SYNCHRONIZATION IN PULSE-COUPLED OSCILLATORS WITH DELAYED EXCITATORY/INHIBITORY COUPLING*

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Abstract. Due to their rich behaviors, pulse-coupled oscillator (PCO) networks have been widely studied. Prior research has focused on systems with excitatory, inhibitory, and mixed excitatory/inhibitory coupling, as well as on systems with and without delays in pulse transmission.

This article focuses on PCO networks with delayed excitatory/inhibitory coupling. We consider a simple phase transition rule and show that the resulting PCO network is a linear time-varying control system with the delays as input disturbances. We define the synchronization error as the length of the arc containing the oscillators' phases. We show that the synchronization error converges exponentially fast to a final value proportional to the maximum transmission delay, under the following sufficient conditions: (i) the coupling strength is sufficiently small, (ii) the network has a globally reachable node, and (iii) the delays are sufficiently small. A corollary to this result is that, when all the delays are zero, the network synchronizes exactly and exponentially fast. We also estimate the rate of convergence, final synchronization error, and basin of attraction of the final state, and analyze special cases where synchronization occurs even in the presence of delays. We then extend the analysis to PCO networks with delayed inhibitory coupling, and identify sufficient conditions for synchronization that are less conservative than those in existing literature.

Key words. pulse-coupled oscillators, clock synchronization, distributed averaging algorithm, sensing digraph

1. Introduction.

1.1. Problem Description.

Coupled oscillator networks consist of individual oscillators (i.e., scalar systems that would each evolve periodically in isolation) and a network of interactions coupling the dynamics of the individual oscillators. We refer to the network as a sensing digraph. The coupling interactions can take place over discrete or continuous time; these two cases are known as pulse coupling and diffusive coupling respectively [17]. In *pulse-coupled oscillator* (PCO) networks, an oscillator sends a pulse to its in-neighbors on the sensing digraph every time it completes an oscillation (i.e., each time it "fires"). Each receiving oscillator experiences a discrete jump in its phase upon reception of the pulse. The phase jump could be forward or backward and depends on the oscilletor's current phase. This jump is defined by a so-called *phase transition rule*. The network is a hybrid system, since its state, comprising the phases of all the oscillators, varies continuously with time except when any oscillator receives a pulse.

PCO network models have been studied extensively since they have been useful both for modeling naturally-occurring phenomena and for technological applications. The oscillators are often assumed to be identical or nearly identical. Two behaviors that are exhibited by the networks have been studied extensively: reaching *synchrony* (where all the oscillators have the same phase), and reaching an *asynchronous* state [28] (where the oscillators have distinct phases whose separation remains constant with time), from arbitrary initial phases. Both types of final states are periodic. Loosely speaking, we refer to the process by which the phase separation converges to zero as *synchronization*, and the process by which the phase separation converges to values bounded by a constant small value as *approximate synchronization*.

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We broadly classify PCO network models in the literature according to the type of coupling (excitatory, inhibitory or mixed) and according to the presence or absence of delay in the transmission of pulses. Excitatory coupling refers to coupling where an oscillator experiences an increase in phase upon pulse reception. Inhibitory coupling refers to coupling where an oscillator experiences a decrease in phase upon pulse reception. Mixed excitatory/inhibitory coupling (referred to in [23, 21] as *advance-delay coupling*) refers to coupling where pulses either cause the phase to jump forward ("advance") or backward ("delay") depending on the current value of the phase. *Delay* in the transmission of pulses refers to the duration between the transmission of a pulse (when an oscillator fires) and the reception of the pulse by an in-neighbor of the sending oscillator. ("Delay" and "delayed coupling" will be used to refer only to transmission delay from now on.) By these two criteria, there are six types of PCO networks. Additionally, for each type of network, the behavior differs based on the type of sensing digraph. The most common types that are studied are all-to-all (complete graph with no self loops), and graphs that satisfy the weaker condition of being strongly connected or strongly rooted.

1.2. Motivation.

Occurrences in Nature. One of the earliest phenomena that sparked interest in PCO networks was the synchronization of the lighting patterns of South Asian fireflies [2]. In this system, the flashes of each firefly act as the coupling signal. Another natural phenomenon, the self-synchronization of the pacemaker cells of the heart, was studied by Peskin in [22]. Both of these phenomena have been modeled as PCO networks with excitatory coupling and no delay. Lastly, the electrical signals of neurons have been modeled extensively as PCO networks with delayed inhibitory coupling [26, 12, 3, 1].

Technological Applications. PCO network-based algorithms have been used for clock synchronization for wireless transceivers [31], in cellular mobile radio [29], robotics [4, 24, 33], wireless sensor networks [11, 10, 6, 25, 30], scheduling [9] and management [5].

Modified Versions of PCO Networks. Modified algorithms have been proposed for improved synchronization properties for technological applications. A *refractory period* is a period during which an oscillator becomes unable to receive signals, right after it fires. It is sometimes incorporated into inhibitory systems to eliminate "echo" effects [16, 14, 20]. *Self adjustment* is an instantaneous self-coupling during firing that is sometimes introduced to enable the system to get closer to synchrony [14].

1.3. PCO Network Models in Literature. The behavior of the six types of PCO networks is summarized in Table 1.3. In most cases, the sensing digraph is assumed to be strongly connected though sometimes the type of sensing digraph is further restricted.

Excitatory Couplings. Mirollo and Strogatz [18] considered all-to-all PCO networks with excitatory coupling and no delay; their sufficient conditions guarantee synchrony is reached from almost all initial condition. Numerical simulations in the same paper indicate that also systems defined over a strongly connected digraph reach synchrony (albeit in a slower fashion); this statement remains a conjecture.

Interestingly, the behavior of such systems changes completely with the introduction of even the smallest delay. Such a system reaches an asynchronous state from all initial conditions, as shown in [7, 27]. In [32] it is proved that synchrony in this case is in fact impossible. This makes excitatory couplings unsuitable in practical applications where exact synchronization is required, since small delays are inevitable.

TABLE 1
An abbreviated partial outline of the literature on the six kinds of PCO networks.

	Excitatory coupling	Inhibitory coupling	Mixed excitatory/inhibitory coupling
With no delays	[18]: Sufficient conditions for all-to-all PCOs to asymptotically synchronize from almost all initial conditions	[12, 15]: Coexistence of synchrony and asynchronous states. Sufficient conditions for all-to-all PCOs to asymptotically synchronize from all initial conditions.	[23, 21]: Sufficient conditions for PCOs with GRN to asymptotically synchronize for initial conditions within an half circle.
With delays	[32, 7, 27]: Asynchronous states are reached from all initial conditions and synchronization is impossible.	[28, 27]: Coexistence of synchrony and asynchronous states. Sufficient conditions for synchronization for large delays and small initial synchronization error.	[20]: Similar to above, if network is an undirected cycle, or a directed cycle with refractory periods.

Inhibitory Couplings. For PCO networks with inhibitory coupling and no delay, both synchrony and asynchronous states are possible, depending on the initial conditions. In [15], sufficient conditions are derived under which all-to-all networks reach synchrony from all initial conditions. However, for general networks, asynchronous states can occur as described in [12].

PCO networks with delayed inhibitory coupling also exhibit a coexistence of the asynchronous state and synchrony, depending on the initial conditions as described in [28]. In [27] it is proved that the synchronous state is locally stable if the delay is greater than some threshold. There is no proof however for arbitrarily small delays.

Mixed Excitatory/Inhibitory Couplings. For PCO networks with mixed excitatory/inhibitory couplings and no delay, [23] derives sufficient conditions for synchrony for PCO networks whose sensing digraph has a globally reachable node (GRN). (Note that the sensing digraph is the inverse of the interaction digraph used in [23].) For a specific choice of phase response curve, it is shown in [21] that, in the absence of delays, the system reaches synchrony for all initial conditions if the coupling strength is above a certain threshold. If the coupling strength is below the threshold, it synchronizes only from initial conditions of phases contained within an half circle.

For systems with delays, it is proved in [20] that similar behavior to the above occurs for the special case of undirected cycle graphs. For directed cycle graphs, a refractory period has to be added to one of the oscillators to ensure that exact synchrony is reached.

1.4. Contribution. In this article we consider a simple PCO network model with a piece-wise linear phase transition rule similar to that described by [21]. The contributions of this article are as follows. Firstly, we study the timing behavior of the firings and receptions in the PCO network, and establish finite upper and lower bounds on the the durations between firings of a given oscillator, on the duration taken for a pulse to be received along every edge of the sensing digraph, and on the maximum number of receptions by a given oscillator in a sequence of successive receptions. Secondly, we present a novel method of analysis for PCO networks based on the study of time-varying distributed averaging systems. While this method is applied to a specific model, we hope it may be relevant in the study of other PCO models.

Thirdly, we derive sufficient conditions on the PCO network system for approximate synchronization to occur exponentially fast, in the presence of delays. The sufficient conditions are: (i) the coupling strength is sufficiently small, (ii) the sensing digraph has a GRN, and (iii) the delays are sufficiently small. Among these, the condition on the sensing digraph is less conservative than those in existing literature on PCO networks with delays, which typically require strongly connected digraphs. We also estimate the rate of convergence, the final synchronization error, and basin of attraction of the final asynchronous state.

Our fourth contribution is then to derive sufficient conditions for exact synchronization of a PCO network with no delays, as a special case of the above analysis. These conditions are consistent with the existing contemporary literature such as the work in [23]. We also show that synchronization in this case occurs exponentially fast, which is a new result. As our fifth contribution, we analyze another special case with uniform delays and equal out-degrees, where intermittent synchronization occurs even in the presence of delays. Sixth, we apply our method of analysis to PCO networks with delayed inhibitory coupling, to demonstrate the applicability of the method to other PCO models. We then extend the results of [27] on intermittent synchronization to PCO networks that are not necessarily strongly connected.

1.5. Organization of this Article. The rest of this article is organized as follows. In Section 2 we discuss the PCO network model and related notions that form the background for subsequent sections. In Section 3, we estimate the durations between successive firings of an oscillator, and establish some useful properties of sequences of successive receptions by any oscillator. These properties allow us to rewrite the system dynamics as an linear time-varying (LTV) control system in Section 4. In Section 4, we prove the main results of the article, on approximate synchronization in the presence of sufficiently small delays and on synchronization in the absence of delays. In Section 5, we provide example applications and extensions of the method of PCO network analysis introduced, to the cases of uniform delays and equal out-degrees, and PCO networks with delayed inhibitory coupling. Section 6 contains numerical simulations that verify the analytic results.

2. Oscillator Networks: Model and Related Notions. PCO networks have been studied by researchers in multiple disciplines, so there exists some differing usage of terminology in the literature. For clarity, in this section, we provide the models and definitions that we use. We consider a PCO network with mixed excitatory/inhibitory coupling similar to that described in [21], where the phase of the oscillator jumps toward 0 (or 1, whichever is nearer), by a distance proportional to the shorter arc from its phase to 0, whenever it receives a pulse. In our most general theorem, we study the case with delays, that is, the problem in the second row, third column of Table 1.3.

2.1. PCO Network Model. The *PCO network* consists of n oscillators. Each oscillator $i \in \{1, \dots, n\}$ has a phase that evolves on a circle of unit circumference. The phase, denoted by $\phi_i(t) \in [0, 1[$, is the length of the counter-clockwise arc from the positive horizontal axis to the state of the oscillator, as shown in Figure 2.1. In what follows, we regard the circle of unit circumference equivalent to the interval $[0, 1[$. Each oscillator obeys a hybrid dynamics with continuous-time evolution on the circle, discrete jumps due to pulses, and a discrete *reset* to 0 on firing, when the phase reaches 1. The continuous-time dynamics is described by

$$\dot{\phi}_i(t) = 1. \tag{2.1}$$

The discrete-time dynamics is described as follows:

(i) when the phase ϕ_i reaches 1, it is reset to 0 and the oscillator i sends a pulse to each of its in-neighbors in a *sensing digraph* $\mathcal{G}$;

(ii) assuming a pulse is sent from oscillator i at time t, it is received by its in-neighbor j at time $t + \tau_{ij}$, where τ_{ij} is a non-negative *delay*;

(iii) assuming oscillator i receives a pulse at time t, it jumps to a new phase $\phi_i(t^+)$ according to the following *phase transition rule*:

$$\phi_i(t^+) = \begin{cases} \phi_i(t) - h\phi_i(t), & \text{if } \phi_i(t) \in [0, \frac{1}{2}[, \\ \phi_i(t) + h(1 - \phi_i(t)), & \text{if } \phi_i(t) \in [\frac{1}{2}, 1[, \end{cases} \tag{2.2}$$

where $h \in]0, 1[$ is a *coupling strength*.

An example sensing digraph is illustrated in Figure 2.2. Note that the sensing digraph is an unweighted digraph $\mathcal{G}$ with vertices $\mathcal{V} = \{1, \dots, n\}$ and with edge set defined as follows: (i, j) is an edge if $i \neq j$ and i can receive a pulse from j.

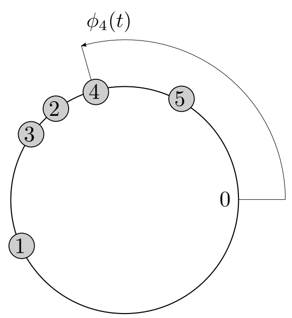

FIG. 2.1. *Phase of oscillator 4, $\phi_4(t)$*

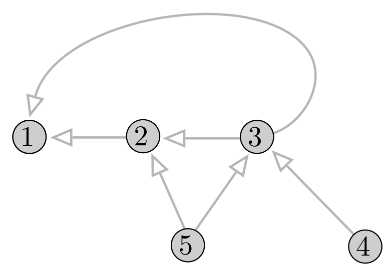

FIG. 2.2. *An example of a sensing digraph, $\mathcal{G}$.*

2.2. Notions related to synchronization.

Distances on the unit-circle. The *counterclockwise arc-length* $\mathrm{dist}_{\mathrm{cc}}(\phi_i, \phi_j)$ is the length of the counter-clockwise arc from ϕ_i and ϕ_j. In the parametrization described above:

$$\mathrm{dist}_{\mathrm{cc}}(\phi_i, \phi_j) = (\phi_j - \phi_i) \bmod 1.$$

Arc length function. Define the *arc-length function* $V_{\text{arc-length}} : \mathbb{T}^n \to [0, 1[$ so that $V_{\text{arc-length}}(\phi)$ is the length of the shortest arc containing every element of ϕ. Note that the arc-length function is independent of the coordinate system used to express the phases, and if $0_n \leq \phi \leq \frac{1}{2}\mathbb{1}_n$, then

$$V_{\text{arc-length}}(\phi) = \max(\phi) - \min(\phi).$$

Arc Subsets of the n-Torus. The vector of the phases of the n oscillators takes values in the n-torus, denoted by $\mathbb{T}^n$, that is, the Cartesian product of n copies of the circle of unit circumference. Given a length $\gamma \in [0, 1[$, the *arc subset* $\Gamma(\gamma) \subseteq \mathbb{T}^n$ is the set of n-tuples $(\phi_1, \dots, \phi_n)$ such that there exists an arc of length *strictly* less than γ containing all $\phi_1, \dots, \phi_n$.

Synchrony. A PCO network is at *synchrony* when all n oscillators in the network have the same phase. In other words, synchony is achieved at time t if $V_{\text{arc-length}}(\phi(t)) = 0$. In what follows, we quantify the *synchronization error* at time t as the length $V_{\text{arc-length}}(\phi(t))$ of the smallest arc containing $\phi(t)$.

Synchronization. A PCO network *asymptotically synchronizes* if the synchronization error converges asymptotically to zero, and *exponentially synchronizes* if the synchronization error converges exponentially to zero. Loosely speaking, we shall say that a PCO network approximately synchronizes if the synchronization error is upper bounded by a small positive constant after a finite transient time.

2.3. Graph-theoretical notions.

Reachability depth and other graph properties. Given the sensing digraph $\mathcal{G}$, let $d_{\max}$ be the *maximum out-degree* of any node, let m be the *number of edges*, and, assuming $\mathcal{G}$ has a GRN, let b be the *reachability depth*, i.e., the maximum distance (number of edges of the shortest directed path) from any node to the GRN. Define the digraph $\mathcal{I}$ with vertices $\mathcal{V}$ and edge set comprising a self loop on every node. If $\mathcal{G}$ has a GRN and reachability depth b, then the GRN can be reached from any node by a path in $\mathcal{G} \cup \mathcal{I}$ that has exactly b edges.

Pulse instants. Let $t_1, t_2, \ldots$ be the ordered sequence of times at which a pulse is received by any one of the oscillators. If two or more pulses are received at the same time, then they are assigned distinct time indices $t_p = t_{p+1} = \ldots$ arbitrarily.

Reception digraph. We define the *reception digraph* at $p \in \mathbb{N}$ as a weighted digraph $\mathcal{G}_p$ with vertices $\mathcal{V} = \{1, \ldots, n\}$ and edge set as follows. Every node corresponding to an oscillator that does not receive a pulse at the pulse instant t_p has a self-loop weighted 1. If at time t_p a pulse is received by oscillator i from oscillator j, then node i has a self-loop weighted $1-h$ and there is an edge from i to j weighted h. We note that each reception digraph can be constructed by taking a sub-graph with just one edge of the sensing digraph, adding self loops on every node, and then weighting all the edges. Examples of reception digraphs are shown in Figure 2.3.

Accordingly, let A_p be the adjacency matrix of $\mathcal{G}_p$. Note that A_p is *row-stochastic*, that is, it has all non-negative elements and the sum of elements of each row is 1.

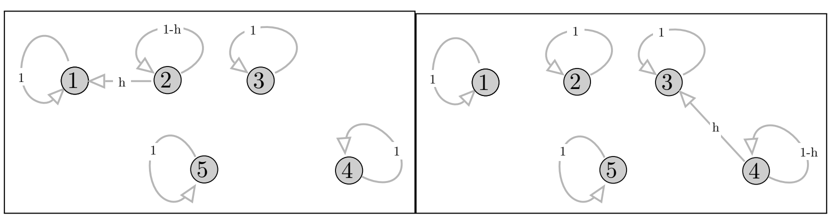

Fig. 2.3. *Two examples of reception digraphs consistent with the sensing digraph in Figure 2.2, where oscillator 2 receives a pulse from oscillator 1 (on the left) and where oscillator 3 receives a pulse from oscillator 4 (on the right).*

3. Durations Between Pulses. In this section we describe the timing of the firings and receptions of pulses in the PCO network, based on the properties of the sensing digraph (namely, degree and number of edges) and the coupling strength. The properties described in this section enable us to write the PCO network system equations as a linear time-varying (LTV) control system sampled at the times when a reception occurs, in subsequent sections.

For a PCO network with constant delays, and coupling strength that is sufficiently small, the duration between successive firings of a given oscillator is upper and lower bounded. A sequence of successive receptions that is sufficiently long must contain at least one reception along each edge of the sensing digraph. Lastly, in a sufficiently long sequence of successive receptions, there is a finite upper bound on the number of receptions by a given oscillator. These statements are formalized quantitatively by Theorem 3.1 below.

THEOREM 3.1 (Upper and lower bounds on receptions by an oscillator in a sequence of successive receptions). *Consider a PCO network with n oscillators, with sensing digraph $\mathcal{G}$ with m edges, with delays τ_{ij} for each edge (i,j), and with coupling strength h. Assume that:*

(A1) *the coupling strength h and the maximum out-degree $d_{\max}$ satisfy $hd_{\max} < 1$.*

Then the following statements hold:

(i) for any oscillator $i \in \{1,\dots,n\}$ and for all $N \in \mathbb{N}$, the duration $T_{i,N}$ between the Nth and $(N+1)$th firings of i satisfies

$$T_{\min} \leq T_{i,N} \leq T_{\max},$$

where $T_{\min} = \frac{1}{2}$ and $T_{\max} = \frac{1}{2} + \frac{1}{1-hd_{\max}}$,

(ii) there exists a duration $\delta \leq \delta_{\max} = 1 + \left\lceil 1 + \frac{2}{1-hd_{\max}} \right\rceil (m-1)$ such that the sensing digraph and the reception digraphs satisfy, for each time index $p \in \mathbb{N}$,

$$\mathcal{G}_p \cup \mathcal{G}_{p+1} \cup \ldots \cup \mathcal{G}_{p+\delta-1} = \mathcal{G} \cup \mathcal{I},$$

(iii) the maximum number of pulses received by each oscillator over an interval of duration $v \in \mathbb{N}$ is

$$\mathcal{N}_\delta(v) = \left\lfloor \frac{v}{\delta} \right\rfloor (\delta - m + d_{\max}) + \min\left\{(v \bmod \delta), (\delta - m + d_{\max})\right\}. \tag{3.1}$$

The proof of statement (i) proceeds as follows. The duration between firings of an oscillator depends on the number of pulses the oscillator receives between its firings. Some pulses move the phase forward, decreasing the duration, and some move the phase backward, increasing the duration to the next firing of the oscillator. The maximum and minimum durations between successive firings of the same oscillator depend on the maximum effect of the pulses. The proofs of Statements (ii) and (iii) follow from statement (i). Note that $\delta \geq m$ since each reception digraph contains only one edge of the sensing digraph, and so, by definition of δ, the function $\mathcal{N}_\delta$ in statement (iii) is always non-negative.

PROOF OF THEOREM 3.1

If an oscillator i receives no pulses from its out-neighbors between its Nth and $(N+1)$th firing, then from the continuous dynamics equation (2.1), $T_{i,N} = 1$. If oscillator i does receive pulses, then each pulse either increases or decreases the time to the next firing according to the discrete dynamics equation (2.2).

We estimate $T_{\min}$ as follows. Equations (2.1) and (2.2) imply that a pulse received by oscillator i when $\phi_i(t) < \frac{1}{2}$ can only *increase* the time to firing of oscillator i. Hence $T_{i,N} \geq \frac{1}{2}$, which is the minimum duration taken for the phase to increase from 0 to $\frac{1}{2}$ (and occurs when no pulses are received). Therefore $T_{\min} = \frac{1}{2}$.

Next, we estimate $T_{\max}$. Note that the duration between any two successive receptions along an edge (i,j) is equal to $T_{j,N}$ for some N in $\mathbb{N}$, since the delays are constant. In a duration Δt, an oscillator i can receive at most

$$\lceil \frac{\Delta t}{T_{\min}} \rceil = \lceil 2\Delta t \rceil$$

pulses from a given out-neighbor, and hence at most $d_i \lceil 2\Delta t \rceil$ pulses in all. From equation (2.2), each pulse results in a phase change of at most $\frac{h}{2}$. Conceivably, each pulse that oscillator i receives could result in a phase change of $\frac{h}{2}$, since the oscillator could receive a pulse when its phase is just less than $\frac{1}{2}$, then move backward by $\frac{h}{2}$, then move forward again to $\frac{1}{2}$ under the continuous dynamics, and then receive another pulse (and so on). Therefore, the total phase change $\Delta\phi_i$ of oscillator i in the duration Δt satisfies the following inequality:

$$\Delta\phi_i \geq \Delta t - \frac{h}{2}(d_i\lceil 2\Delta t\rceil) \geq \Delta t - \frac{h}{2}d_i(2\Delta t + 1) = (1 - hd_i)\Delta t - \frac{h}{2}d_i$$
$$\Longrightarrow \Delta\phi_i \geq (1 - hd_{\max})\Delta t - \frac{h}{2}d_{\max}. \tag{3.2}$$

From the assumption (A1) and equation (3.2) we see that if $\Delta t \geq \frac{1}{(1-hd_{\max})}$, then $\Delta\phi_i \geq \frac{1}{2}$. So the duration taken for the phase to increase from 0 to $\frac{1}{2}$ is less than or equal to $\frac{1}{(1-hd_{\max})}$, and the duration taken for the phase to increase from $\frac{1}{2}$ to 1 is less than or equal to $\frac{1}{2}$ since any pulses it receives during this duration will increase the duration, not decrease it. Therefore, $T_{i,N}$ is less than or equal to $T_{\max} \coloneqq \frac{1}{2} + \frac{1}{(1-hd_{\max})}$. This concludes the proof of statement (i).

To prove statement (ii), we recall that the time between any two successive receptions along the edge (i, j) is $T_{j,N}$ for some N in $\mathbb{N}$. Suppose two successive receptions along (i, j) occur at times t and $t+T_{j,N}$. The number of receptions during the interval $]t, t + T_{j,N}]$ along the other $m-1$ edges is less than or equal to $\left\lceil \frac{T_{j,N}}{T_{\min}} \right\rceil (m-1)$. Hence any sequence of $\delta_{\max} \coloneqq 1 + \left\lceil \frac{T_{\max}}{T_{\min}} \right\rceil (m - 1)$ successive receptions must contain at least one reception along the edge (i, j), and so the edge must appear in a reception digraph at least one of these $\delta_{\max}$ successive pulse instants. Further, every reception digraph has a self loop on every node. This implies that there exists a natural number $\delta \leq \delta_{\max}$ such that the union of the digraphs $\mathcal{G}_p \cup \mathcal{G}_{p+1} \cup \ldots \cup \mathcal{G}_{p+\delta-1}$ is $\mathcal{G} \cup \mathcal{I}$, for all $p \in \mathbb{N}$. This concludes the proof of statement (ii).

Finally, to prove statement (iii) we estimate the number of pulses received by a particular oscillator i during v successive receptions. We note that in a sequence of δ successive firings, at least one reception must occur along each of the $m - d_i$ edges that are not from node i. This implies that the maximum number of pulses that an oscillator may receive in a sequence of δ successive receptions is $\delta - m + d_{\max}$. The maximum number of pulses received by an oscillator in a sequence of v successive receptions, $\mathcal{N}_\delta(v)$ can be estimated by breaking the sequence into parts of length δ. Therefore, we know $\mathcal{N}_\delta(v) = \left\lfloor \frac{v}{\delta} \right\rfloor (\delta - m + d_{\max}) + \min\{v \bmod \delta, (\delta - m + d_{\max})\}$. This concludes the proof of the theorem. □

Note on δ and $\delta_{\max}$. The upper bound $\delta_{\max}$ on the duration δ was calculated without specific knowledge of the sensing digraph, the delays, or the initial conditions of the PCO network, and hence is conservative in some cases. If this specific knowledge is available, tighter bounds may be calculated by repeating the above proof. In some special cases, δ may be found to be equal to m.

4. Synchronization with Delays. In this section we prove that, for initial conditions contained in a small enough arc, approximate synchronization occurs exponentially fast if (i) the coupling strength is sufficiently small, (ii) the sensing digraph has a GRN, and (iii) all the delays are sufficiently small. Under these conditions, the arc length function of the PCO network converges exponentially to a final value that is proportional to the maximum delay. Theorem 4.1 below formalizes these statements.

THEOREM 4.1 (Exponential approximate synchronization with small delays). *Consider a PCO network with n oscillators, with sensing digraph $\mathcal{G}$ with m edges, with delays τ_{ij} for each edge (i,j), and with coupling strength h. Assume that:*

(A1) *the coupling strength h and the maximum out-degree $d_{\max}$ satisfy $hd_{\max} < 1$,*

(A2) *the sensing digraph $\mathcal{G}$ has a globally reachable node, and*

(A3) *the maximum delay $\tau_{\max}$ and the coupling strength h satisfy, for $\Delta = b(\delta+m)$, $\eta = \min\{h, 1-h\}^b(1-h)^{b(3(\delta-m+d_{\max})-1)}$, and b is the reachability depth of $\mathcal{G}$,*

$$\Big(\frac{1-(1-h)^{\mathcal{N}_\delta(\Delta)}}{\eta} + \frac{2}{(1-h)^{\mathcal{N}_\delta(\Delta-1)}} - 1\Big)\tau_{\max} < \frac{1}{2}. \tag{4.1}$$

Then, for each initial condition satisying $\phi(0) \in \Gamma\Big(\frac{1}{2} - \ell_\Delta\tau_{\max}\Big)$, the resulting phase evolution $\phi(t)$, $t \in \mathbb{R}_{\geq 0}$, satisfies

$$V_{\text{arc-length}}(\phi(t)) \leq a^{\lfloor \frac{p}{\Delta} \rfloor}(V_{\text{arc-length}}(\phi(0)) - V_{\text{offset}}) + V_{\text{offset}} + (\ell_\Delta - 1)\tau_{\max}, \tag{4.2}$$

where $p \in \mathbb{N}$ is the largest time index such that $t_p < t$ and where

- *the convergence factor is $a = 1 - \eta \in \,]0,1[$,*
- *the offset value is $V_{\text{offset}} = \frac{1}{\eta}(1-(1-h)^{\mathcal{N}_\delta(\Delta)})\tau_{\max}$,*
- *the maximum contraction factor is $\ell_\Delta = (\frac{2}{(1-h)^{\mathcal{N}_\delta(\Delta-1)}} - 1)$, and*
- *the duration δ and the number of pulses function $\mathcal{N}_\delta$ are as in Theorem 3.1.*

We see that the synchronization error, quantified by the arc-length function, exponentially converges to the final value $V_{\text{offset}} + (\ell_\Delta - 1)\tau_{\max}$. A corollary to Theorem 4.1 is that, when all delays are zero, the arc-length function exponentially decreases to zero for initial conditions contained in an arc of length $\frac{1}{2}$. Therefore, for zero delays, exact synchrony occurs if the the coupling strength is sufficiently small and the sensing digraph has a GRN. This is consistent with a similar result (on asymptotic convergence of the arc-length to zero) that was recently and independently established in [23]. The statement of the corollary below is obtained from Theorem 4.1 by substituting $\tau_{ij} = 0$ for every edge in the sensing digraph.

COROLLARY 4.2 (Exponential synchronization with no delays). *Consider a PCO network with n oscillators, with sensing digraph $\mathcal{G}$ with m edges, with delays $\tau_{ij} = 0$ for each edge (i,j), and with coupling strength h. Assume that:*

(A1) *the coupling strength h and the maximum out-degree $d_{\max}$ satisfy $hd_{\max} < 1$, and*

(A2) *the sensing digraph $\mathcal{G}$ has a globally reachable node.*

Then, for each initial condition satisying $\phi(0) \in \Gamma(\frac{1}{2})$, the resulting phase evolution $\phi(t)$, $t \in \mathbb{R}_{\geq 0}$, satisfies

$$V_{\text{arc-length}}(\phi(t)) \leq a^{\lfloor \frac{p}{\Delta} \rfloor} V_{\text{arc-length}}(\phi(0)), \tag{4.3}$$

where $p \in \mathbb{N}$ is the largest time index such that $t_p < t$ and where

- *the convergence factor is $a = 1 - \eta \in \,]0,1[$,*
- *the duration Δ and the factor η are as in Theorem 4.1, and*
- *the duration δ and the number of pulses $\mathcal{N}_\delta(\Delta)$ are as in Theorem 3.1.*

The proof of Theorem 4.1 is in six steps. In Step 1, beginning at an an arbitrary time step, we transform the phases to a rotating coordinate system, so that the continuous dynamics of the system is eliminated and the discrete dynamics can be written as an affine function of the states, assuming that the phases are contained in a sufficiently small arc for these system equations to be applicable. In Step 2, we introduce the concept of the modified reception digraph (MRD) and rewrite the system dynamics as a discrete-time LTV control system using the MRD adjacency matrix. In Step 3, we study the properties of the MRD and its adjacency matrix, using the analysis of the durations between pulses from the previous section. In Step 4, we analyze the system over some fixed number of time steps, Δ, and find bounds on the homogeneous solution of the LTV control system. We use the properties of the MRD adjacency matrix in this step, based on the method in [19, 8]. In Step 5, we find upper and lower bounds on the particular solution of the system, by analyzing the state one element at a time. This also enables us to validate our assumption that the phases are contained in a sufficiently small arc for the system equations to be applicable. In Step 6, we relate the upper and lower bounds that were found on the solution to the arc-length function, concluding the proof.

PROOF OF THEOREM 4.1

Step 1 - Coordinate Transformation Assume that at some arbitrary start time $p = y$, the following condition is satisfied:

$$\phi(t_p) \in \Gamma\Big(\frac{1}{2} - \ell_\Delta \tau_{\max}\Big). \tag{4.4}$$

Define a rotating coordinate system with its origin $\phi_0(t) \in [0, 1[$ such that the transformed phase vector $\tilde{\phi}(t)$ is given by:

$$\tilde{\phi}_i(t) = \mathrm{dist}_{\mathrm{cc}}\left(\phi_0(t), \phi_i(t)\right), \qquad i \in \{1, \dots, n\}, \tag{4.5}$$

and the rotating coordinate system satisfies the following:

- The origin has the same continuous dynamics as the oscillators, or, $\dot{\phi}_0(t) = 1$.
- When the phase $\phi_0(t)$ reaches 1, it is reset to 0.
- at $t = t_y$, the origin lies within the arc of length $\frac{1}{2} - \ell_\Delta \tau_{\max}$ that also contains $\phi(t_y)$ such that

$$\ell_\Delta \tau_{\max} \mathbb{1}_n \leq \tilde{\phi}(t_y) < \frac{1}{2}\mathbb{1}_n. \tag{4.6}$$

Such an origin can be found since condition (4.4) is satisfied for $p = y$.

Let the origin of the fixed coordinate system be represented in the rotating coordinate system as $\tilde{0}$, given by:

$$\tilde{0}(t) = \mathrm{dist}_{\mathrm{cc}}(\phi_0(t), 0).$$

In the rotating coordinate system, the PCO network has no continuous dynamics or reset. The vector $\tilde{\phi}$ remains constant between receptions.

We will assume that, for all $p \in \mathbb{N}$,

$$\tau_{\max} \mathbb{1}_n \leq \tilde{\phi}(t_p) < \frac{1}{2}\mathbb{1}_n, \tag{4.7}$$

that is, the minimum phase may contract at most by the maximum contraction factor ℓ_Δ, relative to its value at $p = y$. Then it is straightforward to show that the discrete dynamics in the rotating coordinate system is given by:

$$\tilde{\phi}_i(t_{p+1}) = \begin{cases} \tilde{\phi}_i(t_p) - h(\tilde{\phi}_i(t_p) - \tilde{0}(t_p)), & \text{if } i \text{ receives a pulse at time } t_p, \\ \tilde{\phi}_i(t_p), & \text{if } i \text{ does not receive a pulse at time } t_p. \end{cases} \tag{4.8}$$

Equation (4.8) is used in subsequent steps as the PCO network dynamics equation, and the assumption regarding equation (4.7) is validated in Steps 5 and 6.

Step 2 - Rewriting the Discrete Dynamics as an LTV Control System with Input Disturbances

Suppose oscillator i receives a pulse from oscillator j at time t_p. We solve for $\tilde{0}(t_p)$ in terms of the phase of oscillator j and substitute in equation (4.8). Oscillator j fired at time $t_p - \tau_{ij}$, so $\tilde{\phi}_j = \tilde{0}$ at time $t_p - \tau_{ij}$. We solve for $\tilde{0}(t_p)$ as follows:

$$
\begin{aligned}
\tilde{\phi}_j(t_p - \tau_{ij}) &= \tilde{0}(t_p - \tau_{ij}) \\
&= \big(\tilde{0}(t_p) + \tau_{ij}\big) \bmod 1 \\
\implies \tilde{0}(t_p) &= \tilde{\phi}_j(t_p - \tau_{ij}) - \tau_{ij}, \qquad \text{from equation (4.7).}
\end{aligned} \tag{4.9}
$$

Suppose q receptions (by any oscillator) occur in the duration $]t_p - \tau_{ij}, t_p[$. Then, $\tilde{\phi}_j(t_p - \tau_{ij}) = \tilde{\phi}_j(t_{p-q})$, and $\tilde{0}(t_p)$ is given by the following equation:

$$
\tilde{0}(t_p) = \tilde{\phi}_j(t_{p-q}) - \tau_{ij}. \tag{4.10}
$$

The discrete dynamics is rewritten by substituting equation (4.10) in equation (4.8) to obtain the following:

$$
\tilde{\phi}_i(t_{p+1}) = \begin{cases} (1-h)\tilde{\phi}_i(t_p) + h\tilde{\phi}_j(t_{p-q}) - h\tau_{ij}, & \text{if } i \text{ receives a pulse from } j \text{ at time } t_p, \\ \tilde{\phi}_i(t_p) & \text{if } i \text{ does not receive a pulse at time } t_p. \end{cases} \tag{4.11}
$$

Equation (4.11) is a difference equation that may be of order greater than one. From Theorem 3.1 (i), the maximum number of pulses that could have been received along a given edge in the duration $]t_p - \tau_{ij}, t_p[$ is given by

$$
\lceil \frac{\tau_{\max}}{T_{\min}} \rceil = \lceil 2\tau_{\max} \rceil = 1
$$

since assumption (A3) implies that $\tau_{\max} < \frac{1}{2}$. Hence the total number of pulses received along any edge in this duration, q, must be less than or equal to the number of edges in the sensing digraph, m. We can express equation (4.11) as a first order system with the modified state vector $x(t) \in \mathbb{R}^{nm+n}$ where $x(t_p) = \left[\tilde{\phi}(t_p)^T \ldots \tilde{\phi}(t_{p-m})^T\right]^T$. The elements of x with index greater than n represent the 'older' values of the phases. At $t = 0$ these states may be initialized arbitrarily, and for convenience, we initialize each of the 'old' states as the midpoint of the shortest arc containing $\Big(\tilde{\phi}(0)\Big)$.

The discrete dynamics is then described by the following equations (note that we use x_i^p to denote $x_i(t_p)$, for brevity), which are affine in x:

$$
x_i^{p+1} = \begin{cases} (1-h)x_i^p + hx_{j+qm}^p - h\tau_{ij}, & \text{if } i \le n \text{ and oscillator } i \text{ receives a pulse from } j \text{ at time } t_p, \\ x_i^p, & \text{if } i \le n \text{ and oscillator } i \text{ does not receive a pulse at time } t_p, \\ x_{i-m}^p, & \text{if } i > n. \end{cases} \tag{4.12}
$$

The discrete dynamics can be rewritten for $p \in \{y, \ldots, y+v\}$ as an affine function in x. In order to do this, we now introduce the *modified* reception digraph (MRD), whose the adjacency matrix is $\tilde{A}_p$.

At time t_p, we define the modified reception digraph $\tilde{\mathcal{G}}_p$. The $nm+n$ vertices of $\tilde{\mathcal{G}}_p$ are defined as follows: for every node k in $\mathcal{G}_p$, define $m+1$ nodes in $\tilde{\mathcal{G}}_p$: the node k itself, and m *memory* nodes labeled $k^{1-}, k^{2-}, \ldots, k^{m-}$. The node k will be sometimes referred to as k^{0-}. The edge set is given as follows:

- There is an edge weighted 1 from every node k^{w-} to $k^{(w+1)-}$, $w = 0, 1, \ldots, m-1$.
- If node i has a self-loop in $\mathcal{G}_p$, then node i has a self-loop with the same weight in $\tilde{\mathcal{G}}_p$.
- If there is an edge (i,j) for some $i \neq j$ in $\mathcal{G}_p$, then there is an edge (i, j^{q-}) in $\tilde{\mathcal{G}}_p$, where q is the number or receptions that occurred in the duration $]t_p - \tau_{ij}, t_p[$.

An example of an MRD is shown in Figure 4.1. We note that if there is a path from some node j to a node k in $\mathcal{G}_p$, then there must be a path from node j to node k in $\tilde{\mathcal{G}}_p$. Therefore, since we know there is a GRN in $\mathcal{G}_p \cup \mathcal{G}_{p+1} \cup \ldots \cup \mathcal{G}_{p+\delta-1}$ from Theorem 3.1(ii), there must be a GRN in $\tilde{\mathcal{G}}_p \cup \tilde{\mathcal{G}}_{p+1} \cup \ldots \cup \tilde{\mathcal{G}}_{p+\delta-1}$ as well.

The memory nodes store the old states, such that node k^{w-} has the state $\tilde{\phi}_k(t_{p-w})$, where $w \in \{0, 1, \ldots, m\}$. Let the adjacency matrix of $\tilde{\mathcal{G}}_p$ be $\tilde{A}_p$. Then equation (4.12) can be rewritten:

$$x^{p+1} = \tilde{A}_p\, x^p + D_p, \tag{4.13}$$

where the input disturbance term $D_p \in \mathbb{R}^{nm+n}$ is the affine term from equation (4.12) and is defined as follows:

$$[D_p]_i = \begin{cases} -h\tau_{ij}, & \text{if } i \le n \text{ and oscillator } i \text{ receives a pulse from } j \text{ at time } t_p, \\ 0, & \text{if } i \le n \text{ and oscillator } i \text{ does not receive a pulse at time } t_p, \\ & \text{or if } i > n. \end{cases} \tag{4.14}$$

Step 3 - Global Reachability Over Time in the Modified Reception Digraphs

Equation (4.13) is a time-varying distributed averaging algorithm (with an input disturbance term added in) that satisfies the following conditions:

- The adjacency matrices of the MRD are row-stochastic.
- Every element of the adjacency matrix of the MRD belongs to $\{0, h, 1-h, 1\}$.
- There exists a number $\delta \le \delta_{\max}$ such that the union of any δ successive MRD has a GRN.

We show that above three conditions are sufficient for the arc-length $V_{\text{arc-length}}(\tilde{\phi})$ to converge exponentially. First we show that the GRN r of $\mathcal{G}$, is reachable from any node of the MRD over the sequence of $\Delta := b(\delta+m)$ successive MRD, in the following sense: for every $k \in \{1, \ldots, n\}$, there exists a sequence of nodes $\{k, i_1, i_2, \ldots, i_{\Delta-1}, r\}$ such that (k, i_1) is an edge of $\tilde{\mathcal{G}}_p$, (i_w, i_{w+1}) is an edge in $\tilde{\mathcal{G}}_{p+w}$ for $w = 1, \ldots, \Delta-2$ and $\{i_{\Delta-1}, r\}$ is an edge in $\tilde{\mathcal{G}}_{p+\Delta-1}$.

Suppose (i,j) is an edge in $\mathcal{G}$. From Theorem 3.1 (ii), the edge (i,j) must appear in at least one digraph in the sequence $\mathcal{G}_p, \mathcal{G}_{p+1}, \ldots, \mathcal{G}_{p+\delta-1}$. Suppose the edge exists in $\mathcal{G}_{p+w}$, where $0 \le w < \delta$. From the definition of MRD, for some $q \le m$, (i, j^{q-}) must be an edge in $\tilde{\mathcal{G}}_{p+w}$. Then, node j is reachable from i over the duration $\{p, p+m+\delta-1\}$, since the following sequence of $\delta+m+1$ nodes exists:

$$\left\{ \underbrace{i, \ldots, i,}_{w \text{ times}} \quad j^{q-}, j^{(q-1)-}, \ldots, j^{1-}, \quad \underbrace{j, \ldots, j}_{\delta-w+1+m-q \text{ times}} \right\},$$

such that:

- the edges (i,i) exists in the digraphs $\tilde{\mathcal{G}}_p, \tilde{\mathcal{G}}_{p+1}, \ldots, \tilde{\mathcal{G}}_{p+w-1}$ (since there are self-loops on every node of the MRD that is not a memory node),
- the edge (i, j^{q-}) exists in $\tilde{\mathcal{G}}_{p+w}$ by our assumption,
- the edges $(j^{q-}, j^{(q-1)-})$, $(j^{(q-1)-}, j^{(q-2)-})$, …, $(j^{(1-}, j)$ exist in the digraphs $\tilde{\mathcal{G}}_{p+w+1}, \ldots, \tilde{\mathcal{G}}_{p+w+q}$ respectively (since these edges exist in every MRD), and
- the edges (j,j) exist in the digraphs $\tilde{\mathcal{G}}_{p+w+q+1}, \ldots, \tilde{\mathcal{G}}_{p+\delta+m-1}$, (since there are self-loops on every node of the MRD that is not a memory node).

Therefore if (i,j) is an edge in $\mathcal{G}$, then node i is reachable from j over every duration of length $\delta+m$ or greater, in the sequence of MRD. Theorem 3.1(iii) implies that at most $3(\delta - m + d_{\max})$ edges in the above sequence are weighted less than 1 (these are edges associated with receptions by i or j, and $3(\delta - m + d_{\max})$ is the maximum value over w of $\mathcal{N}_\delta(w) + \mathcal{N}_\delta(\delta - w + m)$), and that at most one edge in the above sequence is weighted h (this is the edge from i to j if they are distinct).

Since there is a path of length equal to the reachability depth b from every node k to r in $\mathcal{G} \cup \mathcal{I}$ from the assumption (A2), we can construct a sequence of $b(\delta+m)+1$ nodes starting with k and ending with r, over every sequence of $b(\delta+m)$ successive MRD, by concatenating a sequence of nodes like the one described above for each edge of the path in $\mathcal{G}$ (and discarding the repeated node at the point of concatenation). Hence the node r is reachable from every node k over a every sequence of $b(\delta+m) = \Delta$ successive MRD. We note that at most $3b(\delta - m + d_{\max})$ edges in the above sequence are weighted less than 1, and that at most b edges in the sequence are weighted h.

We use the property of global reachability of r in the MRD to analyze the LTV control system of the PCO network.

Step 4 - Analyzing the Time-Varying Distributed Averaging System

Assume $y = z\Delta$. Equation (4.13) is applied Δ times recursively, to find the new state at time index $z\Delta + \Delta$. Let $\mathcal{A}_z$ denote the product $\tilde{A}_{(z+1)\Delta-1}\tilde{A}_{(z+1)\Delta-2} \ldots \tilde{A}_{z\Delta}$, and let $\mathcal{X}^{(z+1)\Delta}$ be the *particular solution* of the difference equation, that is, the solution to the linear system with the zero initial condition $x^{z\Delta} = 0$. Then, applying equation (4.13) recursively Δ times to calculate $x^{(z+1)\Delta}$ from x^{Δ} yields:

$$x^{(z+1)\Delta} = \mathcal{A}_z \, x^{z\Delta} + \mathcal{X}^{(z+1)\Delta}. \tag{4.15}$$

Define the sequence V_p for $p \in \mathbb{N}$ as $V_p = \max(x^p) - \min(x^p)$. We use the properties of $\mathcal{A}_z$, which is a product of row-stochastic matrices, to analyze V_p. Note that V_p does not depend on the choice of origin of the rotating coordinate system, as long as this origin is chosen such that all the elements of x lie in $[0, \frac{1}{2}[$. This is important, since we may choose a new rotating coordinate system after every Δ time steps.

Let $a_{ij}(p)$ denote the (i,j) element of the adjacency matrix $\tilde{A}_p$. From a node k, consider the sequence of nodes by which the GRN r can be reached reached: $\{k, i_1, \ldots, i_{\Delta-1}, r\}$ such that (k, i_1) is an edge in $\tilde{\mathcal{G}}_{z\Delta}$,$(i_w, i_{w+1})$ is an edge in $\tilde{\mathcal{G}}_{z\Delta+w}$ for $w = \{1, \ldots, z\Delta + \Delta - 2\}$, and $(i_{\Delta-1}, r)$ is an edge in $\tilde{\mathcal{G}}_{z\Delta+\Delta-1}$. The existence of these edges implies that the corresponding element of the MRD adjacency matrix is non-zero, and hence $a_{k,i_1}(z\Delta) \geq \min\{h, 1-h\}$, $a_{i_w,i_{w+1}}(z\Delta + w) \geq \min\{h, 1-h\}$ for $w = \{1, \ldots, z\Delta + \Delta - 2\}$, and $a_{i_{\Delta-1},r}(z\Delta + \Delta - 1) \geq \min\{h, 1-h\}$. Furthermore, at most $3b(\delta - m + d_{\max})$ of these elements are less than 1, and at most b elements are equal to h.

We note that since $\mathcal{A}_z$ is a product of row-stochastic matrices, $\mathcal{A}_z$ is row-stochastic. The method in [19, 8] uses the observation that the product:

$$a_{k,i_1}(z\Delta)\, a_{i_1,i_2}(z\Delta + 1) \, \ldots \, a_{i_{\Delta-2}i_{\Delta-1}}(z\Delta + \Delta - 2)\, a_{i_{\Delta-1}r}(z\Delta + \Delta - 1)$$

is one term in the expression for $[\mathcal{A}_z]_{kr}$, and that all the other terms are non-negative, to deduce that

$$[\mathcal{A}_z]_{kr} \geq \eta := \min\{h, 1-h\}^b (1-h)^{b(3(\delta - m + d_{\max})-1)} \qquad \text{for all } k \in \{1, \ldots, n\}.$$

and therefore:

$$\begin{aligned} \max(\mathcal{A}_z x^{z\Delta}) &\leq \eta\, x_r^{z\Delta} + (1-\eta)\max(x^{z\Delta}), \qquad \text{and} \\ \min(\mathcal{A}_z x^{z\Delta}) &\geq \eta\, x_r^{z\Delta} + (1-\eta)\min(x^{z\Delta}), \\ \text{since } \left[\mathcal{A}_z x^{z\Delta}\right]_k &= [\mathcal{A}_z]_{kr}\, x_r^{z\Delta} + \sum_{\substack{j \,\in\, \mathcal{V}, \\ j \neq r}} [\mathcal{A}_z]_{kj}\, x_j^{z\Delta}. \end{aligned}$$

$$\Longrightarrow\ V_{z\Delta+\Delta} \leq (1-\eta)V_{z\Delta} + \max \mathcal{X}^{(z+1)\Delta} - \min \mathcal{X}^{(z+1)\Delta} \qquad \text{from (4.15).} \tag{4.16}$$

We now require the upper and lower bounds on the particular solution $\mathcal{X}$, to substitute into equation (4.16).

Step 5 - Bounds on the Particular Solution

Equation (4.13) shows that the particular solution $\mathcal{X}$ depends on the input disturbance terms D_p. By definition, every element of D_p is non-positive. Then, equation (4.12) implies that for the ith element of x_y, the following inequality holds:

$$\max x^p \geq x_i^{p+1} \geq \begin{cases} (1-h)x_i^p - h\tau_{\max} & \text{if } i \leq n \text{ and } i \text{ receives a pulse at time } t_p \\ \min x^p & \text{if } i \leq n \text{ and } i \text{ does not receive a pulse at time } t_p, \text{ or } i > n. \end{cases} \tag{4.17}$$

To find bounds on the states after some $v \leq \Delta - 1$ time steps from y, we apply the above equation recursively v times on an element of x_y (noting that a given oscillator can receive at most $\mathcal{N}_\delta(v)$ pulses over v time steps, from Theorem 3.1(iii)):

$$\begin{aligned} \max x^y \geq x_i^{y+v} &\geq (1-h)^{\mathcal{N}_\delta(v)} \min x^y - \Big(1 - (1-h)^{\mathcal{N}_\delta(v)}\Big)\tau_{\max} \\ &\geq (1-h)^{\mathcal{N}_\delta(v)} \ell_\Delta \tau_{\max} - \Big(1 - (1-h)^{\mathcal{N}_\delta(v)}\Big)\tau_{\max} && \text{from equation (4.6)} \\ &= \Big(2(1-h)^{\mathcal{N}_\delta(v) - \mathcal{N}_\delta(\Delta-1)} - 1\Big)\tau_{\max} && \text{from the definition of } \ell_\Delta \text{ in Theorem 4.1} \end{aligned} \tag{4.18}$$

$$\geq \tau_{\max} \qquad \text{since the function } \mathcal{N}_\delta \text{ is increasing.} \tag{4.19}$$

The above enables us to validate our assumption regarding equation (4.7) for $p \in \{z\Delta+1, \ldots, z\Delta+\Delta-1\}$, by induction, as follows. If equation (4.6) is satisfied for $y = z\Delta$ and equation (4.7) is satisfied for $p \in \{z\Delta+1, \ldots, z\Delta+v-1\}$, then the above analysis implies that equation (4.7) is satisfied for $p = z\Delta + v$. Hence, by induction, (4.7) is satisfied for all $v \leq \Delta - 1$, and our use of the system equations (4.13) for Δ time steps is valid.

The equation (4.19) can be used to find the maximum and minimum elements of $\mathcal{X}^{(z+1)\Delta}$. Substituting $y = z\Delta$ and $v = \Delta$ and the zero initial condition $x_y = 0_{n+nm}$ in equation (4.18) for $i \in \{1, \ldots, n+nm\}$ yields

$$0 \geq \mathcal{X}_i^{(z+1)\Delta} \geq -\Big(1 - (1-h)^{\mathcal{N}_\delta(v)}\Big)\tau_{\max}. \tag{4.20}$$

Substituting the above in equation (4.16) yields

$$
\begin{aligned}
& V_{z\Delta+\Delta} \leq (1-\eta)V_{z\Delta} + \left(1-(1-h)^{\mathcal{N}_\delta(v)}\right)\tau_{\max} \\
& \implies V_{z\Delta} - V_{\text{offset}} \leq a^z(V_0 - V_{\text{offset}}),
\end{aligned} \tag{4.21}
$$

with V_{offset} and a as defined in Theorem 4.1. The above relation gives an upper bound on V_N when N is a multiple of Δ. We find an upper bound on V_N when N is not a multiple of Δ, by substituting $p = \Delta\lfloor \frac{N}{\Delta} \rfloor$ and $v = N - \Delta\lfloor \frac{N}{\Delta} \rfloor$ in equation (4.19):

$$
\begin{aligned}
& \max x^{\Delta\lfloor \frac{N}{\Delta} \rfloor} \geq x_i^N \geq \tau_{\max} \\
& \implies V_N \leq \max x^{\lfloor \frac{N}{\Delta} \rfloor} - \tau_{\max} \\
& \qquad \leq \ell_\Delta \tau_{\max} + V_{\lfloor \frac{N}{\Delta} \rfloor} - \tau_{\max} \qquad \text{from equation (4.6)} \\
& \qquad = a^{\lfloor \frac{N}{\Delta} \rfloor}(V_0 - V_{\text{offset}}) + V_{\text{offset}} + (\ell_\Delta - 1)\tau_{\max} \qquad \text{from equation (4.21).}
\end{aligned} \tag{4.22}
$$

Step 6 - Estimating the Arc Length

Since the vector $\tilde{\phi}$ is contained within an arc of length $\frac{1}{2}$, the arc-length function $V_{\text{arc-length}}(\phi)$ is given by $\max \tilde{\phi} - \min \tilde{\phi}$. Since the elements of $\tilde{\phi}$ are also elements of x, $V_{\text{arc-length}}(\phi(t_p))$ is less than or equal to V_p. Also, $V_{\text{arc-length}}(\phi(0)) = V_0$, from our choice of initialization of the elements of x. Hence equation (4.22) can be rewritten as equation (4.2).

Lastly, assumption (A3) ensures that $V_{\text{offset}} + \ell_\Delta \tau_{\max}$ is less than $\frac{1}{2}$, which on substitution in equation (4.21) implies that $V_{z\Delta}$ is less than or equal to $\frac{1}{2} - \ell_\Delta \tau_{\max}$. This means that equation (4.4) is satisfied at $z\Delta$ for every $z \in \mathbb{N}$, as long as equation (4.4) is satisfied at $t = 0$ (as was assumed in Theorem 4.1), and hence a new choice of rotating coordinate system can be made that satisfies equation (4.6) at each $z\Delta$. Furthermore, the assumption regarding equation (4.7), which was validated for $p \in \{m\Delta, \ldots, m\Delta + \Delta\}$ in Step 5, is now validated for all time. □

Note on Corollary 4.2. Corollary 4.2 can be quantitatively improved on noting that in the absence of delays, the firing and reception of a pulse happen simultaneously and so q in equation (4.11) is always 0. Hence there is no need to include memory nodes in the MRD, and the reception digraphs themselves can be used in their place. Repeating the proof of Theorem 4.1, Corollary 4.2 can be restated after replacing Δ with $\tilde{\Delta} = b\delta$, and η with $\tilde{\eta} = \min\{h, 1-h\}^b (1-h)^{b(\delta - m + d_{\max} - 1)}$. We explain the details of this improvement in [13].

5. Example Applications and Extensions. The following examples apply and extend the results and analysis method of Section 4 to two interesting cases. We introduce the concept of *intermittent synchronization*: A PCO network intermittently synchronizes if there exists an ordered time sequence that converges asymptotically to zero, and is constructed as follows - the synchronization error is sampled at least once per oscillation (of any oscillator) at instants when no pulse is received.

5.1. Batch behavior and intermittent synchronization. This subsection analyzes a special case of Theorem 4.1 and demonstrates that, even in the presence of delays, intermittent synchronization is possible if the delays are uniform and all nodes of the sensing digraph have equal out-degrees. Since all the assumptions of Theorem 4.1 are satisfied, all the steps in the proof of Theorem 4.1 are applicable here, and approximate synchronization occurs. By further requiring that the delays

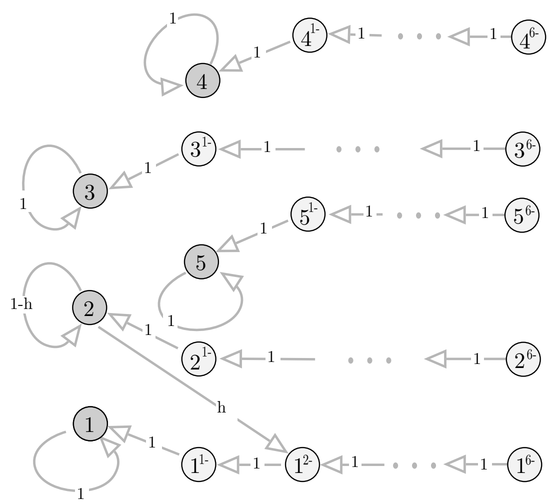

FIG. 4.1. *An example of a modified reception digraph $\tilde{\mathcal{G}}_p$, consistent with the reception digraph on the left in Figure 2.3, where oscillator 2 receives a pulse from oscillator 1 at t_p, and where two receptions have occurred in the duration $]t_p - \tau_{21}, t_p[$ and hence $q = 2$. The lighter-colored nodes are memory nodes.*

be uniform and the sensing digraph have all-equal out-degrees, we can prove the stronger result of intermittent synchronization.

THEOREM 5.1 (Intermittent synchronization with uniform delays and all-equal out-degrees). *Consider a PCO network with n oscillators, with sensing digraph $\mathcal{G}$ with m edges that satisfies all the assumptions of Theorem 4.1. Assume that:*

(A1') *all the delays are uniform and non-zero, that is, $\tau_{ij} = \tau$ for each edge (i, j), and*

(A2') *Each node of $\mathcal{G}$ has the same out-degree d.*

Then, for all initial conditions $\phi(0) \in \Gamma(\min\{\tau, \frac{1}{2} - \ell_{2bm}\tau\})$, the network synchronizes intermittently.

Proof. We will first show that the firings of this system occur in "batches," that is, successive firings by all n oscillators with no receptions in between. Consider a time t when $\phi(t) \in \Gamma(\tau_{\min})$ and all the phases are in the lower two quadrants, and therefore satisfy

$$\frac{1}{2}\mathbb{1}_n \leq \phi(t) \leq \mathbb{1}_n. \tag{5.1}$$

Then, any pulses that had been fired prior to t have been received already, since $\tau_{\max} < \frac{1}{2}$. The next firing occurs at time $t + 1 - \max(\phi(t))$. Since $V_{\text{arc-length}}(\phi(t))$ is less than $\tau_{\min}$, no receptions of pulses occur until all n oscillators have fired their next pulse after time t. This implies that if any oscillator fires between two receptions, at a time $t \in]t_{p-1}, t_p[$, then all n oscillators must fire in the same time interval $]t_{p-1}t_p[$ exactly once. In other words, they fire in a batch. The m pulses that were fired in this batch are received at time indices $\{p, \ldots, p + m - 1\}$. In the time interval $[t_p, t_{p+m-1}]$, one reception occurs along every edge and so each oscillator i receives d pulses.

Let $\tilde{\mathcal{A}}_{p,w}$ denote the matrix product $A_{p+w} \ldots A_p$. Applying equation (4.13)

m times recursively, starting at a time index p that starts a batch, yields

$$\begin{aligned} x^{p+m} &= \tilde{\mathcal{A}}_{p,m-1}x^p + \tilde{\mathcal{A}}_{p+1,m-2}D_p + \ldots + \tilde{\mathcal{A}}_{p+m-1,0}D_{p+m-1} \\ &= \tilde{\mathcal{A}}_{p,m-1}x^p + \mathcal{X}^{p+m}, \end{aligned}$$

where $\mathcal{X}^{p+m}$ is the particular solution evaluated with the zero initial condition $x^p = 0_{nm+m}$. Note that in each of the MRD at time index $w \in \{p+1, \ldots, p+m-1\}$, the value of q is greater than or equal to 1, since at least one reception has occurred (at t_p) between the firing and reception of these pulses. The definitions of the MRD and the state vector x imply that the $n \times n$ matrix formed by the first n rows and columns of A_w is diagonal. Using this observation and the definition of the input disturbance terms, equation (4.14), the particular solution $\mathcal{X}^{p+m}$ is evaluated as $\left(1-(1-h)^d\right)\tau\mathbb{1}_{n+nm}$. Since $\tilde{\mathcal{A}}_{p,m-1}$ is row-stochastic, $\max\left(\tilde{\mathcal{A}}_{p,m-1}x^p\right) \leq \max(x^p)$ and $\min\left(\tilde{\mathcal{A}}_{p,m-1}x^p\right) \geq \min(x^p)$. Hence $V_{p+m} \leq V_p + \max(\mathcal{X}^{p+m}) - \min(\mathcal{X}^{p+m}) = V_p$.

Next, we observe that the GRN is reachable from any node over a sequence of $2bm$ successive MRD, if the batch behavior described above is exhibited over that time. This implies, using reasoning identical to Step 4 in the proof of Theorem 4.1, that $V_{2bm(z+1)} \leq (1-\tilde{\eta})V_{2bmz}$, where $\tilde{\eta} = \min\{h, 1-h\}^b(1-h)^{b(3d-1)}$. Hence, a time sequence of the arc-length function, sampled after each batch, converges asymptotically to zero. In other words, the PCO network intermittently synchronizes. □

5.2. Delayed inhibitory coupling. This subsection demonstrates the applicability of the analysis method of Section 4 to PCO networks with delayed inhibitory coupling as described in [27]. The PCO network model is as described in [27], with the continuous dynamics are given by equation (2.1) and the phase transition rule, assuming oscillator i receives from j, is given by

$$\phi_i(t^+) = g\left(f(\phi_i(t)) - \epsilon_{ij}\right), \tag{5.2}$$

assuming $f(\phi_i(t)) \geq \epsilon_{ij}$, where $f : [0,1[\mapsto [0,1[$, g is f^{-1}, and $f(0) = 0$, $f(1) = 1$, $f' > 0$, and $f'' < 0$.

In [27], intermittent synchronization is proved for strongly connected networks with uniform delays, with initial synchronization error that is sufficiently small. We extend these results to networks that are not necessarily strongly connected but have a GRN. This problem is of the type described in the second row, second column of Table 1.3.

Theorem 5.2 (Intermittent synchronization with inhibitory coupling). *Consider an inhibitory PCO network with n oscillators, with sensing digraph $\mathcal{G}$ with m edges, with pulse strength $\epsilon_{ij} \in]0,1[$ and uniform delay $\tau > \epsilon_{ij}$ for each edge (i,j). Let $d_{\max}$ be the maximum out-degree of any node in $\mathcal{G}$. Assume that:*

(A1") *the sensing digraph $\mathcal{G}$ has a globally reachable node, and*
(A2") *all the pulse strengths are normalized for each oscillator, that is,*

$$\sum_{\substack{j|(i,j) \text{ is} \\ \text{an edge}}} \epsilon_{ij} = \epsilon, \qquad \text{for any } i \text{ in } \{1,\ldots,n\}.$$

Then, for any initial condition the network $\phi(0) \in \Gamma(\omega)$ where $\omega \ll \tau$, the network synchronizes intermittently.

Proof. Analysis similar to that in Section 3 yields finite lower and upper bounds on durations between firings of a given oscillator, and hence statements (ii) and (iii) of

Theorem 3.1 have qualitatively equivalent statements for this system. Proceeding as in the proof of Theorem 4.1, we transform the phases into a rotating coordinate system and rewrite the PCO network dynamics in terms of a state vector x that contains past and present values of the transformed phases. The PCO network dynamics is then given by

$$x^{p+1} = y_p(x^p), \tag{5.3}$$

where $y_p : \mathbb{T}^{nm+n} \mapsto \mathbb{T}^{nm+n}$ is defined by

$$[y_p(x)]_i = \begin{cases} g\left(f(x_i - x_{j+qm} + \tau) - \epsilon_{ij}\right) - \tau, & \text{if } i \leq n \text{ and oscillator } i \text{ receives a pulse from } j \text{ at time } t_p, \\ x_i, & \text{if } i \leq n \text{ and oscillator } i \text{ does not receive a pulse at time } t_p, \\ x_{i-m}, & \text{if } i > n, \end{cases} \tag{5.4}$$

and q is the number of pulses received by any oscillator in the time interval $]t_p - \tau, t_p[$.

It is useful to expand the function y using Taylor's series about the point $x = \tau\mathbb{1}_{n+nm}$, since $V_{\text{arc-length}}(\phi(0)) < \omega$, and so the following condition is satisfied at time t_1:

$$\tau\mathbb{1}_{n+nm} \leq \phi(t_1) \leq (\tau + \omega)\mathbb{1}_{n+nm}. \tag{5.5}$$

Let α^p be the solution of equation (5.3) for zero initial synchronization error, given by

$$\alpha^p = y_p \circ \ldots \circ y_1(\tau\mathbb{1}_{n+nm}),$$

and let β_p be the perturbation of x from this solution so that $x_p = \alpha_p + \beta_p$. Then equation (5.3) is rewritten as

$$\begin{aligned} \alpha^{p+1} + \beta^{p+1} &= y_p(\alpha^p + \beta^p) \\ &= \alpha^{p+1} + \frac{\partial y_p(\alpha^p)}{\partial x}\beta^p + O(\omega^2) \\ \implies \beta^{p+1} &= \frac{\partial y_p(\alpha^p)}{\partial x}\beta^p + O(\omega^2). \end{aligned} \tag{5.6}$$

The Jacobian matrix $\frac{\partial y_p(\alpha^p)}{\partial x}$ is row-stochastic, and equal to the adjacency matrix of the MRD at t_p, weighted by the gradients of y_p, which are shown in [27] to be upper and lower bounded inside $]0, 1[$. Since the synchronization error is sufficiently small, the $O(\omega^2)$ term can be neglected, and batch behavior is exhibited. Analysis identical to the proof of Theorem 5.1 implies that

$$\max(\beta^{(z+1)2bm}) - \min(\beta^{(z+1)2bm}) \leq \hat{a}\left(\max(\beta^{(2zbm)}) - \min(\beta^{2zbm})\right), \tag{5.7}$$

for some $\hat{a}$ in $]0, 1[$. In [27], it is shown that since the pulse weights are normalized, the intermittently-synchronized solution exists, and so $\alpha^{mz} = \tau\mathbb{1}_{n+nm}$ for any $z \in \mathbb{N}$. Therefore, $V_{\text{arc-length}}(\phi(t_{2zbm}))$ is less than or equal to $\max(\beta^{(2zbm)}) - \min(\beta^{2zbm})$, and converges exponentially to 0. Hence the PCO network intermittently synchronizes. ∎

6. Numerical Simulations. The very simple PCO networks shown in Figure 6.1 (left) were simulated in MATLAB. The details of the simulation program are provided in [13]. Figure 6.1 (right) shows the synchronization error during approximate synchronization, for differing amounts of delay. Qualitatively, the behavior is as expected, in that the final synchronization error increases with the delays. However, we see that the predicted error according to Theorem 4.1 is conservative, and in simulations of more complex networks (not reported here), the prediction was even more so. Also, convergence occurred even when assumption (A3) was violated, which suggests that the assumptions may be unnecessarily conservative. However, the method used in the proofs of the theorems may be repeated with specific knowledge of the sensing digraph to obtain better bounds on the parameters.

Figure 6.2 shows phases and synchronization error during synchronization, with zero delays, as discussed in Corollary 4.2. Figure 6.3 shows the intermittent synchronization that was predicted by Theorems 5.1 and 5.2. These graphs show decrease in the synchronization error at times other than brief adjustment periods that occur every oscillation, seen as spikes. Interestingly, although the proof of Theorem 5.2 was for small initial synchronization error, the basin of attraction of the final state is seen in simulations to be larger.

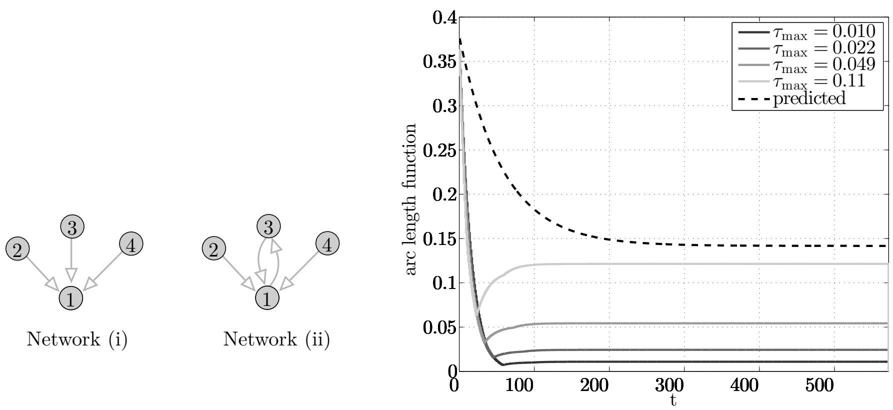

Fig. 6.1. *Left: sensing digraphs used in simulations. Right: synchronization error during approximate synchronization for Network (i), simulated for different values of the delays, with* $h = 0.06$*. The predicted value is calculated from Theorem 4.1, for* $\tau_{\max} = 0.11$*. Interestingly, the approximate synchronization behavior is seen even though assumption* (A3) *is violated for* $\tau_{\max} \geq 0.022$*. This implies that the assumption may be unnecessarily conservative.*

7. Conclusion. We developed a novel method to analyze PCO networks with delayed excitatory/inhibitory coupling. Theorem 3.1 analyzed timing-related properties of the system that enabled us to rewrite the PCO network dynamics as an LTV control system. Theorem 4.1 presented sufficient conditions for the PCO network to approximately synchronize in the presence of small delays, exponentially fast. Corollary 4.2 described a PCO network with all the delays set to zero. Theorem 5.1 studied intermittent synchronization, which may be sufficient for practical applications, for networks with uniform delays and all-equal out-degrees. Theorem 5.2, analyzed a (nonlinear) inhibitory system, and presented sufficient conditions for intermittent synchronization that are less conservative than [27], in that they do not require strong connectivity.

Future work may attempt to find tighter bounds on the rate of convergence, the

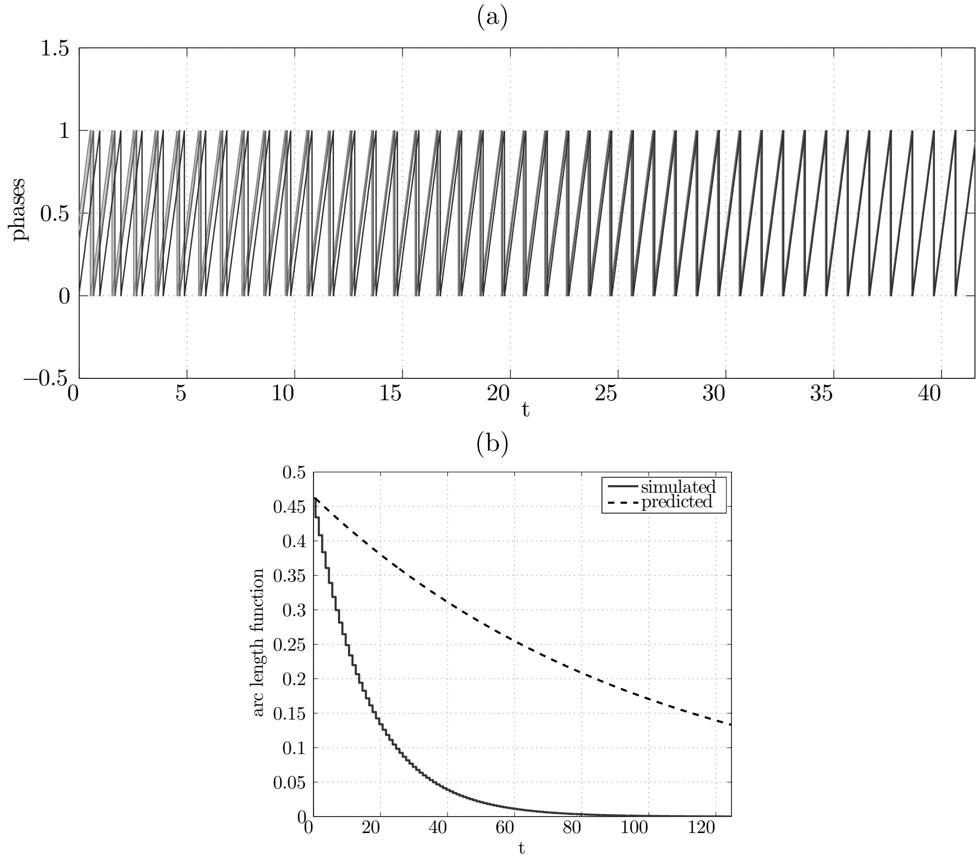

Fig. 6.2. *(a): Phases and (b) synchronization error during synchronization of Network (i), simulated without delays, and with $h = 0.06$. The predicted value is calculated from Corollary 4.2.*

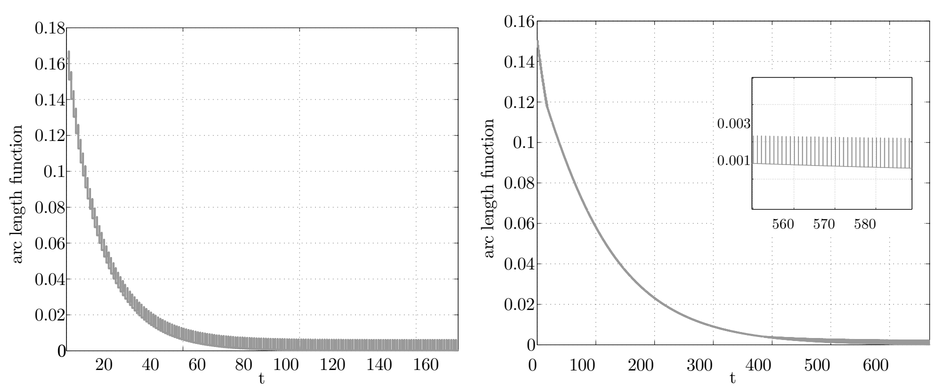

Fig. 6.3. *Synchronization error during intermittent synchronization of Network (ii), simulated for Theorem 5.1 on the left (with $h = .06$ and $\tau = 0.1$), and for Theorem 5.2 on the right (with $\tau = 0.1$, $\epsilon_{ij} = .004$ for every edge, and $g(x)$ defined as $\frac{x(x+.01)}{1.01}$). We see, in the magnified graph in the inset, the spikes that occur every oscillation when the oscillators are adjusting their phases.*

final value of the synchronization error, and the basin of attraction, since our estimates were found to be conservative. Also, the method may be extended to more complex PCO networks that have non-uniform oscillators, time-dependent or random delays, or refractory periods. The less conservative conditions for synchronization, which permit sensing digraphs that are not necessarily strongly connected but have a GRN, may enable improvements in technological applications, such as clock synchronization in sensing and robotic networks, with less heavy requirements on communication.

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