Dynamic Social Balance and Convergent Appraisals via Homophily and Influence Mechanisms

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Abstract

Social balance theory describes allowable and forbidden configurations of the topologies of signed directed social appraisal networks. In this paper, we propose two discrete-time dynamical systems that explain how an appraisal network evolves towards social balance from an initially unbalanced configuration. These two models are based on two different socio-psychological mechanisms respectively: the homophily mechanism and the influence mechanism. Our main theoretical contribution is a comprehensive analysis for both models in three steps. First, we establish the well-posedness and bounded evolution of the interpersonal appraisals. Second, we characterize the set of equilibrium points as follows: for both models, each equilibrium network is composed by an arbitrary number of complete subgraphs satisfying structural balance. Third, under a technical condition, we establish convergence of the appraisal network to a final equilibrium network satisfying structural balance. In addition to our theoretical analysis, we provide numerical evidence that our technical condition for convergence holds for generic initial conditions in both models. Finally, adopting the homophily-based model, we present numerical results on the mediation and globalization of local conflicts, the competition for allies, and the asymptotic formation of a single versus two factions.

Key words: structural balance; nonlinear dynamics; homophily; influence system; faction formation; multi-agent systems.

1 Introduction

Motivation and problem description

Social systems involving friendly/antagonistic relationships between their members are often modeled as signed networks. *Social balance* (also referred to as *structural balance*) theory, which originated from several seminal works by Heider [13,14], characterizes the stable configurations of signed social networks. According to the classic social balanced theory [13,14], in a balanced network, the interpersonal relationships satisfy the four famous Heider's axioms: "The friend of my friend is my friend; the friend of my enemy is my enemy; the enemy of my friend is my enemy; the enemy of my enemy is my friend." While classic studies of social balance focus mainly on the static theory (i.e., the local and global configurations of balanced networks), dynamic social balance theory has attracted much recent interest. In short, dynamic social balance theory aims to explain if and how an initially unbalanced network evolves to a balanced network. Despite recent progress, it remains a valuable open problem to propose dynamic models that are based on natural assumptions and that enjoy desirable boundedness and convergence properties. Such models make it possible for researchers to formulate and study meaningful predictions and control/intervention strategies for the evolution of the social network.

In this paper, we propose two novel discrete-time dynamic models describing the evolution of the interpersonal appraisals towards social balance. For both models, we consider a group of individuals who repeatedly update their interpersonal appraisals via two socio-psychological mechanisms respectively: the homophily mechanism and the influence mechanism. The homophily mechanism means that individuals in a group tend to be friendly to each other if their appraisals of the group members are in agreement (in the sense of signs), and vice versa. On the other hand, the influence mechanism defines an influence process, in which each individual updates its appraisals by assigning positive or negative influences to all the group members. The interpersonal influences are assumed proportional to the corresponding interpersonal appraisals. For both models, our objectives are to characterize their fixed points and their dynamical behavior, with a special emphasis

⋆ This material is based upon work supported by, or in part by, the U.S. Army Research Laboratory and the U.S. Army Research Office under grant number W911NF-15-1-0577.

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on boundedness and convergence properties.

The homophily and influence mechanisms are both well established in the social sciences literature; they have been studied separately in different contexts, e.g., see the seminal work by Lazarsfeld and Merton [21] on the homophily mechanism and the award-winning book by Friedkin and Johnsen on the influence mechanism [10], respectively. These two mechanisms are not necessarily mutually exclusive: in reality, it can be argued that they simultaneously play a role in shaping the evolution of a social network, through surely to varying and distinct degree. It is an open question beyond the scope of this paper to determine conditions under which one phenomenon dominates the other.

Literature review

Following the early works by Heider [13,14], static social balance theory has been extensively studied in the last seven decades. Theoretical studies include the characterization of the structurally balanced configurations for both complete networks [12,4] and arbitrary networks [6,9]; the measure of the degree of balance [3,15]; the concept of clustering and its relation to balance [5,8]; as well as the partitioning algorithms that cut a signed network into multiple clusters [7,20,17]. In addition to the theoretical contributions, numerous empirical studies have been conducted for different types of social systems, including social systems at the national level [26,24], at the group level [18,28], and at the individual level [11].

In the last decade, researchers have started to incorporate dynamical systems into the social balance theory, aiming to explain how a signed network evolves to a structurally balanced state. Early works include the discrete-time *local triad dynamics* and *constrained triad dynamics* on complete graphs, proposed by Antal et al. [1,2]. These two models do not always converge to social balance as they suffer from the existence of so-called *jammed states*, i.e., unbalanced equilibria. Radicchi et al. [27] extend the LTD model to arbitrary graphs. Van De Rijt [30] proposes a network game model in which each individual minimizes the number of the unbalanced triads involving itself, by changing the signs of its outlinks; this model evolves to social balance if each individual is allowed to change the signs of multiple links simultaneously. A similar network game model, allowing the adding and deleting of links, is proposed by Malekzadeh et al. [22]. In all the models introduced in this paragraph, the link weights in the signed networks are assumed to only take values from the set $\{-1, 0, 1\}$.

Our models are related to the continuous-time dynamic social balance models studied by Kułakowski et al. [19], Marvel et al. [23], and Traag et al. [29], as well as the discrete-time model proposed by Jia et al. [16]. In these models, the link weights can take arbitrary real values. The model proposed by Kułakowski et al. [19] is based on an influence-like mechanism. The theoretical analysis of this model by Marvel et al. [23] and Traag et al. [29] reveals that, from a specific set of initial conditions, the system first reaches a structurally balanced state and then diverges to unbounded interpersonal appraisals in finite time. In [19], the authors also modify their original model by imposing a predetermined upper bound R of the interpersonal appraisals so that the evolution of the system remains bounded. Rigorous analysis by Wongkaew et al. [31] shows that, in the modified model, the interpersonal appraisals achieve social balance in finite time and the magnitudes of all appraisals converge to the predetermined upper bound R, if the initial appraisals are all lower bounded from $-R$. Traag et al. [29] propose and analyze a continuous-time model based on the homophily mechanism. Similar to the first model proposed by Kułakowski et al. [19], the homophily-based model also reaches social balance and then diverges to unbounded interpersonal appraisals in finite time. Recently, Jia et al. [16] propose and analyze a discrete-time model based on the relaxation of the classic Heider's social balance theory and a modified influence mechanism with convergence to a generalized notion of structural stability.

Contributions

The contributions of this paper are manifold. Firstly, we propose two novel discrete-time dynamic social balance models and establish their well-posedness and bounded evolution properties. These two models explain the evolution of the interpersonal appraisal networks towards the classic Heider's social balance [13] via two sociologically-grounded processes respectively: the homophily mechanism and the influence mechanism. In both models, the appraisal networks are represented by their associated adjacency matrices, i.e., the appraisal matrices. For the homophily-based model, we prove that, the appraisal matrix is well-defined and uniformly bounded at any time, if each row of the initial appraisal matrix has at least one nonzero item. For the influence-based model, we prove that the well-posedness and bounded evolution are guaranteed if the initial appraisal matrix is a symmetric matrix left multiplied by the diagonal matrix with positive diagonal entries. In addition, both our models are invariant under scaling, i.e., if a solution is scaled by a constant, then it remains a solution.

Secondly, we fully characterize the equilibrium sets and the asymptotic behavior of both models. The analyses of the two models are performed in analogous ways. We prove that, for both models, any appraisal network in the equilibrium state is composed of an arbitrary number of isolated subgraphs, each of which satisfies social balance. Moreover, we prove that, for the homophily-based model, in each of such subgraphs, all individuals' appraisals have the same magnitude, while, for the influence-based model, in each subgraph of the equilibrium appraisal network, the individuals reach consensus, in the sense of magnitude, on the appraisals of each individual. Finally, for both models, under a technical condition, we establish the convergence of the appraisal networks to structurally balanced complete graphs.

Thirdly, in addition to the comprehensive theoretical analysis, we further investigate our models by numerical simulations. We provide numerical evidence that our technical condition for the convergence of the appraisal networks to balanced complete graphs holds for generic initial conditions. Moreover, for the homophily-based model, numerical results on the emergence of multi-clique social balance states and their behavior under perturbations reveals some realistic and strategic insights on the escalation and mediation of conflicts. Finally, we numerically investigate the effect of the initial appraisal distribution on the formation of factions, that is, whether an appraisal network eventually evolves to two antagonistic factions or an all-friendly network.

In summary, our paper is the first to propose discrete-time dynamic social balance models, for both the homophily and influence mechanisms, and to establish, through a comprehensive theoretical analysis, that the evolution of appraisals is bounded and convergent from generic initial conditions in appropriate sets. Compared with the continuous-time homophily-based and influence-based models analyzed in [29] and [23] respectively, our models enjoy the desirable property of convergent appraisals, (as opposed to the undesirable property of finite-time divergence). Compared with the model proposed in [19] with bounded evolution, (1) our models do not rely on any predetermined bound to prevent divergence and (2) the asymptotic appraisals in our models are determined by the initial condition rather than the predetermined bound.

Organization

The rest of the paper is organized as follows. Section 2 introduces some notations and basic concepts. Section 3 and Section 4 are the theoretical analyses of our homophily-based and influence-based models respectively. Section 5 provides further discussions and numerical results. Section 6 gives the conclusion. Auxiliary lemmas and proofs are provided in the Appendix.

2 Notations and basic concepts

2.1 Notations

Let $\mathbb{1}_n$ ($\mathbb{0}_n$ resp.) denote the all-ones (all-zeros resp.) $n \times 1$ column vector. Let $\mathbb{R}$ and $\mathbb{Z}_{\geq 0}$ denote the set of real numbers and non-negative integers, respectively. Let $\succ$ and $\prec$ denote "entry-wise greater than" and "entry-wise less than" respectively. For any $X \in \mathbb{R}^{m\times n}$, denote by X_{ij} the (i,j)-th entry of X. Let $|X|$ denote the entry-wise absolute value of X, i.e., each (i,j)-th entry of $|X|$ is equal to $|X_{ij}|$. Let $\operatorname{sign}(X) \in \{-1, 0, +1\}^{m\times n}$ denote the entry-wise sign of X, i.e., for any i and j, $\operatorname{sign}(X_{ij}) = +1$ when $X_{ij} > 0$, $\operatorname{sign}(X_{ij}) = -1$ when $X_{ij} < 0$ and $\operatorname{sign}(X_{ij}) = 0$ when $X_{ij} = 0$. Define the *max norm* of X by $\|X\|_{\max} = \max_{i,j} |X_{ij}|$. Let X_{i*} (X_{*i} resp.) denote the row (column resp.) vector corresponding to the i^{th} row (column resp.) of the matrix X. Let $G(X)$ denote the directed and weighted graph associated with the adjacency matrix X.

Note that, unlike in the traditional definition of weighted graphs, in this paper we allow the presence of links with negative weights. That is, if $X_{ij} < 0$ for some i and j, then the directed link (i,j) in graph $G(X)$ has negative weight equal to X_{ij}. We assume that there is no link from i to j whenever $X_{ij} = 0$. The terms graph and network are assumed interchangeable.

The following sets will be used throughout this paper:

$$\mathcal{S}_{\text{nz-row}} = \{X \in \mathbb{R}^{n\times n} \mid \text{for every } i, X_{i*} \neq \mathbb{0}_n^\top\}, \tag{1}$$

$$\mathcal{S}^+_{\text{s-symm}} = \{X \in \mathbb{R}^{n\times n} \mid \operatorname{sign}(X) = \operatorname{sign}(X)^\top \text{ and } X_{ii} > 0 \text{ for every } i\}, \tag{2}$$

$$\mathcal{S}^+_{\text{rs-symm}} = \{X \in \mathcal{S}^+_{\text{s-symm}} \mid \text{there exists } \gamma \succ \mathbb{0}_n \text{ such that } \operatorname{diag}(\gamma)X = X^\top \operatorname{diag}(\gamma)\}, \tag{3}$$

$$\mathcal{S}^+_{\text{symm}} = \{X \in \mathcal{S}^+_{\text{s-symm}} \mid X = X^\top\}. \tag{4}$$

In other words, $\mathcal{S}_{\text{nz-row}}$ is the set of matrices with at least one non-zero entry in each row, while $\mathcal{S}^+_{\text{s-symm}}$ is the set of sign-symmetric matrices with positive diagonals. Simple calculations show that any $X \in \mathcal{S}^+_{\text{rs-symm}}$ can be written as the product of a diagonal matrix with positive diagonal entries with a symmetric matrix with positive diagonal entries. By definition, we have

$$\mathcal{S}^+_{\text{symm}} \subset \mathcal{S}^+_{\text{rs-symm}} \subset \mathcal{S}^+_{\text{s-symm}} \subset \mathcal{S}_{\text{nz-row}}.$$

In addition, one can easily check by definition that, the sets $\mathcal{S}^+_{\text{s-symm}}$, $\mathcal{S}^+_{\text{rs-symm}}$, and $\mathcal{S}^+_{\text{symm}}$ are all invariant under permutations. That is, given any $X \in \mathcal{S}^+_{\text{s-symm}}$ (or $X \in \mathcal{S}^+_{\text{rs-symm}}$ and $X \in \mathcal{S}^+_{\text{symm}}$ resp.) and a permutation matrix P, we have $PXP^\top \in \mathcal{S}^+_{\text{s-symm}}$ (or $PXP^\top \in \mathcal{S}^+_{\text{rs-symm}}$ and $PXP^\top \in \mathcal{S}^+_{\text{symm}}$ resp.).

2.2 Appraisal matrices and social balance

Given a group of n agents, the network of interpersonal appraisals among the agents is given by the *appraisal matrix* $X \in \mathbb{R}^{n\times n}$. The sign of X_{ij} determines whether agent i's appraisal of j is positive, i.e., i "likes" j, or negative, i.e., i "dislikes" j. The magnitude of X_{ij} represents the intensity of the sentiment. When $X_{ij} = 0$, the appraisal is one of indifference. The diagonal entry X_{ii} represents agent i's self-appraisal. The directed and weighted graph $G(X)$ associated to X as the adjacency matrix is referred to as the *appraisal network*.

Social balance is a specific property of complete appraisal networks, defined as follows.

Definition 2.1 (social balance [12,14])**.** *An appraisal network $G(X)$ satisfies social balance, or, equivalently, is structurally balanced, if the associated appraisal matrix is such that all of its entries are non-zero and the following conditions are satisfied for all $i, j, k \in \{1, \dots, n\}$:*

(S1) *positive self-appraisals:* $X_{ii} > 0$,

(S2) *positive triads:* $\operatorname{sign}(X_{ij}) \operatorname{sign}(X_{jk}) \operatorname{sign}(X_{ki}) = 1$.

Proposition 2.2 (Equivalent conditions for social balance)**.** *For any $X \in \mathbb{R}^{n\times n}$ such that all of its entries are non-zero, $G(X)$ satisfies social balance if and only if it satisfies* (S1) *and*

(S3) $\operatorname{sign}(X_{i*}) = \pm \operatorname{sign}(X_{j*})$, *for all* $i, j \in \{1, \dots, n\}$.

Moreover, for $G(X)$ satisfying social balance, X is sign-symmetric, i.e., $\operatorname{sign}(X) = \operatorname{sign}(X)^\top$.

Proof. Suppose that (S1) and (S3) hold. For any $i, j \in \{1, \dots, n\}$, we have $\operatorname{sign}(X_{i*}) = \delta \operatorname{sign}(X_{j*})$, where δ is either -1 or 1. Therefore, $\operatorname{sign}(X_{ij}) \operatorname{sign}(X_{ji}) = \delta^2 \operatorname{sign}(X_{jj}) \operatorname{sign}(X_{ii}) = 1$, i.e., $\operatorname{sign}(X_{ij}) = \operatorname{sign}(X_{ji})$. Moreover, for any k, since $\operatorname{sign}(X_{ij}) = \delta \operatorname{sign}(X_{jj})$ and

$\operatorname{sign}(X_{jk}) = \delta \operatorname{sign}(X_{ik})$, we have

$$\begin{aligned}&\operatorname{sign}(X_{ij}) \operatorname{sign}(x_{jk}) \operatorname{sign}(X_{ki})\\ &\qquad = \delta^2 \operatorname{sign}(X_{jj}) \operatorname{sign}(X_{ik}) \operatorname{sign}(X_{ki}) = 1.\end{aligned}$$

Therefore, (S1) and (S3) imply (S1) and (S2), as well as the sign symmetry of X.

Now suppose (S1) and (S2) hold. The sign symmetry of X is obtained by letting $k = j$ in (S2). Moreover, due to the sign symmetry and (S2), we obtain $\operatorname{sign}(X_{ij}) \operatorname{sign}(X_{jk}) \operatorname{sign}(X_{ik}) = 1$, which in turn implies that $\operatorname{sign}(X_{ik}) \operatorname{sign}(X_{jk})$ does not depend on k and is equal to $\operatorname{sign}(X_{ij}) \in \{-1, 1\}$. Therefore, $\operatorname{sign}(X_{i*}) = \pm \operatorname{sign}(X_{j*})$ for any i and j. This concludes the proof. $\square$

According to [12], a structurally balanced appraisal network either has only one *faction* in which the interpersonal appraisals are all positive, or is composed of two antagonistic factions such that individuals in the same faction positively appraise each other while all the inter-faction appraisals are negative.

3 Homophily-based Model

In this and the next sections, we propose and analyze two dynamic social balance models respectively. These two models are distinct in the microscopic individual interaction mechanisms. In this section, we propose our first model: the *homophily-based model* (HbM), and analyze its dynamical behavior.

Definition 3.1 (Homophily-based model)**.** *Given an initial appraisal matrix $X(0) \in \mathcal{S}^+_{\text{s-symm}} \subset \mathbb{R}^{n\times n}$, the homophily-based model is defined by:*

$$X(t+1) = \operatorname{diag}(|X(t)|\mathbb{1}_n)^{-1} X(t) X^\top(t). \tag{5}$$

Remark 3.2 (Interpretation)**.** *Equation* (5) *updates the appraisals based on what can be considered as the homophily mechanism. For any $i, j \in \{1, \dots, n\}$, agent i's appraisal of agent j at time step $t+1$ depends on to what extend they are in agreement with each other on the appraisals of all the agents in the group. For any $k \in \{1, \dots, n\}$, if $\operatorname{sign}(X_{ik}(t)) = \operatorname{sign}(X_{jk}(t))$, then the term $X_{ik}(t) X_{jk}(t)$ contributes positively to $X_{ij}(t+1)$, and vice versa.*

The proposition below presents some useful results on the finite-time behavior of the homophily-based model.

Proposition 3.3 (Invariant set and finite-time behavior of HbM)**.** *Consider the dynamical system* (5) *and define $f_{\text{homophily}}(X) = \operatorname{diag}(|X|\mathbb{1}_n)^{-1} X X^\top$. Pick $X_0 \in \mathcal{S}^+_{\text{s-symm}}$. The following statements hold:*

(i) the map $f_{\text{homophily}}$ is well-defined for any $X \in \mathcal{S}_{\text{nz-row}}$ and maps $\mathcal{S}^+_{\text{s-symm}}$ to $\mathcal{S}^+_{\text{s-symm}}$;

(ii) the solution $X(t)$, $t \in \mathbb{Z}_{\geq 0}$, to equation (5) *from initial condition $X(0) = X_0$ is unique and well-defined;*

(iii) the max norm of any solution $X(t)$ satisfies

$$\|X(t+1)\|_{\max} \leq \|X(t)\|_{\max} \leq \|X(0)\|_{\max};$$

(iv) for any $c \neq 0$, the trajectory $cX(t)$ is the solution to equation (5) *from initial condition $X(0) = cX_0$.*

Proof. For simplicity, denote $X^+ = f_{\text{homophily}}(X)$. For any $X \in \mathcal{S}_{\text{nz-row}}$, since, for any i and j, $X^+_{ij} = \frac{1}{\|X_{i*}\|_1} \sum_k X_{ik} X_{jk}$ and $\|X_{i*}\|_1 > 0$, $f_{\text{homophily}}(X)$ is well-defined. Moreover,

$$X^+_{ii} = \frac{1}{\|X_{i*}\|_1} \sum_k X_{ik} X_{ik} = \frac{\|X_{i*}\|_2^2}{\|X_{i*}\|_1} > 0, \quad \text{and}$$

$$X^+_{ij} = \frac{\|X_{j*}\|_1}{\|X_{i*}\|_1} X^+_{ji}, \quad \text{for any } i \text{ and } j.$$

Therefore, $f_{\text{homophily}}$ maps $\mathcal{S}^+_{\text{s-symm}}$ to $\mathcal{S}^+_{\text{s-symm}}$. This concludes the proof of statement (i). Statements (ii) is a direct consequence of statement (i). In addition,

$$\begin{aligned}|X^+_{ij}| &\leq \frac{1}{\|X_{i*}\|_1} \sum_{k=1}^n |X_{ik} X_{jk}| \leq \frac{1}{\|X_{i*}, \|_1} \sum_{k=1}^n |X_{ik}||X_{jk}|,\\ &\leq \max_k |X_{jk}| \leq \|X\|_{\max}\end{aligned}$$

immediately leads to statement (iii). Finally, statement (iv) is obtained by replacing $X(t)$ with $cX(t)$ on the right-hand side of equation (5). $\square$

In fact, for any $X(0) \in \mathcal{S}_{\text{nz-row}}$, we have $X(1) \in \mathcal{S}^+_{\text{s-symm}}$ and, thus, $X(t) \in \mathcal{S}^+_{\text{s-symm}}$ for any $t \geq 1$. Therefore, the set of initial conditions of the system can be extended to the set $\mathcal{S}_{\text{nz-row}}$. However, without loss of generality, we still consider $\mathcal{S}^+_{\text{s-symm}}$ as the domain of system (5). In addition, according to Proposition 3.3, for any $X(0) \in \mathcal{S}^+_{\text{s-symm}}$, the solution $X(t)$ to equation (9) is uniformly upper bounded for all $t \in \mathbb{Z}_{\geq 0}$. This is a desired property compared with some previous models, in which $X(t)$ diverge in finite time [23,29].

The theorem below characterizes the set of fixed points of system (5).

Theorem 3.4 (Fixed points and balance)**.** *Consider the dynamical system* (5) *in domain $\mathcal{S}^+_{\text{s-symm}}$. Define*

$$\begin{aligned}&Q_{\text{homophily}}\\ &\quad = \Big\{ PYP^\top \in \mathcal{S}^+_{\text{s-symm}} \,\Big|\, P \text{ is a permutation matrix,}\\ &\qquad\quad Y \text{ is a block diagonal matrix with blocks of}\\ &\qquad\quad \text{the form } \alpha b b^\top,\ \alpha > 0, b \in \{-1, +1\}^m,\ m \leq n \Big\}.\end{aligned}$$

Then we have that,

(i) $Q_{\text{homophily}}$ is the set of all the fixed points of (5)*,*

(ii) for any $X \in Q_{\text{homophily}}$, $G(X)$ is composed by isolated complete subgraphs that satisfy social balance.

Proof. We first prove that any $X^* \in Q_{\text{homophily}}$ is a fixed point of system (5). For any $\alpha > 0$ and $b \in \{-1, +1\}^n$, the matrix $Y = \alpha b b^\top$ satisfies

$$f_{\text{homophily}}(Y) = \operatorname{diag}(n\alpha \mathbb{1}_n)^{-1} \alpha^2 b b^\top b b^\top = \alpha b b^\top = Y.$$

Now suppose that Y is a block diagonal matrix, i.e., $Y = \operatorname{diag}(Y^{(1)}, \dots, Y^{(K)})$, where each $Y^{(i)}$ is a $n_i \times n_i$ matrix of the form $\alpha_i b^{(i)} b^{(i)\top}$, with $\alpha_i > 0$, $b^{(i)} \in \{-1, +1\}^{n_i}$, and $n_1 + \cdots + n_K = n$. One can check that, as long as

$$Y^{(i)} = \mathrm{diag}(|Y^{(i)}|\mathbb{1}_n)^{-1} Y^{(i)} Y^{(i)\top} \tag{6}$$

for any $i \in \{1, \ldots, K\}$, Y is a fixed point of system (5). Since $Y^{(i)} = \alpha_i b^{(i)} b^{(i)\top}$, we know that equation (6) is satisfied for any i. Therefore, Y is a fixed point of system (5). Moreover, given any fixed point Y of system (5), for any permutation matrix $P \in \mathbb{R}^{n\times n}$, we have

$$\begin{aligned} PYP^\top &= P\,\mathrm{diag}(|Y|\mathbb{1}_n)^{-1} YY^\top P^\top \\ &= \mathrm{diag}(|PYP^\top|\mathbb{1}_n)^{-1}(PYP^\top)(PYP^\top)^\top \\ &= f_{\mathrm{homophily}}(PYP^\top). \end{aligned}$$

Therefore, any $X^* \in Q_{\mathrm{homophily}}$ is a fixed point of (5).

Now we prove by induction that $Q_{\mathrm{homophily}}$ is the set of all the fixed points of system (5). For the trivial case of $n = 1$, $Q_{\mathrm{homophily}}$ represents the set of all the positive scalars and one can easily check that any positive scalar X is a fixed point of system (5) with $n = 1$. Suppose statement (i) holds for any system with dimension $\tilde{n} < n$.

For system (5) with dimension n, suppose X is a fixed point, i.e., $X = f_{\mathrm{homophily}}(X)$, which implies that,

$$X_{ij} = \frac{1}{\|X_{i*}\|_1} \sum_{k=1}^{n} X_{ik} X_{jk} = \frac{\|X_{j*}\|_1}{\|X_{i*}\|_1} X_{ji}, \text{ for any } i \neq j.$$

Therefore, for any $i, j \in \{1, \ldots, n\}$ and $j \neq i$, we have $\mathrm{sign}(X_{ij}) = \mathrm{sign}(X_{ji})$. In addition, since $X_{ii} = \sum_{k=1}^{n} X_{ik}^2 / \|X_{i*}\|_1$, we have $X_{ii} > 0$ for any i.

Let X^+ denote $f_{\mathrm{homophily}}(X)$ for simplicity. Since $X = X^+$, we have $\|X\|_{\max} = \|X^+\|_{\max}$, which implies that there exists i, j such that $|X^+_{ij}| = \|X\|_{\max}$. We discuss two cases which together include all the possible X's.

Case 1: there exists i such that $|X_{ii}| = \|X\|_{\max}$ and $|X_{ij}| = 0$ for any $j \neq i$. Since $\mathrm{sign}(X_{ij}) = \mathrm{sign}(X_{ji})$, we have $X_{ji} = 0$ for any $j \neq i$. In addition, since $X_{ii} > 0$, $X_{ii} = \|X\|_{\max}$. Therefore, there exists a permutation matrix P such that

$$PXP^\top = \begin{bmatrix} \|X\|_{\max} & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{X}_{(n-1)\times(n-1)} \end{bmatrix},$$

Since $PXP^\top$ is also a fixed point of system (5), one can check that $\tilde{X}$ satisfies $\tilde{X} = \mathrm{diag}(|\tilde{X}|\mathbb{1}_n)^{-1}\tilde{X}\tilde{X}^\top$. Therefore, $\tilde{X}$ is a fixed point of system (5) with dimension $n-1$. Since we have assumed that statement (i) holds for dimension $\tilde{n} < n$, there exists an $(n-1)\times(n-1)$ permutation matrix $\tilde{P}$ and a block diagonal $\tilde{Y}$, with blocks of the form $\alpha bb^\top$, where $\alpha > 0$, $b \in \{-1, +1\}^m$, $m < n-1$, such that $\tilde{X} = \tilde{P}\tilde{Y}\tilde{P}^\top$. Therefore,

$$X = P^\top \begin{bmatrix} 1 & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{P} \end{bmatrix} \begin{bmatrix} \|X\|_{\max} & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{Y} \end{bmatrix} \begin{bmatrix} 1 & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{P} \end{bmatrix}^\top P.$$

The matrix

$$P^\top \begin{bmatrix} 1 & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{P} \end{bmatrix}$$

is also a permutation matrix. Therefore $X \in Q_{\mathrm{homophily}}$.

Case 2: there exists $j \neq i$ such that $|X^+_{ij}| = \|X\|_{\max}$. We first define some notations used in the following proof: For any k, let $\theta_k = \{\ell \mid X_{k\ell} \neq 0\}$ and $|\theta_k|$ be the cardinality of the set θ_k. Note that, since $X = f_{\mathrm{homophily}}(X) \in \mathcal{S}^+_{\text{s-symm}}$, k is always in θ_k and $X_{kk} > 0$. Let $X_{\ell*,\theta_k} \in \mathbb{R}^{1\times|\theta_k|}$ be the ℓ-th row vector of X with all the $X_{\ell p}$ entries such that $p \notin \theta_k$ removed.

We point out a general result that, for any k and ℓ, if

$$|X^+_{k\ell}| = \frac{1}{\|X_{k*}\|_1}\Big|\sum_{p=1}^{n} X_{kp}X_{\ell p}\Big| = \|X\|_{\max},$$

then, for the second equality to hold, X must satisfy that: 1) $\theta_k \subset \theta_l$; 2) $|X_{\ell p}| = \|X\|_{\max}$ for any $p \in \theta_k$; 3) $\mathrm{sign}(X_{\ell*,\theta_k}) = \pm\,\mathrm{sign}(X_{k*,\theta_k})$. Therefore, for the i, j indexes such that $|X^+_{ij}| = \|X\|_{\max}$ and $i \neq j$, we have: $|X_{jk}| = \|X\|_{\max}$, for any $k \in \theta_i$; $\theta_i \subset \theta_j$; and $\mathrm{sign}(X_{j*,\theta_i}) = \pm\,\mathrm{sign}(X_{i*,\theta_i})$. Since $i \in \theta_i$ and $X^+ = X$, we obtain $|X^+_{ji}| = |X_{ji}| = \|X\|_{\max}$. Therefore, $|X^+_{ik}| = |X_{ik}| = \|X\|_{\max}$, for any $k \in \theta_j$, and $\theta_j \subset \theta_i$, which in turn leads to $\theta_i = \theta_j$ and $|X^+_{ik}| = |X_{ik}| = \|X\|_{\max}$ for any $k \in \theta_i$. Therefore, for any $k \in \theta_i$, $|X^+_{ik}| = \|X\|_{\max}$, which implies $|X_{k\ell}| = \|X\|_{\max}$ for any $l \in \theta_i$. Since $|X^+_{k\ell}| = |X_{k\ell}|$, we further obtain that $\theta_k \subset \theta_l$ and $\mathrm{sign}(X_{k*,\theta_k}) = \pm\,\mathrm{sign}(X_{\ell*,\theta_k})$. Moreover, due to the fact that the indexes k and l are interchangeable, we conclude that, for any $k, l \in \theta_i$: a) $\theta_k = \theta_l = \theta_i$; b) $|X_{k\ell}| = \|X\|_{\max}$; c) $\mathrm{sign}(X_{k*}) = \pm\,\mathrm{sign}(X_{\ell*})$.

If $|\theta_i| = n$, let $\alpha = X_{11}$ and $b = \mathrm{sign}(X_{1*})^\top$, then we have $X = \alpha bb^\top$. If $|\theta_i| < n$, there exists a permutation matrix P such that

$$PXP^\top = \begin{bmatrix} X^{(\theta_i)} & \mathbb{0}_{|\theta_i|\times(n-|\theta_i|)} \\ \mathbb{0}_{(n-|\theta_i|)\times|\theta_i|} & \tilde{X} \end{bmatrix},$$

where $X^{(\theta_i)}$ is a $|\theta_i| \times |\theta_i|$ matrix. Moreover, $X^{(\theta_i)} = \|X\|_{\max} bb^\top$, where $b = \mathrm{sign}(X_{i*,\theta_i})^\top$. Following the same line of argument for Case 1, we know that $\tilde{X}$ is of the form $\tilde{P}\tilde{Y}\tilde{P}^\top$ and thereby $X \in Q_{\mathrm{homophily}}$. This concludes the proof for statement (i).

For any $X^* \in Q_{\mathrm{homophily}}$, there exists a permutation matrix P and a block diagonal matrix $Y = \mathrm{diag}(Y^{(1)}, \ldots, Y^{(K)})$ such that $X^* = PYP^\top$. Note that $G(Y)$ has exactly the same topology as $G(X)$, but with the nodes reindexed. Therefore, we only need to analyze the structure of $G(Y)$. One can easily check that Y has positive diagonals (and is also sign-symmetric). Moreover, $G(Y)$ is made up of K isolated complete subgraphs. Therefore, for any triad (j, k, l) in $G(Y)$, there exists $i \in \{1, \ldots, K\}$ such that nodes j, k, l are all in the subgraph $G(Y^{(i)})$ with the adjacency matrix $Y^{(i)} = \big(Y^{(i)}_{jk}\big)_{n_i\times n_i}$. Suppose $Y^{(i)} = \alpha_i b^{(i)} b^{(i)\top}$, where $b^{(i)} = (b^{(i)}_1, \ldots, b^{(i)}_{n_i})^\top$. We have $Y^{(i)}_{jk} Y^{(i)}_{k\ell} Y^{(i)}_{\ell j} = \alpha_i^3 {b^{(i)}_j}^2 {b^{(i)}_k}^2 {b^{(i)}_l}^2 > 0$. Therefore, every triad is positive in graph $G(Y)$. We conclude that any

$X^* \in Q_{\text{homophily}}$ is associated with a graph $G(X^*)$ composed by isolated complete subgraphs that satisfy social balance. This concludes the proof for statement (ii). $\square$

Remark 3.5. *Since X being a fixed point of $f_{\text{homophily}}$ implies that X is sign-symmetric and has positive diagonal, $Q_{\text{homophily}}$ is actually the set of all the fixed points of $f_{\text{homophily}}$ in $\mathcal{S}_{\text{nz-row}}$.*

Now we present the main results on the convergence of the appraisal matrix to social balance.

Theorem 3.6 (Convergence and social balance in HbM)**.** *Consider the homophily-based model given by equation* (5)*. The following statements hold:*

(i) Each fixed point of rank one in $Q_{\text{homophily}}$ is locally stable.

For any $X(0) \in \mathcal{S}^+_{\text{s-symm}}$ such that $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$, we have that:

(ii) there exists $X^* \in Q_{\text{homophily}}$ of rank one such that $\lim_{t\to\infty} X(t) = X^*$, and

(iii) there exists $T > 0$ such that $G(X(t))$ satisfies social balance for all $t \geq T$.

Proof. We start by proving the following two claims. For any given $t_0 \geq 0$, if all the entries of $X(t_0)$ are non-zero and $G(X(t_0))$ satisfies social balance, then,

C.1) for any $t \geq t_0$, $G(X(t))$ satisfies social balance and $\operatorname{sign}(X(t)) = \operatorname{sign}(X(t_0))$;

C.2) there exists $\alpha > 0$ and $b \in \{-1, +1\}^n$, depending on $X(t_0)$, such that $X(t)$ converges to $\alpha b b^\top$ as $t \to \infty$.

To prove claim C.1), it suffices to prove that $G(X(t_0+1))$ satisfies social balance and $\operatorname{sign}(X(t_0+1)) = \operatorname{sign}(X(t_0))$, as the cases for $t \geq t_0 + 1$ follow by induction. For any i and j, since $G(X(t_0))$ satisfies social balance, according to Proposition 2.2, we have $\operatorname{sign}(X_{i*}(t_0)) = \pm \operatorname{sign}(X_{j*}(t_0))$. In addition, we have $X_{jj}(t_0) > 0$ for any j. Therefore,

$$\begin{aligned}\operatorname{sign}\big(X_{ij}(t_0{+}1)\big) &= \operatorname{sign}\Big(\frac{1}{\|X_{i*}(t_0)\|_1}\sum_{k=1}^n X_{ik}(t_0)X_{jk}(t_0)\Big)\\ &= \operatorname{sign}\big(X_{ij}(t_0)X_{jj}(t_0)\big) = \operatorname{sign}\big(X_{ij}(t_0)\big),\end{aligned}$$

for any i and j. This concludes the proof for claim C.1).

For any $t \geq t_0$, since $G(X(t))$ satisfies social balance,

$$|X_{ij}(t{+}1)| = \frac{1}{\|X_{i*}(t)\|_1}\sum_{k=1}^n |X_{ik}(t)||X_{jk}(t)| \text{ for any } i, j,$$

which leads to the following two inequalities: $\min_{k,\ell}|X_{k\ell}(t{+}1)| \geq \min_{k,\ell}|X_{k\ell}(t)|$; $\max_{k,\ell}|X_{k\ell}(t{+}1)| \leq \max_{k,\ell}|X_{k\ell}(t)|$. Therefore, $\min_{k,\ell}|X_{k\ell}(t)|$ is non-decreasing and upper bounded by $\max_{k,\ell}|X_{k\ell}(t_0)|$, while $\max_{k,\ell}|X_{k\ell}(t)|$ is non-increasing and lower bounded by $\min_{k,\ell}|X_{k\ell}(t_0)|$, which in turn implies that there exists $0 < \underline{\alpha} \leq \overline{\alpha}$, depending on $X(t_0)$, such that

$$\lim_{t\to\infty}\min_{k,\ell}|X_{k\ell}(t)| = \underline{\alpha}, \quad \text{and} \quad \lim_{t\to\infty}\max_{k,\ell}|X_{k\ell}(t)| = \overline{\alpha}.$$

Moreover, suppose $\max_{k,\ell}|X_{k\ell}(t)| > \min_{k,\ell}|X_{k\ell}(t)|$, for some $t \geq t_0$, and $|X_{pq}(t)| = \min_{k,\ell}|X_{k\ell}(t)|$. We have

$$\begin{aligned}|X_{jp}(t{+}1)| &= \frac{1}{\|X_{j*}(t)\|_1}\sum_{k=1}^n |X_{jk}(t)||X_{pk}(t)|\\ &< \max_{k,\ell}|X_{k\ell}(t)|, \quad \text{and}\\ |X_{ij}(t{+}2)| &= \frac{1}{\|X_{i*}(t{+}1)\|_1}\sum_{k=1}^n |X_{ik}(t{+}1)||X_{jk}(t{+}1)|\\ &< \max_{k,\ell}|X_{k\ell}(t)|, \end{aligned} \tag{7}$$

for any i and j. Let $V_1 : \mathbb{R}^{n\times n} \to \mathbb{R}_{\geq 0}$ be defined as:

$$V_1(X) = \max_{k,\ell}|X_{k\ell}| - \min_{k,\ell}|X_{k\ell}|.$$

Due to inequality (7), for any $t \geq t_0$, $0 \leq V_1(X(t+2)) < V_1(X(t))$ as long as $V_1(X(t)) > 0$. Therefore, $V_1(X(t))$ converges to 0 as $t \to \infty$, which implies $\underline{\alpha} = \overline{\alpha} = \alpha > 0$. For any i, j and any $t \geq t_0$, since $\min_{k,\ell}|X_{k\ell}(t)| \leq |X_{ij}(t)| \leq \max_{k,\ell}|X_{k\ell}(t)|$, we conclude that $\lim_{t\to\infty}|X_{ij}(t)| = \alpha$. Moreover, since $\operatorname{sign}(X(t)) = \operatorname{sign}(X(t_0))$ for any $t \geq t_0$, we have $\lim_{t\to\infty} X(t) = \alpha b b^\top$, where $b = \operatorname{sign}(X_{1*}(t_0))^\top$. This concludes the proof for claim C.2).

Now we prove statement (i), i.e., each $X^* \in Q_{\text{homophily}}$ with rank 1 is locally stable. Let $X^* = \alpha b b^\top$, where $\alpha > 0$ and $b \in \{-1, +1\}^n$. For any matrix $\Delta \in \mathbb{R}^{n\times n}$ such that $\delta = \max_{i,j}|\Delta_{ij}| < \alpha$, we have $\operatorname{sign}(X^* + \Delta) = \operatorname{sign}(X^*)$. Due to claim C.1) and the proof of claim C.2), we know that, for $X(0) = X^* + \Delta$, $X(t)$ satisfies that, for any $t \geq 0$: (1) $\operatorname{sign}(X(t)) = \operatorname{sign}(X(0)) = \operatorname{sign}(X^*)$; (2) $\alpha - \delta \leq \min_{i,j}|X_{ij}(t)| \leq \max_{i,j}|X_{ij}(t)| \leq \alpha + \delta$. Therefore, for any i and j, $X_{ij}(t)$ is of the form $\alpha_{ij}(t)\operatorname{sign}(X^*_{ij})$, where $0 < \alpha - \delta \leq \alpha_{ij}(t) \leq \alpha + \delta$. We thereby have

$$\begin{aligned}\|X(t) - X^*\|_{\max} &= \max_{ij}\big|\alpha_{ij}(t)\operatorname{sign}(X^*_{ij}) - \alpha \operatorname{sign}(X^*_{ij})\big|\\ &= \max_{ij}|\alpha_{ij}(t) - \alpha| \leq \delta.\end{aligned}$$

Therefore, for any $\epsilon > 0$, there exists $\delta = \min\{\frac{\alpha}{2}, \frac{\epsilon}{2}\}$ such that, for any $X(0)$ satisfying $\|X(0) - X^*\|_{\max} < \delta$, $\|X(t) - X^*\|_{\max} < \epsilon$ for any $t \geq 0$, i.e., X^* is locally stable.

For the rest of the proof, we proceed to prove the statements (ii) and (iii) of the theorem. For simplicity, denote $X^+ = f_{\text{homophily}}(X)$. Firstly, one can easily check that $f_{\text{homophily}}(X)$ is continuous for any $X \in \mathcal{S}^+_{\text{s-symm}}$. Secondly, for any given $X(0) \in \mathcal{S}^+_{\text{s-symm}}$, according to Proposition 3.3, $\|X(t)\|_{\max} \leq \|X(0)\|_{\max}$ for any $t \in \mathbb{Z}_{\geq 0}$. In addition, $\liminf_{t\to\infty}\min_{i,j}|X_{ij}(t)| = \delta$ for some $\delta > 0$ implies that there exists $\tilde{t} \in \mathbb{Z}_{\geq 0}$ such that $\min_{i,j}|X_{ij}(t)| \geq \delta/2$ for any $t \geq \tilde{t}$. Therefore, the set

$$G_c = \Big\{X \in \mathcal{S}^+_{\text{s-symm}} \,\Big|\, \min_{i,j}|X_{ij}| \geq \delta/2, \\ \|X\|_{\max} \leq \|X(0)\|_{\max}\Big\}$$

is a compact subset of $\mathcal{S}^+_{\text{s-symm}}$ and $X(t) \in G_c$ for any $t \geq \tilde{t}$. Thirdly, define $V_2(X) = \|X\|_{\max}$. The function

V_2 is continuous on $\mathcal{S}^+_{\text{s-symm}}$ and, according to Proposition 3.3, satisfies $V_2(X^+) - V_2(X) \leq 0$ for any $X \in \mathcal{S}^+_{\text{s-symm}}$. According to the extended LaSalle invariance principle presented in Theorem 2 of [25], $X(t)$ converges to the largest invariant set M of the set $E = \{X \in G_c \mid V_2(X^+) - V_2(X) = 0\}$.

Now we characterize the largest invariant set M. For any $X \in M \subset E$, $V_2(X^+) = V_2(X) = \|X\|_{\max}$. Suppose $|X^+_{ij}| = \max_{k,\ell}|X^+_{k\ell}|$. Since

$$\begin{aligned} |X^+_{ij}| &= \frac{1}{\|X_{i*}\|_1}\left|\sum_{\ell=1}^{n} X_{i\ell}X_{j\ell}\right| \\ &\leq \frac{1}{\|X_{i*}\|_1}\sum_{\ell=1}^{n}|X_{i\ell}||X_{j\ell}| \leq \max_{\ell}|X_{j\ell}|, \end{aligned} \tag{8}$$

we need all of these inequalities to hold with equality and $\max_{\ell}|X_{j\ell}| = \|X\|_{\max}$. Since $X \in G_c$ implies $|X_{k\ell}| > 0$, for any $k, \ell \in \{1, \dots, n\}$, X must satisfy that

(a) X_{i*} and X_{j*} have the same or opposite sign pattern, i.e., $\operatorname{sign}(X_{i*}) = \pm \operatorname{sign}(X_{j*})$,

(b) All entries of X_{j*} have the magnitude $\|X\|_{\max}$.

Therefore, for any $X \in E$, there exist some i and j such that the aforementioned conditions (a) and (b) hold. Moreover, since the set M is invariant, $X \in M$ implies $X^+ \in M \subset E$, which in turn implies that there exists a $\tilde{j}$ such that, for any p, $|X^+_{\tilde{j}p}| = \|X^+\|_{\max} = \|X\|_{\max}$. Following the same argument on the conditions such that the inequalities (8) become equalities, we know that, for any p, $\operatorname{sign}(X_{\tilde{j}*}) = \pm \operatorname{sign}(X_{p*})$ and $|X_{pk}| = \|X\|_{\max}$ for any k. As these relationships hold for any p, we conclude that for any $i, j \in \{1, \dots, n\}$, X_{i*} and X_{j*} must have the same or the opposite sign pattern. Let $\alpha = \|X\|_{\max}$ and $b = \operatorname{sign}(X^\top_{1*})$. Each row of X is thereby equal to either $\alpha b^\top$ or $-\alpha b^\top$. Therefore, X is of the form $X = \alpha c b^\top$, where $c \in \{-1, 1\}^n$. Moreover, since all the diagonal entries of X are positive, the column vector c must satisfy $c_i b_i = 1$ for any i, which implies $c = b$. Therefore, $X = \alpha b b^\top$. In addition, since any matrix X of the form $\alpha b b^\top$, with $\alpha > 0$ and $b \in \{-1, 1\}^n$, is a fixed point of system (5), we conclude that

$$M = \left\{X = \alpha b b^\top \mid \frac{\delta}{2} \leq \alpha \leq \|X(0)\|_{\max}, b \in \{-1, 1\}^n\right\},$$

which is a compact subset of $\mathcal{S}^+_{\text{s-symm}}$.

For any $\hat{X} \in M$, since $\hat{X}$ satisfies social balance (see Theorem 3.4) and $\min_{i,j} \hat{X}_{ij} \geq \delta/2 > 0$, there exists an open neighbor set defined as $\mathcal{U}(\hat{X}) = \{X = \hat{X} + \Delta \mid \|\Delta\|_{\max} < \min_{i,j} \hat{X}_{ij}\}$ such that any $X \in \mathcal{U}(\hat{X})$ satisfies social balance. According to Hein-Borel theorem, there exists a finite set $\{\hat{X}_1, \dots, \hat{X}_K\} \subset M$ such that $M \subset \cup_{k=1}^K \mathcal{U}(\hat{X}_k)$. Since $\cup_{k=1}^K \mathcal{U}(\hat{X}_k)$ is an open set, there exists $\epsilon > 0$ such that the neighbor set of M, defined as

$$\mathcal{U}(M, \epsilon) = \{X \in \mathcal{S}^+_{\text{s-symm}} \mid \|X - M\|_{\max} < \epsilon\},$$

satisfies that $\mathcal{U}(M, \epsilon) \subset \cup_{k=1}^K \mathcal{U}(\hat{X}_k)$ and thereby any $X \in \mathcal{U}(M, \epsilon)$ satisfies social balance.

Since $X(t) \to M$ as $t \to \infty$, there exists $T \in \mathbb{Z}_{\geq 0}$ such that $X(t) \in \mathcal{U}(M, \epsilon)$ for any $t \geq T$. Therefore, $X(t)$ satisfies social balance for any $t \geq T$, which proves statement (iii). Moreover. Statement (ii) follows from claim C.2) and Theorem 3.4. □

Extensive simulation results indicate that, under generic initial conditions $X(0) \in \mathcal{S}_{\text{nz-row}}$, every entry of the solution $|X(t)|$ is uniformly lower bounded by a positive number for all $t > 0$. This numerical result will be discussed in details in Section 5.

4 Influence-based Model

In this section, we propose and analyze our second model: the *influence-based model* (IbM).

Definition 4.1 (Influence-based model). *Given an initial appraisal matrix $X(0) \in \mathcal{S}^+_{\text{rs-symm}} \subset \mathbb{R}^{n \times n}$, the influence-based model is defined by:*

$$X(t+1) = \operatorname{diag}(|X(t)|\mathbb{1}_n)^{-1} X(t) X(t). \tag{9}$$

Remark 4.2 (Interpretation). *The evolution of the appraisal matrix given by equation (9) can be interpreted as an influence process. The associated time-varying influence matrix $W(t)$ is constructed by $W(t) = \operatorname{diag}(|X(t)|\mathbb{1}_n)^{-1} X(t)$. That is, the influence any agent i assigns to agent j is assumed to be proportional to i's appraisal of j. We allow negative influences. For any i and k, if agent i is has a positive appraisal of agent k, then agent k's positive (negative resp.) appraisal of j contributes positively (negatively resp.) to agent i's appraisal of j at the next time step, and vice versa.*

Next, we present some results on the invariant set and finite-time behavior of the influence-based model.

Proposition 4.3 (Finite-time Properties of the IbM). *Consider the dynamical system (9) and define $f_{\text{influence}}(X) = \operatorname{diag}(|X|\mathbb{1}_n)^{-1} X X$. Pick any $X_0 \in \mathcal{S}^+_{\text{rs-symm}}$. The following statements hold:*

(i) the map $f_{\text{influence}}$ is well-defined for any $X \in \mathcal{S}_{\text{nz-row}}$ and maps $\mathcal{S}^+_{\text{rs-symm}}$ to $\mathcal{S}^+_{\text{rs-symm}}$;

(ii) the solution $X(t)$, $t \in \mathbb{Z}_{\geq 0}$, to equation (9) from initial condition $X(0) = X_0$ is unique and well-defined;

(iii) the max norm of $X(t)$ satisfies

$$\|X(t+1)\|_{\max} \leq \|X(t)\|_{\max} \leq \|X(0)\|_{\max};$$

(iv) for any $c \neq 0$, the trajectory $cX(t)$ is the solution to equation (9) from initial condition $X(0) = cX_0$.

Proof. Denote $X^+ = f_{\text{influence}}(X)$ for simplicity. Following the same argument as in the proof of Proposition 3.3, we know that $f_{\text{influence}}$ is well-defined for any $X \in \mathcal{S}_{\text{nz-row}}$. For any $X \in \mathcal{S}^+_{\text{rs-symm}}$, there exists $\gamma \succ \mathbb{0}_n$ such that $\operatorname{diag}(\gamma) X = X^\top \operatorname{diag}(\gamma)$. Therefore,

$$X^+_{ii} = \frac{1}{\|X_{i*}\|_1}\sum_k X_{ik}X_{ki} = \frac{1}{\|X_{i*}\|_1}\sum_k \frac{\gamma_i}{\gamma_k} X^2_{ik} > 0, \text{ and}$$

$$X^+_{ij} = \frac{1}{\|X_{i*}\|_1}\frac{\gamma_j}{\gamma_i}\sum_k X_{jk}X_{ki} = \frac{\|X_{j*}\|_1\gamma_j}{\|X_{i*}\|_1\gamma_i} X^+_{ji}.$$

Let $\tilde{\gamma} = \operatorname{diag}\big(|X|\mathbb{1}_n\big)\gamma$, then we have $\operatorname{diag}(\tilde{\gamma})X = X^\top \operatorname{diag}(\tilde{\gamma})$. Therefore, $X^+ = f_{\text{influence}}(X) \in \mathcal{S}^+_{\text{rs-symm}}$. This concludes the proof of statement (i). Statements (ii) is a direct consequence of statement (i). Moreover,

$$\big|X^+_{ij}\big| = \frac{1}{\|X_{i*}\|_1}\Big|\sum_k X_{ik}X_{kj}\Big| \leq \frac{1}{\|X_{i*}\|_1}\sum_k |X_{ik}||X_{kj}| \leq \max_k |X_{kj}| \leq \|X\|_{\max}$$

immediately lead to statement (iii). Statement (iv) is a straightforward observation obtained from equation (9). □

Notice that, unlike the homophily-based model, $f_{\text{influence}}$ is not well-defined for all $X \in \mathcal{S}_{\text{nz-row}}$. For example,

$$X(0) = \begin{bmatrix} 1 & 2 \\ -0.5 & -1 \end{bmatrix} \in \mathcal{S}_{\text{nz-row}}$$

leads to $X(1) \notin \mathcal{S}_{\text{nz-row}}$ and $f_{\text{influence}}(X(1))$ is not defined. Instead, we consider $\mathcal{S}^+_{\text{rs-symm}}$ as the domain of system (9). According to Proposition 4.3, for any $X(0) \in \mathcal{S}^+_{\text{rs-symm}}$, the solutions $X(t)$ to equation (9) is uniformly upper bounded, which is a desired property, that the previous models in [23,29] do not have.

The following theorem characterizes the set of fixed points of the map $f_{\text{influence}}$ in $\mathcal{S}^+_{\text{rs-symm}}$.

Theorem 4.4 (Fixed points and social balance)**.** *Consider system* (9) *in domain* $\mathcal{S}^+_{\text{rs-symm}}$*. Define*

$$\begin{aligned} &Q_{\text{influence}} \\ &= \Big\{PYP^\top \in \mathcal{S}^+_{\text{rs-symm}} \,\Big|\, P \text{ is a permutation matrix,} \\ &\quad Y \text{ is a block diagonal matrix with blocks of the} \\ &\quad \text{form } \operatorname{sign}(w)w^\top,\ w \in \mathbb{R}^m \text{ and } |w| \succ \mathbb{0}_m, m \leq n\Big\}. \end{aligned}$$

Then the following statements hold:

(i) $Q_{\text{influence}}$ *is the set of all the fixed points of system* (9) *in domain* $\mathcal{S}^+_{\text{rs-symm}}$*,*

(ii) for any $X \in Q_{\text{influence}}$*,* $G(X)$ *is composed by isolated complete subgraphs that satisfy social balance.*

Proof. We first prove that any $X^* \in Q_{\text{influence}}$ is a fixed point of system (9). For any $w \in \mathbb{R}^n$ such that $|w| \succ \mathbb{0}_n$, the matrix $Y = \operatorname{sign}(w)w^\top$ satisfies

$$\begin{aligned} &f_{\text{influence}}(Y) \\ &\quad= \operatorname{diag}(|\operatorname{sign}(w)w^\top|\mathbb{1}_n)^{-1}(\operatorname{sign}(w)w^\top)(\operatorname{sign}(w)w^\top) \\ &\quad= \operatorname{sign}(w)w^\top = Y. \end{aligned}$$

Therefore, $Y = \operatorname{sign}(w)w^\top$ is a fixed point of system (9). Now suppose that Y is a block diagonal matrix, i.e., $Y = \operatorname{diag}(Y^{(1)}, \dots, Y^{(K)})$, where each $Y^{(i)}$ is a $n_i \times n_i$ matrix of the form $\operatorname{sign}(w^{(i)}){w^{(i)}}^\top$, with $|w^{(i)}| \succ \mathbb{0}_{n_i}$, and $n_1 + \cdots + n_K = n$. One can check that, as long as

$$Y^{(i)} = \operatorname{diag}(|Y^{(i)}|\mathbb{1}_n)^{-1}Y^{(i)}Y^{(i)} \tag{10}$$

for any $i \in \{1, \dots, K\}$, Y is a fixed point of system (9). Since $Y^{(i)} = \operatorname{sign}(w^{(i)}){w^{(i)}}^\top$, following the same line of argument for the case in which Y only has one block, we know that equation (10) holds for any i. Therefore, Y is a fixed point of system (9). Moreover, given any fixed point Y, for any permutation matrix $P \in \mathbb{R}^{n\times n}$, since

$$\begin{aligned} PYP^\top &= P\operatorname{diag}(|Y|\mathbb{1}_n)^{-1}YYP^\top \\ &= \operatorname{diag}(|PYP^\top|\mathbb{1}_n)^{-1}(PYP^\top)(PYP^\top) \\ &= f_{\text{influence}}(PYP^\top), \end{aligned}$$

any $X^* \in Q_{\text{influence}}$ is a fixed point of $f_{\text{influence}}$.

For any $X^* \in Q_{\text{influence}}$, there exists a permutation matrix P and a block diagonal matrix Y in the form $\operatorname{diag}(Y^{(1)}, \dots, Y^{(K)})$ such that $X^* = PYP^\top$. One can easily check that Y has positive diagonals and is sign-symmetric. Moreover, $G(Y)$ is made up of K isolated complete subgraphs. Therefore, for any triad (j,k,l) in $G(Y)$, there exists $i \in \{1, \dots, K\}$ such that nodes j, k, l are all in the subgraph $G(Y^{(i)})$ with the adjacency matrix $Y^{(i)} = \big(Y^{(i)}_{jk}\big)_{n_i\times n_i}$. Suppose $Y^{(i)} = \operatorname{sign}(w^{(i)}){w^{(i)}}^\top$, where $w^{(i)} = (w^{(i)}_1, \dots, w^{(i)}_{n_i})^\top$. We have

$$Y^{(i)}_{jk}Y^{(i)}_{k\ell}Y^{(i)}_{\ell j} = |w^{(i)}_j||w^{(i)}_k||w^{(i)}_l| > 0.$$

Therefore, every triad is positive in graph $G(Y)$. Since $G(Y)$ has exactly the same topology as $G(X)$, but just with the nodes reindexed. We conclude that any $X^* \in Q_{\text{influence}}$ is associated with a graph $G(X^*)$ composed by isolated complete subgraphs that satisfy social balance. This concludes the proof for statement (ii).

Now we prove that $Q_{\text{influence}}$ contains all the fixed points of system (9) in $\mathcal{S}^+_{\text{rs-symm}}$. We the notations θ_i and X_{j*,θ_i} in the same way as defined in the proof of Theorem 3.4, and, in addition, define X_{*j,θ_i} as the j-th column of X with all the k-th entry such that $k \notin \theta_i$ removed.

Now we prove by induction that $Q_{\text{influence}}$ is actually the set of all the fixed point of system (9). One can check that the trivial case of $n = 1$ is true. Suppose statement (i) holds for any system with dimension $\tilde{n} < n$.

For system (9) with dimension n, suppose $X \in \mathcal{S}^+_{\text{rs-symm}}$ is a fixed point of the system (9), i.e., $X = f_{\text{influence}}(X)$. For any given j,

$$\begin{aligned} |X_{ij}| &= \frac{1}{\|X_{i*}\|_1}\Big|\sum_k X_{ik}X_{kj}\Big| \leq \frac{1}{\|X_{i*}\|_1}\sum_k |X_{ik}||X_{kj}| \\ &\leq \max_k |X_{kj}|, \quad \text{for any } i, \end{aligned}$$

and there exists some i such that $|X_{ij}| = \max_k |X_{kj}|$. Now we discuss two cases that cover all the possible X's.

Case 1: $|X_{jj}| = \max_k |X_{kj}|$ and $|X_{ij}| < \max_k |X_{kj}|$ for any $i \neq j$. Since $X \in \mathcal{S}^+_{\text{rs-symm}}$, X is sign-symmetric,

$$|X_{jj}| = \frac{1}{\|X_{j*}\|_1}\sum_k |X_{jk}||X_{kj}| = \max_k |X_{kj}|.$$

Due to the second equality in the equations above, $|X_{kj}| = \max_\ell |X_{\ell j}|$ for any $k \in \theta_j$. Therefore, in Case 1, $i \notin \theta_j$ for any $i \neq j$, which in turn implies that

$X_{ji} = X_{ij} = 0$ for any $i \neq j$. As the consequence, there exists a permutation matrix P such that

$$PXP^\top = \begin{bmatrix} X_{jj} & \mathbb{0}_{n-1}^\top \\ \mathbb{0}_{n-1} & \tilde{X} \end{bmatrix},$$

where $\tilde{X}$ is an $(n-1) \times (n-1)$ matrix. Following the same line of argument in the Case 1 of the proof of Theorem 3.4, we conclude that $X \in Q_{\text{influence}}$.

Case 2: there exists $i \neq j$ such that $|X_{ij}| = \max_k |X_{kj}|$. For such i, we have $j \in \theta_i$. In addition, the equality below

$$|X_{ij}| = \frac{1}{\|X_{i*}\|_1}\Big|\sum_k X_{ik}X_{kj}\Big| = \max_k |X_{kj}|$$

leads to the following two results:

R.1) $\operatorname{sign}(X_{i*,\theta_i}) = \pm\operatorname{sign}(X_{*j,\theta_i}^\top)$;

R.2) $|X_{kj}| = \max_\ell |X_{\ell j}|$ for any $k \in \theta_i$.

Result R.2) and $j \in \theta_i$ lead to $|X_{jj}| = \max_\ell |X_{\ell j}|$. Therefore, for any $k \in \theta_j$, $|X_{kj}| = \max_\ell |X_{\ell j}|$. Moreover, since X is sign-symmetric, for any $k \notin \theta_j$, $X_{jk} = X_{kj} = 0$.

For any $i \in \theta_j$, since $|X_{ij}| = \frac{1}{\|X_{i*}\|_1}\Big|\sum_k X_{ik}X_{kj}\Big|$, $|X_{ij}| = \max_\ell |X_{\ell j}|$ implies that $|X_{kj}| = \max_\ell |X_{\ell j}|$ for any $k \in \theta_i$. Since $k \notin \theta_j$ leads to $k \notin \theta_i$, we have $\theta_i \subset \theta_j$.

For any given $i \in \theta_j$, since $X \in \mathcal{S}^+_{\text{rs-symm}}$, we know that $X_{ii} > 0$ and $X_{ji} > 0$. Apply the same argument for the j-th column in Case 2 to the i-th column, we conclude that $X_{ii} = \max_\ell |X_{\ell i}|$ and $|X_{ji}| = \max_\ell |X_{\ell i}|$, the latter of which in turn implies that $|X_{ki}| = \max_\ell |X_{\ell i}|$ for any $k \in \theta_j$. Moreover, since $X_{ii} = \max_\ell |X_{\ell i}|$ leads to $|X_{ki}| = \max_\ell |X_{\ell i}|$ for any $k \in \theta_i$ and $X_{ik} = X_{ki} = 0$ for any $k \notin \theta_i$, we have $\theta_j \subset \theta_i$. Since we already get $\theta_i \subset \theta_j$, we conclude that $\theta_j = \theta_i$ for any $i \in \theta_j$. In addition, due to Result R.1) and the facts that $\theta_i = \theta_j$ and X is sign-symmetric, we obtain that $\operatorname{sign}(X_{i*}, \theta_j) = \operatorname{sign}(X_{*i,\theta_j}^\top) = \pm\operatorname{sign}(X_{*j,\theta_j}^\top)$ for all $i \in \theta_j$.

Taking together all the results we have obtained for Case 2, we conclude that, for any given j in Case 2: (1) $|X_{kj}| = \max_\ell |X_{\ell j}|$ for any $k \in \theta_j$ and $X_{kj} = X_{jk} = 0$ for any $k \notin \theta_j$; (2) For any $i \in \theta_j$, $\theta_i = \theta_j$. In addition, $|X_{ki}| = \max_\ell |X_{\ell i}|$ for any $k \in \theta_j$ and $X_{ki} = X_{ik} = 0$ for any $k \notin \theta_j$; (3) For any $i \in \theta_j$, $\operatorname{sign}(X_{*i}) = \operatorname{sign}(X_{*j})$. Denote by $|\theta_j|$ the cardinality of θ_j and define the $|\theta_j| \times |\theta_j|$ matrix $X^{(\theta_j)} = \operatorname{sign}(w^{(\theta_j)})w^{(\theta_j)\top}$, where $w^{(\theta_j)} = X_{j*,\theta_j}^\top$. There exists a permutation matrix P such that

$$PXP^\top = \begin{bmatrix} X^{(\theta_j)} & \mathbb{0}_{|\theta_j|\times(n-|\theta_j|)} \\ \mathbb{0}_{(n-|\theta_j|)\times|\theta_j|} & \tilde{X} \end{bmatrix}.$$

Following the line of argument in Case 1, we have $X \in Q_{\text{influence}}$. This concludes the proof for statement (i). ☐

Remark 4.5. *By carefully examining the proof for Theorem 4.4, one can observe that $Q_{\text{influence}}$ is actually the set of all the fixed point of the map $f_{\text{influence}}$ in $\mathcal{S}^+_{\text{s-symm}}$. However, the set $Q_{\text{influence}}$ does not contain all the fixed points in $\mathcal{S}_{\text{nz-row}}$. For example, let $X = \alpha bb^\top$ for some $\alpha > 0$ and $b \in \{-1,+1\}^n$. Then, pick one $i \in \{1,\dots,n\}$ and set $X_{*i} = \mathbb{0}_n$. It can be easily verified that $X = f_{\text{influence}}(X)$ but $X \notin Q_{\text{influence}}$.*

Now we present the main results on the convergence of the appraisal network to social balance.

Theorem 4.6 (Convergence and social balance in the IbM)**.** *Consider the influence-based model given by equation* (9)*. The following statements hold:*

(i) Each fixed point of rank one in $Q_{\text{influence}}$ is locally stable.

For any $X(0) \in \mathcal{S}^+_{\text{rs-symm}}$ such that $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$,

(ii) there exists $X^* \in Q_{\text{influence}}$ of rank one such that $\lim_{t\to\infty} X(t) = X^*$, and

(iii) there exists $T > 0$ such that $G(X(t))$ satisfies social balance for all $t \geq T$.

Proof. We start by proving the following two claims. For any given $t_0 \geq 0$, if all the entries of $X(t_0)$ are non-zero and $G(X(t_0))$ satisfies social balance, then,

C.1) for any $t \geq t_0$, $G(X(t))$ satisfies social balance and $\operatorname{sign}(X(t)) = \operatorname{sign}(X(t_0))$;

C.2) there exists $w \in \mathbb{R}^n \setminus \{\mathbb{0}_n\}$, depending on $X(t_0)$, such that $X(t)$ converges to $\operatorname{sign}(w)w^\top$ as $t \to \infty$.

Claim C.1) is proved in the same way as in the proof of Theorem 3.6. For any $t \geq t_0$, since $G(X(t))$ satisfies social balance,

$$|X_{ij}(t+1)| = \frac{1}{\|X_{i*}(t)\|_1}\sum_{\ell=1}^n |X_{i\ell}(t)||X_{\ell j}(t)|,$$

for any i and j. Therefore, for a given j, the previous expression leads to the following two inequalities: $\min_\ell |X_{\ell j}(t+1)| \geq \min_\ell |X_{\ell j}(t)|$; $\max_\ell |X_{\ell j}(t+1)| \leq \max_\ell |X_{\ell j}(t)|$. Therefore, $\min_\ell |X_{\ell j}(t)|$ is non-decreasing and upper bounded by $\max_\ell |X_{\ell j}(t_0)|$, while $\max_\ell |X_{\ell j}(t)|$ is non-increasing and lower bounded by $\min_\ell |X_{\ell j}(t_0)|$, which in turn implies that there exists $0 < \underline{\omega} \leq \overline{\omega}$, depending on $X(t_0)$, such that

$$\lim_{t\to\infty} \min_\ell |X_{\ell j}(t)| = \underline{\omega}, \quad \text{and} \quad \lim_{t\to\infty} \max_\ell |X_{\ell j}(t)| = \overline{\omega}.$$

Suppose, at some time $t \geq t_0$, $\min_\ell |X_{\ell j}(t)| < \max_\ell |X_{\ell j}(t)|$ and $|X_{pj}(t)| = \max_\ell |X_{\ell j}(t)|$. For any i,

$$|X_{ij}(t+1)| \geq \frac{\sum_{k\neq p} |X_{ik}(t)| \min_\ell |X_{\ell j}(t)| + |X_{ip}(t)||X_{pj}(t)|}{\|X_{i*}(t)\|_1} > \min_\ell |X_{\ell j}(t)|.$$

Therefore, $\min_\ell |X_{\ell j}(t+1)| > \min_\ell |X_{\ell j}(t)|$ and, similarly, $\max_\ell |X_{\ell j}(t+1)| < \max_\ell |X_{\ell j}(t)|$. As the consequence, $\max_\ell |X_{\ell j}(t)| - \min_\ell |X_{\ell j}(t)|$ is strictly decreasing as long as $\min_\ell |X_{\ell j}(t)| < \max_\ell |X_{\ell j}(t)|$. Therefeore, $\underline{\omega} = \overline{\omega} > 0$, which implies that $|X_{ij}|$ converges for any i and j, and the magnitude of the entries of X in the same column converge to the same value. In addition, since $\operatorname{sign}(X(t)) = \operatorname{sign}(X(t_0))$ for all $t \geq t_0$, we conclude that

$X(t)$ converges to a matrix in the form $\mathrm{sign}(w)w^\top$. This concludes the proof for claim C.2).

Now we prove statement (i), i.e., each $\hat{X} \in Q_{\mathrm{influence}}$ with rank 1 is locally stable. Let $\hat{X} = \mathrm{sign}(w)w^\top$, where $|w| \succ \mathbb{0}_n$. For any matrix $\Delta \in \mathbb{R}^{n\times n}$ such that for any $k \in \{1,\dots,n\}$, $\delta_k = \max_i |\Delta_{ik}| < |w_k|$, we have $\mathrm{sign}(\hat{X}_{*k} + \Delta_{*k}) = \mathrm{sign}(\hat{X}_{*k})$. Due to claim C.1) and the proof of claim C.2), we know that, for $X(0) = \hat{X} + \Delta$, $X(t)$ satisfies that, for any $t \geq 0$,

(1) $\mathrm{sign}(X(t)) = \mathrm{sign}(X(0)) = \mathrm{sign}(\hat{X})$;
(2) $|w_k| - \delta_k \leq \min_i |X_{ik}(t)| \leq \max_i |X_{ik}(t)| \leq |w_k| + \delta_k$.

Therefore, for any i, $X_{ik}(t)$ is of the form $\alpha_{ik}(t)\mathrm{sign}(\hat{X}_{ik})$, where $0 < |w_k| - \delta_k \leq \alpha_{ik}(t) \leq |w_k| + \delta_k$. We have

$$\begin{aligned}\left\|X(t) - \hat{X}\right\|_{\max} &= \max_{ij} \left|\alpha_{ij}(t)\,\mathrm{sign}(\hat{X}_{ij}) - |w_j|\,\mathrm{sign}(\hat{X}_{ij})\right| \\ &= \max_{ij} \left|\alpha_{ij}(t) - |w_j|\right| \leq \delta,\end{aligned}$$

where $\delta = \max_k \delta_k$. Therefore, for any $\epsilon > 0$, there exists $\delta = \min\{\frac{\max_k |w_k|}{2}, \frac{\epsilon}{2}\}$ such that, for any $X(0)$ satisfying $\|X(0) - X^*\|_{\max} < \delta$, $\|X(t) - X^*\|_{\max} < \epsilon$ for any $t \geq 0$. That is, $\hat{X}$ is locally stable.

Now we proceed to prove the statements (ii) and (iii) of the theorem. For simplicity, denote $X^+ = f_{\mathrm{influence}}(X)$. Firstly, one can easily check that $f_{\mathrm{influence}}(X)$ is continuous for any $X \in \mathcal{S}^+_{\text{rs-symm}}$. Secondly, for any $X(0) \in \mathcal{S}^+_{\text{rs-symm}}$ and any $k \in \{1,\dots,n\}$, according to the proof of Proposition 4.3, $\|X_{*k}(t)\|_{\max} \leq \|X_{*k}(0)\|_{\max}$ for any $t \in \mathbb{Z}_{\geq 0}$. In addition, $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$ implies that there exists $\delta > 0$ and $\tilde{t} \in \mathbb{Z}_{\geq 0}$ such that $\min_{i,j} |X_{ij}(t)| \geq \delta/2$ for any $t \geq \tilde{t}$. Therefore, the set

$$G_c = \Big\{X \in \mathcal{S}^+_{\text{rs-symm}} \,\Big|\, \min_{i,j} |X_{ij}| \geq \delta/2, \text{ and, for any } k, \|X_{*k}\|_{\max} \leq \|X_{*k}(0)\|_{\max}\Big\}$$

is a compact subset of $\mathcal{S}^+_{\text{rs-symm}}$ and $X(t) \in G_c$ for any $t \geq \tilde{t}$. Thirdly, define $V_2(X_{*k}) = \|X_{*k}\|_{\max}$. The function V_2 is continuous on $\mathcal{S}^+_{\text{rs-symm}}$ and, according to the proof of Proposition 4.3, satisfies $V_2(X^+_{*k}) - V_2(X_{*k}) \leq 0$ for any $X \in \mathcal{S}^+_{\text{rs-symm}}$. According to the extended LaSalle invariance principle presented in Theorem 2 of [25], we conclude that, given any $X(0) \in \mathcal{S}^+_{\text{symm}}$ such that $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| = \delta$, $X(t)$ converges to the largest invariantset M of the set $E = \{X \in G_c \mid V_2(X^+_{*k}) - V_2(X_{*k}) = 0 \text{ for any } k\}$.

Now we characterize the largest invariant set M. For any $X \in M \subset E$ and $k \in \{1,\dots,n\}$, $V_2(X^+_{*k}) = V_2(X_{*k}) = \|X_{*k}\|_{\max}$. Suppose $|X^+_{ik}| = \max_\ell |X^+_{\ell k}|$. Since

$$\begin{aligned}|X^+_{ik}| &= \frac{1}{\|X_{i*}\|_1} \left|\sum_{\ell=1}^n X_{i\ell}X_{\ell k}\right| \\ &\leq \frac{1}{\|X_{i*}\|_1} \sum_{\ell=1}^n |X_{i\ell}||X_{\ell k}| \leq \max_\ell |X_{\ell k}|,\end{aligned} \tag{11}$$

we need all of these inequalities to hold with equality Since $X \in G_c \subset \mathcal{S}^+_{\text{rs-symm}}$ implies $|X_{j\ell}| > 0$, for any $j, \ell \in \{1,\dots,n\}$, X must satisfy that

(a) X_{i*} and X_{*k} have the same or opposite sign pattern, i.e., $\mathrm{sign}\,(X_{*k}) = \mathrm{sign}\,(X_{k*}) = \pm\,\mathrm{sign}\,(X_{i*})$,
(b) All entries of X_{*k} have magnitude $\|X_{*k}\|_{\max}$.

Therefore, for any $X \in E$ and k, there exist some i such that the aforementioned conditions (a) and (b) hold. Moreover, since the set M is invariant, $X \in M$ implies $X^+ \in M \subset E$, which in turn implies that, for any p, $|X^+_{pk}| = \left\|X^+_{*k}\right\|_{\max} = \|X_{*k}\|_{\max}$. Following the same argument on the conditions such that the inequalities (11) become strict equalities, we know that, for any p, $\mathrm{sign}\,(X_{p*}) = \pm\,\mathrm{sign}\,(X^\top_{*k})$ and $|X_{pk}| = \|X_{*k}\|_{\max}$ for any k. Using these relationships, we conclude that for any i and j, X_{i*} and X_{j*} must have the same or the opposite sign pattern, and that $|X_{ij}| = \|X_{*j}\|_{\max}$. Let $w = X^\top_{1*}$. Each row of X is thereby equal to either $w^\top$ or $-w^\top$. Therefore, X is of the form $X = cw^\top$, where $c \in \{-1,1\}^n$. Moreover, since all the diagonal entries of X are positive, the column vector c must satisfy $c_i w_i = 1$ for any i, which implies $c = \mathrm{sign}(w)$. Therefore, $X = \mathrm{sign}(w)w^\top$. Thus, since any matrix X of the form $\mathrm{sign}(w)w^\top$, with $|w| \succ \mathbb{0}_n$, is a fixed point of system (9), we conclude that

$$\begin{aligned}M = \{&X = \mathrm{sign}(w)w^\top \mid \delta/2 \leq w_i \leq \|X(0)\|_{\max}, \\ &w \in \mathbb{R}^n \setminus \{\mathbb{0}_n\}, \text{ for any } i \in \{1,\dots,n\}\},\end{aligned}$$

which is a compact subset of $\mathcal{S}^+_{\text{rs-symm}}$. Following the same line of argument in the proof of Theorem 3.6, we conclude that there exists $\epsilon > 0$ such that any X in the neighbor set $\mathcal{U}(M,\epsilon)$ satisfies social balance.

Since $X(t) \to M$ as $t \to \infty$, there exists $T \in \mathbb{Z}_{\geq 0}$ such that $X(t) \in \mathcal{U}(M,\epsilon)$ for any $t \geq T$. Therefore, $X(t)$ satisfies social balance for any $t \geq T$, which proves statement (iii). Moreover, according to claim C.2) and Theorem 4.4, there exists $X^* = \mathrm{sign}(w)w^\top$, which is a matrix in the set $Q_{\mathrm{influence}}$ with rank one, such that $X(t) \to X^*$ as $t \to \infty$, concluding the proof for statement (ii). □

Extensive simulation results indicate that, under generic initial conditions $X(0) \in \mathcal{S}^+_{\text{rs-symm}}$, every entry of $|X(t)|$ is uniformly strictly lower bounded from 0 for all $t > 0$. This numerical result is further discussed in Section 5.

5 Further discussion and numerical simulations

5.1 Generic convergence to rank-one appraisal matrix

According to Theorem 3.6, for any $X(0) \in \mathcal{S}^+_{\text{s-symm}}$ such that $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$, in the homophily-based model, the solution $X(t)$ converges to some rank-one matrix of the form $\alpha bb^\top$. In this subsection, we use the Monte Carlo method to numerically verify that $\liminf_{t\to\infty} \min_{i,j} |X_{ij}(t)| > 0$ holds for generic initial conditions in $\mathcal{S}_{\text{nz-row}}$. By generic initial condition, we mean each of $X(0)$'s entries is selected independently and uniformly at random from a support of positive measure. We consider the support to be $[-a, a]$, where $a > 0$. For any randomly generated $X(0) \in \mathcal{S}_{\text{nz-row}}$, define the random variable $Z : \mathcal{S}_{\text{nz-row}} \to \{0, 1\}$ as

(i) There exists $\delta > 0$ such that $\min_{i,j} |X_{ij}(t)| \geq \delta$ for any $t \in \{100, \dots, 10000\}$;

(ii) $Z(X(0)) = 0$ otherwise.

Let $p = \mathbb{P}[Z(X(0)) = 1]$. For N independent random samples $Z_1, \dots, Z_N$, in each of which $X(0) \in \mathcal{S}_{\text{nz-row}}$ is a generic initial condition, define $\hat{p}_N = \sum_{i=1}^N Z_i/N$. For any accuracy $1 - \varepsilon \in (0, 1)$ and confidence level $1-\xi \in (0,1)$, $|\hat{p}_N - p| < \varepsilon$ with probability greater than $1-\xi$ if the Chernoff bound is satisfied: $N \geq \frac{1}{2\varepsilon^2} \log \frac{2}{\xi}$. For $\varepsilon = \xi = 0.01$, the bound is satisfied by $N = 27000$. We ran the 27000 independent simulations of the homophily-based model with $n = 8$ and $a = 20$, and found that $\hat{p} = 1$. Then, we conclude that for any generic initial condition $X(0) \in \mathcal{S}_{\text{nz-row}}$, with 99% confidence level, there is at least 0.99 probability that every entry of $|X(t)|$ is lower bounded by a positive scalar for all $t \in \{100, \dots, 10000\}$.

The Monte Carlo method under the same settings is applied to the influence-based model, except that now the generic initial conditions $X(0) \in \mathcal{S}^+_{\text{rs-symm}} \subset \mathbb{R}^{n\times n}$ is generated by the following steps: 1) Randomly and independently generate the diagonal and the upper triangular entries of a matrix $\hat{X} \in \mathbb{R}^{n\times n}$, according to some uniform distribution; 2) Let $\hat{X}_{ij} = \hat{X}_{ji}$ for any $i > j$; 3) Randomly and independently generate the entries of a $n \times 1$ vector γ, according to some uniform distribution with some positive interval as the support; 4) Let $X(0) = \operatorname{diag}(\gamma)\hat{X}$. Not surprisingly, we obtained the same results as the homophily-based model. That is, for any generic initial condition $X(0) \in \mathcal{S}_{\text{nz-row}}$, with 99% confidence level, there is at least 0.99 probability that every entry of $|X(t)|$ is uniformly strictly lower bounded from 0 for all $t \in \{100, \dots, 10000\}$.

5.2 *Multi-clique social balance and perturbation*

1) The initial conditions leading to multi-clique social balance: Despite the generic convergence to complete graphs, for both the homophily-based and the influence-based models, there exists some special initial conditions leading to the *multi-clique social balance*. By *clique* we mean an isolated subgraph (Negative links are counted as links), and by multi-clique social balance we mean that the appraisal network consists of multiple cliques and each of them satisfies social balance. For example, let

$$X(0) = \begin{bmatrix} 1 & 1 & 1 \\ 0.5 & -1 & 0.5 \\ -0.5 & 1 & -0.5 \end{bmatrix}, \quad \hat{X}(0) = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & -2 \\ 0 & -2 & -1 \end{bmatrix}.$$

For the homophily-based model, the initial condition $X(0)$ eventually results in the formation of two isolated cliques with node sets $\{1\}$ and $\{2, 3\}$ respectively. For both the homophily-based and the influence-based models, the initial condition $\hat{X}(0)$ results in the formation of two isolated cliques with node sets $\{2\}$ and $\{1, 3\}$.

2) Multi-clique social balance under perturbation: For the homophily model, extensive simulation observations indicate that the multi-clique social balance is unstable under perturbations. With some links added to the multi-clique structurally balanced network, the perturbed network eventually converges to a single-clique structurally balanced state, see Fig. 1 as a concrete example. The following two examples illustrate the behavior of multi-clique social balance under perturbations.

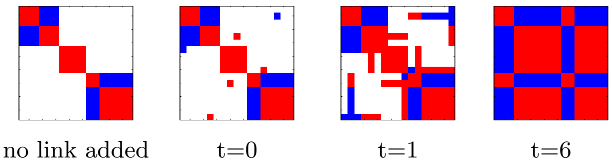


Fig. 1. Visualization of the evolution of the appraisal matrix under perturbations. For each entry, the red color indicates a positive appraisal, while the blue color indicates a negative one. The white color indicate no appraisal. The appraisal network has 17 nodes and is initially in a multi--clique structurally balanced state with three isolated balanced cliques. With 6 links (4 positive and 2 negative links) added to the network, the appraisal network evolves to a single-clique structurally balanced state after 6 iterations.

Example 1: (Globalization of local conflicts) Consider the appraisal network with two isolated cliques. Each clique is made up of two antagonistic factions. Clique 1 has two factions, with node sets V_1 and V_2 respectively, and Clique 2 also has two factions, with node sets V_3 and V_4 respectively. Suppose one link with weight ϵ is added from one node in V_1 to one node in V_3. We find that the perturbed appraisal network always recovers to a complete and structurally balanced network such that:

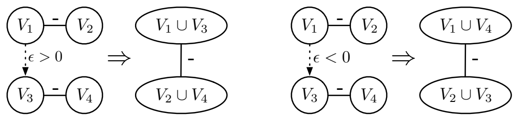
(i) It is composed of two antagonistic factions;

(ii) If $\epsilon > 0$, the two factions are $V_1 \cup V_3$ and $V_2 \cup V_4$;

(iii) If $\epsilon < 0$, the two factions are $V_1 \cup V_4$ and $V_2 \cup V_3$.

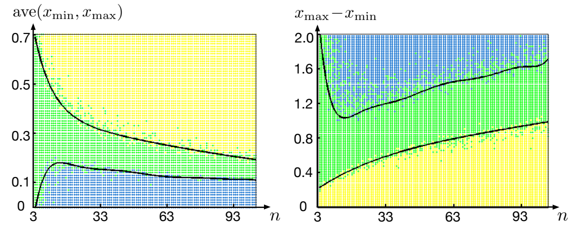
Figure 2 visualizes the behavior described above. In reality, such behavior could be interpreted as the escalation of local conflicts. In the example above, the two original conflicting relations, i.e., V_1 v.s. V_2 and V_3 v.s. V_4, are escalated into global conflicts between two reunified factions $V_1 \cup V_2$ and $V_3 \cup V_4$, once a node in V_1 builds a connection with V_3. One real example of such phenomena is the formation of the globalized conflicts between the Axis and the Ally in World War II, after the Nazi German allied with the Imperial Japan.

Example 2: (Competition for ally and mediation of con-



(a) $\epsilon > 0$ (b) $\epsilon < 0$

Fig. 2. Visual illustration of the behavior of the multi-clique social balance with the addition of one inter-clique link.



(a) Case 1 (b) Case 2

Fig. 3. Formation of factions under different initial condition distributions. The blue color indicates the presence of two factions in all the 30 random samples, while the yellow color indicates the presence of one factions in all of the samples. The green color indicates any other case.

flicts) Consider an appraisal network with two isolated cliques: Clique 1 with two antagonistic factions $V_1 = \{1, \dots, n_1\}$ and $V_2 = \{n_1+1, \dots, n_1+n_2\}$, and Clique 2 with only one faction $V_3 = \{n_1+n_2+1, \dots, n_1+n_2+n_3\}$. Suppose the appraisal matrix associated with Clique 1 is given by $\alpha b b^\top$, where $b = (\mathbb{1}_{n_1}^\top, -\mathbb{1}_{n_2}^\top)^\top$, and $\alpha > 0$ represents the sentiment strength inside Clique 1. Similarly, the appraisal matrix associated with Clique 2 is given by $\hat{\alpha}\hat{b}\hat{b}^\top$, where $\hat{b} = \mathbb{1}_{n_3}$ and $\hat{\alpha} > 0$ represents the sentiment strength inside Clique 2. Imagine then that both cliques V_1 and V_2 would aim to ally with V_3 in order to grow the number of their members. Accordingly, suppose that, in order to ally with V_3, each node in V_1 builds a bilateral link with each node in V_3, with link weight $\epsilon_1 > 0$, while each node in V_2 builds a bilateral link with each node in V_3 with weight $\epsilon_2 > 0$. With all these links added, the associated appraisal matrix takes the following form:

$$X(0) = \begin{bmatrix} \alpha \mathbb{1}_{n_1}\mathbb{1}_{n_1}^\top & -\alpha \mathbb{1}_{n_1}\mathbb{1}_{n_2}^\top & \epsilon_1 \mathbb{1}_{n_1}\mathbb{1}_{n_3}^\top \\ -\alpha \mathbb{1}_{n_2}\mathbb{1}_{n_1}^\top & \alpha \mathbb{1}_{n_2}\mathbb{1}_{n_2}^\top & \epsilon_2 \mathbb{1}_{n_2}\mathbb{1}_{n_3}^\top \\ \epsilon_1 \mathbb{1}_{n_3}\mathbb{1}_{n_1} & \epsilon_2 \mathbb{1}_{n_3}\mathbb{1}_{n_1}^\top & \hat{\alpha} \mathbb{1}_{n_3}\mathbb{1}_{n_3}^\top \end{bmatrix}.$$

Along the evolution of $X(t)$ determined by $X(0)$, we obtain the following numerical results.

(i) If $\epsilon_1 n_1 > \epsilon_2 n_2$, i.e., faction V_1 takes greater effort than V_2 in allying with V_3, then faction V_1 gains at least one ally, either V_2 or V_3, which is a situation more in favor of V_1 than V_2. Moreover, the following conditions:

$$\epsilon_1 n_1 - \epsilon_2 n_2 \geq \hat{\alpha}\epsilon_2 n_3/\alpha \quad \text{and} \quad \epsilon_1 \epsilon_2 n_3 \leq \alpha^2(n_1 + n_2)$$

guarantee that V_1 ally with V_3; This argument also holds when all the subscripts 1 and 2 are switched;

(ii) If $\epsilon_1\epsilon_2 n_3 \leq \alpha^2(n_1+n_2)$, then V_3 eventually gains at least one ally. That is, V_3 avoids the situation in which V_1 and V_2 end up allying with each other against V_3;

(iii) Any of the following conditions guarantees the non-existence of any negative link in the asymptotic state of the appraisal network: (1) $\epsilon_1\epsilon_2 n_3 \geq \alpha^2(n_1+n_2)$ and $\epsilon_1 n_1 - \epsilon_2 n_2 = 0$; (2) $\epsilon_1\epsilon_2 n_3 \geq \alpha^2(n_1+n_2)$ and $0 < \epsilon_1 n_1 - \epsilon_2 n_2 \leq \epsilon_2 \hat{\alpha} n_3$; (3) $\epsilon_1\epsilon_2 n_3 \geq \alpha^2(n_1+n_2)$ and $0 < \epsilon_2 n_2 - \epsilon_1 n_1 \leq \epsilon_1 \hat{\alpha} n_3$. Notice that the inequality

$$\epsilon_1\epsilon_2 n_3 \geq \alpha^2(n_1+n_2)$$

is required for all the three sufficient conditions. The right-hand side of the inequality above reflects the "scale" of the conflicts between factions V_1 and V_2, while the left-hand side is V_1 and V_2's average efforts in allying with V_3, multiplied by the size of V_3. From the three sufficient conditions, we learn that, the larger the size of V_3, the more capable it is of mediating the conflicts between V_1 and V_2. In addition, V_1 and V_2's strong willingness to ally with V_3, as well as the sentiment strength inside V_3, i.e., $\hat{\alpha}$, also help mediate the conflicts.

5.3 *Distribution of initial conditions and formation of factions in the homophily-based model*

We investigate numerically, for the homophily-based model, the relation between the initial condition distribution and the formation of factions. The question of interest is whether the appraisal network evolves to only one faction or two antagonistic factions. We randomly and independently sample the entries of $X(0)$ from the uniform distribution with support $[x_{\min}, x_{\max}]$. The quantity $x_{\max} - x_{\min}$ indicates how spread out are the possible values taken by the initial appraisals, while $\text{ave}(x_{\min}, x_{\max}) = (x_{\max}+x_{\min})/2$ indicates how the initial appraisals are biased towards being positive. Given $[x_{\min}, x_{\max}]$, we independently generate 30 random samples of the initial condition $X(0)$ and count how many factions appear at $X(500)$. The simulations are conducted under two different set-ups:

Case 1: We set $x_{\max} - x_{\min} = 2$ and change the values of $\text{ave}(x_{\min}, x_{\max})$ and the number of agents. Since any $X(0)$ and $-X(0)$ lead to the same $X(1)$ and $X(t)$ thereafter, we only consider different values of $\text{ave}(x_{\min}, x_{\max}) \geq 0$. Figure 3(a) shows that, for fixed network size, the smaller the value of $\text{ave}(x_{\min}, x_{\max})$, the more likely is to find two antagonistic factions; for fixed value of $\text{ave}(x_{\min}, x_{\max})$, the larger the network size, the more likely that only one faction emerges.

Case 2: We set $\text{ave}(x_{\min}, x_{\max}) = 1$ and change $x_{\max} - x_{\min}$. Figure 3(b) shows that, for fixed network size n, the larger $x_{\max} - x_{\min}$, the more likely to find two antagonistic factions; For fixed $x_{\max} - x_{\min}$, the larger the network size, the more likely that only one faction emerges.

6 Conclusion

This paper proposes two novel discrete-time dynamical models for the bounded evolution of interpersonal appraisal networks towards social balance. Under a technical condition, theoretical analysis shows that both models exhibit asymptotic convergence to structurally balanced networks. Each model uses different social updating mechanisms for updating the appraisals and, as a result, the asymptotic balanced states are qualitatively different between the two models. Numerical study indicates how the final emergence of factions in the social network is sensitive to the initial distribution of appraisals among its agents. Moreover, our models admits the existence of two or more isolated cliques in the final structure of the evolved social network, and simulation results reveal interesting sociological phenomena when they are under certain classes of perturbations. Possible future research directions include a better understanding of the influence-based model for arbitrary initial conditions, a validation of the proposed models with laboratory and/or field data, the study of asynchronous models with pairwise updates, and the study of conditions and cases in which one socio-psychological mechanism dominates the other.

References

[1] T. Antal, P. L. Krapivsky, and S. Redner. Dynamics of social balance on networks. *Physical Review E*, 72(3):036121, 2005. doi:10.1103/PhysRevE.72.036121.

[2] T. Antal, P.L. Krapivsky, and S. Redner. Social balance on networks: The dynamics of friendship and enmity. *Physica D: Nonlinear Phenomena*, 224(1):130–136, 2006. doi:10.1016/j.physd.2006.09.028.

[3] D. Cartwright and T. C. Gleason. The number of paths and cycles in a digraph. *Psychometrika*, 31:179–199, 1966. doi:10.1007/BF02289506.

[4] D. Cartwright and F. Harary. Structural balance: a generalization of Heider's theory. *Psychological Review*, 63(5):277, 1956. doi:10.1037/h0046049.

[5] J. A. Davis. Clustering and structural balance in graphs. *Human Relations*, 20:181–187, 1967. doi:10.1177/001872676702000206.

[6] W. de Nooy. The sign of affection: Balance-theoretic models and incomplete signed digraphs. *Social Networks*, 21(3):269–286, 1999. doi:10.1016/S0378-8733(99)00012-X.

[7] P. Doreian and D. Krackhardt. Pre-transitive balance mechanisms for signed networks. *Journal of Mathematical Sociology*, 25(1):43–67, 2001. doi:10.1080/0022250X.2001.9990244.

[8] P. Doreian and A. Mrvar. A partitioning approach to structural balance. *Social Networks*, 18:149–168, 1996. doi:10.1016/0378-8733(95)00259-6.

[9] D. Easley and J. Kleinberg. *Networks, Crowds, and Markets: Reasoning About a Highly Connected World*. Cambridge University Press, 2010.

[10] N. E. Friedkin and E. C. Johnsen. *Social Influence Network Theory: A Sociological Examination of Small Group Dynamics*. Cambridge University Press, 2011.

[11] R. Guha, R. Kumar, P. Raghavan, and A. Tomkins. Propagation of trust and distrust. In *Int. Conference on World Wide Web*, New York, USA, May 2004. doi:10.1145/988672.988727.

[12] F. Harary. On the notion of balance of a signed graph. *Michigan Mathematical Journal*, 2(2):143–146, 1953. doi:10.1307/mmj/1028989917.

[13] F. Heider. Social perception and phenomenal causality. *Psychological Review*, 51(6):358–374, 1944. doi:10.1037/h0055425.

[14] F. Heider. Attitudes and cognitive organization. *The Journal of Psychology*, 21(1):107–112, 1946. doi:10.1080/00223980.1946.9917275.

[15] N. M. Henley, R. B. Horsfall, and C. B. De Soto. Goodness of figure and social structure. *Psychological Review*, 76:194–204, 1969. doi:10.1037/h0027358.

[16] P. Jia, N. E. Friedkin, and F. Bullo. The coevolution of appraisal and influence networks leads to structural balance. *IEEE Transactions on Network Science and Engineering*, 3(4):286–298, 2016. doi:10.1109/TNSE.2016.2600058.

[17] M. Kim and K. S. Candan. SBV-Cut: Vertex-cut based graph partitioning using structural balance vertices. *Data & Knowledge Engineering*, 72:285–303, 2012. doi:10.1016/j.datak.2011.11.004.

[18] M. G. King. Structural balance, tension, and segregation in a university group. *Human Relations*, 17:221–225, 1964. doi:10.1177/001872676401700303.

[19] K. Kułakowski, P. Gawroński, and P. Gronek. The Heider balance: A continuous approach. *International Journal of Modern Physics C*, 16(05):707–716, 2005. doi:10.1142/S012918310500742X.

[20] J. Kunegis, S. Schmidt, A. Lommatzsch, J. Lerner, E. W. De Luca, and S. Albayrak. Spectral analysis of signed graphs for clustering, prediction and visualization. In *SIAM International Conference on Data Mining*, Columbus, USA, April 2010. doi:10.1137/1.9781611972801.49.

[21] P. F. Lazarsfeld and R. K. Merton. Friendship as a social process: A substantive and methodological analysis. In M. Berger and T. Abel, editors, *Freedom and Control in Modern Society*, volume 18, pages 18–66. Van Nostrand, 1954.

[22] M. Malekzadeh, M. Fazli, P. Jalaly Khalidabadi, H. R. Rabiee, and M. A. Safari. Social balance and signed network formation games. In *Proceedings of 5th KDD workshop on Social Network Analysis (SNA-KDD)*, San Diego, USA, August 2011.

[23] S. A. Marvel, J. Kleinberg, R. D. Kleinberg, and S. H. Strogatz. Continuous-time model of structural balance. *Proceedings of the National Academy of Sciences*, 108(5):1771–1776, 2011. doi:10.1073/pnas.1013213108.

[24] H. B. McDonald and R. Rosecrance. Alliance and structural balance in the international system: A reinterpretation. *Journal of Conflict Resolution*, 29(1):57–82, 1985. doi:10.1177/0022002785029001004.

[25] W. Mei and F. Bullo. LaSalle invariance principle for discrete-time dynamical systems: A concise and self-contained tutorial, 2017. arXiv preprint. URL: https://arxiv.org/abs/1710.03710.

[26] M. Moore. An international application of Heider's balance theory. *European Journal of Social Psychology*, 8:401–405, 1978. doi:10.1002/ejsp.2420080313.

[27] F. Radicchi, D. Vilone, S. Yoon, and H. Meyer-Ortmanns. Social balance as a satisfiability problem of computer science. *Physical Review E*, 75:026106, 2007. doi:10.1103/PhysRevE.75.026106.

[28] C. M. Rawling and N. E. Friedkin. The structural balance theory of sentiment networks: Elaboration and test. *American Journal of Sociology*, 123(2):510–548, 2017. doi:10.1086/692757.

[29] V. A. Traag, P. Van Dooren, and P. De Leenheer. Dynamical models explaining social balance and evolution of cooperation. *PLOS ONE*, 8(4):e60063, 2013. doi:10.1371/journal.pone.0060063.

[30] A. van de Rijt. The micro-macro link for the theory of structural balance. *Journal of Mathematical Sociology*, 35(1-3):94–113, 2011. doi:10.1080/0022250X.2010.532262.

[31] S. Wongkaew, M. Caponigro, K. Kułakowski, and A. Borzì. On the control of the Heider balance model. *The European Physical Journal Special Topics*, 224(17):3325–3342, Dec 2015. doi:10.1140/epjst/e2015-50087-9.