{V} = \{1, 2, \dots, n\}$ denote the set of nodes, and \mathcal{E} denote the set of edges where an ordered pair $(j, i) \in \mathcal{E}$ if and only if $A_{ij} > 0$. The digraph associated with A is defined by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. A sequence $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k)$ of edges is called a directed path from node i_1 to node i_k . \mathcal{G} contains a directed spanning tree if there exists a root node i such that i has a directed path to j for any node $j \neq i$.

We need the following lemma in our results.

Lemma 4.1 (Lemma 3.3 in [35]): Given a matrix $A \in \mathbb{R}^{n \times n}$, where for any $i \in \mathcal{V}$, $A_{ii} \leq 0, A_{ij} \geq 0$ for $j \neq i$, and $\sum_{j=1}^n A_{ij} = 0$, then A has at least one zero eigenvalue and all of the non-zero eigenvalues have negative real parts. Furthermore, A has exactly one zero eigenvalue if and only if the directed graph associated with A contains a directed spanning tree.

From Corollary 4.1 and Lemma 4.1 we have the following result.

Corollary 4.3: Consider the system (2) or (32) satisfying (A1) and (A2). Assume that P is a row-stochastic matrix and $u = \mathbf{0}_{n \times 1}$. Then $x(s)$ asymptotically reaches consensus in mean square for any initial state if and only if the digraph associated with P contains a directed spanning tree.

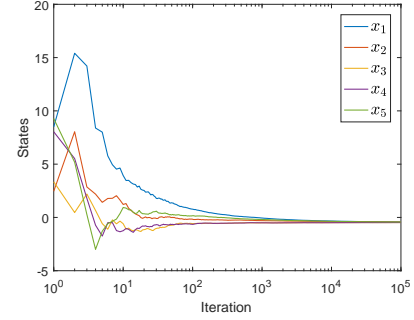
Proof: Let $A = P - I_n$ and ‘ \leftrightarrow ’ denote the ‘if and only if’. The digraph associated with P contains a directed spanning tree \leftrightarrow the digraph associated with A contains a directed spanning tree $\xrightarrow{\text{Lemma 4.1}}$ A has exactly one zero eigenvalue, and all the non-zero eigenvalues have negative real parts \leftrightarrow P has $n - 1$ eigenvalues whose real parts are all less than 1 $\xrightarrow{\text{Corollary 4.1}}$ $x(s)$ asymptotically reaches consensus in mean square for any initial state, where the last two ‘ \leftrightarrow ’ uses the hypothesis that P is a row-stochastic matrix which has at least one eigenvalue that is equal to 1. \square

Corollary 4.3 coincides with the consensus condition for the continuous-time consensus protocol with time-invariant interaction topology (Theorem 3.8 in [35]).

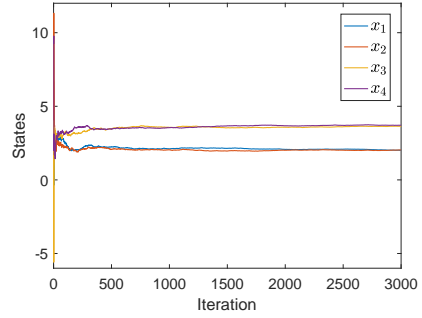
If P is not a row-stochastic matrix, the consensus may be also reached. For example, let

$$P := \begin{bmatrix} 0.5 & 0.3 & 0 & 0.3 & -0.1 \\ -0.1 & 0.3 & 0.3 & 0 & 0.5 \\ 0 & 0.2 & 0.4 & 0.5 & -0.1 \\ 0.1 & 0 & 0.6 & 0.4 & -0.1 \\ 0.1 & -0.1 & 0.1 & 0.3 & 0.6 \end{bmatrix}. \quad (36)$$

The eigenvalues of P are $1, 0.5708, -0.2346, 0.4319 + 0.3270i, 0.4319 - 0.3270i$. By Corollary 4.1 $x(s)$ asymptotically reaches consensus in mean square.



(a) Consensus



(b) Group consensus

Fig. 1. Consensus and group consensus under linear SA protocol (2)

Different from consensus, the group consensus does not require that the sum of each row of P equals 1. For example, if

$$P = \begin{bmatrix} 0.3 & 0.5 & 0.5 & -0.4 \\ 0.5 & 0.3 & -0.4 & 0.5 \\ -0.1 & 0.5 & 0.4 & 0.4 \\ 0.5 & -0.1 & 0.4 & 0.4 \end{bmatrix}, \quad (37)$$

then $P[1, 1, 2, 2]^T = [1, 1, 2, 2]^T$, and the eigenvalues of P are $1, 0.6, -0.1 + 0.728i, -0.1 - 0.728i$. Let $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$, by Theorem 4.1 $x(s)$ can asymptotically reach $\{S_1, S_2\}$ -group consensus in mean square for any initial state.

In the following, we simulate system (2) to show consensus and group consensus using P matrices in (36) and (37) respectively. For $s \geq 0$, $P(s)$ and $u(s)$ are generated by i.i.d. matrix and vector with mean P and $\mathbf{0}_{n \times 1}$ respectively. We set the gain function $a(s) = \frac{1}{s}$. From Fig. 1, we can see that consensus and group consensus are reached as guaranteed by Corollary 4.1 and Theorem 4.1, respectively.

B. An extension to multidimensional linear SA algorithms

Our results in Section III can be extended to multidimensional linear SA algorithms in which the state of each agent is a m -dimensional vector. The dynamics is, for all $s \geq 0$

$$X(s+1) = (1 - a(s))X(s) + a(s)[P(s)X(s)C^T(s) + U(s)], \quad (38)$$

where $X(s) \in \mathbb{R}^{n \times m}$ is the state matrix, $P(s) \in \mathbb{R}^{n \times n}$ is still an interaction matrix, $C \in \mathbb{R}^{m \times m}$ is an interdependency matrix, and $U(s) \in \mathbb{R}^{n \times m}$ is an input matrix.

The system (38) can be transformed to one dimensional system (2) by the following way:

Given a pair of matrices $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{p \times q}$, their Kronecker product is defined by

$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1m}B \\ \cdots & \cdots & \cdots \\ A_{n1}B & \cdots & A_{nm}B \end{bmatrix} \in \mathbb{R}^{np \times mq}.$$

Let $Q(s) := P(s) \otimes C(s)$. From (38) we have

$$\begin{aligned} X_{ij}(s+1) &= (1 - a(s))X_{ij}(s) \\ &+ a(s) \left[\sum_{k_1, k_2} P_{ik_1}(s)X_{k_1, k_2}(s)C_{jk_2}(s) + U_{ij}(s) \right]. \end{aligned} \quad (39)$$

for any $s \geq 0$, $1 \leq i \leq n$, and $1 \leq j \leq m$. Let

$$y(s) := (X_{11}(s), \dots, X_{1m}(s), \dots, X_{n1}(s), \dots, X_{nm}(s))^\top$$

and

$$v(s) := (U_{11}(s), \dots, U_{1m}(s), \dots, U_{n1}(s), \dots, U_{nm}(s))^\top$$

be the vector in \mathbb{R}^{nm} transformed from the matrices $X(s)$ and $U(s)$ respectively. By (39) we have

$$\begin{aligned} y_{(i-1)m+j}(s+1) &= X_{ij}(s+1) \\ &= (1 - a(s))X_{ij}(s) \\ &+ a(s) \left[\sum_{k_1, k_2} P_{ik_1}(s)X_{k_1, k_2}(s)C_{jk_2}(s) + U_{ij}(s) \right] \\ &= (1 - a(s))y_{(i-1)m+j}(s) + \\ &a(s) \left[\sum_{k_1, k_2} Q_{(i-1)m+j, (k_1-1)m+k_2}(s)y_{(k_1-1)m+k_2}(s) \right. \\ &\quad \left. + v_{(i-1)m+j}(s) \right], \end{aligned}$$

which implies

$$y(s+1) = (1 - a(s))y(s) + a(s)[Q(s)y(s) + v(s)].$$

The system (40) has the same form as the system (2), so the results in Section III can be applied to the multidimensional linear SA algorithms.

C. SA Friedkin-Johnsen model over time-varying interaction network

The Friedkin-Johnsen (FJ) model proposed by [14] considers a community of n social actors (or agents) whose opinion column vector is $x(s) = (x_1(s), \dots, x_n(s))^\top \in \mathbb{R}^n$ at time s . The FJ model also contains a row-stochastic matrix of interpersonal influences $P \in \mathbb{R}^{n \times n}$ and a diagonal matrix of actors' susceptibilities to the social influence $\Lambda \in \mathbb{R}^{n \times n}$ with $\mathbf{0}_{n \times n} \leq \Lambda \leq I_n$. The state of the FJ model is updated by

$$x(s+1) = \Lambda P x(s) + (I_n - \Lambda)x(0), \quad s = 0, 1, \dots \quad (40)$$

By [31], if $\mathbf{0}_{n \times n} \leq \Lambda < I_n$, then

$$\lim_{s \rightarrow \infty} x(s) = (I_n - \Lambda P)^{-1}(I_n - \Lambda)x(0). \quad (41)$$

However, if the interpersonal influences are affected by noise, then the system (40) may not converge.

The FJ model (40) was extended to the multidimensional case in [15], [31]. The multidimensional FJ model still contains n individuals, but each individual has beliefs on m truth statements. Let $X(s) \in \mathbb{R}^{n \times m}$ be the matrix of n individuals' beliefs on m truth statements at time s . Following [15], it is updated by

$$X(s+1) = \Lambda P X(s) C^\top + (I_n - \Lambda)X(0) \quad (42)$$

for $s = 0, 1, \dots$, where $\Lambda, P \in \mathbb{R}^{n \times n}$ are the same matrices in (40), and $C \in \mathbb{R}^{m \times m}$ is a row-stochastic matrix of interdependencies among the m truth statements. The convergence of system (42) has been analyzed in [31]. Similar to (40) it is easy to see that if system (42) is affected by noise, then it will not converge. We will adopt the stochastic-approximation method to smooth the effects of the noise.

Proposition 4.1: Consider the system

$$\begin{aligned} X(s+1) &= (1 - a(s))X(s) + a(s)[\Lambda(s)P(s)X(s)C(s)^\top \\ &\quad + (I_n - \Lambda(s))X(0)], \end{aligned} \quad (43)$$

for $s = 0, 1, \dots$, where $\Lambda(s) \in \mathbb{R}^{n \times n}$, $P(s) \in \mathbb{R}^{n \times n}$ and $C(s) \in \mathbb{R}^{m \times m}$ are independent matrix sequence with invariant expectation Λ, P , and C respectively. Assume $E\|\Lambda(s)\|_2^2$, $E\|P(s)\|_2^2$, and $E\|C(s)\|_2^2$ are uniformly bounded. Suppose P and C are row-stochastic matrix, and $\mathbf{0}_{n \times n} \leq \Lambda < I_n$, and the gain function $a(s)$ satisfies (A2). Then for any initial state, $X(s)$ converges to X^* in mean square, where X^* is the unique solution of the equation

$$X = \Lambda P X C^\top + (I_n - \Lambda)X(0). \quad (44)$$

Proof: Since P and C are row-stochastic matrices, $P \otimes C$ is still a row-stochastic matrix. Together with the condition that $\mathbf{0}_{n \times n} \leq \Lambda < I_n$, we have that the sum of each row of $(\Lambda P) \otimes C$ is less than 1. Thus, using the Geršgorin Disk Theorem we obtain $\tilde{\rho}_{\max}((\Lambda P) \otimes C) < 1$. Let $Q := (\Lambda P) \otimes C$, $U(s) := (I_n - \Lambda(s))X(0)$,

$$y(s) := (X_{11}(s), \dots, X_{1m}(s), \dots, X_{n1}(s), \dots, X_{nm}(s))^\top,$$

$$v(s) := (U_{11}(s), \dots, U_{1m}(s), \dots, U_{n1}(s), \dots, U_{nm}(s))^\top,$$

and $v := E v(s)$. By Proposition 3.1 and the transformation from (38) to (40), we obtain that $y(s)$ converges to $(I_{mn} - Q)^{-1}v$ in mean square.

It remains to discuss the relation between $(I_{mn} - Q)^{-1}v$ and X^* . Let

$$y^* := (X_{11}^*, \dots, X_{1m}^*, \dots, X_{n1}^*, \dots, X_{nm}^*)^\top \in \mathbb{R}^{nm}.$$

By (44), similar to (40) we have $y^* = Q y^* + v$, which has a unique solution $y^* = (I_{mn} - Q)^{-1}v$ since $I_{mn} - Q$ is an invertible matrix by $\tilde{\rho}_{\max}(Q) < 1$. Thus, with the fact that $y(s)$ converges to $(I_{mn} - Q)^{-1}v$ in mean square we obtain that $X(s)$ converges to X^* in mean square. \square

Remark 4: According to Theorem 3.1 and Proposition 3.1, the conditions of $\Lambda(s), P(s)$ and $C(s)$ in Proposition 4.1 can be further relaxed for convergence, such as P and C are not row-stochastic matrices, and $\mathbf{0}_{n \times n} \leq \Lambda < I_n$ may be extended to $\Lambda < \mathbf{0}_{n \times n}$ or $\Lambda \geq I_n$.

V. CONCLUSION

In this paper, we study a time-varying linear dynamical system, where the state of the system features persistent oscillation and does not converge. We consider a stochastic approximation-based approach and obtain necessary and sufficient conditions to guarantee mean-square convergence. Our theoretical results largely extend the conditions on the spectrum of the expectation of the system matrix and thus can be applied in a much broader range of applications. We also derived the convergence rate of the system. To illustrate the theoretical results, we applied them in two different applications: group consensus in multi-agent systems and FJ model with time-varying interactions in social networks.

This work leaves various problems for future research. First, the system matrix and input are assumed to have constant expectations in this paper. However, it would be more interesting, yet challenging, to study systems with time-varying expectation of the system matrix and input. Second, we only considered linear dynamical systems in this paper. How and whether the proposed framework can be extended to non-linear system are important and intriguing questions. Finally, we have illustrated our results in two different application scenarios; there are other possible applications such as gossip algorithms for consensus.

APPENDIX A

Lemma A.1: Suppose the non-negative real number sequence $\{y_s\}_{s \geq 1}$ satisfies

$$y_{s+1} \leq (1 - a_s)y_s + b_s, \quad (45)$$

where $b_s \geq 0$ and $a_s \in [0, 1)$ are real numbers. If $\sum_{s=1}^{\infty} a_s = \infty$ and $\lim_{s \rightarrow \infty} b_s/a_s = 0$, then $\lim_{s \rightarrow \infty} y_s = 0$ for any $y_1 \geq 0$.

Proof: Repeating (45) we obtain

$$y_{s+1} \leq y(1) \prod_{t=1}^s (1 - a_t) + \sum_{i=1}^s b_i \prod_{t=i+1}^s (1 - a_t).$$

Here we define $\prod_{t=i}^s (\cdot) := 1$ when $i > s$. From the hypothesis $\sum_{t=1}^{\infty} a_t = \infty$ we have $\prod_{t=1}^{\infty} (1 - a_t) = 0$. Thus, to obtain $\lim_{s \rightarrow \infty} y_s = 0$ we just need to prove that

$$\lim_{s \rightarrow \infty} \sum_{i=1}^s b_i \prod_{t=i+1}^s (1 - a_t) = 0. \quad (46)$$

Since $\lim_{s \rightarrow \infty} b_s/a_s = 0$, for any real number $\varepsilon > 0$, there exists an integer $s^* > 0$ such that $b_s \leq \varepsilon a_s$ when $s \geq s^*$. Thus,

$$\begin{aligned} & \sum_{i=1}^s b_i \prod_{t=i+1}^s (1 - a_t) \\ & \leq \sum_{i=1}^{s^*-1} b_i \prod_{t=i+1}^s (1 - a_t) + \sum_{i=s^*}^s \varepsilon a_i \prod_{t=i+1}^s (1 - a_t) \\ & = \sum_{i=1}^{s^*-1} b_i \prod_{t=i+1}^s (1 - a_t) + \varepsilon \left(1 - \prod_{t=s^*}^s (1 - a_t) \right) \\ & \rightarrow \varepsilon \text{ as } s \rightarrow \infty, \end{aligned} \quad (47)$$

where the first equality uses the classic equality

$$\sum_{t=s^*}^s c_t \prod_{k=t+1}^s (1 - c_k) = 1 - \prod_{t=s^*}^s (1 - c_t) \quad (48)$$

with $\{c_t\}$ being any complex numbers, which can be obtained by induction. Here we define $\prod_{k=s_1}^{s_2} (\cdot) = 1$ if $s_2 < s_1$. Let ε decrease to 0, then (47) is followed by (46). \square

APPENDIX B PROOF OF THEOREM 3.2

We prove this theorem under the following three cases:

Case I: $\tilde{\rho}_{\max}(P) < 1$. Define $\theta(s)$, A and A_2 as in the proof of Theorem 3.1 but with $r = 0$. Set $V(\theta) := \theta^* A \theta$ for any $\theta \in \mathbb{C}^n$, where θ^* denotes the conjugate transpose of θ . We remark that $A_2 = A \in \mathbb{C}^{n \times n}$ under the case $r = 0$, so that, by (19), we have

$$\begin{aligned} & \mathbb{E}[V(\theta(s+1))] \\ & \leq \left(1 - \frac{\alpha}{\rho(A)(s+\beta)\gamma} \right) \mathbb{E}[V(\theta(s))] + O\left(\frac{1}{(s+\beta)^{2\gamma}} \right). \end{aligned} \quad (49)$$

Set

$$\Phi(s, i) := \prod_{k=i}^s \left(1 - \frac{\alpha}{\rho(A)(k+\beta)\gamma} \right)$$

and define $\prod_{k=i}^s (\cdot) := 1$ if $s < i$. We compute

$$\begin{aligned} \Phi(s, i) & = O\left(\exp \left[\sum_{k=i}^s -\frac{\alpha}{\rho(A)(k+\beta)\gamma} \right] \right) \\ & = O\left(\exp \left(\int_i^s -\frac{\alpha}{\rho(A)(k+\beta)\gamma} dk \right) \right) \\ & = \begin{cases} O\left(\left(\frac{s+\beta}{i+\beta} \right)^{-\alpha/\rho(A)} \right), & \text{if } \gamma = 1, \\ O\left(\exp \left(\frac{-\alpha}{(1-\gamma)\rho(A)} [(s+\beta)^{1-\gamma} - (i+\beta)^{1-\gamma}] \right) \right), & \text{if } \frac{1}{2} < \gamma < 1. \end{cases} \end{aligned} \quad (50)$$

Also, using (49) repeatedly we obtain

$$\begin{aligned} & \mathbb{E}[V(\theta(s+1))] \\ & \leq \Phi(s, 0) \mathbb{E}[V(\theta(0))] + \sum_{i=0}^s \Phi(s, i+1) O\left(\frac{1}{(i+\beta)^{2\gamma}} \right). \end{aligned} \quad (51)$$

Assume $\alpha \geq \rho(A)$. We first consider the case that $\gamma = 1$. From (50) and (51) we have

$$\mathbb{E}[V(\theta(s+1))] = o\left(\frac{1}{s} \right) + O\left(\sum_{i=0}^s \frac{(s+\beta)^{-\alpha/\rho(A)}}{(i+\beta)^{2-\frac{\alpha}{\rho(A)}}} \right) = O\left(\frac{1}{s} \right). \quad (52)$$

For the case when $\gamma \in (\frac{1}{2}, 1)$, we take $b = \frac{\alpha}{(1-\gamma)\rho(A)}$, and from (50) and (51) we can obtain

$$\begin{aligned}
& \mathbb{E}[V(\theta(s+1))] \\
&= e^{-b(s+\beta)^{1-\gamma}} \cdot O\left(1 + \sum_{i=0}^s \frac{e^{b(i+\beta)^{1-\gamma}}}{(i+\beta)^{2\gamma}}\right) \\
&= e^{-b(s+\beta)^{1-\gamma}} \cdot O\left(\sum_{i=0}^s \sum_{k=0}^{\infty} \frac{b^k (i+\beta)^{(1-\gamma)k-2\gamma}}{k!}\right) \\
&= e^{-b(s+\beta)^{1-\gamma}} \cdot O\left(\sum_{k=0}^{\infty} \frac{b^k}{k!} \sum_{i=0}^s (i+\beta)^{(1-\gamma)k-2\gamma}\right) \\
&= e^{-b(s+\beta)^{1-\gamma}} \cdot O\left(\sum_{k=0}^{\infty} \frac{b^k (s+\beta)^{(1-\gamma)k-2\gamma+1}}{k![(1-\gamma)k-2\gamma+1]}\right) \\
&= \frac{e^{-b(s+\beta)^{1-\gamma}}}{(s+\beta)^\gamma} \cdot O\left(\sum_{k=0}^{\infty} \frac{b^{k+1} (s+\beta)^{(1-\gamma)(k+1)}}{(k+1)!}\right) \\
&= O(s^{-\gamma}). \tag{53}
\end{aligned}$$

By (52) and (53), we have $\mathbb{E}[V(\theta(s))] = O(s^{-\gamma})$ for $\frac{1}{2} < \gamma \leq 1$. Combing this with the definition of $\theta(s)$ yields our result.

Case II: $\tilde{\rho}_{\max}(P) = 1$. Let $\theta(s)$, $\underline{\theta}(s)$, $\bar{\theta}(s)$, $\bar{\theta}(\infty)$, H , y and z be the same variables as in the proof of Theorem 3.1. With (19) and following the similar process from (49) to (53), we have $\mathbb{E}\|\underline{\theta}(s)\|_2^2 = O(s^{-\gamma})$. Also, from (22) we have

$$\begin{aligned}
\mathbb{E}\|\bar{\theta}(\infty) - \bar{\theta}(s)\|_2^2 &= O\left(\sum_{k=s}^{\infty} a^2(k)\right) = O\left(\sum_{k=s}^{\infty} a^2(k)\right) \\
&= O\left(\sum_{k=s}^{\infty} \frac{1}{(s+\beta)^{2\gamma}}\right) = O\left(\frac{1}{s^{2\gamma-1}}\right).
\end{aligned}$$

Since $x(s) = H^{-1}[\theta(s) + z]$ and $H^{-1}y$ is a mean square limit of $x(s)$, the arguments above imply

$$\begin{aligned}
\mathbb{E}\|x(s) - H^{-1}y\|_2^2 &= \max\{O(s^{-\gamma}), O(s^{1-2\gamma})\} \\
&= O(s^{1-2\gamma}).
\end{aligned}$$

Case III: $\tilde{\rho}_{\min}(P) \geq 1$. The protocol (2) is written as

$$x(s+1) = x(s) + \frac{\alpha}{(s+\beta)^\gamma} [(I_n - P(s))x(s) - u(s)].$$

Because $\tilde{\rho}_{\max}(I_n - P) \leq 0$, arguments similar to that for Cases I) and II) yield our result.

APPENDIX C PROOF OF THEOREM 3.3

i) As same as Subsection III-B, the Jordan normal form of H is

$$D = \text{diag}(J_1, \dots, J_k) = HPH^{-1}.$$

We also set $y(s) := Hx(s)$, $v(s) := Hu(s)$, $D(s) := HPH^{-1}$, $D = \mathbb{E}D(s) = HPH^{-1}$, and $v = Ev(s) = Hu$.

By (8) and (A1) we have

$$\begin{aligned}
\mathbb{E}y(s+1) &= \mathbb{E}[\mathbb{E}[y(s+1) | y(s)]] \\
&= \mathbb{E}y(s) + a(s)[(D - I_n)Ey(s) + v]. \tag{54}
\end{aligned}$$

Let $B(s) := I_n + a(s)(D - I_n)$. Using (54) repeatedly we obtain

$$\mathbb{E}[y(s+1)] = B(s) \cdots B(0)y(0) + \sum_{t=0}^s a(t)B(s) \cdots B(t+1)v. \tag{55}$$

We will continue the proof under the following two cases:

Case I: $\tilde{\rho}_{\max}(P) > 1$. Without loss of generality we assume $\text{Re}(\lambda_1(P)) > 1$. Let J_1 be a Jordan block in D corresponding to $\lambda_1(P)$. Let m_1 be the row index of D corresponding to the last line of J_1 , i.e.,

$$D_{m_1} = (0, \dots, 0, \lambda_1(P), 0, \dots, 0). \tag{56}$$

Then by (55)

$$\begin{aligned}
& \mathbb{E}[y_{m_1}(s+1)] \\
&= y_{m_1}(0) \prod_{t=0}^s [1 - a(t)[1 - \lambda_1(P)]] + \frac{v_{m_1}}{1 - \lambda_1(P)} \\
&\quad \times \sum_{t=0}^s a(t)[1 - \lambda_1(P)] \prod_{k=t+1}^s (1 - a(k)[1 - \lambda_1(P)]) \\
&= y_{m_1}(0) \prod_{t=0}^s (1 - a(t)[1 - \lambda_1(P)]) + \frac{v_{m_1}}{1 - \lambda_1(P)} \\
&\quad \times \left(1 - \prod_{t=0}^s (1 - a(t)[1 - \lambda_1(P)])\right), \tag{57}
\end{aligned}$$

where the last equality uses the equality (48). Since $\sum_s a(s) = \infty$,

$$\begin{aligned}
& \prod_{t=0}^{\infty} |1 - a(t)[1 - \lambda_1(P)]|^2 \\
&\geq \prod_{t=0}^{\infty} \{1 + 2a(t)[\text{Re}(\lambda_1(P)) - 1]\} = \infty.
\end{aligned}$$

Hence, from (57), if $y_{m_1}(0) \neq \frac{v_{m_1}}{1 - \lambda_1(P)}$, then

$$\lim_{s \rightarrow \infty} |\mathbb{E}[y_{m_1}(s)]| = \infty, \tag{58}$$

which implies $\lim_{s \rightarrow \infty} \mathbb{E}\|x(s)\|_2^2 = \infty$.

Case II: $\tilde{\rho}_{\max}(P) = 1$. Under this case we consider the following three situations:

(a) There is an eigenvalue $\lambda_j(P) = 1 + \text{Im}(\lambda_j(P))i$ with $\text{Im}(\lambda_j(P)) \neq 0$, where $\text{Im}(\lambda_j(P))$ denotes the imaginary part of $\lambda_j(P)$. Similar to (56), we can choose a row $D_{j'}$ of D which is equal to $(0, \dots, 0, \lambda_j(P), 0, \dots, 0)$. Similar to (57), we have

$$\begin{aligned}
\mathbb{E}[y_{j'}(s+1)] &= y_{j'}(0) \prod_{t=0}^s [1 - a(t)[1 - \lambda_j(P)]] \\
&\quad + \frac{v_{j'}}{1 - \lambda_j(P)} \cdot \left(1 - \prod_{t=0}^s [1 - a(t)[1 - \lambda_j(P)]]\right). \tag{59}
\end{aligned}$$

We write

$$\begin{aligned}
1 - a(t)[1 - \lambda_j(P)] &= 1 + a(t)\text{Im}(\lambda_j(P))i \\
&= r_t e^{i\varphi_t} = r_t (\cos \varphi_t + i \sin \varphi_t),
\end{aligned}$$

where $r_t = \sqrt{1 + a^2(t)\text{Im}^2(\lambda_j(P))}$ and

$$\begin{aligned} \varphi_t &= \arctan[a(t)\text{Im}(\lambda_j(P))] \\ &= a(t)\text{Im}(\lambda_j(P)) + \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} [a(t)\text{Im}(\lambda_j(P))]^{2k+1}, \end{aligned} \quad (60)$$

so

$$\prod_{t=0}^s (1 - a(t)[1 - \lambda_j(P)]) = \exp\left(i \sum_{t=0}^s \varphi_t\right) \prod_{t=0}^s r_t. \quad (61)$$

Assume $y_{j'}(0) \neq \frac{v_{j'}}{1 - \lambda_j(P)}$. Since $\sum_{t=0}^{\infty} a(t) = \infty$, equations (59), (61), and (60) imply

$$\overline{\lim}_{s \rightarrow \infty} \overline{\lim}_{s_2 \rightarrow \infty} |\mathbb{E}[y_{j'}(s_2) - y_{j'}(s)]| > 0. \quad (62)$$

Next we consider the convergence of $x(s)$. Because $x(s) = H^{-1}y(s)$, using Jensen's inequality we have

$$\begin{aligned} \mathbb{E}\|x(s_2) - x(s)\|_2^2 &= \mathbb{E}\|H^{-1}[y(s_2) - y(s)]\|_2^2 \\ &\geq \sigma_n^2(H^{-1})\mathbb{E}\|y(s_2) - y(s)\|_2^2 \\ &\geq \sigma_n^2(H^{-1})\mathbb{E}|y_{j'}(s_2) - y_{j'}(s)|^2 \\ &\geq \sigma_n^2(H^{-1})|\mathbb{E}[y_{j'}(s_2) - y_{j'}(s)]|^2, \end{aligned} \quad (63)$$

where $\sigma_n(H^{-1}) = \inf_{\|x\|_2=1} \|H^{-1}x\|_2$ denotes the least singular value of H^{-1} . Because H^{-1} is invertible, we have $\sigma_n(H^{-1}) > 0$. Hence, by (62) and (63), we obtain

$$\overline{\lim}_{s \rightarrow \infty} \overline{\lim}_{s_2 \rightarrow \infty} \mathbb{E}\|x(s_2) - x(s)\|_2^2 > 0.$$

By the Cauchy criterion (see [21, page 58]), $x(s)$ is not mean square convergent.

(b) The geometric multiplicity of the eigenvalue 1 is less than its algebraic multiplicity. By (a), we only need to consider the case when any eigenvalue of P with 1 as real part has zero imaginary part. Thus, the Jordan normal form D contains a Jordan block

$$J_j = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}_{m_j \times m_j}$$

with $m_j \geq 2$. Let j' be the row index of D corresponding to the second line from the bottom of J_j . It can be computed that

$$[B(s) \cdots B(t)]_{j', j'+1} = \sum_{k=t}^s a(k).$$

Since $\sum_{k=0}^{\infty} a(k) = \infty$, from (55), there are some initial states such that $\lim_{s \rightarrow \infty} |\mathbb{E}[y_{j'}(s)]| = \infty$, which is followed by $\lim_{s \rightarrow \infty} \mathbb{E}\|x(s)\|_2^2 = \infty$.

(c) There is a left eigenvector ξ^T of P corresponding to the eigenvalue 1 such that $\xi^T u \neq 0$. By (2) and (A1) we have

$$\begin{aligned} \xi^T \mathbb{E}x(s+1) &= (1 - a(s))\xi^T \mathbb{E}x(s) + a(s)[\xi^T P \mathbb{E}x(s) + \xi^T u] \\ &= \xi^T \mathbb{E}x(s) + a(s)\xi^T u \\ &= \cdots = \xi^T x(0) + \sum_{k=0}^s a(k)\xi^T u, \end{aligned}$$

which implies $\lim_{s \rightarrow \infty} \mathbb{E}\|x(s)\|_2^2 = \infty$ by $\sum_{k=0}^{\infty} a(k) = \infty$.

ii) It can be obtained by the similar method as i).

APPENDIX D PROOF OF THEOREM 3.4

We prove our result by contradiction: Suppose that there exists a real number sequence $\{a(s)\}_{s \geq 0}$ independent with $\{x(s)\}$ such that

$$\lim_{s \rightarrow \infty} \mathbb{E}\|x(s) - b\|_2^2 = 0. \quad (64)$$

We assert that $\lim_{s \rightarrow \infty} a(s) = 0$. This assertion will be proved still by contradiction: Assume that there exists a subsequence $\{a(s_k)\}_{k \geq 0}$ which does not converge to zero. Let $\tilde{P}(s) := P(s) - P$ and $\tilde{u}(s) = u(s) - u$ for any $s \geq 0$, then by (2), (A1) and (28) we have

$$\begin{aligned} &\mathbb{E}\left[\|x(s_k+1) - b\|_2^2 \mid x(s_k)\right] \\ &= \mathbb{E}\left[\|\xi + a(s_k)(\tilde{P}(s_k)x(s_k) + \tilde{u}(s_k))\|_2^2 \mid x(s_k)\right] \\ &= \|\xi\|_2^2 + a^2(s_k)\mathbb{E}\left[\|\tilde{P}(s_k)x(s_k) + \tilde{u}(s_k)\|_2^2 \mid x(s_k)\right] \\ &\geq a^2(s_k)c_3, \end{aligned} \quad (65)$$

where

$$\xi := (1 - a(s_k))x(s_k) + a(s_k)(Px(s_k) + u) - b.$$

From (65) we know that $\mathbb{E}\|x(s_k+1) - b\|_2^2$ will not converge to 0 as k grows to infinity, which is in contradiction with (64).

Since $x(0) \neq b$, to guarantee the convergence of $x(s)$, the gain function $\{a(s)\}_{s \geq 0}$ must at least contain one non-zero element. Also, from (65), we can obtain that the number of the non-zero elements in the sequence $\{a(s)\}_{s \geq 0}$ must be infinite. Thus, together with the assertion of $\lim_{s \rightarrow \infty} a(s) = 0$, there exists an integer $s^* > 0$ such that $a(s^* - 1) \neq 0$, $\{a(i)\}_{i=0}^{s^*-2}$ contains non-zero element, and

$$2|a(s)(1 - \text{Re}(\lambda_j(P)))| < 1, \quad \forall s \geq s^*, 1 \leq j \leq n. \quad (66)$$

Let $A(s) := (1 - a(s))I_n + a(s)P(s)$. By (2) we have

$$x(s+1) = A(s)x(s) + a(s)u(s), \quad s \geq s^*.$$

By (A1), we obtain

$$\begin{aligned} &\mathbb{E}[x(s+1) \mid x(s^*)] - (I_n - P)^{-1}u \\ &= [I_n - a(s)(I_n - P)]\mathbb{E}[x(s) \mid x(s^*)] \\ &\quad + a(s)u - (I_n - P)^{-1}u \\ &= [I_n - a(s)(I_n - P)](\mathbb{E}[x(s) \mid x(s^*)] - (I_n - P)^{-1}u) \\ &= \cdots = \left(\prod_{k=s^*}^s \mathbb{E}[A(k)]\right)(x(s^*) - (I_n - P)^{-1}u), \end{aligned}$$

which implies

$$\begin{aligned} \mathbb{E}[x(s+1) \mid x(s^*)] &= H^{-1} \left(\prod_{i=s^*}^s [I_n - a(i)(I_n - D)] \right) \\ &\quad \times H(x(s^*) - (I_n - P)^{-1}u) + (I_n - P)^{-1}u \end{aligned} \quad (67)$$

from (5). Set

$$\begin{aligned} z(s) &:= \left(\prod_{i=s^*}^s [I_n - a(i)(I_n - D)] \right) H \\ &\quad \cdot (x(s^*) - (I_n - P)^{-1}u) + H(I_n - P)^{-1}u - Hb. \end{aligned} \quad (68)$$

Using Jensen's inequality and (67) we have

$$\begin{aligned} \mathbb{E}[\|x(s+1) - b\|_2^2 | x(s^*)] &\geq \|\mathbb{E}[(x(s+1) - b) | x(s^*)]\|_2^2 \\ &= \|\mathbb{E}[(x(s+1) | x(s^*)) - b]\|_2^2 \\ &= \|H^{-1}z(s)\|_2^2 \geq \sigma_n^2(H^{-1})\|z(s)\|_2^2, \end{aligned} \quad (69)$$

where $\sigma_n(H^{-1}) = \inf_{\|x\|_2=1} \|H^{-1}x\|_2$ denotes the least singular value of H^{-1} . Because H^{-1} is invertible, $\sigma_n(H^{-1}) > 0$. Define

$$w_j(s) := \prod_{i=s^*}^s (1 - a(i)[1 - \lambda_j(P)]) \quad (70)$$

and

$$M_j := \prod_{i=s^*}^{\infty} [I_{m_j} - a(i)(I_{m_j} - J_j)]. \quad (71)$$

We can compute that

$$\begin{aligned} |w_j(s)|^2 &= \prod_{i=s^*}^s |1 - a(i)[1 - \lambda_j(P)]|^2 \\ &= \prod_{i=s^*}^s \{1 - 2a(i)[1 - \operatorname{Re}(\lambda_j(P))] \\ &\quad + a^2(i)[1 - 2\operatorname{Re}(\lambda_j(P)) + |\lambda_j(P)|^2]\}. \end{aligned}$$

From this and (66) we have $w_j(s) \neq 0$ for any finite s . Also, if $w_j(\infty) = 0$, then $[1 - \operatorname{Re}(\lambda_j(P))] \sum_{i=s^*}^{\infty} a(i) = \infty$. Hence, by (A4) or (A4'), there exists a Jordan block J_{j_1} associated with the eigenvalue $\lambda_{j_1}(P)$ such that $w_{j_1}(\infty) \neq 0$ and (27) holds. Because M_{j_1} is an upper triangular matrix whose diagonal elements are all $w_{j_1}(\infty) \neq 0$, we can obtain the least singular value

$$\sigma_{m_{j_1}}(M_{j_1}) > 0.$$

Also, by (69) and (26), we obtain

$$\begin{aligned} \mathbb{E}\|x(\infty) - b\|_2^2 &= \mathbb{E}[\mathbb{E}\|x(\infty) - b\|_2^2 | x(s^*)] \\ &\geq \sigma_n^2(H^{-1})\mathbb{E}\|z(\infty)\|_2^2 \\ &\geq \sigma_n^2(H^{-1})\mathbb{E}\|z(\infty) - Ez(\infty)\|_2^2 \\ &\geq \sigma_n^2(H^{-1})\mathbb{E}\|\tilde{I}_{j_1}[z(\infty) - Ez(\infty)]\|_2^2 \\ &= \sigma_n^2(H^{-1})\mathbb{E}\|\tilde{I}_{j_1}\left(\prod_{i=s^*}^{\infty} [I_n - a(i)(I_n - D)]\right) \\ &\quad \times H(x(s^*) - Ex(s^*))\|_2^2 \\ &= \sigma_n^2(H^{-1})\mathbb{E}\|M_{j_1}\tilde{I}_{j_1}H(x(s^*) - Ex(s^*))\|_2^2 \\ &\geq \sigma_n^2(H^{-1})\sigma_{m_{j_1}}^2(M_{j_1}) \\ &\quad \times \mathbb{E}\|\tilde{I}_{j_1}H(x(s^*) - Ex(s^*))\|_2^2. \end{aligned} \quad (72)$$

Using (2) and (27) we have

$$\begin{aligned} &\mathbb{E}\left\{\|\tilde{I}_{j_1}H(x(s^*) - Ex(s^*))\|_2^2 | x(s^* - 1)\right\} \\ &= a^2(s^* - 1)\mathbb{E}\left\{\|\tilde{I}_{j_1}H\tilde{P}(s^* - 1)x(s^* - 1) \right. \\ &\quad \left. + \tilde{I}_{j_1}H\tilde{u}(s^* - 1)\|_2^2 | x(s^* - 1)\right\} \\ &\geq a^2(s^* - 1)(c_1\|x(s^* - 1)\|_2^2 + c_2). \end{aligned} \quad (73)$$

Because c_1 and c_2 cannot be zero at the same time, we consider the case when $c_2 > 0$ first. With the fact that $a(s^* - 1) \neq 0$ and (73) we obtain

$$\begin{aligned} &\mathbb{E}\|\tilde{I}_{j_1}H(x(s^*) - Ex(s^*))\|_2^2 \\ &= \mathbb{E}\left\{\mathbb{E}\|\tilde{I}_{j_1}H(x(s^*) - Ex(s^*))\|_2^2 | x(s^* - 1)\right\} > 0. \end{aligned}$$

Substituting this into (72) yields $\mathbb{E}\|x(\infty) - b\|_2^2 > 0$, which is contradictory with (64).

For the case when $c_1 > 0$, by (72) and (73), we have

$$\begin{aligned} &\mathbb{E}\|x(\infty) - b\|_2^2 \\ &\geq \sigma_n^2(H^{-1})\sigma_{m_{j_1}}^2(M_{j_1})a^2(s^* - 1) \\ &\quad \cdot \mathbb{E}\left(\|\tilde{I}_{j_1}H\tilde{P}(0)x(s^* - 1)\|_2^2 | x(s^* - 1)\right) \\ &\geq \sigma_n^2(H^{-1})\sigma_{m_{j_1}}^2(M_{j_1})a^2(s^* - 1)c_1\mathbb{E}\|x(s^* - 1)\|_2^2. \end{aligned} \quad (74)$$

Because $\{a(i)\}_{i=0}^{s^*-2}$ contains non-zero elements, we set s' to be the biggest number such that $s' \leq s^* - 2$ and $a(s') \neq 0$. By (65) we have

$$\begin{aligned} \mathbb{E}\|x(s^* - 1)\|_2^2 &= \mathbb{E}\|x(s' + 1)\|_2^2 \\ &\geq a^2(s')EE\left[\|\tilde{P}(s')x(s') + \tilde{u}(s')\|_2^2 | x(s')\right] \\ &\geq a^2(s')c_3 > 0. \end{aligned}$$

Substituting this into (74) we get $\mathbb{E}\|x(\infty) - b\|_2^2 > 0$, which is contradictory with (64).

APPENDIX E

PROOF OF THEOREM 3.5

Similar to the proof of Theorem 3.4 we prove our result by contradiction: Suppose that there exists a real number sequence $\{a(s)\}_{s \geq 0}$ independent with $\{x(s)\}$ such that (64) holds. Since $x(0) \neq b$, by (64) $\{a(s)\}_{s \geq 0}$ must contain non-zero elements. We consider the following three cases respectively to deduce the contradiction:

Case I: The condition i) is satisfied. Similar to the proof of Theorem 3.4, we first prove $\lim_{s \rightarrow \infty} a(s) = 0$ by contradiction: Suppose there exists a subsequence $\{a(s_k)\}$ that does not converge to zero. For the case when $b \neq \mathbf{0}_{n \times 1}$, by (64), there exists a time $s_1 \geq 0$ such that

$$\mathbb{E}\|x(s) - b\|_2^2 \leq \frac{1}{4}\|b\|_2^2, \quad \forall s > s_1. \quad (75)$$

Because for any $x(s_k)$,

$$\begin{aligned} \|b\|_2^2 &\leq (\|x(s_k)\|_2 + \|b - x(s_k)\|_2)^2 \\ &\leq 2(\|x(s_k)\|_2^2 + \|b - x(s_k)\|_2^2), \end{aligned}$$

(75) is followed by

$$\mathbb{E}\|x(s_k)\|_2^2 \geq \frac{1}{2}\|b\|_2^2 - \mathbb{E}\|x(s_k) - b\|_2^2 \geq \frac{1}{4}\|b\|_2^2 \quad (76)$$

for large k . By (65), (29) and (76) we obtain

$$\begin{aligned} &\mathbb{E}\|x(s_k + 1) - b\|_2^2 \geq a^2(s_k)\mathbb{E}\|\tilde{P}(s_k)x(s_k)\|_2^2 \\ &= a^2(s_k)\mathbb{E}\|H^{-1}H\tilde{P}(s_k)x(s_k)\|_2^2 \\ &\geq a^2(s_k)\sigma_n^2(H^{-1})\mathbb{E}\|H\tilde{P}(s_k)x(s_k)\|_2^2 \\ &\geq a^2(s_k)\sigma_n^2(H^{-1})c_1\mathbb{E}\|x(s_k)\|_2^2 \\ &\geq \frac{1}{4}a^2(s_k)\sigma_n^2(H)c_1\|b\|_2^2, \end{aligned} \quad (77)$$

which is contradictory with (64).

For the case when $b = \mathbf{0}_{n \times 1}$, by (65) and (77), we have

$$\begin{aligned} \mathbb{E}\|x(s_k + 1)\|_2^2 &\geq \mathbb{E}\|(1 - a(s_k))x(s_k) + a(s_k)(Px(s_k) + u)\|_2^2 \\ &\quad + a^2(s_k)\sigma_n^2(H^{-1})c_1\mathbb{E}\|x(s_k)\|_2^2. \end{aligned} \quad (78)$$

If $\|(1 - a(s_k))I_n + a(s_k)P\|_2\mathbb{E}\|x(s_k)\|_2 > \frac{1}{2}\|a(s_k)u\|_2$, by (78) and Jensen's inequality we have

$$\begin{aligned} \mathbb{E}\|x(s_k + 1)\|_2^2 &\geq a^2(s_k)\sigma_n^2(H^{-1})c_1(\mathbb{E}\|x(s_k)\|_2)^2 \\ &\geq \frac{a^4(s_k)\sigma_n^2(H^{-1})c_1\|u\|_2^2}{4\|(1 - a(s_k))I_n + a(s_k)P\|_2^2} \\ &\rightarrow 0 \text{ if } a(s_k) \rightarrow 0. \end{aligned} \quad (79)$$

Otherwise,

$$\begin{aligned} &\mathbb{E}\|(1 - a(s_k))x(s_k) + a(s_k)(Px(s_k) + u)\|_2 \\ &\geq \|a(s_k)u\|_2 - \mathbb{E}\|(1 - a(s_k))x(s_k) + a(s_k)Px(s_k)\|_2 \\ &\geq \|a(s_k)u\|_2 - \mathbb{E}\|(1 - a(s_k))I_n + a(s_k)P\|_2\|x(s_k)\|_2 \\ &\geq \frac{1}{2}\|a(s_k)u\|_2, \end{aligned}$$

Hence, using (78) and Jensen's inequality again, we obtain

$$\begin{aligned} \mathbb{E}\|x(s_k + 1)\|_2^2 &\geq (\mathbb{E}\|(1 - a(s_k))x(s_k) + a(s_k)(Px(s_k) + u)\|_2)^2 \\ &\geq \|a(s_k)u\|_2^2/4. \end{aligned} \quad (80)$$

Combining (79) and (80) yields $\mathbb{E}\|x(s_k + 1)\|_2^2$. This quantity does not converge to zero, which is in contradiction with (64). By summarizing the arguments above we prove the assertion of $\lim_{s \rightarrow \infty} a(s) = 0$.

Because $\lim_{s \rightarrow \infty} a(s) = 0$ and because $\{a(s)\}_{s \geq 0}$ contains non-zero elements, there exists an integer $s^* > 0$ such that $a(s^* - 1) \neq 0$ and (66) holds. Define $w_j(s)$ and M_j by (70) and (71) respectively. With the arguments similar to the proof of Theorem 3.4, we can find a Jordan block J_{j_1} associated with the eigenvalue $\lambda_{j_1}(P)$ such that $w_{j_1}(\infty) \neq 0$ and (29) holds. Similar to (74) we obtain

$$\mathbb{E}\|x(\infty) - b\|_2^2 \geq \sigma_n^2(H)\sigma_{m_{j_1}}^2(M_{j_1})a^2(s^* - 1)c_1\mathbb{E}\|x(s^* - 1)\|_2^2. \quad (81)$$

By (78) we have that if $\mathbb{E}\|x(s)\|_2^2 > 0$, then $\mathbb{E}\|x(s + 1)\|_2^2 > 0$ for any $a(s) \in \mathbb{R}$. Then with the condition $x(0) \neq \mathbf{0}_{n \times 1}$, we have $\mathbb{E}\|x(s^* - 1)\|_2^2 > 0$. Using this and (81) we get $\mathbb{E}\|x(\infty) - b\|_2^2 > 0$, which is contradictory with (64).

Case II: The condition ii) is satisfied. Since $\{a(s)\}_{s \geq 0}$ contains non-zero elements, we define s_1 to be the first s such that $a(s) \neq 0$. Then $x(s_1 + 1) = a(s_1)u \neq b$ almost surely. Let $s_1 + 1$ be the initial time and by the same arguments as in Case I we obtain $\mathbb{E}\|x(\infty) - b\|_2^2 > 0$.

Case III: The condition iii) is satisfied. If $x(0) = \mathbf{0}_{n \times 1}$, we obtain $\mathbb{E}\|x(s)\|_2^2 = 0$ for any $s \geq 0$, which is contradictory with (64). Thus, we just need to consider the case when $x(0) \neq \mathbf{0}_{n \times 1}$. Since $\{a(s)\}_{s \geq 0}$ contains non-zero elements, we define s_1 to be the first s such that $a(s) \neq 0$.

Set $x^* := x_1 + 1$. Define $w_j(s)$ and M_j by (70) and (71) respectively. If $\lambda_j(P)$ is not a real number, then $w_j(s)$ cannot be equal to 0 for any finite s . By the similar arguments as in the proof of Theorem 3.4, there exists a Jordan block J_{j_1}

associated with the eigenvalue $\lambda_{j_1}(P)$ such that $w_{j_1}(\infty) \neq 0$ and (29) holds. By (81) we have

$$\begin{aligned} \mathbb{E}\|x(\infty) - b\|_2^2 &\geq \sigma_n^2(H^{-1})\sigma_{m_{j_1}}^2(M_{j_1})a^2(s^* - 1)c\mathbb{E}\|x(s^* - 1)\|_2^2 \\ &= \sigma_n^2(H^{-1})\sigma_{m_{j_1}}^2(M_{j_1})a^2(s_1)c\|x(0)\|_2^2 > 0, \end{aligned}$$

which is contradictory with (64).

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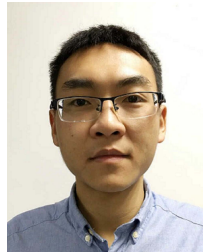
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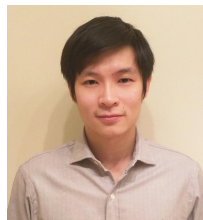
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