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Social Power Dynamics over Switching and Stochastic Influence Networks

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***Abstract*—The DeGroot-Friedkin (DF) model is a recently-proposed dynamical description of the evolution of individuals' self-appraisal and social power in a social influence network. Most studies of this system and its variations have so far focused on models with a time-invariant influence network.**

This paper proposes novel models and analysis results for DF models over switching influence networks, and with or without environment noise. First, for a DF model over switching influence networks, we show that the trajectory of the social power converges to a ball centered at the equilibrium reached by the original DF model. For the DF model with memory on random interactions, we show that the social power converges to the equilibrium of the original DF model almost surely. Additionally, this paper studies a DF model which contains random interactions and environment noise, and has memory on the self-appraisal. We show that such a system converges to an equilibrium or a set almost surely. Finally, as a by-product, we provide novel results on the convergence rates of the original DF model and convergence results for a continuous-time DF model.

***Index Terms*—DeGroot-Friedkin model, stochastic approximation, social networks, social power evolution, opinion dynamics**

I. Introduction

Models for the dynamics of opinions and social power: Over the past decades, social networks have drawn tremendous attention from both academia and industry. The study of opinion dynamics aims to characterize and understand how individuals' opinions form and evolve over time through interactions with their peers. The first mathematical model for opinion dynamics was proposed by French in [9] with further refinements by Harary [16]. This model is based on distributed opinion averaging and is now widely referred to as the DeGroot model [8]. Closely-related important variations include the Friedkin-Johnsen affine model [12], [13] and the Hegselmann-Krause bounded-confidence model [17].

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Recently, by combining the DeGroot model of opinion dynamics and a reflected appraisal mechanism [6], [10], Jia *et al.* [20] proposed a DeGroot-Friedkin (DF) model to describe the evolution of individuals' self-appraisal and social power (i.e., influence centrality) along an issue sequence. This influence network model combines two steps. First, individuals update their opinions on each issue as in the DeGroot averaging model, where an interaction matrix characterizes the relative interpersonal influence among the individuals. Second, based on the opinion averaging outcome, individuals update their self-appraisal via a reflection appraisal mechanism. In other words, individuals' self-appraisals on the current issue are elevated or dampened depending upon their influence centrality (i.e., social power) on the prior issue. Under an assumption that the relative interaction matrix is constant, irreducible, and row-stochastic, Jia *et al.* [20] proved the convergence of individuals' self-appraisals in the DF model.

Since its introduction, the DF model has attracted a lot of interest. Two articles study the DF model with varying assumptions on the interaction matrix. First, Jia *et al.* [19] extend the convergence results to the setting of reducible interaction matrices. Second, Ye *et al.* [27] show that, if the interaction matrix switches in a periodic manner, then individuals' self-appraisals have a periodic solution. Additionally, several other dynamical models have been proposed and analyzed. Mirtabatabaei *et al.* extended the DF model to include stubborn agents who have attachment to their initial opinions in [22]. Xu *et al.* [26] proposed a modified DF model, where the social power is updated without waiting for the opinion consensus on each issue, i.e., the local estimation of social power is truncated; a complete analysis of convergence and equilibria properties was given when the interaction matrix is doubly stochastic. Considering time-varying doubly stochastic influence matrix, Xia *et al.* [25] investigated the convergence rate of the modified DF model, which was proven to converge exponentially fast. A continuous-time self-appraisal model was introduced by Chen *et al.* in [4]. It is worth noting that, all the above existing works assume the interaction matrix either is constant or has some special time-varying structure, like double stochasticity or periodicity [25], [27].

Empirical evidence motivating new models: Empirical evidence in support of the DeGroot model for opinion dynamics is provided in [2] and support of the reflected appraisal mechanism over issue sequences is provided in [10], [11]. The data in [11] establishes that (i) the interaction matrix in the influence network is not constant along the issue sequence, (ii) the reflected appraisal mechanism is indeed observed, whereby prior influence centrality predict future self-appraisals, and

(iii) a predictor of self-appraisal that is even better than prior social power is cumulative prior social power (i.e, the average of prior influence centrality scores over the issue sequence). In other words, individuals learn “their place in a social group” via an accumulation of experiences rather than over a single episode. It is worth mentioning that a similar learning mechanism based on the averages of prior outcomes is widely adopted [18] in game theory and economics to model human behavior.

Motivated by the available empirical evidence, this paper proposes and characterizes several DF models subject to switching influence networks, and also the environment noise. Additionally, we incorporate memory in our models so that, for example, individuals may update their self-appraisal based on cumulative prior influence centrality.

Useful tools: In what follows, we adopt useful stochastic models and analysis methods from the field of stochastic approximation; these models and methods were originally aimed at optimization and root-finding problems with noisy data. The earliest methods of stochastic approximation were proposed by Robbins and Monro [23] and aimed to solve a root finding problem. During more than sixty years of development, stochastic approximation methods have attracted a lot of interest due to many applications such as the study of reinforcement learning [24], consensus protocols in multi-agent systems [3], and fictitious play in game theory [18]. For general noisy processes and algorithms, a very powerful stochastic approximation tool is the so-called “ordinary differential equations (ODE) method” (see Chapter 5 in [21]), which transforms the analysis of asymptotic properties of a discrete-time stochastic process into the analysis of a continuous-time deterministic process.

Statement of contributions: This paper proposes and analyzes multiple novel DF models with varying assumptions on interaction and memory. First, we investigate a DF model with switching interactions, i.e., we assume that the interpersonal interaction matrix is time-varying. Under such a model, we establish convergence results under both relevant settings, i.e., when the digraph corresponding to the interaction matrix is or is not a star graph. In the former case, the trajectory of social power converges to autocracy; in the latter case, the social power converges into a ball centered at the equilibrium point reached by the original DF model. Second, as a by-product of this analysis, we establish convergence rates for the original DF model for both settings (with or without star topology).

Third, we consider a DF model with memory on the random interaction matrix. In such a model the self-appraisal of each individual is updated in the same manner as that in the original DF model, but we assume the individual has memory on the interaction weights assigned to others. For such a model we show, using a stochastic approximation method, that the impact of the stochasticity on the interaction matrix disappears asymptotic. In other words, we prove that, for this model, the social power converges to the same equilibrium point reached by the original DF model almost surely.

Fourth, we study a DF model which contains random interactions and environment noise, and has memory on the self-appraisal. In this model, each individual remembers his/her self-appraisal of last time (modeling for example the concept of cumulative prior social power). While this model is quite different from the DF model with memory on the interaction matrix, we again establish using stochastic approximation methods (and under certain technical conditions) that the adoption of memory leads to a vanishing effect of switch and noise and that the system converges to an equilibrium point or a set almost surely. Fifth and finally, we also propose and characterize a novel continuous-time DF model.

Organization: We review the original DF model in Section II. Section III contains the convergence rate results for the DF model and a new continuous-time DF model. We propose the DF models with switching and stochastic interactions in Section IV. A DF model with random interactions, environment noise, and self-appraisal memory is analyzed in Section V. Section VI concludes the paper.

Notations: A nonnegative matrix is row-stochastic (resp. doubly stochastic) if its row sums are equal to 1 (resp., its row and column sums are equal to 1). The digraph $\mathcal{G}(M)$ associated to a nonnegative matrix $M = \{m_{ij}\}_{i,j\in\{1,\dots,n\}}$ is defined as follows: the node set is $\{1,\dots,n\}$; there is a directed edge (i,j) from node i to node j if and only if $m_{ij} > 0$. The nonnegative matrix M is irreducible if its associated digraph is strongly connected. The n-simplex Δ_n is $\{x \in [0,1]^n \mid \sum_{i=1}^n x_i = 1\}$ and its interior is $\Delta_n^o = \{x \in (0,1)^n \mid \sum_{i=1}^n x_i = 1\}$. Let $\mathbf{e}_i \in \mathbb{R}^n$ be the row vector whose i-th component is 1 and whose other components are 0. For $v \in \mathbb{R}^n$, let $\|v\|_\infty := \max_{1\le i\le n} |v_i|$ denote its infinity norm. For a matrix $M \in \mathbb{R}^{n\times n}$, let $\|M\|_{\max} := \max_{1\le i,j\le n} |M_{ij}|$ denote the maximum norm. Given two sequences of positive numbers $\{g_1(t)\}$ and $\{g_2(t)\}$, we say $g_1(t) = o(g_2(t))$ if $\lim_{t\to\infty} g_1(t)/g_2(t) = 0$, and $g_1(t) = O(g_2(t))$ if there exist two positive constants a and t_0 such that $g_1(t) \le a g_2(t)$ for all $t \ge t_0$. Let $\mathbb{Z}_{\ge 0}$ denote the set of nonnegative integers.

II. Review of original DF model

The original DF model was proposed by Jia *et al.* in [20]. The model considers a group of $n \ge 3$ individuals who discuss a sequence of issues under the DeGroot model. The column vector $y(s,t) \in \mathbb{R}^n$ denoting the individuals' opinions over issue s evolves according to the following formula

$$y(s,t+1) = W(s)y(s,t),$$

where $W(s) \in \mathbb{R}^{n\times n}$ is a row-stochastic influence matrix over issue s. Then, for each individual i, her opinion is updated via a convex combination

$$y_i(s,t+1) = W_{ii}(s)y_i(s,t) + \sum_{j=1,j\neq i}^{n} W_{ij}(s)y_j(s,t).$$

Here $W_{ii}(s)$ denotes the self-appraisal of individual i, and $W_{ij}(s) = (1 - W_{ii}(s))C_{ij}$ for all $i \neq j$, where the coefficient C_{ij} is the relative interpersonal weight that individual i accords to individual j. Throughout this paper, the square matrix C is a *relative interaction matrix*, that is row-stochastic and zero-diagonal.

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Denoting $W_{ii}(s)$ by $x_i(s)$ for simplicity as in [20], the influence matrix $W(s)$ can then be decomposed as

$$W(s) = \text{diag}[x(s)] + (I_n - \text{diag}[x(s)])C.$$

If $W(s)$ is an irreducible row-stochastic matrix, according to Perron-Frobenius theorem $W(s)$ has a unique *dominant left eigenvector* $\pi(W(s)) \in \mathbb{R}^n$, which is a row vector satisfying $\pi(W(s)) = \pi(W(s))W(x(s))$, $\pi_i(W(s)) \geq 0$ for all $i \in \{1,\dots,n\}$, and $\sum_{i=1}^n \pi_i(W(s)) = 1$. Under some assumptions on C, the opinion vector $y(s,t)$ asymptotically reaches consensus, i.e., $\lim_{t\to\infty} y(s,t) = [\pi(W(s))y(s,0)]\mathbf{1}_n$.

Let $x(s) := (x_1(s),\dots,x_n(s))$ be a row vector. To deal with the evolution of $x(s)$ across issues, a reflected appraisal mechanism is adopted as follows,

$$x(s+1) = \pi(W(s)).$$

The meaning of this equation is that individuals' self weights on current issue are their relative influence centrality (i.e., social power) over prior issue. In summary, given an interaction matrix C, the *DF model* is given by [20]

$$\begin{cases} W(x(s)) = \text{diag}[x(s)] + (I_n - \text{diag}[x(s)])C, \\ x(s+1) = \pi(W(x(s))). \end{cases} \quad (1)$$

We adopt the same assumptions on C as in [20], i.e., we assume that C is irreducible. According to the Perron-Frobenius theorem, C has a unique dominant left eigenvector $c := (c_1,\dots,c_n)$ with $c_i > 0$ for all $i \in \{1,\dots,n\}$, and $\sum_{i=1}^n c_i = 1$.

Lemma II.1 (Lemma 2.2 in [20]: Explicit formulation of DF model): Assume $n \geq 2$ and $C \in \mathbb{R}^{n\times n}$ is a row-stochastic, irreducible, and zero-diagonal matrix whose dominant left eigenvector is c. Then, for any $x \in \Delta_n$, the dominant left eigenvector of the matrix $\text{diag}[x] + (I_n - \text{diag}[x])C$ is

$$\begin{cases} \mathbf{e}_i & \text{if } x = \mathbf{e}_i \text{ for all } i = 1,\dots,n, \\ \left(\frac{c_1}{1-x_1},\dots,\frac{c_n}{1-x_n}\right)/\sum_{i=1}^n \frac{c_i}{1-x_i} & \text{otherwise.} \end{cases}$$

Let $\mathcal{G}(C)$ be the digraph associated with C. The dynamics of the DF model (1) depend on the topology of $\mathcal{G}(C)$ and a certain topology, namely the star topology or star network, has to be discussed separately. The star topology is shown in Fig. 1. A star network has a unique center node when $n \geq 3$.

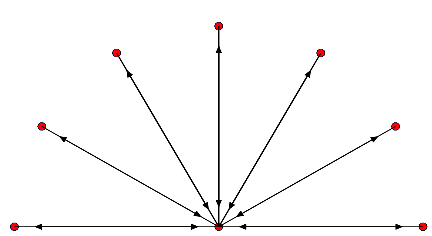

Fig. 1. A digraph $\mathcal{G}$ has a star topology if there exists a node i, called the *center node*, such that all directed edges of $\mathcal{G}$ are either from or to node i.

We start by reviewing a preliminary result.

Lemma II.2 (Lemma 2.3 in [20]: Eigenvector centrality for relative interaction matrices): For $n \geq 3$, let C be row-stochastic, irreducible, and zero-diagonal whose dominant left eigenvector is c and associated digraph is $\mathcal{G}(C)$. Then

(i) if $\mathcal{G}(C)$ is not a star network, then $c_i \in (0,1/2)$ for all $i \in \{1,\dots,n\}$; and
(ii) if $\mathcal{G}(C)$ is a star network and let node i be its center node, then $c_i = 1/2$, and $c_j \in (0,1/2)$ for $j \neq i$.

Convergence results for the DF model (1) have been provided in the cases when $\mathcal{G}(C)$ is or is not a star graph respectively.

Lemma II.3 (Lemma 3.2 in [20]: DF model with star topology): For $n \geq 3$, consider the DF model (1) with row-stochastic, irreducible, and zero-diagonal interaction matrix C. If the digraph associated with C is a star network with center node i, then

(i) (Equilibria) the equilibrium points of (1) are the autocratic vertices $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$.
(ii) (Convergence property) for any $x(0) \in \Delta_n\backslash\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$, $\lim_{s\to\infty} x(s) = \mathbf{e}_i$.

Lemma II.4 (Theorem 4.1 in [20]: DF model without star topology): For $n \geq 3$, consider the DF model (1) with row-stochastic, irreducible, and zero-diagonal interaction matrix C. Assume the digraph associated with C is not a star network and let c be the dominant left eigenvector of C. Then

(i) (Equilibria) the equilibrium points of (1) are $\{\mathbf{e}_1,\dots,\mathbf{e}_n, x^*\}$, where x^* is the unique solution in Δ_n^0 of the following equation with respect to x:

$$x = \left(\frac{c_1}{1-x_1},\dots,\frac{c_n}{1-x_n}\right)/\sum_{i=1}^n \frac{c_i}{1-x_i}; \quad (2)$$

(ii) (Convergence property) for any $x(0) \in \Delta_n\backslash\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$, $\lim_{s\to\infty} x(s) = x^*$.

III. New results on discrete-time and continuos-time DF models

A. Convergence rate of the original DF model

This subsection establishes the convergence rate of the original DF model; these results are also useful for the subsequent analysis.

Lemma III.1 (Lemma F.1 in [20]): Suppose $n \geq 3$ and let $x^* \in \Delta_n^o$ be the equilibrium point appearing in Lemma II.4. For any $x \in \Delta_n^o$, assume $\frac{x_i}{x_i^*} = \max_{1\leq k\leq n} \frac{x_k}{x_k^*}$ and $\frac{x_j}{x_j^*} = \min_{1\leq k\leq n} \frac{x_k}{x_k^*}$. Then,

$$\frac{1-x_j^*}{1-x_j} \leq \frac{1-x_k^*}{1-x_k} \leq \frac{1-x_i^*}{1-x_i}, \quad \text{for all } k \in \{1,\dots,n\}.$$

The following two novel lemmas are key results for the analysis of convergence rate in the cases of star topology and non star topology respectively.

Lemma III.2: Suppose $n \geq 3$ and let c be the dominant left eigenvector of C, where $C \in \mathbb{R}^{n\times n}$ is a row-stochastic, irreducible, and zero-diagonal matrix. If the digraph associated with C is a star network with center node i, then for any $x \in \Delta_n^o$,

$$\frac{\frac{c_i}{1-x_i}}{\sum_{j=1}^n \frac{c_j}{1-x_j}} > x_i + x_i(1-x_i)^2\big(1 - 2\max_{j\neq i} c_j\big).$$

Proof. By Lemma II.2 we have $c_i = 1/2$ so that $\sum_{j\neq i} c_j = 1/2$. Then,

$$\begin{aligned}\frac{\frac{c_i}{1-x_i}}{\sum_{j=1}^n \frac{c_j}{1-x_j}} &= \frac{\frac{1}{2(1-x_i)}}{\frac{1}{2(1-x_i)} + \frac{1}{2x_i} + \sum_{j\neq i}\left(\frac{c_j}{1-x_j} - \frac{c_j}{x_i}\right)} \\ &= \frac{\frac{1}{2(1-x_i)}}{\frac{1}{2(1-x_i)x_i} + \sum_{j\neq i}\left(\frac{c_j}{1-x_j} - \frac{c_j}{x_i}\right)}.\end{aligned} \tag{3}$$

Let $c' = \max_{j\neq i} c_j$. Then,

$$\begin{aligned}\sum_{j\neq i}\left(\frac{c_j}{1-x_j} - \frac{c_j}{x_i}\right) &= -\sum_{j\neq i}\frac{c_j(1-x_i-x_j)}{(1-x_j)x_i} \\ &< -\sum_{j\neq i}\frac{c_j(1-x_i-x_j)}{x_i} = -\frac{1-x_i}{2x_i} + \sum_{j\neq i}\frac{c_j x_j}{x_i} \\ &\le -\frac{1-x_i}{2x_i} + \frac{c'(1-x_i)}{x_i} = -\frac{(1-x_i)(1-2c')}{2x_i}.\end{aligned}$$

Substituting this into (3) we have

$$\begin{aligned}\frac{\frac{c_i}{1-x_i}}{\sum_{j=1}^n \frac{c_j}{1-x_j}} &> \frac{\frac{1}{2(1-x_i)}}{\frac{1}{2(1-x_i)x_i} - \frac{(1-x_i)(1-2c')}{2x_i}} \\ &= \frac{x_i}{1-(1-x_i)^2(1-2c')} \\ &> x_i + x_i(1-x_i)^2(1-2c').\end{aligned} \tag{4}$$

□

Lemma III.3: Suppose $n \ge 3$ and let c be the dominant left eigenvector of C, where $C \in \mathbb{R}^{n\times n}$ is a row-stochastic, irreducible, and zero-diagonal matrix. If the digraph associated with C is not a star network, then, for any $x \in \Delta_n^o$,

$$\max_{i\neq j}\frac{c_i(1-x_j)x_j^*}{c_j(1-x_i)x_i^*} \le 1 + \left(\max_{i\neq j}\frac{x_i x_j^*}{x_j x_i^*} - 1\right)\max_{i\neq j}\frac{x_i^*}{1-x_j^*},$$

where $x^* \in \Delta_n^o$ is the equilibrium point defined in Lemma II.4.

Proof. According to (2) we get

$$\frac{c_i}{(1-x_i^*)x_i^*} = \frac{c_j}{(1-x_j^*)x_j^*} \tag{5}$$

for any $1 \le i, j \le n$. Hence,

$$\frac{c_i(1-x_j)x_j^*}{c_j(1-x_i)x_i^*} = \frac{1-x_j}{1-x_i}\cdot\frac{c_i/x_i^*}{c_j/x_j^*} = \frac{1-x_j}{1-x_i}\cdot\frac{1-x_i^*}{1-x_j^*}. \tag{6}$$

Without loss of generality, we assume

$$\frac{x_1}{x_1^*} \le \frac{x_2}{x_2^*} \le \cdots \le \frac{x_n}{x_n^*}, \tag{7}$$

then by Lemma III.1, we have

$$\frac{1-x_1^*}{1-x_1} \le \frac{1-x_k^*}{1-x_k} \le \frac{1-x_n^*}{1-x_n}, \quad \text{for all } 1 \le k \le n.$$

Substituting this inequality into (6), we obtain

$$\begin{aligned}\max_{i\neq j}\frac{c_i(1-x_j)x_j^*}{c_j(1-x_i)x_i^*} &= \frac{(1-x_n^*)/(1-x_n)}{(1-x_1^*)/(1-x_1)} \\ &= \frac{1-x_n^*}{1-x_1^*}\cdot\frac{1-x_1}{1-x_n}.\end{aligned} \tag{8}$$

Let $\delta_k = \frac{x_k/x_k^*}{x_1/x_1^*}$ so that $x_k = \delta_k x_k^* x_1/x_1^*$. By (7) we have $1 = \delta_1 \le \delta_2 \le \cdots \le \delta_n$. Thus,

$$\frac{1-x_1}{1-x_n} = \frac{\sum_{k=2}^n x_k}{\sum_{k=1}^{n-1} x_k} = \frac{\sum_{k=2}^n \delta_k x_k^*}{\sum_{k=1}^{n-1}\delta_k x_k^*} := \frac{z+\delta_n x_n^*}{z + x_1^*}, \tag{9}$$

where $z = \sum_{k=2}^{n-1}\delta_k x_k^*$. From

$$\sum_{k=2}^{n-1} x_k^* \le z \le \delta_n \sum_{k=2}^{n-1} x_k^*,$$

we know

$$\begin{aligned}\frac{z+\delta_n x_n^*}{z+x_1^*} &= 1 + \frac{\delta_n x_n^* - x_1^*}{z + x_1^*} \\ &\le 1 + \max\left\{\frac{\delta_n x_n^* - x_1^*}{\sum_{k=2}^{n-1} x_k^* + x_1^*}, \frac{\delta_n x_n^* - x_1^*}{\delta_n\sum_{k=2}^{n-1} x_k^* + x_1^*}\right\} \\ &= \max\left\{\frac{1-x_1^*+(\delta_n-1)x_n^*}{1-x_n^*}, \right. \\ &\qquad \left.\frac{\delta_n(1-x_1^*)}{\delta_n(1-x_n^*)-(\delta_n-1)x_1^*}\right\} \\ &= \frac{1-x_1^*}{1-x_n^*}\max\left\{1+(\delta_n-1)\frac{x_n^*}{1-x_1^*}, \frac{\delta_n}{\delta_n - \frac{(\delta_n-1)x_1^*}{1-x_n^*}}\right\}.\end{aligned} \tag{10}$$

Let

$$a^* = \max_{i\neq j}\frac{x_i^*}{1-x_j^*} = \max_{i\neq j}\frac{x_i^*}{x_i^* + \sum_{k\neq i,j} x_k^*} < 1,$$

so that

$$\begin{aligned}&\max\left\{1+(\delta_n-1)\frac{x_n^*}{1-x_1^*}, \frac{\delta_n}{\delta_n - \frac{(\delta_n-1)x_1^*}{1-x_n^*}}\right\} \\ &\quad \le \max\left\{1+(\delta_n-1)a^*, \frac{\delta_n}{\delta_n-(\delta_n-1)a^*}\right\} \\ &\quad = 1+(\delta_n-1)a^*.\end{aligned}$$

Substituting this inequality into (10) yields

$$\frac{z+\delta_n x_n^*}{z+x_1^*} \le \frac{1-x_1^*}{1-x_n^*}(1+(\delta_n-1)a^*). \tag{11}$$

Putting together (7), (8), (9) and (11) we obtain

$$\begin{aligned}\max_{i\neq j}\frac{c_i(1-x_j)x_j^*}{c_j(1-x_i)x_i^*} &= \frac{1-x_n^*}{1-x_1^*}\cdot\frac{z+\delta_n x_n^*}{z+x_1^*} \\ &\le 1+(\delta_n-1)a^* = 1+\left(\frac{x_n/x_n^*}{x_1/x_1^*}-1\right)a^* \\ &= 1+\left(\max_{i\neq j}\frac{x_i x_j^*}{x_j x_i^*}-1\right)a^*.\end{aligned}$$

□

The following theorem establishes novel convergence rates for the original DF model.

Theorem III.1 (Convergence rate of the original DF model): For $n \geq 3$, consider the DF model (1) with row-stochastic, irreducible, and zero-diagonal interaction matrix C. Let $\mathcal{G}(C)$ be the digraph associated with C. For any $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$,

(i) if $\mathcal{G}(C)$ is a star network with center node i, then,

$$\|x(s) - \mathbf{e}_i\|_\infty = O(s^{-1});$$

(ii) if $\mathcal{G}(C)$ is not a star network, then,

$$\|x(s) - x^*\|_\infty = O(a^{*s}),$$

where

$$a^* = \max_{i \neq j} \frac{x_i^*}{1 - x_j^*} = \max_{i \neq j} \frac{x_i^*}{x_i^* + \sum_{k \neq i,j} x_k^*} \in (0,1).$$

Proof. (i) First, for any $x(s) \in \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, by Lemma II.1 we have

$$x(s+1) = \Big(\frac{c_1}{1 - x_1(s)}, \dots, \frac{c_n}{1 - x_n(s)}\Big) / \sum_{j=1}^n \frac{c_j}{1 - x_j(s)} \tag{12}$$

belongs to Δ_n^o, and thus $x(s) \in \Delta_n^o$ for all $s \geq 1$. Since node i is the center node of the star network, by Lemma II.2 we have $c_i = 1/2$, and $c_j < 1/2$ for $j \neq i$. Also, by (12) and Lemma III.2 we get for all $s \geq 1$,

$$\begin{aligned} x_i(s+1) &= \frac{c_i}{1 - x_i(s)} \cdot \frac{1}{\sum_{j=1}^n \frac{c_j}{1 - x_j(s)}} \\ &> x_i(s) + x_i(s)(1 - x_i(s))^2 \big(1 - 2 \max_{j \neq i} c_j\big), \end{aligned}$$

which implies $x_i(s+1) > x_i(s) > \dots > x_i(1)$, and

$$\begin{aligned} &1 - x_i(s+1) \\ &\quad < 1 - x_i(s) - x_i(s)(1 - x_i(s))^2 \big(1 - 2 \max_{j \neq i} c_j\big) \\ &\quad \leq 1 - x_i(s) - x_i(1)(1 - x_i(s))^2 \big(1 - 2 \max_{j \neq i} c_j\big). \end{aligned} \tag{13}$$

Set

$$\alpha := \max \Big\{ 2(1 - x_1(s)), \frac{1}{x_i(1)(1 - 2\max_{j \neq i} c_j)} \Big\}.$$

We will prove $1 - x_i(s) < \frac{\alpha}{s}$ for all $s \geq 1$ by induction. First, $1 - x_i(2) < 1 - x_i(1) \leq \frac{\alpha}{2}$. Also, if $1 - x_i(s) < \frac{\alpha}{s}$ holds for some $s \geq 2$, then the inequality (13) implies

$$\begin{aligned} 1 - x_i(s+1) &< 1 - x_i(s) - \frac{(1 - x_i(s))^2}{\alpha} \\ &< \frac{\alpha}{s} - \frac{\alpha}{s^2} < \frac{\alpha}{s+1}, \end{aligned}$$

where the second inequality uses that the maximum value of $z - \frac{z^2}{\alpha}$ in the interval of $[0, \frac{\alpha}{s}]$ with $s \geq 2$ is reached at $z = \frac{\alpha}{s}$. By induction we get $1 - x_i(s) < \frac{\alpha}{s}$ for all $s \geq 1$. Finally, for any $j \neq i$ we get $x_j(s) < 1 - x_i(s) < \frac{\alpha}{s}$ for any $s \geq 1$.

(ii) With the same arguments as those used in (i), we have $x(s) \in \Delta_n^o$ for all $s \geq 1$. By (12) and Lemma III.3, we get that, for any $s \geq 1$,

$$\begin{aligned} \max_{i \neq j} \frac{x_i(s+1)/x_i^*}{x_j(s+1)/x_j^*} - 1 &\leq \left(\max_{i \neq j} \frac{x_i(s)/x_i^*}{x_j(s)/x_j^*} - 1 \right) a^* \\ &\leq \dots \leq \left(\max_{i \neq j} \frac{x_i(1)/x_i^*}{x_j(1)/x_j^*} - 1 \right) a^{*s}. \end{aligned} \tag{14}$$

Because $\sum_{i=1}^n x_i^* = \sum_{i=1}^n x_i(s) = 1$, we have $\min_j \frac{x_j(s)}{x_j^*} \leq 1$ for any $s \geq 0$. Thus, from (14) we have

$$\max_i \frac{x_i(s)}{x_i^*} - 1 \leq \max_{i \neq j} \frac{x_i(s)/x_i^*}{x_j(s)/x_j^*} - 1 = O(a^{*s}). \tag{15}$$

This inequality and the fact that $x_i^* > 0$, for all $i \in \{1, \dots, n\}$, together implied the claimed statement. □

B. A continuous-time DF model

We here introduce a continuous-time DF model, which is novel in its own and whose analysis will be used later.

Let c denote the normalized left dominant eigenvector of an irreducible interaction matrix C and define $g : \Delta_n \to \Delta_n$ by

$$g(x) = \begin{cases} 0, & \text{if } x \in \{\mathbf{e}_1, \dots, \mathbf{e}_n\}; \\ -x + \Big(\frac{c_1}{1 - x_1}, \dots, \frac{c_n}{1 - x_n}\Big) / \sum_{i=1}^n \frac{c_i}{1 - x_i}, & \text{otherwise}. \end{cases}$$

Assume that the graph associated with C is not a star network. The *continuous-time DF model* is

$$\dot{x}(\tau) = g(x(\tau)), \quad s \in \mathbb{R}_{\geq 0}. \tag{16}$$

Lemma III.4 (Well-posedness of the continuous-time DF model): For $n \geq 3$, pick $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then the solution to the continuous-time DF model (16) satisfies $x(\tau) \in \Delta_n^o$ for all $\tau > 0$.

Proof: We start by showing that, for any $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, there exists $\tau_0 > 0$ such that $x(\tau) \in (0,1)^n$ for any $\tau \in (0, \tau_0]$. In fact, this result holds obviously for $x(0) \in \Delta_n^o$. When $x(0) \in \Delta_n \backslash (\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \cup \Delta_n^o)$, if $x_i(0) = 0$, then by (16) we have

$$\lim_{\tau \to 0^+} \frac{x_i(\tau) - x_i(0)}{\tau} = g_i(0) = \frac{c_i}{\sum_{j=1}^n \frac{c_j}{1 - x_j(0)}} > 0,$$

which implies $x(\tau) \in (0,1)^n$ for small positive τ.

Next we show $x_i(\tau)$ cannot leave the interval $(0,1)$ for any $\tau > \tau_0$ and $i \in \{1, \dots, n\}$. Let $c_{\max} := \max_{i \in \{1, \dots, n\}} c_i$. At time τ, assume without loss of generality that $x_1(\tau) = x_{\max}(\tau)$ and $x_n(\tau) = x_{\min}(\tau)$. Since the network we consider here is not a star network, by Lemma II.2, $c_{\max} < 1/2$. If $x_{\min} \geq 0$ and $x_{\max}(\tau) = x_1(\tau) \in [\frac{c_{\max}}{1 - c_{\max}}, 1)$, then by (16)

$$\begin{aligned} \dot{x}_1(\tau) &= \frac{c_1}{(1 - x_1(\tau)) \sum_{i=1}^n \frac{c_i}{1 - x_i(\tau)}} - x_1(\tau) \\ &< \frac{c_1}{(1 - x_1(\tau))(\frac{c_1}{1 - x_1(\tau)} + 1 - c_1)} - x_1(\tau) \\ &= \frac{c_1 - (1 - c_1)x_1(\tau)}{\frac{c_1}{1 - x_1(\tau)} + 1 - c_1} \leq 0, \end{aligned}$$

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which implies $x_{\max}(\tau)$ will decrease. Thus, $x_{\max}(\tau)$ will not be larger than

$$\max\Big\{x_{\max}(\tau_0), \frac{c_{\max}}{1-c_{\max}}\Big\} := b_1 < 1.$$

At the same time, if $x_{\min}(\tau) = x_n(\tau) \le \frac{c_{\min}(1-b_1)}{(n-2)c_{\max}}$, then

$$\begin{aligned}\dot{x}_n(\tau) &= \frac{c_n}{(1-x_n(\tau))\sum_{i=1}^n \frac{c_i}{1-x_i(\tau)}} - x_n(\tau)\\ &> \frac{c_n}{(1-x_n(\tau))(\frac{c_n}{1-x_n(\tau)} + \frac{c_{\max}(n-2)}{1-b_1})} - x_n(\tau)\\ &= \frac{c_n - \frac{c_{\max}(n-2)x_n}{1-b_1}}{\frac{c_n}{1-x_n(\tau)} + \frac{c_{\max}(n-2)}{1-b_1}} \ge 0,\end{aligned}$$

which implies $x_{\min}(\tau)$ will increase. Collecting these two properties we obtain $x(\tau) \in (0,1)^n$ for any $\tau > 0$.

Let $S(\tau) := \sum_{i=1}^n x_i(\tau)$. By (16) we get

$$\dot{S}(\tau) = 1 - S(\tau).$$

Solving this ODE yields $S(\tau) = b_2 e^{-\tau} + 1$. With the initial condition $S(0) = 1$ we get $S(\tau) \equiv 1$. Thus, we have $x(\tau) \in \Delta_n^o$ for any $s\tau > 0$. □

We next consider the convergence properties of this system and establish a continuous-time version of Lemma II.4.

Lemma III.5 (Convergence of continuous-time DF model): For $n \ge 3$, consider the continuous-time DF model (16) with row-stochastic, irreducible, and zero-diagonal interaction matrix C. Assume the digraph associated with C is not a star network and let c be the dominant left eigenvector of C. Then

(i) (Equilibria) the equilibrium points of (16) are $\{\mathbf{e}_1, \dots, \mathbf{e}_n, x^*\}$, where x^* is the unique solution in Δ_n^0 of the equation (2);

(ii) (Convergence property) for any $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, $\lim_{\tau\to\infty} x(\tau) = x^*$.

Proof: Define the Lyapunov function $V(\tau)$ by

$$V(\tau) := \log \max_{i\ne j} \frac{x_i(\tau)/x_i^*}{x_j(\tau)/x_j^*}.$$

Let $\bar{I}(\tau)$ denote the index set $\{(i,j)\}$ in which the maximum value of $\frac{x_i(\tau)/x_i^*}{x_j(\tau)/x_j^*}$ is reached. For any $\tau > 0$, if $|\bar{I}(\tau)| = 1$, without loss of generality, we assume $\bar{I}(\tau) = (1, n)$. Then by (16) we have

$$\begin{aligned}\dot{V}(\tau) &= \frac{d}{ds} \log \frac{x_1(\tau)/x_1^*}{x_n(\tau)/x_n^*} = \frac{\dot{x}_1(\tau)}{x_1(\tau)} - \frac{\dot{x}_n(\tau)}{x_n(\tau)}\\ &= \frac{1}{\sum_{i=1}^n \frac{c_i}{1-x_i(\tau)}}\Big(\frac{c_1}{(1-x_1(\tau))x_1(\tau)} - \frac{c_n}{(1-x_n(\tau))x_n(\tau)}\Big).\end{aligned} \tag{17}$$

Also, by Lemma III.3 we have

$$\begin{aligned}\frac{\frac{c_1}{(1-x_1(\tau))x_1(\tau)}}{\frac{c_n}{(1-x_n(\tau))x_n(\tau)}} &= \frac{\frac{c_1}{(1-x_1(\tau))x_1^*}}{\frac{c_n}{(1-x_n(\tau))x_n^*}} \cdot \frac{\frac{x_1^*}{x_1(\tau)}}{\frac{x_n^*}{x_n(\tau)}}\\ &\le \Big(1 + (\frac{x_1(\tau)/x_1^*}{x_n(\tau)/x_n^*} - 1)r\Big) \cdot \frac{x_1^*/x_1(\tau)}{x_n^*/x_n(\tau)}\\ &= \big(1 + (e^{V(\tau)} - 1)r\big)e^{-V(\tau)}\\ &= 1 - (1-r)(1-e^{-V(\tau)}) \le 1,\end{aligned} \tag{18}$$

where $r = \max_{i\ne j} \frac{x_i^*}{1-x_j^*} = \max_{i\ne j} \frac{x_i^*}{x_i^* + \sum_{k\ne i,j} x_k^*} < 1$. Substituting (18) into (17) and using Lemma III.4, we get

$$\dot{V}(\tau) \le -\frac{\frac{c_n}{(1-x_n(\tau))x_n(\tau)}}{\sum_{i=1}^n \frac{c_i}{1-x_i(\tau)}}(1-r)(1-e^{-V(\tau)}) \le 0, \tag{19}$$

where $\dot{V}(\tau) = 0$ if and only if $V(\tau) = 0$.

For the case when $|\bar{I}(\tau)| > 1$, the derivative of $V(\tau)$ may not exist because its left derivative may not be equal to its right derivative. Therefore, we use the Dini derivative instead. For any $\tau_0 \ge 0$, define

$$D^+V(\tau_0) := \limsup_{\tau\to\tau_0^+} \frac{V(\tau) - V(\tau_0)}{\tau - \tau_0}.$$

From Danskin's Lemma [7], it can be deduced that

$$\begin{aligned}D^+V(\tau) &= \max_{(i,j)\in\bar{I}(\tau)} \frac{d}{ds} \log \frac{x_i(\tau)/x_i^*}{x_j(\tau)/x_j^*}\\ &\le -(1-r)(1-e^{-V(\tau)}) \min_{(i,j)\in\bar{I}(\tau)} \frac{\frac{c_j}{(1-x_j(\tau))x_j(\tau)}}{\sum_{k=1}^n \frac{c_k}{1-x_k(\tau)}}\\ &\le 0,\end{aligned} \tag{20}$$

where the second line relies upon (19). Also, $D^+V(\tau) = \dot{V}(\tau)$ if $V(\tau)$ is differentiable. By Theorem 1.13 in [15] we have that $V(\tau)$ is decreasing in $[0,\infty)$, which implies that $\lim_{\tau\to\infty} V(\tau)$ exists. If $\lim_{\tau\to\infty} V(\tau) = v > 0$, then, because $\max_{i\in\{1,\dots,n\}} \frac{x_i(\tau)}{x_i^*} \ge 1$, we have $\liminf_{\tau\to\infty} \min_{i\in\{1,\dots,n\}} \frac{x_i(\tau)}{x_i^*} \ge e^{-v} > 0$. Together with (20), there exists a constant $\epsilon > 0$ such that $D^+V(\tau) \le -\epsilon$ for all large τ, which implies $\lim_{\tau\to\infty} V(\tau) = -\infty$. Thus, we have $\lim_{\tau\to\infty} V(\tau) = 0$, which implies $\lim_{\tau\to\infty} x(\tau) = x^*$ because $\sum_{i=1}^n x_i^* = 1 = \sum_{i=1}^n x(\tau)$ for any $\tau \ge 0$. □

IV. DF models with switching and stochastic interactions

This section considers the case of time-varying relative interaction matrices. We first consider a DF model with switching interaction and then propose a novel DF model with memory on random interactions.

A. The DF model with switching interactions

Let $\{C(s) \in \mathbb{R}^{n\times n}\}_{s\in\mathbb{Z}_{\ge 0}}$ denote a sequence of relative interaction matrices, that is, a sequence of row-stochastic matrices with zero diagonal. Given such a sequence, the *DF model with switching interactions* is given by

$$\begin{cases} W(x(s)) = \text{diag}[x(s)] + (I_n - \text{diag}[x(s)])C(s),\\ x(s+1) = \pi(W(x(s))).\end{cases} \tag{21}$$

Let $\{\mathcal{G}(C(s))\}_{s\in\mathbb{Z}_{\ge 0}}$ be the sequence of digraph associated with the sequence $\{C(s)\}_{s\in\mathbb{Z}_{\ge 0}}$. We will consider the cases when every graph $\mathcal{G}(C(s))$ in $\{\mathcal{G}(C(s))\}_{s\in\mathbb{Z}_{\ge 0}}$ is a star network with fixed center node, or $\{\mathcal{G}(C(s))\}_{s\in\mathbb{Z}_{\ge 0}}$ is not a sequence of fixed star network.

First, we suppose $\{\mathcal{G}(C(s))\}_{s\in\mathbb{Z}_{\ge 0}}$ is a sequence of star networks with a common center node i as described in the following assumption.

Assumption 1 (Sequence of relative interaction matrices with star topology): The sequence of relative interaction matrices $\{C(s) \in \mathbb{R}^{n\times n}\}_{s\in\mathbb{Z}_{\geq 0}}$ has the properties that $\{\mathcal{G}(C(s))\}_{s\in\mathbb{Z}_{\geq 0}}$ is a sequence of star networks with common center node i, and that there exists a constant $\varepsilon > 0$ such that $C_{ij}(s) \geq \varepsilon$ for all $j \neq i$ and $s \geq 0$.

Proposition IV.1 (Convergence and convergence rate of the DF model over star topologies with switching weights): For $n \geq 3$, consider a sequence of relative interaction matrices satisfing Assumption 1 with common center node i, and the corresponding DF model with switching interactions (21). Then

(i) the system (21) has an equilibrium point $\mathbf{e}_i$,
(ii) for any initial condition $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, the solution $x(s)$ converges to $\mathbf{e}_i$ with a rate of $O(s^{-1})$.

Proof: The proof of statement (i) is identical to the proof of the corresponding statement in Lemma II.3 in [20]; we do not report it here in the interest of brevity.

Regarding statement (ii), because each relative interaction matrix $C(s)$ is irreducible, $C(s)$ has a dominant left eigenvector $c(s) = (c_1(s), \dots, c_n(s))$. Assumption 1 and Lemma II.2 imply $c_i(s) = 1/2$, and $c_j(s) \geq \epsilon$ for $j \neq i$, where ϵ is a positive constant depending on ε in Assumption 1. Similar to (13) we have

$$\begin{aligned} 1 - x_i(s+1) & \\ &< 1 - x_i(s) - x_i(1)(1 - x_i(s))^2 \big(1 - 2\max_{j\neq i} c_j(s)\big) \\ &\leq 1 - x_i(s) - x_i(1)(1 - x_i(s))^2 \Big(1 - 2\Big(\frac{1}{2} - \epsilon\Big)\Big) \\ &= 1 - x_i(s) - x_i(1)(1 - x_i(s))^2 2\epsilon, \quad \text{for all } s \geq 1. \end{aligned}$$

Similar to the proof of Theorem III.1(i) we get $\|x(s) - \mathbf{e}_i\|_\infty = O(1/s)$. □

For the case when $\{\mathcal{G}(C(s))\}_{s\in\mathbb{Z}_{\geq 0}}$ is not a sequence of star network, the DF model with switching interactions (21) may not converge to an equilibrium point. However, if there exists a row-stochastic, zero-diagonal, and irreducible matrix C such that the difference between every $C(s)$ of the sequence $\{C(s)\}_{s\in\mathbb{Z}_{\geq 0}}$ and C is sufficiently small, then the trajectories converge to a ball centered around the equilibrium reached by the original DF model (1). For any issue s, let c and $c(s)$ be the dominant left eigenvectors of C and $C(s)$, respectively, and define $\xi(s) := c(s) - c$.

By Lemma II.1, for any $x(s) \in \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, the DF model with switching interactions (21) has the following form:

$$x(s+1) = \Big(\frac{c_1 + \xi_1(s)}{1 - x_1(s)}, \dots, \frac{c_n + \xi_n(s)}{1 - x_n(s)}\Big) \Big/ \sum_{i=1}^{n} \frac{c_i + \xi_i(s)}{1 - x_i(s)}. \tag{22}$$

So, in order to investigate the DF model with switching interactions (21), we can analyze the system (22) instead. We next present a third assumption.

Assumption 2 (Sequence of relative interaction matrices with small variations): The sequence of relative interaction matrices $\{C(s) \in \mathbb{R}^{n\times n}\}_{s\in\mathbb{Z}_{\geq 0}}$ has the following property: there exists an irreducible relative interaction matrix C such that $\mathcal{G}(C)$ is not a star network and, for all $s \geq 0$ and $i \in \{1, \dots, n\}$,

$$\frac{|c_i(s) - c_i|}{c_i} \leq r \qquad \Longleftrightarrow \qquad |\xi_i(s)| \leq r c_i,$$

where $r \in (0, \frac{1-a^*}{1+a^*})$ is a constant with $a^* = \max_{i\neq j} \frac{x_i^*}{1-x_j^*} < 1$ and $x^* = x^*(C) \in \Delta_n^o$ denotes the equilibrium point of the DF model (1), as established in Lemma II.4.

Remark IV.1: We here elaborate on the sequences $\{C(s)\}_{s\in\mathbb{Z}_{\geq 0}}$ satisfying Assumption 2. Loosely speaking, because the dominant eigenvalue of C is simple, if $C(s) - C$ is sufficiently small, then the left dominant eigenvector of $C(s)$ is close to that of C. Indeed, Funderlic and Meyer [5] review various perturbation bounds for the left dominant eigenvector of a row-stochastic matrix. Specifically, [14, Subsection 3.4] states $\|\xi(s)\|_\infty \leq \kappa(C)\|C(s) - C\|_\infty$, where $\kappa(C)$ is an appropriate function of C. Therefore, if

$$\|C(s) - C\|_\infty \leq \frac{1}{\kappa(C)} r c_{\min}, \text{ with } c_{\min} = \min_{1\leq j\leq n} c_j,$$

then $\|\xi(s)\|_\infty \leq r c_{\min}$ and, in turn, $\frac{|\xi_i(s)|}{c_i} \leq \frac{\|\xi(s)\|_\infty}{c_{\min}} \leq r$. □

We are now ready to state the main result of this subsection.

Theorem IV.1 (Convergence of the DF model with switching non-star topologies): For $n \geq 3$, consider a sequence of relative interaction matrices satisfing Assumption 2 and the corresponding DF model with switching interactions (21). Then for any $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$

$$\limsup_{s\to\infty} \max_{i\neq j} \frac{x_i(s)/x_i^*}{x_j(s)/x_j^*} \leq 1 + \frac{2r}{1 - r - (1+r)a^*}, \tag{23}$$

where x^*, a^*, r are defined in Assumption 2.

Proof: Let $D_s = \max_{i\neq j} \frac{x_i(s)/x_i^*}{x_j(s)/x_j^*}$ and $D^* = 1 + \frac{2r}{1-r-(1+r)a^*}$. By (22) and Assumption 2, we get $x(s) \in \Delta_n^o$ for any $s \geq 1$, and

$$\frac{x_i(s+1)}{x_j(s+1)} = \frac{(c_i + \xi_i(s))(1 - x_j(s))}{(c_j + \xi_j(s))(1 - x_i(s))}, \tag{24}$$

so that

$$\begin{aligned} D_{s+1} &= \max_{i\neq j} \frac{(c_i + \xi_i(s))(1 - x_j(s))x_j^*}{(c_j + \xi_j(s))(1 - x_i(s))x_i^*} \\ &\leq \max_{i\neq j} \frac{(c_i + rc_i)(1 - x_j(s))x_j^*}{(c_j - rc_j)(1 - x_i(s))x_i^*} \\ &\leq \frac{1+r}{1-r}(1 + (D_s - 1)a^*), \end{aligned} \tag{25}$$

where the last inequality uses Lemma III.3.

If $D_s \leq D^*$, then by (25)

$$\begin{aligned} D_{s+1} - D^* &\leq \frac{1+r}{1-r}(1 + (D^* - 1)a^*) - D^* \\ &= \left(\frac{1+r}{1-r}a^* - 1\right) D^* + \frac{1+r}{1-r}(1 - a^*) \\ &= \frac{(1+r)a^* - 1 + r}{1-r} \cdot \frac{1 + r - (1+r)a^*}{1 - r - (1+r)a^*} \\ &\quad + \frac{1+r}{1-r}(1 - a^*) = 0, \end{aligned} \tag{26}$$

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which implies $D_{s+1} \leq D^*$. If $D_s > D^*$, then by (25) and (26)

$$\begin{aligned} D_{s+1} - D^* &\leq \frac{1+r}{1-r} a^* D_s + \frac{1+r}{1-r}(1-a^*) - D^* \\ &= \frac{1+r}{1-r} a^*(D_s - D^*) + \left(\frac{1+r}{1-r}a^* - 1\right) D^* \\ &\quad + \frac{1+r}{1-r}(1-a^*) \\ &= \frac{1+r}{1-r} a^*(D_s - D^*). \end{aligned} \tag{27}$$

Because $\frac{1+r}{1-r}a^* < 1$, combining (26) and (27) yields our result. □

Remark IV.2: The bound in Theorem IV.1 can be written in a more conservative and explicit form as follows. Because $\sum_{i=1}^n x_i^* = \sum_{i=1}^n x_i(s) = 1$, we have $\min_j \frac{x_j(s)}{x_j^*} \leq 1$ for any $s \geq 0$. Thus, from (23)

$$\begin{aligned} &\limsup_{s\to\infty} \max_i \frac{x_i(s)}{x_i^*} - 1 \\ &\quad \leq \limsup_{s\to\infty} \max_{i\neq j} \frac{x_i(s)/x_i^*}{x_j(s)/x_j^*} - 1 \leq \frac{2r}{1-r-(1+r)a^*}, \end{aligned}$$

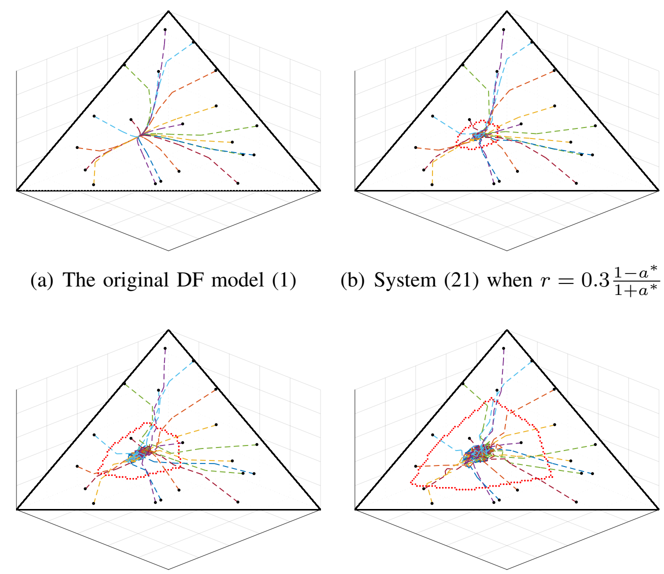
which implies

$$\limsup_{s\to\infty} \|x(s) - x^*\|_\infty \leq \frac{2r \max_i x_i^*}{1-r-(1+r)a^*}. \qquad \square$$

To visualize the result of Theorem IV.1, we consider a three-node network with relative interaction matrix:

$$C = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

In order to show the effect of $\{\xi(s)\}$ on the radius of the convergence ball, we generate $\{\xi(s)\}$ that satisfies Assumption 2 and simulate the DF model under switching interactions using (22). The convergence results under different r are shown in Fig. 2. As predicted, all the trajectories eventually converge into the ball whose boundary are marked with red dots. The radius of the convergence ball depends on r. Our simulation results suggest the existence of a potential tighter bound than (23).



(a) The original DF model (1) (b) System (21) when $r = 0.3\frac{1-a^*}{1+a^*}$

(c) System (21) when $r = 0.5\frac{1-a^*}{1+a^*}$ (d) System (21) when $r = 0.7\frac{1-a^*}{1+a^*}$

Fig. 2. Illustrating the convergence result of Theorem IV.1 with $a^* = 0.639$.

B. A DF model with memory on stochastic interactions

As shown in the last section, the DF model with switching interactions (21) does not converge to an equilibrium point in general. We now consider a DF model where the sequence of interaction matrices is a stochastic process, which individuals observe and filter.

Assumption 3 (Stochastic relative interaction matrices with constant conditional expectation): The sequence of interaction matrices $\{C(s)\}_{s\in\mathbb{Z}_{\geq 0}}$ is generated by a stochastic process with the following properties:

(i) each $C(s)$ takes values in the set of row-stochastic, zero-diagonal, and irreducible matrices, and

(ii) there exists a relative interaction matrix C such that $\mathcal{G}(C)$ is not a star network, and

$$\mathbb{E}\left[C(s)\,|\,C(0),\ldots,C(s-1)\right] = C.$$

We next introduce a sequence of scalar numbers that is deterministic and that satisfies the standard tapering stepsize assumption, stated as follows.

Assumption 4 (Tapering step size sequence): The deterministic sequence $\{a(s) \in \mathbb{R}\}_{s\in\mathbb{Z}_{\geq 0}}$ satisfies

(i) $a(s) \in [0,1)$ for any $s \geq 0$;
(ii) $\sum_{s=0}^{\infty} a(s)^2 < \infty$;
(iii) $\sum_{s=0}^{\infty} a(s) = \infty$.

Our modeling approach is to use a stochastic approximation algorithm to describe the evolution of the interaction matrix as follows. Given a sequence of relative interaction matrices as in Assumption 3 and stepsizes as in Assumption 4, consider the sequence $\{\overline{C}(s)\}_{s\in\mathbb{Z}_{\geq 0}}$ defined by, for all $s \geq 0$,

$$\overline{C}(s+1) := (1-a(s))\overline{C}(s) + a(s)C(s+1), \tag{28}$$

with a deterministic relative interaction matrix $\overline{C}(0)$. The *DF model with memory on random interactions* is given by equation (28) combined with

$$\begin{cases} W(x(s)) = \mathrm{diag}[x(s)] + (I_n - \mathrm{diag}[x(s)])\overline{C}(s), \\ x(s+1) = \pi(W(x(s))). \end{cases} \tag{29}$$

We remark that, in the iteration (28), each individual only remembers the influence weights assigned to others.

Theorem IV.2 (Convergence of the DF model with memory on random interactions): For $n \geq 3$, consider a stochastic sequence of relative interaction matrices satisfying Assumption 3 and stepsizes as in Assumption 4 with expected relative

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interaction matrix C, and the corresponding system (28)-(29). Let $x^* = x^*(C) \in \Delta_n^o$ be the equilibrium point of the DF model (1) with relative interaction matrix C (see Lemma II.4).

Then for any $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, the solution $x(s)$ of the system (29) converges to x^* a.s.

Proof: We start by applying Theorem 2.2 in [1] to equation (28). First, note that

$$\begin{aligned} \overline{C}(s+1) &= \overline{C}(s) + a(s)(C(s+1) - \overline{C}(s)), \\ \iff X(s+1) &= X(s) + a(s)(-X(s) + M(s+1)), \end{aligned}$$

with $X(n) = \overline{C}(s) - C$ and $M(s+1) = C(s+1) - C$. This final expression matches equation (1.1) in [1] with $h(X(s)) = -X(s)$. Note that the four assumptions in Theorem 2.2 in [1] are satisfied, because, adopting the notation in [1], (1) the conditions (A1) on the function h are satisfied by our $h(x) = -x$; (2) the conditions (A2) on the martingale property and boundedness of $M(s)$ are satisfied by Assumption 3(ii) and because $C(s) - C$ takes values in a bounded set; (3) the stepsize sequence is tapering by Assumption 4; and (4) the conditions on ODE (1.2) are satisfied because 0 is the unique globally asymptotically stable equilibrium of $\dot{x}(t) = -x$. Therefore, Theorem 2.2 in [1] implies $X(s) \to 0$ a.s. as $s \to \infty$, that is, $\overline{C}(s+1) \to C$ a.s. as $s \to \infty$.

Note that Assumption 3 implies that the relative interaction matrix C is irreducible; let c be its left dominant eigenvector. Also note that $\overline{C}(s)$ is a.s. a row-stochastic, zero-diagonal, and irreducible matrix for all $s \geq 0$. Then the Perron-Frobenius Theorem implies that $\overline{C}(s)$ has a left dominant eigenvector $c + \xi(s)$ a.s. with $c_i + \xi_i(s) > 0$ for any $i \in \{1, \dots, n\}$. Let Ω' be the set of events for which $C(s)$ is a row-stochastic, zero-diagonal, and irreducible matrix for all $s \geq 0$ and $\lim_{s\to\infty} \overline{C}(s) = C$. Because $\lim_{s\to\infty} \overline{C}(s) = C$ a.s. we have $\mathbb{P}[\Omega'] = 1$. Also, for any sample in Ω', by Subsection 3.4 in [5] or Theorem 2.3 in [14], we obtain

$$\lim_{s\to\infty} \xi(s) = \mathbf{0}. \tag{30}$$

Finally, we show $x(s)$ converges to x^* for any sample in Ω'. By (22) and the fact that $c + \xi(s) > 0$, we know that $x(s) \in \Delta_n^o$ for any $s \geq 1$. Similar to (25), we have

$$\begin{aligned} D_{s+1} &= \max_{i \neq j} \frac{(c_i + \xi_i(s))(1 - x_j(s))x_j^*}{(c_j + \xi_j(s))(1 - x_i(s))x_i^*} \\ &= \max_{i \neq j} \frac{(1 + \xi_i(s)/c_i)c_i(1 - x_j(s))x_j^*}{(1 + \xi_j(s)/c_j)c_j(1 - x_i(s))x_i^*} \\ &\leq \frac{1 + \xi_i(s)/c_i}{1 + \xi_j(s)/c_j}(1 + (D_s - 1)a^*), \end{aligned} \tag{31}$$

where D_s and a^* are defined as in the proof of Theorem IV.1. By (31) we can get

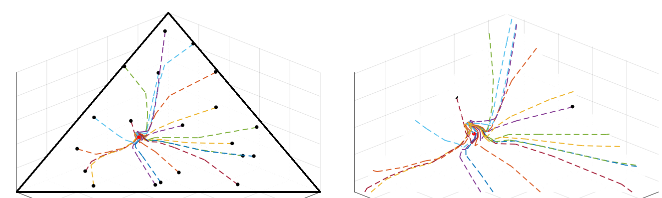
$$D_{s+1} - 1 \leq \frac{\frac{\xi_i(s)}{c_i} - \frac{\xi_j(s)}{c_j}}{1 + \frac{\xi_j(s)}{c_j}} + \frac{1 + \frac{\xi_i(s)}{c_i}}{1 + \frac{\xi_j(s)}{c_j}}(D_s - 1)a^*.$$

Set

$$f_s := \frac{\frac{\xi_i(s)}{c_i} - \frac{\xi_j(s)}{c_j}}{1 + \frac{\xi_j(s)}{c_j}} \quad \text{and} \quad g_s := 1 - \frac{1 + \frac{\xi_i(s)}{c_i}}{1 + \frac{\xi_j(s)}{c_j}}a^*.$$

By (30) there exists a constant $s_0 > 0$ such that $g_s \in (0, 1)$ for all $s \geq s_0$, $\sum_{s=s_0}^{\infty} g_s = \infty$, and $\lim_{s\to\infty} f_s/g_s = 0$. By the Lemma C.1 in Appendix C, we get $\lim_{s\to\infty}(D_s - 1) = 0$, which implies $\lim_{s\to\infty} x(s) \to x^*$ by $\sum_{i=1}^n x_i(s) = 1 = \sum_{i=1}^n x_i^*$. Our result implies that $\mathbb{P}[\Omega'] = 1$. □

To illustrate the convergence of the DF model with interaction memory, we simulate the same network with the same initial conditions as those used in the last subsection. The interaction matrix $C(s)$ is generated as follows: for any $s \geq 0$, we let $\widetilde{N}(s)$ be an $n \times n$ matrix with the same zero/non-zero pattern as C and we select the nonzero elements $\widetilde{N}_{ij}(s)$ uniformly and independently distributed in $[-C_{ij}, C_{ij}]$. For any $i \neq j$, set $C_{ij}(s) := C_{ij} + \widetilde{N}_{ij}(s) - \frac{1}{n}\sum_{k=1}^n \widetilde{N}_{ik}(s)$. We then scale (i.e., multiply by an appropriate constant) each row of $N(s)$ so as to guarantee that $C(s)$ is row-stochastic. This way we know $\mathbb{E}\left[C(s) \,|\, C(0), \dots, C(s-1)\right] = C$ is satisfied. We can observe from Fig. 3 that, after some oscillation, $x(s)$ converges to the same equilibrium of the original DF model as established by Theorem IV.2.



(a) The DF model with interaction memory (29) (b) Zoom-in of Fig. 3(a)

Fig. 3. Illustrating the convergence result of Theorem IV.2

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V. A DF MODEL WITH STOCHASTIC INTERACTIONS, ENVIRONMENT NOISE, AND SELF-APPRAISAL MEMORY

This section considers a DF model where the sequence of interaction matrices is a stochastic process and there are noise and memory on self-appraisals. As before, we also adopt a stochastic approximation model to include memory in the system and, as a byproduct, asymptotically eliminate the impact of interaction randomness and environment noise.

The *DF model with random interactions, environment noise, and self-appraisal memory* is given by

$$\begin{cases} W(x(s)) = \text{diag}[x(s)] + (I_n - \text{diag}[x(s)])C(s), \\ x(s+1) = (1 - a(s))x(s) + a(s)\big[\pi(W(x(s))) + \zeta(s)\big], \end{cases} \tag{32}$$

where $\{C(s)\}_{s\in\mathbb{Z}_{>0}}$ satisfies Assumption 3 and $\{\zeta(s)\}_{s\in\mathbb{Z}_{>0}}$ is a stochastic process denoting the environment noise.

Theorem V.1 (Convergence of the DF model with random interactions, environment noise, and self-appraisal memory): For $n \geq 3$, consider a stochastic sequence of relative interaction matrices satisfying Assumption 3 and stepsizes as in Assumption 4, and the corresponding system (32). Assume:

(i) along every solution $\{x(s) \in \Delta_n \backslash \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}\}_{s \in \mathbb{Z}_{>0}}$ and at every time $s \geq 0$, we have $\pi(W(x(s))) + \zeta(s) \in \Delta_n^o$ a.s.,

(ii) there exists a vector $\tilde{c} \in \Delta_n^o$ with $\max_i \tilde{c}_i < \frac{1}{2}$ such that, along every solution $\{x(s) \in \Delta_n \backslash \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}\}_{s \in \mathbb{Z}_{>0}}$, the sequence $\{\beta(s) \in \mathbb{R}^n\}_{s \in \mathbb{Z}_{\geq 0}}$ defined by

$$\beta(s) = \mathbb{E}\left[\pi(W(x(s))) + \zeta(s) \,|\, x(0), C(t), \zeta(t), t < s\right] - \Big(\frac{\tilde{c}_1}{1 - x_1(s)}, \ldots, \frac{\tilde{c}_n}{1 - x_n(s)}\Big) \Big/ \sum_{i=1}^{n} \frac{\tilde{c}_i}{1 - x_i(s)},$$

satisfies a.s.

$$\sum_{s=0}^{\infty} a(s) \|\beta(s)\|_\infty < \infty. \tag{33}$$

Then, for any $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, $x(s)$ converges to $\{\tilde{x}^*, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ a.s., where $\tilde{x}^*$ is the unique solution in Δ_n^o of the equation

$$x = \Big(\frac{\tilde{c}_1}{1 - x_1}, \ldots, \frac{\tilde{c}_n}{1 - x_n}\Big) \Big/ \sum_{i=1}^{n} \frac{\tilde{c}_i}{1 - x_i}. \tag{34}$$

It is important to clarify that the condition (33) is complex to verify in general, since the dynamics are highly nonlinear and the condition depends on evolution of the state $x(s)$. However, it can be checked that condition (33) is a weaker condition for some special cases. For example, if $C(s)$ converges sufficiently quickly to a constant matrix C a.s. and $\mathbb{E}[\zeta(s)|\zeta(t), t < s]$ converges to zero sufficiently quickly, then the summability condition (33) is satisfied with $\tilde{c}$ equal to the dominant left eigenvector of C, and system (32) converges.

In what follows we present some simulation results for the reduced Krackhardt's advice network with $n = 17$. The interaction matrix $C(s)$ is generated as before in the simulation after Theorem IV.2. The environment noise $\zeta(s)$ is also generated in a similar way so that $\pi(W(x(s))) + \zeta(s)$ is still in Δ_n^o. The sequence $\{a(s)\}_{s \in \mathbb{Z}_{>0}}$ is the harmonic sequence such that Assumption 4 holds. The condition (33) is verified numerically in Fig. 4, where it is shown that $x(s)$ converges to $\tilde{x}^*$ in this case.

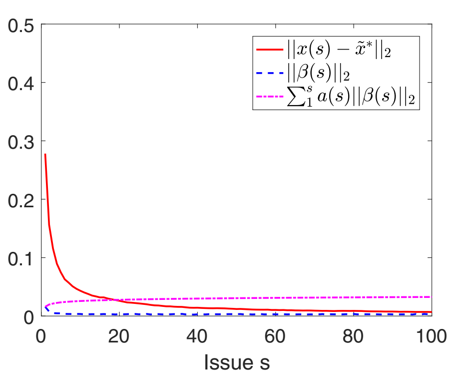

Fig. 4. Convergence of $x(s)$ and verification of equation (33).

Theorem V.1 gives some conditions to guarantee that $x(s)$ converges to a set. If we add further assumptions $x(s)$ can a.s. converge to a fixed point.

Theorem V.2 (Convergence of the DF model with random interactions, environment noise and self-appraisal memory): For the system (32), assume all conditions in Theorem V.1 are satisfied. In addition, assume there exist constants $c' \in (0, \frac{1}{2})$, $p_1 \in (0, 1)$, $d_1 > 0$, and $\gamma > 1$ such that for any $s \geq 0$, $x(s) \in \Delta_n^o$, and $i \in \{1, \ldots, n\}$,

$$\mathbb{P}\Big[\pi_i(W(x(s))) + \zeta_i(s) \leq \frac{c'}{c' + (1 - c')(1 - x_i(s))} \,\Big|\, x(s)\Big] \geq p_1, \tag{35}$$

and a.s.

$$\pi_i(W(x(s))) + \zeta_i(s) \leq x_i(s) + d_1(1 - x_i(s))^\gamma. \tag{36}$$

If the tapering step size sequence $\{a(s)\}_{s \in \mathbb{Z}_{>0}}$ satisfies $a(s) = d_2/(s+1)$ with $d_2 \in (0, 1]$, then, for any $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, the solution $x(s)$ converges a.s. to $\widetilde{x}^*$, i.e., the solution to equation (34).

Remark V.1: At the cost of a more complex analysis, it is possible to obtain a version of Theorem V.2 for general tapering step size sequences satisfying Assumption 4. □

It is important to clarify that the conditions (35) and (36) are complex to verify. We provide the following assumption, which is sufficient for conditions (35) and (36).

Assumption 5 (Random relative interaction matrices and environment noise): For any $s \geq 0$ and $x(s) \in \Delta_n \backslash \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, assume:

(i) there exists constants $d_1 > 0$ and $\gamma > 1$ such that $\zeta_i(s) \leq d_1(1 - x_i(s))^\gamma$ a.s. for $i \in \{1, \ldots, n\}$;

(ii) there exist two constants $p_1, \delta \in (0, 1)$ such that

$$\mathbb{P}\big[\|C(s)\|_{\max} \leq \delta \,|\, x(s)\big] \geq p_1.$$

Corollary V.1: For the system (32), assume all conditions in Theorem V.1 and Assumption 5 are satisfied. If the tapering step size sequence $\{a(s)\}_{s \in \mathbb{Z}_{>0}}$ satisfies $a(s) = \frac{d_2}{s+1}$ with $d_2 \in (0, 1]$, then, for any $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, the solution $x(s)$ a.s. converges to $\widetilde{x}^*$, i.e., the solution to equation (34).

The analysis of the DF model with random interactions and environment noise (32) is much more complicated than that for the system (29). Therefore, to prove Theorem V.1, we adopt the so-called ODE method of stochastic approximation. The proofs of Theorem V.2 and Corollary V.1 are postponed to Appendices A and B respectively.

A. *Preliminary: A basic stochastic approximation theorem*

We here review a basic convergence theorem for the ODE method in stochastic approximation, taken from Chapter 5.2 in [21]. Let $\theta(t) \in \mathbb{R}^n$ be a state vector updated by

$$\theta(t+1) = \Pi_H\big(\theta(t) + a(t)Y(t)\big), \tag{37}$$

where $\Pi_H(\cdot)$ is the projection onto a constraint set $H = \{\theta \in \mathbb{R}^n \mid \underline{b}_i \leq \theta_i \leq \bar{b}_i, i \in \{1, \ldots, n\}\}$ with $\underline{b}_i < \bar{b}_i$ being constants, and $\{a(t)\}_{s \in \mathbb{Z}_{>0}}$ is the tapering step size sequence. The projection Π_H is to restrict $\theta(t)$ to a bounded region and has the following property: if $\theta \in H$ then $\Pi_H(\theta) = \theta$. Assume:

(i) $\sup_{t\geq 0} \mathbb{E}[\|Y(t)\|_\infty^2] < \infty$;
(ii) $a(t) \geq 0, \sum_{t=0}^\infty a(t) = \infty$, and $\sum_{t=0}^\infty a^2(t) < \infty$; and
(iii) there is a continuous function $h(\cdot)$ of θ and random variables $\beta(t)$ such that $\sum_{t\geq 0} a(t)\|\beta(t)\|_\infty < \infty$ and

$$\mathbb{E}[Y(t)\,|\,\theta_0, Y(i), i < t] = h(\theta(t)) + \beta(t).$$

A fundamental method to analyze the system (37) is to construct an ODE whose dynamics are projected onto H:

$$\dot{\theta}(\tau) = h(\theta(\tau)) + z(\theta(\tau)), \tag{38}$$

where $z(\theta(\tau))$ is the projection or constraint term, i.e., the minimal term needed to keep $\theta(\tau)$ in H. If $x(\tau)$ is an interior point of H, then $z(x(\tau)) = \mathbf{0}$. Let L_H be the set of limit points of the ODE (38), i.e., $L_H := \{\theta \in H : h(\theta) + z(\theta) = \mathbf{0}\}$. The following lemma builds a connection between the protocol (37) and the ODE (38).

Lemma V.1 (Theorem 5.2.1 in [21]): For system (37), suppose the conditions i), ii) and iii) hold. For the ODE (38), let L_H^1 be a subset of L_H and let A_H be a set that is locally asymptotically stable in the sense of Lyapunov. If for any initial state not in L_H^1 the trajectory of (38) goes to A_H, then the system (37) converges to $L_H^1 \cap A_H$ a.s. for any initial state.

B. Proof of Theorem V.1

We start by verifying that system (32) satisfies the conditions of Lemma V.1. Let

$$Y(s) := \pi(W(x(s))) + \zeta(s) - x(s),$$

then by (32) we get

$$x(s+1) = x(s) + a(s)Y(s). \tag{39}$$

By Assumption 4 we have $x(s) \in \Delta_n$ a.s. for all $s \geq 0$, so (39) is a special form of the system (37) when we choose $H = [-1, 2]^n \supset \Delta_n$. The conditions (i) in Subsection V-A is guaranteed by the fact that Δ_n is a bounded set, and (ii) in Subsection V-A is guaranteed by Assumption 4.

Replacing the vector c in the definition of the function g in Section III-B by the vector $\tilde{c}$ in (33), we get

$$\mathbb{E}\left[Y(s)\,|\,x(0), C(t), \zeta(t), t < s\right] = g(x(s)) + \beta(s)$$

for any $x(0) \in \Delta_n \backslash \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $s \geq 0$, where $\beta(s)$ satisfies $\sum_{s=0}^\infty a(s)\|\beta(s)\|_\infty < \infty$ a.s. by (33). Moreover, a.s. $x(s)$ cannot go out of Δ_n and $g(x)$ is continuous for $x \in \Delta_n$, so we can take Δ_n as the full space and the condition (iii) in Subsection V-A still holds.

It remains to verify the conditions of Lemma V.1 for the ODE (38). From Lemma III.4 (replacing c with $\tilde{c}$) and Lemma III.5 (replacing c and x^* with $\tilde{c}$ and $\tilde{x}^*$), we know that the solution of the ODE (16) converges and never goes to the boundary of $H = [-1, 2]^n$, and it is a special form of the ODE (38). Thus, $x(s)$ in both the systems (39) and (16) cannot leave Δ_n a.s., and we can take Δ_n as the full space. By Lemma II.4 the solution set of the equation $g(x) = 0$ in Δ_n is $\{\tilde{x}^*, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Also, for the ODE (16), from Lemma III.5 we get $\tilde{x}^*$ is locally asymptotically stable, and for any initial state $x(0) \notin \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the trajectory goes to $\tilde{x}^*$. By Lemma V.1, we obtain that $x(s)$ in protocol (39) converges to $\{\tilde{x}^*, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ a.s.

VI. Conclusion

This paper introduces multiple versions of the DeGroot-Friedkin model. We consider switching interaction matrices and individual memories and, in doing so, we generalize the original deterministic DF model to more realistic and richer models. Fluctuation and memory are natural phenomena when investigating the dynamics of opinions and appraisal over social networks. Exact evaluations and absence of reflection are less likely to occur in real human world.

We have presented several novel analysis results for these variations of the original DF model. First, we have derived the convergence rate for the original DF model when the associated digraph is or is not a star network, and we applied these results to analyze the DF models with switching interactions. We proved that the original DF model has an exponentially fast convergence rate when the digraph is not a star network. Then, we proposed a DF model with switching interactions. Again, two cases were considered: in the star network case, the social power converges to autocracy; while in the non-star network case, individuals' social power can only converge into a ball centered around the equilibrium point of the original DF model with the same relative interaction matrix. Thirdly, we proposed a DF model with memory on random interactions and proved that this new model converges to the same equilibrium of the original DF model almost surely. Finally, a stochastic-approximation DF model was proposed by considering random interactions, environment noise, and self-appraisal memory simultaneously. For this most general, complicated, and realistic model, we proved that, under appropriate technical assumptions, the trajectories of individuals' social power also converges.

Much work still remains to be done. First, one can simplify condition (33) in Theorem V.1 which is admittedly not easy to verify. Secondly, it would be interesting and valuable to extend the analysis to reducible graphs as Jia *et. al* considered in [19]. Furthermore, we can also consider similar extensions to other representative appraisal models in the literature such as the modified DF model in [26] and the continuous-time self-appraisal model in [4].

Appendix A
Proof of Theorem V.2

From Theorem V.1, to obtain our result, we just need to show that a.s. $x(s)$ cannot converge to $\mathbf{e}_i$ for any $1 \leq i \leq n$. We will prove this result by contradiction. Without loss of generality we assume $\mathbb{P}[\lim_{s\to\infty} x(s) = \mathbf{e}_1] > 0$, then for any $\varepsilon > 0$ we have

$$\lim_{s\to\infty} \mathbb{P}\left[\bigcap_{s'=s}^{\infty} \{x_1(s') > 1 - \varepsilon\}\right] \geq \mathbb{P}\left[\lim_{s\to\infty} x(s) = \mathbf{e}_1\right] > 0. \tag{40}$$

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By (40), there exists a constant $s_0 > 0$ such that

$$\mathbb{P}\left[\bigcap_{s=s_0}^{\infty}\{x_1(s) > 1-\varepsilon\}\right] > 0. \tag{41}$$

We will show (41) does not hold for some s_0 and ε. Let

$$I_s := I_{\{\pi_1(W(x(s)))+\zeta_1(s)\le \frac{c'}{c'+(1-c')(1-x_1(s))}\}}, \tag{42}$$

where c' is the same constant appearing in (35). By the Lemma C.2 in Appendix C, there exists a constant $p_2 > 0$ such that for any $s \ge 0, \Delta_s > 0$ and $x(s) \in \Delta_n^o$,

$$\mathbb{P}\left[\sum_{s'=s}^{s+\Delta_s-1} I_{s'} \le \frac{p_1\Delta_s}{2}\right] \le e^{-p_2\Delta_s}, \tag{43}$$

where p_1 is the same constant appearing in (35). We set $s_0 := \lceil \max\{d_1d_2\gamma, (1-2c')d_2(\gamma-1), 4/p_2\}\rceil$ and

$$\varepsilon := \min\left\{\frac{1-2c'}{2(1-c')}, [(2\gamma-1)d_1d_2]^{-\frac{1}{\gamma-1}}, \left(\frac{p_1(1-2c')}{2(2-p_1)d_1}\right)^{\frac{1}{\gamma-1}}\right\}.$$

For large s and $i \ge 1$, set A_i^s to be the event that $N_i^s > \frac{p_1}{2}\lceil\frac{2}{p_2}\log s\rceil$ with

$$N_i^s := \sum_{s'=s_0+(i-1)\lceil\frac{2}{p_2}\log s\rceil}^{s_0+i\lceil\frac{2}{p_2}\log s\rceil-1} I_{s'}.$$

From (43) we can get

$$\begin{aligned}\mathbb{P}\Bigg[\bigcap_{i=1}^{\lfloor\frac{p_2 s}{2\log s}\rfloor} A_i^s\Bigg] &\ge \left(1-e^{-p_2\lceil\frac{2}{p_2}\log s\rceil}\right)^{\lfloor\frac{p_2 s}{2\log s}\rfloor}\\ &= 1-\frac{p_2}{2s\log s}+o(s^{-2}).\end{aligned} \tag{44}$$

Let $y(s_0) = x_1(s_0)$ and

$$y(s'+1) = y(s') + \frac{d_2}{s'+1}\begin{cases} -(1-2c')(1-y(s')), & \text{if } I_{s'}=1\\ d_1(1-y(s'))^\gamma, & \text{otherwise}\end{cases}$$

for $s' \ge s_0$. We show that if $\bigcap_{s'=s_0}^{\infty}\{x_1(s') > 1-\varepsilon\}$ happens, then a.s. $y(s') \ge x_1(s')$ for all $s' \ge s_0$. First, $y(s_0) = x_1(s_0)$; if $y(s') \ge x_1(s')$, then for the case when $I_{s'} = 1$, by (42), we have

$$\begin{aligned}y(s'+1)-x_1(s'+1) \ge \frac{d_2}{s'+1}\Big(&y(s')-(1-2c')(1-y(s'))\\ &-\frac{c'}{c'+(1-c')(1-x_1(s'))}\Big).\end{aligned} \tag{45}$$

Also, since $y(s') > 1-\varepsilon \ge \frac{1}{2(1-c')}$,

$$\begin{aligned}y(s')-&\frac{c'}{c'+(1-c')(1-x_1(s'))}\\ &\ge y(s')-\frac{c'}{c'+(1-c')(1-y(s'))}\\ &=\frac{-c'+y(s')(1-c')}{c'+(1-c')(1-y(s')}(1-y(s'))\\ &>\frac{-c'+\frac{1}{2}}{c'+(1-c')(1-\frac{1}{2(1-c')})}(1-y(s'))\\ &=(1-2c')(1-y(s')).\end{aligned}$$

Substituting this into (45) we have $y(s'+1) \ge x_1(s'+1)$. Also, for the case when $I_{s'} = 0$, by (36) and Lemma C.3 i) we have a.s.

$$\begin{aligned}y(s'+1)&-x_1(s'+1)\\ &\ge \frac{d_2}{s'+1}\Big(d_1(1-y(s'))^\gamma - d_1(1-x_1(s'))^\gamma\Big) \ge 0.\end{aligned}$$

By induction we get $y(s') \ge x_1(s')$ a.s. for all $s' \ge s_0$.

Next we estimate the maximum possible value of $y(s')$. Set $s_1 := s_0 + \lfloor\frac{p_2 s}{2\log s}\rfloor\lceil\frac{2}{p_2}\log s\rceil$. Let $z(s_0) = x_1(s_0)$. For $s_0 \le s' < s_1$, set

$$\widetilde{s'} := (s'-s_0) \mod \lceil\frac{2}{p_2}\log s\rceil$$

and let

$$z(s'+1) = z(s') + \frac{d_2}{s'+1}\cdot \begin{cases} d_1(1-z(s'))^\gamma, & \text{if } \widetilde{s'} < \lceil\frac{2}{p_2}\log s\rceil - N_i^s,\\ -(1-2c')(1-z(s')), & \text{if } \widetilde{s'} \ge \lceil\frac{2}{p_2}\log s\rceil - N_i^s.\end{cases} \tag{46}$$

We will show that if $\bigcap_{s'=s_0}^{\infty}\{y(s') > 1-\varepsilon\}$ happens, then $y(s') \le z(s')$ for $s_0 \le s' \le s_1$. In fact, by Lemma C.3 iii) in Appendix C it can be deduced directly that $y(s') \le z(s')$ for $s' \in (s_0, s_0+\lceil\frac{2}{p_2}\log s\rceil]$, and by Lemma C.3 i) and iii) in Appendix C we can get $y(s') \le z(s')$ for $s' \in (s_0+\lceil\frac{2}{p_2}\log s\rceil, s_0+2\lceil\frac{2}{p_2}\log s\rceil]$. By repeating this process we get $y(s') \le z(s')$ for all $s_0 \le s' \le s_1$.

In the following part we estimate the value of $z(s_1)$ under the events $\bigcap_{i=1}^{\lfloor\frac{p_2 s}{2\log s}\rfloor} A_i^s$ and $\bigcap_{s'=s_0}^{s_1}\{z(s') > 1-\varepsilon\}$. Let

$$\delta_1 := d_1\varepsilon^{\gamma-1} > d_1(1-z(s'))^{\gamma-1}.$$

and $\bar{z}(s') := 1-z(s')$. For any $i \in [1, \lfloor\frac{p_2 s}{2\log s}\rfloor]$ and $s' \in [s_0+(i-1)\lceil\frac{2}{p_2}\log s\rceil, s_0+i\lceil\frac{2}{p_2}\log s\rceil - N_i^s)$, by (46) we get

$$\begin{aligned}\bar{z}(s'+1) &= \bar{z}(s') - \frac{d_2d_1}{s'+1}\bar{z}(s')^\gamma\\ &> \bar{z}(s') - \frac{d_2\delta_1}{s_0+(i-1)\lceil\frac{2}{p_2}\log s\rceil}\bar{z}(s'),\end{aligned}$$

and then

$$\begin{aligned}\bar{z}\big(s_0+i\lceil\frac{2}{p_2}\log s\rceil - N_i^s\big) &> \bar{z}\big(s_0+(i-1)\lceil\frac{2}{p_2}\log s\rceil\big)\\ &\times\left(1-\frac{d_2\delta_1}{s_0+(i-1)\lceil\frac{2}{p_2}\log s\rceil}\right)^{\lceil\frac{2}{p_2}\log s\rceil-N_i^s}.\end{aligned} \tag{47}$$

Similarly, for $s' \in [s_0+i\lceil\frac{2}{p_2}\log s\rceil - N_i^s, s_0+i\lceil\frac{2}{p_2}\log s\rceil)$, by (46) we get

$$\begin{aligned}\bar{z}(s'+1) &= \bar{z}(s') + \frac{d_2(1-2c')}{s'+1}\bar{z}(s')\\ &\ge \bar{z}(s') + \frac{d_2(1-2c')}{s_0+i\lceil\frac{2}{p_2}\log s\rceil}\bar{z}(s')\end{aligned}$$

and then

$$\begin{aligned}\bar{z}\big(s_0+i\lceil\frac{2}{p_2}\log s\rceil\big) &\ge \bar{z}\big(s_0+i\lceil\frac{2}{p_2}\log s\rceil - N_i^s\big)\\ &\times\left(1+\frac{d_2(1-2c')}{s_0+i\lceil\frac{2}{p_2}\log s\rceil}\right)^{N_i^s}.\end{aligned} \tag{48}$$

From (47), (48), and the assumption that $\bigcap_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} A_i^s$ happens we get

$$\bar{z}(s_1) > \bar{z}(s_0) \prod_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} \left[\left(1+\frac{d_2(1-2c')}{s_0+i\lceil \frac{2}{p_2}\log s\rceil}\right)^{N_i^s} \times \left(1-\frac{d_2\delta_1}{s_0+(i-1)\lceil \frac{2}{p_2}\log s\rceil}\right)^{\lceil \frac{2}{p_2}\log s\rceil - N_i^s} \right]$$
$$> \bar{z}(s_0) \prod_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} \left[\left(1+\frac{d_2(1-2c')}{s_0+i\lceil \frac{2}{p_2}\log s\rceil}\right)^{\frac{p_1}{2}\lceil \frac{2}{p_2}\log s\rceil} \times \left(1-\frac{d_2\delta_1}{s_0+(i-1)\lceil \frac{2}{p_2}\log s\rceil}\right)^{\lceil \frac{2}{p_2}\log s\rceil - \frac{p_1}{2}\lceil \frac{2}{p_2}\log s\rceil} \right] \quad (49)$$

We can compute that

$$\sum_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} \frac{p_1}{2}\lceil \frac{2}{p_2}\log s\rceil \log\left(1+\frac{d_2(1-2c')}{s_0+i\lceil \frac{2}{p_2}\log s\rceil}\right)$$
$$= \frac{p_1}{2}\lceil \frac{2}{p_2}\log s\rceil \sum_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} \frac{d_2(1-2c')}{s_0+i\lceil \frac{2}{p_2}\log s\rceil}(1+o(1))$$
$$= \frac{p_1 d_2(1-2c')}{2}\big(\log s - \log\log s\big)(1+o(1)),$$

and similarly

$$\sum_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} \left(\lceil \frac{2}{p_2}\log s\rceil - \frac{p_1}{2}\lceil \frac{2}{p_2}\log s\rceil\right) \cdot \log\left(1-\frac{d_2\delta_1}{s_0+(i-1)\lceil \frac{2}{p_2}\log s\rceil}\right)$$
$$\geq -\left(1-\frac{p_1}{2}\right)d_2\delta_1\left[\frac{2\log s}{p_2 s_0} + \big(\log s - \log\log s\big)\right](1+o(1))$$
$$\geq -\left(1-\frac{p_1}{2}\right)d_2\delta_1\left[\frac{3}{2}\log s - \log\log s\right](1+o(1)),$$

so by the condition that

$$\varepsilon \leq \left(\frac{p_1(1-2c')}{2(2-p_1)d_1}\right)^{\frac{1}{\gamma-1}} \iff \delta_1 \leq \frac{p_1(1-2c')}{2(2-p_1)}$$
$$\iff \left(1-\frac{p_1}{2}\right)\frac{3d_2\delta_1}{2} \leq \frac{3p_1 d_2(1-2c')}{8}$$

and (49), we get $1-\lim_{s\to\infty} z(s_1) = \lim_{s\to\infty}\bar{z}(s_1) = \infty$. Combining this equality with (44), we obtain

$$\mathbb{P}\Big[\bigcap_{s'=s_0}^{\infty}\{x_1(s')>1-\varepsilon\}\Big] \leq \lim_{s\to\infty}\Bigg\{\mathbb{P}\Big[\Big[\bigcap_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} A_i^s\Big]^c\Big] + \mathbb{P}\Big[\Big[\bigcap_{s'=s_0}^{\infty}\{x_1(s')>1-\varepsilon\}\Big]\cap\Big[\bigcap_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} A_i^s\Big]\Big]\Bigg\}$$
$$\leq \lim_{s\to\infty}\mathbb{P}\Big[\Big[\bigcap_{s'=s_0}^{\infty}\{y(s')>1-\varepsilon\}\Big]\cap\Big[\bigcap_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} A_i^s\Big]\Big]$$
$$\leq \lim_{s\to\infty}\mathbb{P}\Big[\Big[\bigcap_{s'=s_0}^{\infty}\{z(s')>1-\varepsilon\}\Big]\cap\Big[\bigcap_{i=1}^{\lfloor \frac{p_2 s}{2\log s}\rfloor} A_i^s\Big]\Big]$$
$$= 0,$$

which is contradictory with (41).

Appendix B
The Proof of Corollary V.1

Let $c(s) = (c_1(s),\dots,c_n(s))$ denote the left dominant eigenvector of $C(s)$, then by Assumption 3 and Lemma II.2 we have $c_i(s)\leq 1/2$ for any $i\in\{1,\dots,n\}$ a.s. Then, for any $x(s)\in\Delta_n^0$, by Lemma II.1 and Assumption 5 i), we have a.s.

$$\pi_i(W(x(s)))+\zeta_i(s)$$
$$\leq \frac{c_i(s)/(1-x_i(s))}{\sum_{j=1}^n c_j(s)/(1-x_j(s))} + d_1(1-x_i(s))^\gamma$$
$$< \frac{1}{2-x_i(s)} + d_1(1-x_i(s))^\gamma$$
$$= \frac{(1-x_i(s))^2}{2-x_i(s)} + x_i(s) + d_1(1-x_i(s))^\gamma$$
$$< x_i(s) + (1+d_1)(1-x_i(s))^{\min\{\gamma,2\}}, \quad (50)$$

which implies the condition (36) holds.

By the proof of Theorem V.2 we need to verify the condition (35) when $x(s)$ is close to $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$. We consider the case that $x_i(s)\geq 1-(2(1+\delta)d_1)^{\frac{-1}{\gamma-1}}$. First, if $\|C(s)\|_{\max}\leq\delta<1$, then

$$c_i(s) = \sum_{j\neq i} c_j(s)C_{ji}(s) \leq (1-c_i(s))\delta \iff c_i(s)\leq\frac{\delta}{1+\delta}. \quad (51)$$

Let $c' := \frac{1+2\delta}{2(1+\delta)}$. If $c_i(s)\leq\frac{\delta}{1+\delta}$ holds, then similar to (50) we have

$$\pi_i(W(x(s)))+\zeta_i(s)$$
$$< \frac{\delta/(1+\delta)}{\frac{\delta}{1+\delta}+(1-\frac{\delta}{1+\delta})(1-x_i(s))} + d_1(1-x_i(s))^\gamma$$
$$= \frac{c'}{c'+(1-c')(1-x_i(s))} + d_1(1-x_i(s))^\gamma - \Big[c'\Big(1-\frac{\delta}{1+\delta}\Big)-\frac{\delta}{1+\delta}(1-c')\Big](1-x_i(s))$$
$$\leq \frac{c'}{c'+(1-c')(1-x_i(s))}. \quad (52)$$

By Assumption 5 ii), (51) and (52) we get (35) holds.

Appendix C
Some Lemmas

Lemma C.1: Suppose the non-negative real number sequence $\{y_s\}_{s\geq 1}$ satisfies

$$y_{s+1}\leq(1-a_s)y_s+b_s, \quad (53)$$

where $b_s\geq 0$ and $a_s\in[0,1)$ are real numbers. If $\sum_{s=1}^\infty a_s=\infty$ and $\lim_{s\to\infty} b_s/a_s=0$, then $\lim_{s\to\infty} y_s=0$ for any $y_1\geq 0$.

Proof: Repeating (53) we get

$$y_{s+1}\leq y(1)\prod_{t=1}^s(1-a_t)+\sum_{i=1}^s b_i\prod_{t=i+1}^s(1-a_t).$$

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Here we define $\prod_{t=i}^{s}(\cdot) := 1$ when $i > s$. Because $\sum_{t=1}^{\infty} a_t = \infty$, we have $\prod_{t=1}^{\infty}(1-a_t) = 0$. Then, we just need to prove that

$$\lim_{s\to\infty} \sum_{i=1}^{s} b_i \prod_{t=i+1}^{s} (1-a_t) = 0. \tag{54}$$

Since $\lim_{s\to\infty} b_s/a_s = 0$, for any real number $\varepsilon > 0$, there exists an integer $s^* > 0$ such that $b_s \le \varepsilon a_s$ when $s \ge s^*$. Thus,

$$\begin{aligned}\sum_{i=1}^{s} b_i \prod_{t=i+1}^{s} (1-a_t) \\ &\le \sum_{i=1}^{s^*-1} b_i \prod_{t=i+1}^{s} (1-a_t) + \sum_{i=s^*}^{s} \varepsilon a_i \prod_{t=i+1}^{s} (1-a_t) \\ &= \sum_{i=1}^{s^*-1} b_i \prod_{t=i+1}^{s} (1-a_t) + \varepsilon\Big(1 - \prod_{t=s^*}^{s}(1-a_t)\Big) \\ &\to \varepsilon \quad \text{as} \quad s\to\infty, \end{aligned} \tag{55}$$

where the first equality uses the classical equality

$$\sum_{i=1}^{t} c_i \prod_{j=i+1}^{t} (1-c_j) = 1 - \prod_{i=1}^{t}(1-c_i)$$

with $\{c_i\}$ being any complex numbers, which can be obtained by induction. Let ε decrease to 0, then (55) implies (54). □

Lemma C.2: There exists a constant $p_2 > 0$ such that for any $s \ge 0$, $s_1 > 0$ and $x(s) \in \Delta_n^o$,

$$\mathbb{P}\left[\sum_{s'=s}^{s+s_1-1} I_{s'} \le \frac{p_1 s_1}{2}\right] \le e^{-p_2 s_1},$$

where I_s is defined by (42) and p_1 is the same constant appearing in (35).

Proof: From (35) we have for any $s \ge 0$ and $x(s) \in \Delta_n^o$,

$$\mathbb{P}\left[I_s = 1\right] \ge p_1. \tag{56}$$

By Markov's inequality we have for any $s \ge 0$, $s_1 > 0, \theta > 0$ and $x(s) \in \Delta_n^o$,

$$\begin{aligned}\mathbb{P}\left[\sum_{s'=s}^{s+s_1-1} I_{s'} \le \frac{p_1 s_1}{2}\right] \\ &= \mathbb{P}\left[\exp\Big(-\theta \sum_{s'=s}^{s+s_1-1} I_{s'}\Big) \ge e^{-\theta p_1 s_1/2}\right] \\ &\le e^{\frac{\theta p_1 s_1}{2}} \mathbb{E}\Big[\exp\Big(-\theta \sum_{s'=s}^{s+s_1-1} I_{s'}\Big)\Big] \\ &= e^{\frac{\theta p_1 s_1}{2}} \mathbb{E}\Big[\prod_{s'=s}^{s+s_1-1} e^{-\theta I_{s'}}\Big]. \end{aligned} \tag{57}$$

Also, we can get

$$\begin{aligned}\mathbb{E}\Big[\prod_{s'=s}^{s+s_1-1} e^{-\theta I_{s'}}\Big] \\ &= \sum_{z_1,z_2} z_1 z_2 \mathbb{P}\Big[\prod_{s'=s}^{s+s_1-2} e^{-\theta I_{s'}} = z_1, e^{-\theta I_{s+s_1-1}} = z_2\Big] \\ &= \sum_{z_1,z_2} z_1 \mathbb{P}\Big[\prod_{s'=s}^{s+s_1-2} e^{-\theta I_{s'}} = z_1\Big] \\ &\quad \cdot z_2 \mathbb{P}\Big[e^{-\theta I_{s+s_1-1}} = z_2 | \prod_{s'=s}^{s+s_1-2} e^{-\theta I_{s'}} = z_1\Big] \\ &= \sum_{z_1} z_1 \mathbb{P}\Big[\prod_{s'=s}^{s+s_1-2} e^{-\theta I_{s'}} = z_1\Big)\Big(1 - (1-e^{-\theta}) \\ &\quad \cdot \mathbb{P}\Big[I_{s+s_1-1} = 1 | \prod_{s'=s}^{s+s_1-2} e^{-\theta I_{s'}} = z_1\Big]\Big) \\ &\le \mathbb{E}\Big[\prod_{s'=s}^{s+s_1-2} e^{-\theta I_{s'}}\Big)\Big(1-(1-e^{-\theta})p_1\Big] \\ &\le \cdots \le \Big(1-(1-e^{-\theta})p_1\Big)^{s_1}, \end{aligned} \tag{58}$$

where the first inequality uses (56). For small positive θ,

$$\frac{\theta p_1}{2} + \log\big(1-(1-e^{-\theta})p_1\big) = -\frac{\theta p_1}{2} + O(\theta^2),$$

so that we can choose suitable $\theta > 0$ and obtain

$$e^{\frac{\theta p_1 s_1}{2}}\Big(1-(1-e^{-\theta})p_1\Big)^{s_1} \le e^{-\frac{\theta p_1 s_1}{3}}.$$

Combining this with (57) and (58) yields

$$\begin{aligned}\mathbb{P}\left[\sum_{s'=s}^{s+s_1-1} I_{s'} \le \frac{p_1 s_1}{2}\right] \\ &\le e^{\frac{\theta p_1 s_1}{2}}\Big(1-(1-e^{-\theta})p_1\Big)^{s_1} \le e^{-\frac{\theta p_1 s_1}{3}}. \end{aligned}$$

□

Lemma C.3: For any $s \ge \max\{d_1 d_2 \gamma, (1-2c')d_2(\gamma-1)\}$ and $x \in [1-[(2\gamma-1)d_1 d_2]^{-\frac{1}{\gamma-1}}, 1)$, define

$$f_s(x) := x + \frac{d_2 d_1 (1-x)^{\gamma}}{s},$$

$$g_s(x) := x - \frac{d_2}{s}(1-2c')(1-x),$$

where d_1, d_2, γ, c' are the same constants appearing in Theorem V.2. Then:
i) f_s and g_s are strictly monotonically increasing functions.
ii) $(f_{s+1} \circ g_s)(x) := f_{s+1}(g_s(x)) < (g_{s+1} \circ f_s)(x)$.
iii) Let $s_2 \ge s_1 > 0$ be arbitrarily given, and set

$$H := h_{s+s_2-1} \circ h_{s+s_2-2} \circ \cdots \circ h_s,$$

where $h_{s'} (s \le s' \le s+s_2-1)$ equals to $f_{s'}$ or $g_{s'}$ and the total number of f is not larger than s_1. If

$$(h_{s'} \circ h_{s'-1} \circ \cdots \circ h_s)(x) \ge 1 - [(2\gamma-1)d_1 d_2]^{-\frac{1}{\gamma-1}}$$

for any $s' \in [s, s+s_2)$, then

$$H(x) \leq (g_{s+s_2-1} \circ \cdots \circ g_{s+s_1} \circ f_{s+s_1-1} \circ \cdots \circ f_s)(x). \quad (59)$$

Proof: i) For any $0 < x_1 < x_2 < 1$, we set $\Delta := x_2 - x_1$ and then

$$\begin{aligned} f_s(x_2) - f_s(x_1) & \\ &= \Delta + \frac{d_1 d_2}{s}(1-x_1)^\gamma \Big[\Big(1 - \frac{\Delta}{1-x_1}\Big)^\gamma - 1\Big] \\ &= \Delta + \frac{d_1 d_2}{s}(1-x_1)^\gamma \sum_{i=1}^{\infty} \binom{\gamma}{i} \Big(-\frac{\Delta}{1-x_1}\Big)^i \\ &> \Delta - \frac{d_1 d_2}{s}\gamma(1-x_1)^{r-1}\Delta > 0 \end{aligned}$$

when $s \geq d_1 d_2 \gamma$. Also, $g_s(x_2) < g_s(x_1)$ holds obviously.
ii) Set $a := 1 - 2c'$ and $z := 1 - x$, then we have

$$\begin{aligned} g_{s+1}&(f_s(x)) \\ &= x + \frac{d_2 d_1 (1-x)^\gamma}{s} - \frac{d_2 a}{s+1}\Big(1 - x - \frac{d_2 d_1 (1-x)^\gamma}{s}\Big) \\ &= x + \frac{d_2 d_1 z^\gamma}{s} - \frac{d_2 a}{s+1}\Big(z - \frac{d_2 d_1 z^\gamma}{s}\Big) \end{aligned} \quad (60)$$

and

$$\begin{aligned} f_{s+1}&(g_s(x)) \\ &= x - \frac{d_2 a}{s}(1-x) + \frac{d_1 d_2}{s+1}\Big(1 - x + \frac{d_2 a}{s}(1-x)\Big)^\gamma \\ &= x - \frac{d_2 a z}{s} + \frac{d_1 d_2}{s+1}\Big(z + \frac{d_2 a}{s} z\Big)^\gamma. \end{aligned} \quad (61)$$

Also, if $s \geq d_2 a(\gamma - 1)$, then

$$\begin{aligned} \Big(1 + \frac{d_2 a}{s}\Big)^\gamma &= 1 + \frac{d_2 a \gamma}{s} + \frac{d_2 a}{s} \sum_{i=2}^{\infty} \binom{\gamma}{i} \Big(\frac{d_2 a}{s}\Big)^{i-1} \\ &< 1 + \frac{d_2 a \gamma}{s} + \frac{d_2 a}{s} \sum_{i=2}^{\infty} \frac{\gamma}{i!} < 1 + \frac{2 d_2 a \gamma}{s}. \end{aligned} \quad (62)$$

By (60), (61) and (62) we have

$$\begin{aligned} g_{s+1}&(f_s(x)) - f_{s+1}(g_s(x)) \\ &= \Big(\frac{1}{s} + \frac{d_2 a}{s(s+1)} - \frac{1}{s+1}\big(1 + \frac{d_2 a}{s}\big)^\gamma\Big) d_1 d_2 z^\gamma \\ &\quad + \Big(\frac{1}{s} - \frac{1}{s+1}\Big) d_2 a z \\ &> \frac{1 + d_2 a - 2\gamma d_2 a}{s(s+1)} d_1 d_2 z^\gamma + \frac{d_2 a z}{s(s+1)} \\ &> \frac{d_2 a z}{s(s+1)}\big(-(2\gamma - 1) d_1 d_2 z^{\gamma-1} + 1\big) \geq 0, \end{aligned}$$

where the last inequality uses the condition that $x \geq 1 - [(2\gamma - 1)d_1 d_2]^{-\frac{1}{\gamma-1}}$.
iii) Let s^* be total number of f in H. For the case when $s_2 = s_1 = s^*$, (59) holds obviously.

For the case when $s_2 > s_1$ and $s^* = s_1$, if H is not equal to $g_{s+s_2-1} \circ \cdots \circ g_{s+s_1} \circ f_{s+s_1-1} \circ \cdots \circ f_s$, we can find $s' \in [s, s+s_2-2]$ such that $h_{s'} = g_{s'}$ and $h_{s'+1} = f_{s'+1}$. Then by ii) we have

$$\begin{aligned} (f_{s'+1} &\circ g_{s'} \circ h_{s'-1} \circ \cdots \circ h_s)(x) \\ &= (f_{s'+1} \circ g_{s'})\left[(h_{s'-1} \circ \cdots \circ h_s)(x)\right] \\ &< (g_{s'+1} \circ f_{s'})\left[(h_{s'-1} \circ \cdots \circ h_s)(x)\right] \\ &= (g_{s'+1} \circ f_{s'} \circ h_{s'-1} \circ \cdots \circ h_s)(x). \end{aligned}$$

Combining this with i) yields

$$\begin{aligned} \big(h_{s+s_2-1} &\circ \cdots \circ h_{s'+2} \circ f_{s'+1} \circ g_{s'} \circ h_{s'-1} \circ \cdots \circ h_s\big)(x) \\ &= (h_{s+s_2-1} \circ \cdots \circ h_{s'+2}) \\ &\qquad \big[(f_{s'+1} \circ g_{s'} \circ h_{s'-1} \circ \cdots \circ h_s)(x)\big] \\ &< (h_{s+s_2-1} \circ \cdots \circ h_{s'+2}) \\ &\qquad \big[(g_{s'+1} \circ f_{s'} \circ h_{s'-1} \circ \cdots \circ h_s)(x)\big] \\ &= \big(h_{s+s_2-1} \circ \cdots \circ h_{s'+2} \\ &\qquad \circ g_{s'+1} \circ f_{s'} \circ h_{s'-1} \circ \cdots \circ h_s\big)(x). \end{aligned}$$

Repeating the above process we get (59).

For the case when $s^* < s_1 \leq s_2$, by the above discussion we have

$$\begin{aligned} \big(h_{s+s_2-1} &\circ h_{s+s_2-2} \circ \cdots \circ h_s\big)(x) \\ &\leq (g_{s+s_2-1} \circ \cdots \circ g_{s+s^*} \circ f_{s+s^*-1} \circ \cdots \circ f_s)(x) \\ &< (g_{s+s_2-1} \circ \cdots \circ g_{s+s_1} \circ f_{s+s_1-1} \circ \cdots \circ f_s)(x), \end{aligned}$$

where the last inequality uses i) and $g_{s'}(x) < f_{s'}(x)$. □

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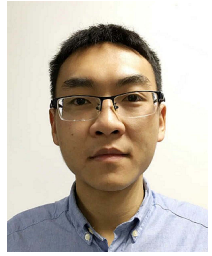

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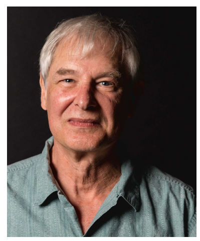

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