OPINION DYNAMICS AND SOCIAL POWER EVOLUTION OVER REDUCIBLE INFLUENCE NETWORKS*

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Abstract. Our recent work [28] proposes the DeGroot-Friedkin dynamical model for the analysis of social influence networks. This dynamical model describes the evolution of self-appraisals in a group of individuals forming opinions in a sequence of issues. Under a strong connectivity assumption, the model predicts the existence and semi-global attractivity of equilibrium configurations for self-appraisals and social power in the group.

In this paper, we extend the analysis of the DeGroot-Friedkin model to two general scenarios where the interpersonal influence network is not necessarily strongly connected and where the individuals form opinions with reducible relative interactions. In the first scenario, the relative interaction digraph is reducible with globally reachable nodes; in the second scenario, the condensation of the relative interaction digraph has multiple aperiodic sinks. For both scenarios, we provide the explicit mathematical formulations of the DeGroot-Friedkin dynamics, characterize their equilibrium points, and establish their asymptotic attractivity properties. This work completes the study of the DeGroot-Friedkin model with most general social network settings and predicts that, under all possible interaction topologies, the emerging social power structures are determined by the individuals' eigenvector centrality scores.

Key words. opinion dynamics, reflected appraisal, influence networks, mathematical sociology, network centrality, dynamical systems, coevolutionary networks

AMS subject classifications. 91D30, 91C99, 37A99, 93A14, 91B69

1. Introduction.

Originated from structural social psychology, the development of social networks has a long history combining concepts from psychology, sociology, anthropology, and mathematics. Recently, motivated by the popularity of online social networks and encouraged by large corporate and government investments, social networks have attracted extensive research interest from natural and engineering sciences. Though classic studies on social networks mainly focused on static analyses of social structures [15, 42], much ongoing interest in this field lies on dynamic models [1, 26, 31, 40] and includes, for example, the study of opinion formation [2, 6, 12, 21, 34, 38], social learning [3, 23], social network sensing [41] and information propagation [16, 30, 36].

Among the investigations of social networks, opinion dynamics draw considerable attention as it focuses on the basic problem of how individuals are influenced by the presence of others in a social group [4]. In particular, the available empirical evidence suggests that individuals update their opinions as convex combinations of their own and others' displayed opinions, based on interpersonal accorded weights. This convex combination mechanism is considered as a fundamental "cognitive algebra" of heterogeneous information [5] and appears in the early seminal works by French [18], Harary [24], and DeGroot [14].

Related to the field of opinion dynamics, the theory of social influence networks [21] presents a formalization of the social process of attitude change via endogenous interpersonal influence among a social group. This theory focuses on the

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evolution of self-appraisal, social power (i.e., influence centrality) and interpersonal influence for a group of individuals who discuss and form opinions about multiple issues. In particular, social power evolves when individuals' accorded interpersonal influence is modified in positive correspondence with their prior relative control over group issue outcomes. Such a *reflected appraisal* mechanism was summarized by Friedkin [19] and validated by empirical data [20]: individuals' self-appraisals are elevated or dampened based upon their relative power and their influence accorded to others.

Our recent work [28] introduces the DeGroot-Friedkin model, that is, a theoretical model of social influence network evolution that combines (i) the averaging rule by DeGroot [14] to describe opinion formation processes on a single issue and (ii) the reflected appraisal mechanism by Friedkin [19] to describe the dynamics of individuals' self-appraisals and social power across an issue sequence. Given a constant set of irreducible relative interpersonal weights (i.e., a strongly connected relative interaction network), the DeGroot-Friedkin model predicts the evolution of the influence network and the opinion formation process. This nonlinear model shows that the social power ranking among individuals is asymptotically equal to their centrality ranking, that social power tends to accumulate at the top of the hierarchy, and that an autocratic (resp. democratic) power structure arises when the centrality scores are maximally non-uniform (resp. uniform). In other words, the results for the DeGroot-Friedkin model suggest that influence networks evolve toward a concentration of social power over issue outcomes.

This article aims to extend the previous work on the DeGroot-Friedkin model to social groups associated with *reducible* relative interaction digraphs and complete the characterization of the DeGroot-Friedkin dynamical system in the most general network settings. The consideration of reducible networks is a very useful extension of the mathematical treatment evolving social networks, because many real social groups and networks are not strongly connected. Reducibility is encouraged by homophily and the existence of multiple stubborn agents. Thus, this article moves towards greater realism and widens the scope of analysis. It is interesting and meaningful to investigate whether the social power configurations converge in general and whether the social power accumulates regardless of the strong connectivity of the networks. In particular, we consider two classes of reducible networks: (i) the associated digraph of the relative interaction network is reducible with globally reachable nodes (i.e., there exist some individuals in such a social network to which any other individual accords positive influence weight directly or indirectly through the network); (ii) the associated digraph of the relative interaction network does not have any globally reachable nodes and its associated condensation digraph has multiple aperiodic sinks. The main technical difficulties arise twofold. First, we need to redefine the DeGroot-Friedkin model on reducible networks, as the central systemic parameters, the centrality scores may include zero value on the digraphs of case (i) above, or the centrality scores are not well defined for the whole network on the digraphs of case (ii). Second, as the DeGroot-Friedkin dynamical systems appear in different mathematical formations in reducible digraphs compared to the original work [28], we have to analyze and re-examine the existence and convergence properties of the equilibria for the new nonlinear systems.

The main contributions of this paper are as follows. We analyze the DeGroot-Friedkin model on two classes of reducible social networks, provide the explicit and concise mathematical formulations of the reflected appraisal mechanism for both cases, and characterize the existence and asymptotic convergence properties of their equilibrium points. In particular, for the first class of reducible networks (with globally reachable nodes), we show that the DeGroot-Friedkin model has equilibrium points and convergence properties that are similar to those of the strongly connected networks. The final values of social power are independent of the initial states and depend uniquely upon the relative interpersonal weights or, more precisely, upon the eigenvector centrality scores generated from these weights. For the second class of reducible networks (without globally reachable nodes), the social power equilibrium still uniquely depends upon the relative interaction digraph. Precisely, at equilibrium, the sink components in the associated digraphs share all social power whereas the remaining nodes have zero power. This unique equilibrium is globally attractive. Moreover, to our best knowledge, the convergence of the DeGroot model on networks without globally reachable nodes has been little discussed in the literature. Once again, our results are consistent with the "iron law of oligarchy" postulate [33] in social organizations about the concentration of social power. Finally, we numerically illustrate our results by applying the DeGroot-Friedkin model to the Sampson's monastery network, that is, a well-known example of a reducible network.

Paper organization. The rest of the paper is organized as follows. Section 2 briefly reviews the DeGroot-Friedkin model and its dynamical properties in strongly connected social networks. Section 3 includes the main results: subsection 3.1 characterizes the DeGroot-Friedkin model in reducible networks with globally reachable nodes; subsection 3.2 characterizes the DeGroot-Friedkin model in reducible networks without globally reachable nodes and presents a numerical study of the DeGroot-Friedkin model on Sampson's monastery network. Section 4 contains our conclusions and all proofs are in the Appendices.

Notation. For a vector $x \in \mathbb{R}^n$, $x \geq 0$ and $x > 0$ denote component-wise inequalities, and x^T denote its transpose. We adopt the shorthands $\mathbb{1}_n = [1, \dots, 1]^T$ and $\mathbb{0}_n = [0, \dots, 0]^T$. For $i \in \{1, \dots, n\}$, we let $\mathbb{e}_i$ be the ith basis vector with all entries equal to 0 except for the i-th entry equal to 1. Given $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, we let $\mathrm{diag}(x)$ denote the diagonal $n \times n$ matrix whose diagonal entries are $x_1, \dots, x_n$. The n-simplex Δ_n is the set $\{x \in \mathbb{R}^n \mid x \geq 0,\ \mathbb{1}_n^T x = 1\}$; recall that the vertices of the simplex are the vectors $\{\mathbb{e}_1, \dots, \mathbb{e}_n\}$. A non-negative matrix is *row-stochastic* (respectively, *doubly-stochastic*) if all its row sums are equal to 1 (respectively, all its row and column sums are equal to 1). For a non-negative matrix $M = \{m_{ij}\}_{i,j \in \{1,\dots,n\}}$, the *associated digraph* $G(M)$ of M is the directed graph with node set $\{1, \dots, n\}$ and with edge set defined as follows: (i, j) is a directed edge if and only if $m_{ij} > 0$. A non-negative matrix M is *irreducible* if its associated digraph is strongly connected; a non-negative matrix is *reducible* if it is not irreducible. An irreducible matrix M is *aperiodic* if it has only one eigenvalue of maximum modulus. A node of a digraph is *globally reachable* if it can be reached from any other node by traversing a directed path. A *sink* in a digraph is a node without outgoing edges. A subgraph H is a *strongly connected component* of a digraph G if H is strongly connected and any other subgraph of G strictly containing H is not strongly connected. The *condensation digraph* $D(G)$ of G is defined as follows: the nodes of $D(G)$ are the strongly connected components of G, and there exists a directed edge in $D(G)$ from node H_1 to node H_2 if and only if there exists a directed edge in G from a node of H_1 to a node of H_2. G has a globally reachable node if and only if $D(G)$ has a single sink.

2. Preliminary studies of the DeGroot-Friedkin model.

In this section we will briefly introduce the previous work on the DeGroot-Friedkin model [28]. The mathematical formation of the model and its equilibrium and convergence properties for irreducible social networks will be applied in section 3 as a starting point.

2.1. The DeGroot-Friedkin model. The DeGroot-Friedkin model was motivated by the DeGroot's opinion dynamics model on a single issue and the Friedkin's reflected appraisal model over a sequence of issues.

As discussed in the Introduction, the available empirical evidence and independent work by investigators from different disciplines have formulated opinion dynamics as convex combination mechanisms of heterogeneous information. One well-known model for opinion dynamics is the *DeGroot model* [14]. Consider a group of $n \geq 2$ individuals, each individual updates its opinion based upon others' displayed opinions via the DeGroot model

$$y(t+1) = Wy(t), \quad t = 0, 1, 2, \dots. \tag{1}$$

Here the vector $y \in \mathbb{R}^n$ represents the individuals' opinions. A row-stochastic weight matrix $W = [w_{ij}] \in \mathbb{R}^{n \times n}$ describes the social influence network among the individuals, which satisfies $w_{ij} \in [0, 1]$ for all $i, j \in \{1, \dots, n\}$ and $\sum_{j=1}^{n} w_{ij} = 1$ for all i. This row-stochastic weight matrix assumption is inherited from the DeGroot model [14] and is consistent with Friedkin's reflected appraisal model [19]. For interpersonal weights defined on real numbers, including negative numbers, the reader may be referred to the topic on balance theory [11,25] and our recent work [27], but we do not do so here. Each w_{ij} represents the interpersonal (influence) weight accorded by individual i to individual j. In particular, w_{ii} represents individual i's *self-weight* (self-appraisal). For simplicity of notation, we adopt the shorthand $x_i = w_{ii}$. Because $1 - x_i$ is the aggregated allocation of weights to others, the *influence matrix* W is decomposed as

$$W(x) = \mathrm{diag}(x) + (I_n - \mathrm{diag}(x))C, \tag{2}$$

where the matrix C is called *relative interaction matrix* such that the coefficients c_{ij} are the *relative interpersonal weights* that individual i accords to other individuals, and $c_{ii} = 0$. It is easy to verify that $w_{ij} = (1 - x_i)c_{ij}$, and C is row-stochastic with zero diagonal as W is row-stochastic.

If C is irreducible, then, by applying the Perron-Frobenius Theorem, the influence matrix $W(x)$ admits a unique normalized left eigenvector $w(x)^T \geq 0$ associated with the eigenvalue 1, such that $w(x) \in \Delta_n$. We call $w(x)^T$ the *dominant left eigenvector* of $W(x)$ and it satisfies $\lim_{t\to\infty} W(x)^t = \mathbb{1}_n w(x)^T$. Moreover, the DeGroot process (1) converges to an opinion consensus

$$\lim_{t\to\infty} y(t) = \Big(\lim_{t\to\infty} W(x)^t\Big) y(0) = \big(w(x)^T y(0)\big) \mathbb{1}_n. \tag{3}$$

That is, the individuals' opinions converge to a common value equal to a convex combination of their initial opinions $y(0)$, where the coefficients $w(x)$ mathematically describe each individual's relative control, i.e., the ability to control issue outcomes. As claimed by Cartwright [10], this relative control is precisely a manifestation of individual social power.

Different from the DeGroot model defined on a single issue, the DeGroot-Friedkin model focuses on the evolution of social power over an issue sequence, which is inspired by the fact that social groups, like firms, deliberative bodies of government and other associations of individuals, may be constituted to deal with sequences of issues. Considering a group of $n \geq 2$ individuals who discuss an issue sequence $s \in \mathbb{Z}_{\geq 0}$, the individuals' opinions about each issue s are described by the DeGroot model

$$y(s, t+1) = W(x(s))y(s, t), \tag{4}$$

with given initial conditions $y_i(s,0)$ for each individual i. By assuming an issue-independent C, the self-weights $s \mapsto x(s)$ evolve from issue to issue via Friedkin's *reflected appraisal* model [19]. The Friedkin model assumes that the self-weight of an individual is updated, after each issue discussion, equal to the relative control over the issue outcome. That is

$$x(s+1) = \Big(\lim_{t\to\infty} W(x(s))^t\Big)^T \mathbb{1}_n/n = w(x(s)), \tag{5}$$

where $w(x(s))^T$ is the dominant left eigenvector of the influence matrix $W(x(s))$. Notice that, for issue $s \geq 1$, the self-weight vector $x(s)$ necessarily takes value inside Δ_n. It is therefore natural to assume that $x(s)$ takes value inside Δ_n for all issues.

By integrating the Friedkin model with the DeGroot model, we have

DEFINITION 1 (DeGroot-Friedkin model [28]). *Consider a group of $n \geq 2$ individuals discussing a sequence of issues $s \in \mathbb{Z}_{\geq 0}$. Let the row-stochastic zero-diagonal irreducible matrix C be the relative interaction matrix encoding the relative interpersonal weights among the individuals. The DeGroot-Friedkin model for the evolution of the self-weights $s \mapsto x(s) \in \Delta_n$ is*

$$x(s+1) = w(x(s)),$$

where $w(x(s)) \in \Delta_n$ and $w(x(s))^T$ is the dominant left eigenvector of the influence matrix $W(x(s))$,

$$W(x(s)) = \operatorname{diag}(x(s)) + (I_n - \operatorname{diag}(x(s)))C.$$

Let $c^T = [c_1, \dots, c_n]$ be the dominant left eigenvector of C. The explicit expression for the DeGroot-Friedkin model with irreducible C is established as follows.

LEMMA 2 (Explicit formulation of the DeGroot-Friedkin model [28]). *For $n \geq 2$, let c^T be the dominant left eigenvector of the relative interaction matrix $C \in \mathbb{R}^{n\times n}$ that is row-stochastic, zero-diagonal and irreducible. The DeGroot-Friedkin model is equivalent to $x(s+1) = F(x(s))$, where $F : \Delta_n \to \Delta_n$ is a continuous map defined by*

$$F(x) = \begin{cases} \mathbb{e}_i, & \textit{if } x = \mathbb{e}_i \textit{ for all } i \in \{1,\dots,n\}, \\ \Big(\dfrac{c_1}{1-x_1}, \dots, \dfrac{c_n}{1-x_n}\Big)^T \Big/ \displaystyle\sum_{i=1}^n \dfrac{c_i}{1-x_i}, & \textit{otherwise.} \end{cases} \tag{6}$$

Note that we regard c_i as an appropriate eigenvector centrality score of individual i in the digraph with adjacency matrix C, as the classic definition of eigenvector centrality score [7], i.e., the dominant right eigenvector of C, is not informative here. Lemma 2 implies that the dominant left eigenvector c^T of the relative interaction matrix C plays a key role in the DeGroot-Friedkin model. Eigenvector centrality and its variations has been widely applied in social networks and other realistic networks to determine the importance of individuals (see e.g., [17, 29, 37]). Google's PageRank algorithm [8] is also closely related to this concept. We refer the reader to [22] for a extensive survey of eigenvector centrality. This paper together with the original paper on the DeGroot-Friedkin model [28] claim eigenvector centrality as the elementary driver of social power evolution in sequences of opinion formation processes. It is also noted that the psychological assumption that C is issue-independent is relaxed in our recent work [20] and in the work [43].

2.2. Influence dynamics with irreducible relative interactions. The equilibrium and convergence properties of a DeGroot-Friedkin dynamical system associated with an irreducible relative interaction matrix C is briefly introduced in this subsection.

Given $n = 2$, C is always doubly-stochastic and, for any $(x_1, x_2)^T \in \Delta_2$ with strictly-positive components, F satisfies $F((x_1, x_2)^T) = (x_1, x_2)^T$. We therefore discard the trivial case $n = 2$ for the following statements.

LEMMA 3 (DeGroot-Friedkin behavior with star topology [28]). *For $n \geq 3$, consider the DeGroot-Friedkin dynamical system $x(s+1) = F(x(s))$ defined by a relative interaction matrix $C \in \mathbb{R}^{n\times n}$ that is row-stochastic, irreducible, and has zero diagonal. If C has star topology with center node 1, then*

(i) ***(Equilibria:)*** *the equilibrium points of F are $\{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, and*
(ii) ***(Convergence property:)*** *for all non-autocratic initial conditions $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, the self-weights $x(s)$ and the social power $w(x(s))$ converge to the autocratic configuration $\mathbb{e}_1$ as $s \to \infty$.*

That is to say, for a DeGroot-Friedkin model associated with star topology, the autocrat is predicted to appear on the center node.

THEOREM 4 (DeGroot-Friedkin behavior with stochastic interactions [28]). *For $n \geq 3$, consider the DeGroot-Friedkin dynamical system $x(s+1) = F(x(s))$ defined by a relative interaction matrix $C \in \mathbb{R}^{n\times n}$ that is row-stochastic, irreducible, and has zero diagonal. Assume that the digraph associated to C does not have star topology and let c^T be the dominant left eigenvector of C. Then*

(i) ***(Equilibria:)*** *the equilibrium points of F are $\{\mathbb{e}_1, \dots, \mathbb{e}_n, x^*\}$, where x^* lies in the interior of the simplex Δ_n and the ranking of the entries of x^* is equal to the ranking of the eigenvector centrality scores c, and*
(ii) ***(Convergence property:)*** *for all non-autocratic initial conditions $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, the self-weights $x(s)$ and the social power $w(x(s))$ converge to the equilibrium configuration x^* as $s \to \infty$.*

The DeGroot-Friedkin model in strongly connected networks predicts that the self-weight and social power for each individual asymptotically converges along the sequence of opinion formation processes, the equilibrium social power ranking among individuals coincides their eigenvector centrality ranking (that is to say, the entries of x^* have the same ordering as that of c: if the centrality scores satisfy $c_i > c_j$, then the equilibrium social power x^* satisfies $x_i^* > x_j^*$, and if $c_i = c_j$, then $x_i^* = x_j^*$), and the social power accumulation arises over issue discussions (see Proposition 4.2 in [28]). The power accumulation is most evident in the star topology case: the center individual has all social power.

3. Influence dynamics with reducible relative interactions. The main results in the previous work [28] (as repeated in section 2) rely on the assumption that the relative interaction matrix C is irreducible, i.e., the associated digraph is strongly connected. However, this assumption does not always hold and we may confront situations where C is reducible so that the social influence network is not strongly connected. We consider three exclusive cases for a reducible C.

In subsection 3.1 we assume that the matrix C is reducible and its associated digraph has globally reachable nodes. Then C admits a unique dominant left eigenvector, the DeGroot opinion dynamics (4) are always convergent, and the analysis of the DeGroot-Friedkin model is essentially similar to that for an irreducible matrix C.

In subsection 3.2 we assume that the matrix C is reducible and its associated condensation digraph has multiple aperiodic sinks. In this case, the modeling analysis for the DeGroot-Friedkin influence dynamics is not directly applicable because C has a left eigenvector with eigenvalue 1 corresponding to each sink. In our analysis below, we show that the DeGroot opinion dynamics (4) always converge , so that the DeGroot-Friedkin dynamics are well posed. We then establish the existence, uniqueness and attractivity of an equilibrium point even for this general setting.

Finally, we do not analyze the third case where C has neither globally reachable nodes nor aperiodic sinks (in its associated condensation digraph). This third case is similar to the second case (analyzed in subsection 3.2) with, however, the added complication that the convergence of DeGroot opinion dynamics depends upon the value of the self-weights. Because the aperiodicity assumption does not appear to be overly restricting, we find this final third case is least interesting.

3.1. Reducible relative interactions with globally reachable nodes. In this subsection we generalize Theorem 4 to the setting of reducible C with globally reachable nodes. Recall that C is reducible if and only if $G(C)$ is not strongly connected. Without loss of generality, assume that the globally reachable nodes are $\{1,\dots,g\}$, for $g \leq n$, and let $G(C_g)$ be the subgraph induced by the globally reachable nodes. One can show that there does not exist a row-stochastic matrix C with zero diagonal and a globally reachable node; if $g = 1$, then, by assuming that node 1 is the only globally reachable node, the self-weights converge to $x^* = x(1) = w(0) = \mathbb{e}_1$ for any initial conditions even if C is not well defined. We therefore assume $g \geq 2$ in the following. For simplicity of analysis, we also assume that the subgraph $G(C_g)$ is aperiodic (otherwise, the dynamics of opinions about a single issue may exhibit oscillations and not converge). Under these assumptions the DeGroot opinion dynamics is always convergent. Indeed, the matrix C admits a unique dominant left eigenvector c^T with the property that $c_1,\dots,c_g$ are strictly positive and $c_{g+1},\dots,c_n$ are zero. Moreover, for $x \in \Delta_n \setminus \{\mathbb{e}_1,\dots,\mathbb{e}_n\}$, there exists a unique $w(x) \in \Delta_n$ such that $w(x)^T W(x) = w(x)^T$, $w_{g+1}(x) = \cdots = w_n(x) = 0$, and $\lim_{t\to\infty} W(x)^t = \mathbb{1}_n w(x)^T$. In other words, opinion consensus is always achieved and the individuals who are not globally reachable in $G(C)$ have no influence on the final opinion. Consequently, the DeGroot-Friedkin model is well defined via the reflected appraisal mechanism (5).

LEMMA 5 (DeGroot-Friedkin model with reachable nodes). *For $n \geq g \geq 2$, consider the DeGroot-Friedkin dynamical system $x(s+1) = F(x(s))$ associated with a relative interaction matrix $C \in \mathbb{R}^{n\times n}$ which is row-stochastic, reducible and with zero diagonal. Let c^T be the dominant left eigenvector of C and let $\{1,\dots,g\}$ be the globally reachable nodes of $G(C)$. Assume that the globally reachable subgraph $G(C_g)$ is aperiodic. Then the map $F : \Delta_n \to \Delta_n$ satisfies*

$$F(x) = \begin{cases} \mathbb{e}_i, & \text{if } x = \mathbb{e}_i, i \in \{1,\dots,g\}, \\ \big(d_{1i},\dots,d_{gi},0,\dots,0,d_{ii},0,\dots 0\big)^T, & \text{if } x = \mathbb{e}_i, i \in \{g+1,\dots,n\}, \\ \Big(\dfrac{c_1}{1-x_1},\dots,\dfrac{c_g}{1-x_g},0,\dots,0\Big)^T \Big/ \sum_{i=1}^{g} \dfrac{c_i}{1-x_i}, & \text{otherwise}, \end{cases} \quad (7)$$

for appropriate strictly-positive scalars $\{d_{1i},\dots,d_{gi},d_{ii}\}$, $i \in \{g+1,\dots,n\}$. Moreover, the map F is continuous in $\Delta_n \setminus \{\mathbb{e}_{g+1},\dots,\mathbb{e}_n\}$.

The proof of Lemma 5, together with the expression for $\{d_{1i},\dots,d_{gi},d_{ii}\}$, $i \in \{g+1,\dots,n\}$, is presented in Appendix A. Apparently, the irreducible relative interaction case described in Lemma 2 is a special case of Lemma 5 for $g = n$.

Theorem 6 (DeGroot-Friedkin behavior with reachable nodes). *For $n \geq g \geq 2$, consider the DeGroot-Friedkin dynamical system $x(s+1) = F(x(s))$ under the same assumptions as in Lemma 5, described by (7). Then*

(i) *in case $g = 2$, the equilibrium points of F are $\{(\alpha, 1-\alpha, 0, \cdots, 0)^T\}$ for any $\alpha \in [0,1]$, and for all initial conditions $x(0) \in \Delta_n$, the self-weights $x(s)$ and the social power $w(x(s))$ converge to an equilibrium point in at most 2 steps;*

(ii) *in case $g \geq 3$ and $G(C_g)$ has star topology with the center node 1, the equilibrium points of F are $\{\mathbb{e}_1, \ldots, \mathbb{e}_g\}$, and for all initial conditions $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_g\}$, the self-weights $x(s)$ and the social power $w(x(s))$ converge to $\mathbb{e}_1$ as $s \to \infty$;*

(iii) *in case $g \geq 3$ and $G(C_g)$ does not have star topology, the equilibrium points of F are $\{\mathbb{e}_1, \ldots, \mathbb{e}_g, x^*\}$, where $x^* \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$ satisfies: 1) $x_i^* > 0$ for $i \in \{1, \ldots, g\}$ and $x_j^* = 0$ for $j \in \{g+1, \ldots, n\}$, and 2) the ranking of the entries of x^* is equal to the ranking of the eigenvector centrality scores c; moreover, for all initial conditions $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_g\}$, the self-weights $x(s)$ and the social power $w(x(s))$ converge to x^* as $s \to \infty$;*

The social power accumulation occurred in the DeGroot-Friedkin dynamics with irreducible C is also observed here. The following proposition is parallel to an equivalent result for the case of irreducible relative interactions in our previous work [28].

Proposition 7 (Power accumulation with reachable nodes). *Consider the DeGroot-Friedkin dynamical system $x(s+1) = F(x(s))$ under the same assumptions as in Theorem 6 part* (iii). *There exists a unique threshold $c_{\text{thrshld}} := 1 - (\sum_{i=1}^{g} \frac{c_i}{1-x_i^*})^{-1} \in [0,1]$ such that*

(i) *if $c_{\text{thrshld}} < 0.5$, then every individual with a centrality score above the threshold ($c_i > c_{\text{thrshld}}$) has social power larger than its centrality score ($x_i^* > c_i$) and, conversely, every individual with a centrality score below the threshold ($c_i < c_{\text{thrshld}}$) has social power smaller than its centrality score ($x_i^* < c_i$); moreover, individuals with $c_i = c_{\text{thrshld}}$ satisfy $x_i^* = c_i$;*

(ii) *if $c_{\text{thrshld}} \geq 0.5$, then there exists only one individual with social power larger than its centrality score ($x_i^* > c_i$) and all other individuals have $x_i^* < c_i$;*

(iii) *for any individuals $i, j \in \{1, \ldots, g\}$ with centrality scores satisfying $c_i > c_j > 0$, the social power is increasingly accumulated in individual i compared to individual j, that is, $x_i^*/c_i > x_j^*/c_j$.*

Remark 8 (Interpretation of Theorem 6 and Proposition 7). According to Theorem 6, for a reducible row-stochastic C with $m \geq 3$ globally reachable nodes, the vector of self-weights $x(s)$ converges to a unique equilibrium value x^* from all initial conditions, except the autocratic states. This equilibrium value x^* is uniquely determined by the eigenvector centrality score c. Those nodes, which are not globally reachable, have zero self-weights and then zero social power in the equilibrium. If the topology among the globally reachable nodes is a star, then the autocrat is predicted to appear on the center node. Otherwise, if the topology among the globally reachable nodes is not a star, then the entries of x^* corresponding to the globally reachable nodes are strictly positive and have the same ranking as that of c. Moreover, according to Proposition 7, an accumulation of social power is observed in the central nodes of the network. That is, individuals with the large centrality scores have an equilibrium social power that is larger than their respective centrality scores; in turn, the individual with the lowest centrality score has a lower equilibrium social power. Additionally, such a social power accumulation accelerates in the nodes with larger centrality scores. (This property, as described in fact (iii) of Proposition 7, also holds for the DeGroot-Friedkin model with irreducible relative interactions, though it is not explicitly discussed in [28].) This accumulation phenomenon is especially evident for the star topology case: the center individual with $c_i = 0.5$ has all social power and all other individuals have zero social powers. These claims are comparable to the previous results in the irreducible relative interaction case as demonstrated in subsection 2.2, and their proofs are presented in Appendices B and C, respectively.

3.2. Reducible relative interactions with multiple sink components. In this subsection we generalize the treatment of the DeGroot-Friedkin model to the setting of reducible C without globally reachable nodes. Such matrices C have an associated condensation digraph $D(G(C))$ with $K \geq 2$ sinks. Subject to the aperiodicity assumption on each sink, the DeGroot opinion dynamical system still converges for each single issue, even though consensus is not achieved for generic initial opinions.

In what follows, n_k denotes the number of nodes in sink k, $k \in \{1, \dots, K\}$, of the condensation digraph; by construction $n_k \geq 2$. (When $n_k = 1$, the corresponding sink node never changes its opinion in issue discussions, and therefore, its self-weight and social power keep constant.) Assume that the number of nodes in $G(C)$, not belonging to any sink in $D(G(C))$, is m, that is, $\sum_{k=1}^{K} n_k + m = n$. After a permutation of rows and columns, C can be written as

$$C = \begin{bmatrix} C_{11} & 0 & \dots & 0 & 0 \\ 0 & C_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & C_{KK} & 0 \\ C_{M1} & C_{M2} & \dots & C_{MK} & C_{MM} \end{bmatrix}, \tag{8}$$

where the first $(n - m)$ nodes belong to the sinks of $D(G(C))$ and the remaining m nodes do not. By construction each $C_{kk} \in \mathbb{R}^{n_k \times n_k}$, $k \in \{1, \dots, K\}$ is row-stochastic and irreducible. If C_{kk} is also aperiodic, then its dominant left eigenvector $c_{kk}^T = (c_{kk_1}, \dots, c_{kk_{n_k}})$ is unique and positive. Under these assumptions, the matrix C has the following properties: eigenvalue 1 has geometric multiplicity equal to K, the number of sinks in the condensation digraph $D(G(C))$; eigenvalue 1 is strictly larger than the magnitude of all other eigenvalues so that C is semi-convergent. Consequently, C has K dominant left eigenvectors associated with eigenvalue 1, denoted by ${c^k}^T \in \mathbb{R}^n$ for $k \in \{1, \dots, K\}$, with the properties that: $c^k \geq 0$, $\sum_{i=1}^{n} c_i^k = 1$, $c_i^k > 0$ if and only if node i belongs to sink k, and $c_i^k = c_{kk_j}$ for $j = i - \sum_{l=1}^{k-1} n_\ell$. We also denote $x = (x_{11}^T, x_{22}^T, \dots, x_{KK}^T, x_{MM}^T)^T$, where $x_{kk} = (x_{kk_1}, \dots, x_{kk_{n_k}})^T \in \mathbb{R}^{n_k}$ are the self-weights associated with sink k. Similarly, $x_i = x_{kk_j}$ for $j = i - \sum_{l=1}^{k-1} n_\ell$.

As mentioned in the beginning of this subsection, we first prove that the DeGroot opinion dynamics converge for each issue discussion, subject to the assumptions above (see details in the proof of Lemma 9). That is, $\lim_{t\to\infty} W(x(s))^t$ exists for each s, but the limit is not necessarily equal to a rank-1 matrix (different from the previous cases of irreducible relative interactions or reducible relative interactions with globally reachable nodes). The reflected appraisal mechanism (5) still holds here, but the social power $w(x) = (\lim_{t\to\infty} W(x)^t)^T \mathbb{1}_n / n$ does not satisfy the property that $w(x)^T$ is the dominant left eigenvector of $W(x)$. Now we are ready to discuss the DeGroot-Friedkin model with multiple sink components. The proofs of the following results are postponed to Appendices D to F.

Lemma 9 (DeGroot-Friedkin model with multiple sinks). *For $n \geq 4$, consider the DeGroot-Friedkin dynamical system $x(s+1) = F(x(s))$ associated with a relative interaction matrix $C \in \mathbb{R}^{n \times n}$. Assume that the condensation digraph $D(G(C))$ contains $K \geq 2$ aperiodic sinks and that C is written as in equation (8). Then the map $F : \Delta_n \to \Delta_n$ satisfies*

$$F(x) = \begin{cases} \big(d_{1i}, \dots, d_{ni}\big)^T, & \text{if } x = \mathbb{e}_i,\ i \in \{n-m+1, \dots, n\}, \\ \big(F_{11}(x)^T, \dots, F_{KK}(x)^T, 0, \dots, 0\big)^T, & \text{otherwise.} \end{cases} \tag{9}$$

Here the non-negative scalars d_{ji}, $j, i \in \{1, \dots, n\}$ are strictly positive precisely when $j = i$ or j belongs to a sink of $D(G(C))$. The maps $F_{kk} : \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \dots, e_n\} \to \mathbb{R}^{n_k}$, $k \in \{1, \dots, K\}$, are defined by

$$F_{kk}(x) = \begin{cases} \zeta_k(x)\mathbb{e}_i, & \text{if } x_{kk} = \mathbb{e}_i \in \Delta_{n_k}, i \in \{1, \dots, n_k\}, \\ \zeta_k(x)\Big(\dfrac{c_{kk_1}}{1 - x_{kk_1}}, \dots, \dfrac{c_{kk_{n_k}}}{1 - x_{kk_{n_k}}}\Big)^T \Big/ \Big(\displaystyle\sum_{i=1}^{n_k} \dfrac{c_{kk_i}}{1 - x_{kk_i}}\Big), & \text{otherwise,} \end{cases} \tag{10}$$

where the functions $\zeta_k : \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \dots, e_n\} \to \mathbb{R}$, for $k \in \{1, \dots, K\}$, are appropriate positive functions satisfying $\sum_{k=1}^{K} \zeta_k(x) = 1$ for all x. Moreover, F is continuous in $\Delta_n \setminus \{\mathbb{e}_{n-m+1}, \dots, e_n\}$.

Theorem 10 (DeGroot-Friedkin behavior with multiple sinks). *For $n \geq 4$, consider the DeGroot-Friedkin dynamical system $x(s+1) = F(x(s))$ under the same assumptions as in Lemma 9, described by (9) and (10). Then*

- (i) *(**Social power of sinks:**) for all $s \geq 2$, $\zeta_k(x(s))$, the sum of the individual self-weights in each sink $k \in \{1, \dots, K\}$, is constant, i.e., $\zeta_k^* = \zeta_k(x(2))$;*
- (ii) *(**Equilibrium:**) there exists a unique equilibrium point x^* of F satisfying*
  - (ii.1) *if node i, $i \in \{1, \dots, n\}$, does not belong to any sink, then $x_i(s) = x_i^* = 0$ for all $s \geq 2$,*
  - (ii.2) *if node i, $i \in \{1, \dots, n\}$, belongs to sink $k \in \{1, \dots, K\}$ and $n_k = 2$, then $x_i^* = \zeta_k^*/2$, and*
  - (ii.3) *if node i, $i \in \{1, \dots, n\}$, belongs to sink $k \in \{1, \dots, K\}$ and $n_k \geq 3$, then $x_i^* > 0$; moreover, the ranking of the entries of x_{kk}^* is equal to the ranking of the eigenvector centrality scores c_{kk} in the same sink k;*
- (iii) *(**Convergence of Self-weights:**) for all initial conditions $x(0) \in \Delta_n$, the self-weights $x(s)$ and the social power $w(x(s))$ converge to x^* as $s \to \infty$;*

Finally, for all initial conditions $x(0) \in \Delta_n$, at each issue discussion $s \geq 1$, the influence matrix $W(x(s))$ has K dominant left eigenvectors, denoted by ${w^1}^T(s)$, …, ${w^K}^T(s) \in \Delta_n$, with the properties that

- (iv) *(**Convergence of Influence:**) for $k \in \{1, \dots, K\}$ and $i \in \{1, \dots, n\}$, $w_i^k(s) > 0$ if and only if node i belongs to sink k, and $w_i^k(s)$ converges to x_i^*/ζ_k^* as $s \to \infty$ if node i belongs to sink k.*

Note that $w(x(s))$ in fact (iii) of Theorem 10 does not have the property that $w(x(s))^T$ is the dominant left eigenvector of $W(x(s))$.

Remark 11 (Interpretation of Theorem 10). According to Theorem 10, the self-weight equilibrium is still uniquely determined by the relative interactions C. The sink components of $G(C)$ share all social power after at most two issue discussions and the rest nodes have zero power. Moreover, the sink social powers remain constant (uniquely determined by C) after at most three issue discussions. If a sink component includes two nodes, then those nodes have equal social powers in the equilibrium, independent of initial conditions. Otherwise, if a sink component includes at least three nodes, then those nodes have strictly-positive self-weights in the equilibrium (even for the sink component with a star topology) and their self-weights have the same ranking as that of their centrality scores.

Remark 12 (DeGroot-Friedkin behavior with disconnected components). In an extreme case where all entries of one matrix $C_{Mk}, k \in \{1, \dots, K\}$ are equal to 0, the corresponding component associated with C_{kk} is then disconnected from the rest of the network. If such a C_{kk} is row-stochastic, irreducible and aperiodic, then the analysis in Theorem 10 holds similarly. That is to say, for all initial states $x(0) \in \Delta_n$,

(i) the sum of the individual self-weights in the k-th component associated with C_{kk} is equal to n_k/n for all $s \geq 1$ where n_k is the cardinality of the component;

(ii) the equilibrium of the DeGroot-Friedkin dynamics on the k-th component is uniquely determined, and the self-weight x_i of each node i in the component satisfies: 1) if $n_k = 2$, then $\lim_{s\to\infty} x_i(s) = x_i^* = 1/n$; 2) if $n_k \geq 3$, then $\lim_{s\to\infty} x_i(s) = x_i^* > 0$, and for any other node j that belongs to the same component as i, $c_i^k > c_j^k$ implies $x_i^* > x_j^*$ and $c_i^k = c_j^k$ implies $x_i^* = x_j^*$.

Remark 13 (Eigenvector centrality). We may regard $\zeta_k^* c_{kk}$ as the revised individual eigenvector centrality scores in sink k. A node has zero eigenvector centrality score if it does not belong to any sink. When the number of the sinks is $K \geq 2$, we have $\zeta_k^* c_{kk_i} < 0.5$ for any sink $k \in \{1, \dots, K\}$ with at least two nodes. Consequently, the star topology in a sink does not correspond to an equilibrium point on the center vertex as previously discussed in Lemma 3 and Theorem 6.

Furthermore, the social power accumulation is observed by comparing the revised eigenvector centrality scores $\zeta_k^* c_{kk}$ and the equilibrium self-weights x_{kk}^*.

Proposition 14 (Social power accumulation with multiple sinks). *Consider the DeGroot-Friedkin dynamical system $x(s+1) = F(x(s))$ under the same assumptions as in Theorem 10 part* (ii.3). *There exists a unique threshold ${c_{\mathrm{thrshld}}}^k := 1 - (\sum_{i=1}^{n_k} \frac{c_{kk_i}}{1 - x_{kk_i}^*})^{-1}$ such that*

(i) *if ${c_{\mathrm{thrshld}}}^k < 0.5$, then every individual with a revised centrality score above the threshold ($\zeta_k^* c_{kk_i} > {c_{\mathrm{thrshld}}}^k$) has social power larger than its revised centrality score ($x_{kk_i}^* > \zeta_k^* c_{kk_i}$) and, conversely, every individual with a revised centrality score below the threshold ($\zeta_k^* c_{kk_i} < {c_{\mathrm{thrshld}}}^k$) has social power smaller than its revised centrality score ($x_{kk_i}^* < \zeta_k^* c_{kk_i}$); moreover, individuals with $\zeta_k^* c_{kk_i} = {c_{\mathrm{thrshld}}}^k$ satisfy $x_{kk_i}^* = \zeta_k^* c_{kk_i}$;*

(ii) *if ${c_{\mathrm{thrshld}}}^k \geq 0.5$, then there exists only one individual with social power larger than its revised centrality score ($x_{kk_i}^* > \zeta_k^* c_{kk_i}$) and all other individuals have $x_{kk_i}^* < \zeta_k^* c_{kk_i}$.*

(iii) *for any individuals $i, j \in \{1, \dots, n_k\}$ with centrality scores satisfying $c_{kk_i} > c_{kk_j} > 0$, the social power is increasingly accumulated in individual i compared to individual j, that is, $x_{kk_i}^*/c_{kk_i} > x_{kk_j}^*/c_{kk_j}$.*

An example application to Sampson's monastery network. The social interactions among a group of monks in an isolated contemporary American monastery were investigated by Sampson [39]. Based on his observations and experiments, Sampson collected a variety of experimental information on four types of interpersonal relations: Affect, Esteem, Influence, and Sanctioning. Each of 18 respondent monks ranked their first three choices on these relations, where 3 indicates the highest or first choice and 1 indicates the last choice in the presented interaction matrices. Some subjects offered tied ranks for their top five choices. Here we focus on a monastery social structure from the ranking of the most esteemed members in Sampson's empirical data. The underlying empirical matrix has been normalized to conform to the relative interaction matrix C employed in this paper as follows:

$$C = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .125 & 0 & 0 & 0 & .375 & 0 & 0 & .25 & .25 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .33 & .5 & .17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .143 & .428 & 0 & 0 & .143 & 0 & .286 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .167 & 0 & 0 & 0 & 0 & .33 & .5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & .33 & 0 & .5 & 0 & 0 & .167 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .167 & .33 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .22 & .22 & 0 & 0 & 0 & 0 & 0 & 0 & .33 & 0 & .11 & .11 & 0 & 0 & 0 \\
0 & 0 & .3 & .2 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .375 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .25 & .25 & .125 & 0 & 0 & 0 \\
0 & 0 & .5 & 0 & 0 & 0 & 0 & .33 & 0 & 0 & 0 & 0 & .167 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .33 & .5 & 0 & 0 & 0 & .167 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .5 & .33 & 0 & .167 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .375 & .125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .25 & .125 & .125 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .167 & .5 & .33 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .167 & 0 & .5 & .33 & 0 & 0 & 0 \\
.125 & 0 & .25 & .25 & 0 & .375 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

The condensation digraph associated with C includes two sinks: sink 1 consists of the nodes $\{1, 2\}$, and sink 2 consists of the nodes $\{3, \ldots, 15\}$, see Figure 1. The corresponding two dominant left eigenvectors of C are:

$$\begin{aligned}
{c^1}^T &= [0.5\ \ 0.5\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0], \\
{c^2}^T &= [0\ \ 0\ \ 0.1184\ \ 0.2060\ \ 0.0127\ \ 0.0407\ \ 0.0705\ \ 0.1677\ \ 0.0411\ \ 0.0796 \ldots \\
&\qquad 0.0018\ \ 0.0417\ \ 0.1314\ \ 0.0597\ \ 0.0287\ \ 0\ \ 0\ \ 0].
\end{aligned}$$

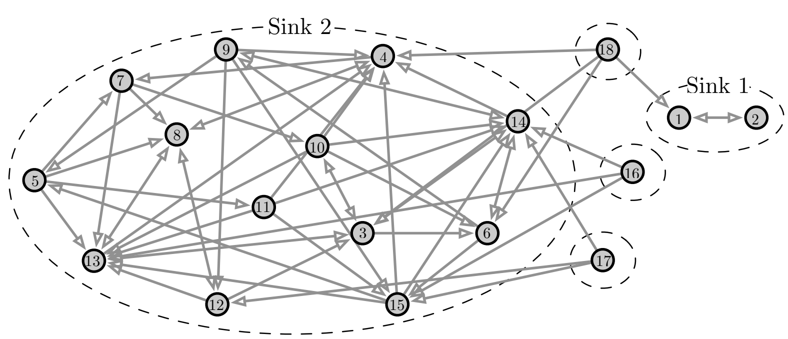

FIG. 1. *Sampson's monastery network*

We simulated the DeGroot-Friedkin model on this monastery network with randomly selected initial states $x(0) \in \Delta_{18}$. The simulation shows that all dynamical trajectories converge to a unique equilibrium self-weight vector x^*, given by

$$\begin{aligned}
x^* = [&0.0590\ \ 0.0590\ \ 0.1029\ \ 0.2009\ \ 0.0100\ \ 0.0328\ \ 0.0583\ \ 0.1547 \ldots \\
&0.0331\ \ 0.0665\ \ 0.0014\ \ 0.0336\ \ 0.1158\ \ 0.0490\ \ 0.0229\ \ 0\ \ 0\ \ 0]^T.
\end{aligned}$$

Meanwhile, $\zeta_1^* = 0.118$, $\zeta_2^* = 0.882$, the revised eigenvector centrality scores, denoted by c^r, can be calculated as follows:

$$c^r = \zeta_1^* c^1 + \zeta_2^* c^2 = [0.0590\ \ 0.0590\ \ 0.1044\ \ 0.1817\ \ 0.0112\ \ 0.0359\ \ 0.0622\ \ 0.1479 \ldots$$
$$0.0363\ \ 0.0702\ \ 0.0016\ \ 0.0368\ \ 0.1159\ \ 0.0527\ \ 0.0253\ \ 0\ \ 0\ \ 0]^T,$$

and the social power accumulation threshold for sink 2 is ${c_{\text{thrshld}}}^2 = 0.1162$.

The dynamical trajectories of 6 selected nodes in Sampson's monastery network (as shown in Figure 1) are illustrated in the first 6 subgraphs of Figure 2, where ten different initial conditions are considered. The trajectories of the summed self-weights in two sinks under the same set of initial conditions are shown in the last two subgraphs of Figure 2.

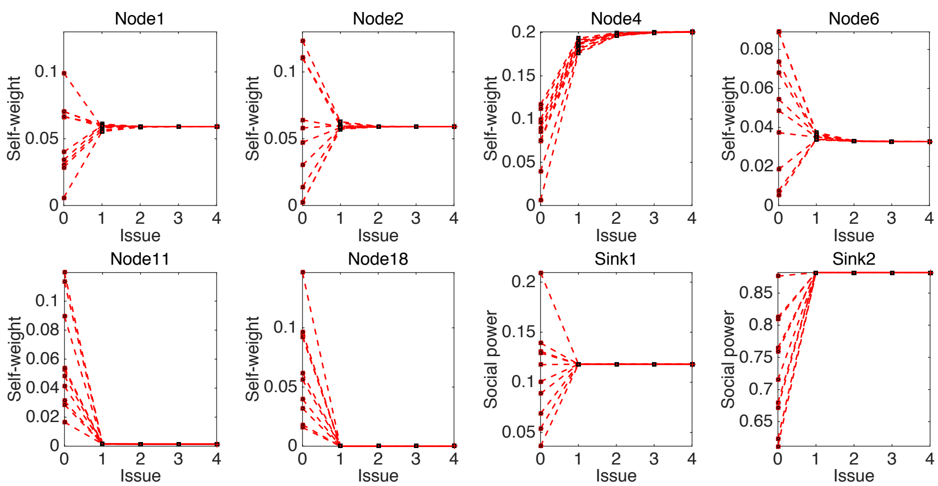

Fig. 2. *DeGroot-Friedkin dynamics for Sampson's monastery network: ten different initial states converge to a unique self-weight configuration* x^* *with the properties that 1) for two nodes* $\{1, 2\}$ *in sink* 1 *with* $n_1 = 2$*, the equilibrium self-weights are strictly positive and equal; 2) for the nodes in sink* 2 *with* $n_2 = 13$*, all equilibrium self-weights are strictly positive and* $x_i^* > x_j^*$ *if and only* $c_i^2 > c_j^2$*, in particular, node* 4 *has the max eigenvector centrality score in the sink, node* 11 *has the min score, and node* 6 *has a score in between; 3) the nodes* $\{16, 17, 18\}$*, which do not belong to any sink, have zero equilibrium self-weights; 4) the convergence of the self-weight sum at each sink occurs in two issues.*

It has been verified in the simulation that, 1) the DeGroot-Friedkin dynamics converge to a unique equilibrium point x^*given any initial condition; 2) all social power is shared by the sinks and each sink's social power remains constant after a few issue discussions; 3) for the nodes in sink k, the ranking of the corresponding entries in x^* is consistent with the centrality score ranking of those nodes in c^k. These observations are consistent with Theorem 10. Moreover, the social power accumulation can also be examined: for $i, j \in \{3, \ldots, 15\}$ in sink 2, $c_i^r > {c_{\text{thrshld}}}^2$ implies $x_i^* > c_i^r$, $c_i^r < {c_{\text{thrshld}}}^2$ implies $x_i^* < c_i^r$, and $x_i^*/x_j^* > c_i^r/c_j^r$ for $c_i^r > c_j^r$. This is consistent with Proposition 14.

4. Conclusion. This article studies the evolution of the influence network in a social group, as the group members discuss and form opinions over a sequence of issues. The paper focuses on reducible networks of relative interactions. The DeGroot-Friedkin model is employed to provide a mechanistic explanation for the evolution of self-appraisal and social power of individuals. This model characterizes the individual self-weights and social power as a function of the individual eigenvector centrality of the relative interaction network. We provide a rigorous mathematical analysis of the DeGroot-Friedkin dynamics on reducible digraphs: we derive the explicit formulations of influence network evolution, characterize the equilibrium points, and establish the convergence properties for two classes of reducible social networks (with or without globally reachable nodes, respectively). The analytical and numerical results in this article complete and confirm the predictions of the DeGroot-Friedkin model on general social influence networks: (i) the individuals' social power ranking is asymptotically equal to their eigenvector centrality ranking, and (ii) social power tends to accumulate in the individuals with higher centrality scores.

The scope of the DeGroot-Friedkin model. The DeGroot-Friedkin model assume that each individual perceives her relative control over discussion outcomes. Subject to this implicit fundamental assumption, the model is most relevant for small to moderate size social groups and is also applicable with some assumptions to large social networks. First, small and moderate-size social groups, e.g., deliberative assemblies, boards of directors, judiciary bodies, and policy making groups, play an important role in modern society. Individuals in such groups are typically able to directly perceive who shaped the discussion and whose opinion had an impact in the final decisions. Therefore, the DeGroot-Friedkin model is well-justified in this setting. Second, as discussed in the our original work on DeGroot-Friedkin model [28], even in large networks, the relative control over discussion outcomes can be perceived by individuals, provided that the individuals are dealing with a common sequence of issues. Consequently, the DeGroot-Friedkin model is applicable in these large social groups. In both cases, the topologies of the influence networks occurred in social groups could be strongly connected, or reducible with or without globally reachable nodes.

Future work. The development of the DeGroot-Friedkin model has motivated various ongoing research directions on social influence networks, that include a refined description of the DeGroot-Friedkin model scope and justification (which was incorporated in [28] and also discussed in [13,43,44]), the extension of the model and analysis to the setting of influence networks with stubborn individuals (e.g., a preliminary work was published in [35]), and the extension of the model and analysis to a more general setting of interpersonal influence. Moreover, the model and its associated analytical techniques may be applicable to other classes of multi-agent network problems.

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Appendix A. Proof of Lemma 5.

Proof. The proof of Lemma 5 is parallel to the proof of Lemma 2. In what follows we mainly focus upon the differences of Lemma 5 compared to the existing results in section 2, and show how to derive the new results from those established theories. We then refer to [28] for supplemental reading. The same strategies are also applied in all the following proofs.

If $G(C)$ contains g, $g \geq 1$, globally reachable nodes $\{1,\dots,g\}$, then the dominant normalized left eigenvector c^T of C exists uniquely satisfying 1) $c_i > 0$ for all $i \in \{1,\dots,g\}$, 2) $c_j = 0$ for all $j \in \{g+1,\dots,n\}$, and 3) $\sum_i^g c_i = 1$. Consequently, F satisfies equation (6) if $x \neq e_i$ for $i \in \{g+1,\dots,n\}$ with the same arguments as in the proof of Lemma 2 (see [28, Appendix B] for details).

If $x = e_i$ for some $i \in \{g+1,\dots,n\}$ (without loss of generality, let $i = n$), then the corresponding $W(x)$ has the form:

$$W(\mathbb{e}_n) = \mathrm{diag}(0,\dots,0,1) + \mathrm{diag}(1,\dots,1,0)C = \begin{bmatrix} C_{\{1,\dots,n-1\}} \\ \mathbb{e}_n^T \end{bmatrix} = \begin{bmatrix} C_{11} & 0 & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

where $C_{\{1,\dots,n-1\}}$ is the $(n-1)\times n$ matrix obtained by removing the last row from C, C_{11} is the $g \times g$ matrix obtained by removing the last $(n-g)$ rows and the last $(n-g)$ columns from C, C_{21}, C_{22} and C_{23} are respectively the $(n-g-1)\times g$, $(n-g-1)\times(n-g-1)$, $(n-g-1)\times 1$ matrices obtained by removing the first g rows and the last row from C. 0 and 1 in the matrix correspond to block matrices with all entries equal to 0 or 1, respectively. The condensation digraph of $G(W(\mathbb{e}_n))$ has at least three nodes, two of which are aperiodic sinks (i.e., the node corresponding to the first m individuals and the node corresponding to individual n).

By linear algebra calculations (see similarly in [32, Chapter 8.3]),

$$\lim_{l\to\infty} W(\mathbb{e}_n)^l = \begin{bmatrix} \mathbb{1}_g(c_1,\dots,c_g) & 0 & 0 \\ (I-C_{22})^{-1}C_{21}\mathbb{1}_g(c_1,\dots,c_g) & 0 & (I-C_{22})^{-1}C_{23} \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

Since $F(x) := \left(\lim_{l\to\infty} W(x)^l\right)^T \mathbb{1}_n/n$ as from equation (5),

$$F(\mathbb{e}_n) = \Big(d_{1n},\dots,d_{gn},0,\dots,0,d_{nn}\Big)^T,$$

where $d_{jn} > 0$ for all $j \in \{1,\dots,g\} \cup \{n\}$ and can be calculated from (12). $F(x)$ is not continuous on these vertices $\{\mathbb{e}_{g+1},\dots,\mathbb{e}_n\}$ since $F_j(x) > 1/n$ if $x = \mathbb{e}_j$ for all $j \in \{g+1,\dots,n\}$, and $F_j(x) = 0$ for any other x. But F is continuous everywhere in the simplex except $\{\mathbb{e}_{g+1},\dots,\mathbb{e}_n\}$, that can be proved in the same way as we did in Lemma 2 (see [28, Appendix B] for details). Moreover, the vertices $\{\mathbb{e}_{g+1},\dots,\mathbb{e}_n\}$ are not in the image of F, that is to say, for all initial conditions $x(0)$, Given F defined in (7), $F(x(s)) \notin \{\mathbb{e}_{g+1},\dots,\mathbb{e}_n\}$ for all $s \geq 1$. ∎

Appendix B. Proof of Theorem 6.

Proof. Fact (i) is from the claim for $n = 2$ discussed in subsection 2.2, and note that $x(1)$ may not be the equilibrium point if $x(0) = \mathbb{e}_i$ for $i \in \{g+1,\dots,n\}$ but $x(s) = x(s+1)$ for all $s \geq 2$. Facts (ii) and (iii) can be directly derived from Lemma 3 and Theorem 4, respectively, because F defined in (7) is exactly the same as F defined in (6) given $x(0) \in \Delta_n \backslash \{\mathbb{e}_1,\dots,\mathbb{e}_g\}$ and $c_j = 0$ for $j \in \{g+1,\dots,n\}$. (See the detailed proofs in [28, Appendices E and F].) ∎

Appendix C. Proof of Proposition 7.

Proof. The social power accumulation fact (i) and (ii) can be deduced from Proposition 4.2 in [28] (see the detailed proof in [28, Appendix G]). The reason is as follows. As F defined in (7) is exactly the same as F defined in (6) given $x(0) \in \Delta_n \backslash \{\mathbb{e}_1,\dots,\mathbb{e}_g\}$ and $c_j = 0$ for $j \in \{g+1,\dots,n\}$, one can check that the analysis remains the same no matter the values of $\{c_{g+1},\dots,c_n\}$ are zero or non-zero. Regarding fact (iii), because $x^* = F(x^*)$ for F defined in (7), we have $x_i^*/x_j^* = \big(c_i/(1-x_i^*)\big)/\big(c_j/(1-x_j^*)\big)$ for $c_i > c_j > 0$. Moreover, $c_i > c_j$ implies $x_i^* > x_j^*$ from fact (iii) of Theorem 6. Hence, $1 - x_i^* < 1 - x_j^*$ implies $x_i^*/x_j^* > c_i/c_j$ or equivalently, $x_i^*/c_i > x_j^*/c_j$.

Appendix D. Proof of Lemma 9.

Proof. Formulation of F: Two cases are considered. First, if $x = \mathbb{e}_i$ and i does not belong to any sink of $D(G(C))$, i.e., $i \in \{n-m+1,\dots,n\}$ (without loss of generality, let $i = n$), then, given C in (8), the influence matrix $W(\mathbb{e}_i)$ is as follows:

$$W(\mathbb{e}_i) = \mathrm{diag}(0,0,\dots,1) + \mathrm{diag}(1,1,\dots,0)C = \begin{bmatrix} C_{\{1,\dots,n-1\}} \\ \mathbb{e}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & 0 & \dots & 0 & 0 & 0 \\ 0 & C_{22} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & C_{KK} & 0 & 0 \\ C_{M1r} & C_{M2r} & \dots & C_{MKr} & C_{MMr1} & C_{MMr2} \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}, \tag{13}$$

where the matrix $[C_{M1r}, \ldots, C_{MMr2}]$ is derived from $[C_{M1}, \ldots, C_{MM}]$ by deleting the last row. It is clear that $W(\mathbb{e}_n)$ in equation (13) has the similar form as in equation (11). By the similar analysis, we have $F(\mathbb{e}_i) = \Big(d_{1i}, \ldots, d_{ni}\Big)^T$ with $d_{ji} > 0$ for j belonging to a sink of $D(G(C))$ or $j = i$, and $d_{ji} = 0$ otherwise.

Second, for a more general $x \in \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \ldots, \mathbb{e}_n\}$, we have

$$W(x) = X + (I_n - X)C = \begin{bmatrix} W_{11}(x) & 0 & \ldots & 0 & 0 \\ 0 & W_{22}(x) & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & W_{KK}(x) & 0 \\ W_{M1}(x) & W_{M2}(x) & \ldots & W_{MK}(x) & W_{MM}(x) \end{bmatrix},$$

where, by denoting $\operatorname{diag}(x_{ii}) = X_{ii}$ for $i \in \{1, \ldots, K, M\}$,

$$X = \operatorname{diag}(x) = \operatorname{diag}\begin{bmatrix} x_{11} \\ x_{22} \\ \vdots \\ x_{KK} \\ x_{MM} \end{bmatrix} := \begin{bmatrix} X_{11} & 0 & \ldots & 0 & 0 \\ 0 & X_{22} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & X_{KK} & 0 \\ 0 & 0 & \ldots & 0 & X_{MM} \end{bmatrix},$$

$W_{kk}(x) = X_{kk} + (I_{n_k} - X_{kk})C_{kk}$, $W_{Mk}(x) = (I_m - X_{MM})C_{Mk}$ for all $k \in \{1, \ldots, K\}$, and $W_{MM}(x) = X_{MM} + (I_m - X_{MM})C_{MM}$. Consequently,

$$\lim_{l \to \infty} W(x)^l = \begin{bmatrix} \mathbb{1}_{n_1} w_{11}^T(x) & 0 & \ldots & 0 & 0 \\ 0 & \mathbb{1}_{n_2} w_{22}^T(x) & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbb{1}_{n_K} w_{KK}^T(x) & 0 \\ N_1(x)\mathbb{1}_{n_1} w_{11}^T(x) & N_2(x)\mathbb{1}_{n_2} w_{22}^T(x) & \ldots & N_K(x)\mathbb{1}_{n_K} w_{KK}^T(x) & 0 \end{bmatrix},$$

where

$$N_k(x) := (I - W_{MM}(x))^{-1} W_{Mk}(x) \quad \text{for all } k \in \{1, \ldots, K\},$$

and in particular

$$N_k(x) = N_k^* := (I - C_{MM})^{-1} C_{Mk}, \quad \text{if } X_{MM} = \mathbb{0}_m.$$

The dominant left eigenvectors $\{w_{kk}^T(x) \in \mathbb{R}^{n_k}, k \in \{1, \ldots, K\}\}$ exist uniquely and positively since the associated matrices $\{W_{kk}(x), k \in \{1, \ldots, K\}\}$ are row-stochastic, aperiodic, irreducible. Moreover,

$$(14) \quad w_{kk}(x) = w_{kk}(x_{kk}) = \begin{cases} \mathbb{e}_j \in \Delta_{n_k}, & \text{if } x_{kk} = \mathbb{e}_j \text{ for all } j \in \{1, \ldots, n_k\}, \\ \dfrac{\left(\frac{c_{kk_1}}{1 - x_{kk_1}}, \ldots, \frac{c_{kk n_k}}{1 - x_{kk n_k}}\right)^T}{\sum_{j=1}^{n_k} \frac{c_{kk_j}}{1 - x_{kk_j}}}, & \text{otherwise,} \end{cases}$$

and $\mathbb{1}_{n_k}^T w_{kk}(x) = 1$ for all $k \in \{1, \ldots, K\}$. According to the reflected appraisal mechanism (5), $F(x) = w(x) := \left(\lim_{l\to\infty} W(x)^l\right)^T \mathbb{1}_n/n$, and hence, we have

$$F(x) = \begin{bmatrix} F_{11}(x) \\ F_{22}(x) \\ \vdots \\ F_{KK}(x) \\ \mathbb{0}_m \end{bmatrix} := \begin{bmatrix} w_{11}(x)\mathbb{1}_{n_1}^T(\mathbb{1}_{n_1} + N_1(x)^T\mathbb{1}_{n_1})/n \\ w_{22}(x)\mathbb{1}_{n_2}^T(\mathbb{1}_{n_2} + N_2(x)^T\mathbb{1}_{n_2})/n \\ \vdots \\ w_{KK}(x)\mathbb{1}_{n_K}^T(\mathbb{1}_{n_K} + N_K(x)^T\mathbb{1}_{n_K})/n \\ \mathbb{0}_m \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} w_{11}(x)(n_1 + \sum_{i=1}^m \sum_{j=1}^{n_1} N_{1_{ij}}(x))/n \\ w_{22}(x)(n_2 + \sum_{i=1}^m \sum_{j=1}^{n_2} N_{2_{ij}}(x))/n \\ \vdots \\ w_{KK}(x)(n_K + \sum_{i=1}^m \sum_{j=1}^{n_K} N_{K_{ij}}(x))/n \\ \mathbb{0}_m \end{bmatrix}.$$

Here $(n_k + \sum_i \sum_j N_{k_{ij}}(x))/n < 1$ for all $k \in \{1, \dots, K\}$ since the row-stochasticity of $W(x)$ implies

$$\sum_{k=1}^K W_{MK}(x) I_{n_k} + W_{MM}(x) I_m = I_m,$$

and since $\rho(W_{MM}(x)) < 1$, we have

$$\sum_{k=1}^K (I_m - W_{MM}(x))^{-1} W_{MK}(x) I_{n_k} = \sum_{k=1}^K N_k(x) I_{n_k} = I_m,$$

which implies that $\sum_{k=1}^K \sum_{i=1}^m \sum_{j=1}^{n_k} N_{k_{ij}}(x) = m$ or equivalently, $\sum_{k=1}^K (n_k + \sum_i \sum_j N_{k_{ij}}(x))/n = 1$, and

$$\sum_i \sum_j N_{k_{ij}}(x) < m = n - \sum_{i=1}^K n_i, \quad \text{for all } k \in \{1, \dots, K\}.$$

Denoting $\zeta_k(x) := (n_k + \sum_i \sum_j N_{k_{ij}}(x))/n$, from (15), the social power $w(x)$ satisfies

$$w(x) := (w_1(x), \dots, w_n(x))^T = \left(\zeta_1(x) w_{11}(x)^T, \dots, \zeta_K(x) w_{KK}(x)^T, \mathbb{0}_m^T\right)^T.$$

Note that $w(x) \in \Delta_n$, and $w_{kk}(x) > 0$ for $k \in \{1, \dots, K\}$ if $x \notin \{\mathbb{e}_1, \dots, \mathbb{e}_{n-m}\}$.

Overall, for $x \in \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \dots, \mathbb{e}_n\}$, $F(x)$ satisfies that each entry $F_j(x) \geq 0$ for all $j \in \{1, \dots, n\}$:

- if j belongs to a sink k, then $F_j(x) = w_j(x) = \zeta_k(x) w_{kk_i}(x)$ for $i = j - \sum_{l=1}^{k-1} n_\ell$ as described in (10). Since $w_{kk_i}(x) \geq 0$ and $\zeta_k(x) > 0$, $F_j(x) \geq 0$;
- if j does not belong to a sink, then $F_j(x) = 0$.

Continuity of F: Next, we show the function F is continuous everywhere except $\{\mathbb{e}_{n-m+1}, \dots, \mathbb{e}_n\}$. First, we claim $w_{kk}(x), k \in \{1, \dots, K\}$, is continuous w.r.t x for $x \in \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \dots, \mathbb{e}_n\}$. By the definition (14), $w_{kk}(x_{kk})$ is continuous w.r.t. all x_{kk} such that $x \in \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \dots, \mathbb{e}_n\}$ (see a similar analysis as in the proof of Lemma 2 [28, Appendix B]). Additionally, since $w_{kk}(x_{kk})$ is continuous w.r.t. x_{kk}, given an $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that if $||x_{kk} - x'_{kk}|| < \delta(\epsilon)$ then $||w_{kk}(x_{kk}) - w_{kk}(x'_{kk})|| < \epsilon$. Moreover, if $||x - x'|| < \delta(\epsilon)$, then $||x_{kk} - x'_{kk}|| < \delta(\epsilon)$. That is to say, for such $\delta(\epsilon)$ satisfying $||x-x'|| < \delta(\epsilon)$, $||w_{kk}(x)-w_{kk}(x')|| = ||w_{kk}(x_{kk})-w_{kk}(x'_{kk})|| < \epsilon$. Hence, $w_{kk}(x)$ is continuous w.r.t. all $x \in \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \ldots, \mathbb{e}_n\}$. Second, $N_k(x)$ is continuous w.r.t. x for all $x \in \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \ldots, \mathbb{e}_n\}$ by its definition.

Overall, by the definition (15), F is continuous for all $x \in \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \ldots, \mathbb{e}_n\}$. The continuity of F on the vertices $\{\mathbb{e}_1, \ldots, \mathbb{e}_{n-m}\}$ inherits from the continuity of $\{w_{kk}\}$ on these vertices. F is not continuous on the vertices $\{\mathbb{e}_{n-m+1}, \ldots, \mathbb{e}_n\}$ since $F_i(x) = d_{ii}$ is strictly greater than $1/n$ if $x \in \{\mathbb{e}_{n-m+1}, \ldots, \mathbb{e}_n\}$, and $F_i(x) = 0$ for any other $x \in \Delta_n$. ∎

Appendix E. Proof of Theorem 10.

Proof. Properties of F: Regarding fact (i), note that for any initial state $x(0) \in \Delta_n$, we always have $x_{MM}(2) = \mathbb{0}_m$ via the mapping F. Then for all $s \geq 2$ and all $k \in \{1, \ldots, K\}$, $N_k(x(s)) = N_k^* = (I - C_{MM})^{-1} C_{Mk}$, and

$$\mathbb{1}_{n_k}^T x_{kk}(s+1) = \mathbb{1}_{n_k}^T w_{kk}(x(s))(n_k + \sum_i \sum_j N_{k_{ij}}(x(s)))/n = (n_k + \sum_i \sum_j N^*_{k_{ij}})/n,$$

which is a constant. That is to say, the sum of the individual social powers in each sink is constant for all $s \geq 2$. We denote

$$\zeta_k^* = (n_k + \sum_i \sum_j N^*_{k_{ij}})/n.$$

Existence of equilibrium points: Regarding fact (ii), from the definition of F, we have $x(s) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$ for all $s \geq 1$ and for all initial states $x(0)$. It is true since 1) if $x(0) \in \{\mathbb{e}_{n-m+1}, \ldots, \mathbb{e}_n\}$, then $1/n < x_i(1) < m/n$ and $x(1) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$; 2) if $x(0) \in \Delta_n \setminus \{\mathbb{e}_{n-m+1}, \ldots, \mathbb{e}_n\}$, then $x(1) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$ by (15).

We may define a set $A = \{x \in \Delta_n \mid m/n \geq x_i \geq 0, i \in \{n-m+1, \ldots, n\}\}$, which is compact. It is clear that $F(A) \subset A$ and $F(x(0)) \in A$ for any $x(0) \in \Delta_n$. By Brouwer fixed-point theorem, there exists at least one equilibrium point $x^* \in A$ and no equilibrium point in $\Delta_n \setminus A$.

For an equilibrium point x^* of F, we have the following properties between c_{kk} and x^*_{kk} for all $k \in \{1, \ldots, K\}$: considering $i, j \in \{1, \ldots, n_k\}$, $n_k \geq 2$,

- if $c_{kk_i} > c_{kk_j}$, then $x^*_{kk_i} > x^*_{kk_j}$.
- if $c_{kk_i} = c_{kk_j}$, then $x^*_{kk_i} = x^*_{kk_j}$.

The proof of the two statements above for $n_k \geq 3$ is the same as the proof of Theorem 4 fact (i) [28, Appendix F]. If $n_k = 2$, then $c_{kk_i} = c_{kk_j} = 1/2$, and we can prove $x^*_{kk_i} = x^*_{kk_j}$ by direct calculations from the equations (14) and (15).

Uniqueness of the equilibrium point: In the following we show the equilibrium point x^* is unique. Given $i \in \{1, \ldots, n\}$, it is clear that

(ii.1) if i does not belong to a sink, then $x_i^* = 0$,
(ii.2) if i belongs to sink k and $n_k = 2$, then $c_{kk_1} = c_{kk_2} = 1/2$ and $x_i^* = \zeta_k^*/2$,
(ii.3) if i belongs to sink k and $n_k = 3$, then assume that there exist two different vectors $x_{kk}, y_{kk} > 0$ such that $\mathbb{1}_{n_k}^T x_{kk} = \mathbb{1}_{n_k}^T y_{kk} = \zeta_k^*$, $w_{kk}(x_{kk}) = x_{kk}$, and $w_{kk}(y_{kk}) = y_{kk}$. Since

$$x_{kk_j}(1 - x_{kk_j}) = \alpha(x_{kk}) c_{kk_j}, \quad y_{kk_j}(1 - y_{kk_j}) = \alpha(y_{kk}) c_{kk_j},$$

with two positive constants $\alpha(x_{kk})$ and $\alpha(y_{kk})$ for all $j \in \{1, \ldots, n_k\}$, we can write $x_{kk_j}(1 - x_{kk_j}) = \gamma y_{kk_j}(1 - y_{kk_j})$ for all $j \in \{1, \ldots, n_k\}$. Without loss of generality, $1 \geq \gamma > 0$.

If $\gamma = 1$, then $x_{kk_j} = y_{kk_j}$ because $x_{kk_j} < \zeta_k^* < 1 - y_{kk_j}$ for all $j \in \{1, \ldots, n_k\}$, which is a contradiction of $x_{kk} \neq y_{kk}$.

If $\gamma < 1$, then, by assuming that $c_{kk_1} = \max\{c_{kk_1}, \ldots, c_{kk_{n_k}}\}$, we have $x_{kk_1} = \max\{x_{kk_1}, \ldots, x_{kk_{n_k}}\}$ and $y_{kk_1} = \max\{y_{kk_1}, \ldots, y_{kk_{n_k}}\}$, which imply $x_{kk_j} < 0.5\zeta_k^*$ and $y_{kk_j} < 0.5\zeta_k^*$ for all $j \in \{2, \ldots, n_k\}$. For all $j \in \{2, \ldots, n_k\}$, the facts $x_{kk_j} + y_{kk_j} < \zeta_k^* < 1$ and $x_{kk_j}(1 - x_{kk_j}) < y_{kk_j}(1 - y_{kk_j})$ together imply $x_{kk_j} < y_{kk_j}$, and hence, $x_{kk_1} > y_{kk_1}$. Moreover, for all $j \in \{2, \ldots, n_k\}$,

$$\frac{x_{kk_j}}{x_{kk_1}} < \frac{y_{kk_j}}{y_{kk_1}} \implies \frac{1 - x_{kk_j}}{x_{kk_1}} < \frac{1 - y_{kk_j}}{y_{kk_1}}. \tag{16}$$

Additionally, we have $\sum_{i=2}^{n} x_{kk_i}(1 - x_{kk_i}) = \gamma \sum_{i=2}^{n} y_{kk_i}(1 - y_{kk_i})$, which, together with the inequality (16), implies that

$$\begin{aligned} \sum_{i=2}^{n} x_{kk_i} x_{kk_1} > \gamma \sum_{i=2}^{n} y_{kk_i} y_{kk_1} &\iff (\zeta_k^* - x_{kk_1}) x_{kk_1} > \gamma(\zeta_k^* - y_{kk_1}) y_{kk_1} \\ &\implies (1 - x_{kk_1}) x_{kk_1} > \gamma(1 - y_{kk_1}) y_{kk_1}. \end{aligned} \tag{17}$$

The statement (17) is from the fact that, since $x_{kk_1} > y_{kk_1}$ and $\gamma < 1$, $(1 - \zeta_k^*) x_{kk_1} > \gamma(1 - \zeta_k^*) y_{kk_1}$, which, however, is a contradiction of the previous hypothesis $x_{kk_j}(1 - x_{kk_j}) = \gamma y_{kk_j}(1 - y_{kk_j})$ for all $j \in \{1, \ldots, n_k\}$. Therefore, if $x = F(x)$, then x is uniquely determined.

Convergence to the equilibrium point: Regarding fact (iii), based upon the analysis above, if i does not belong to a sink, then $x_i(s) = x_i^* = 0$ for all $s \geq 2$. In the rest, we prove the convergence of x_i to the equilibrium point x_i^* for i belonging to a sink k with $n_k \geq 2$.

For each $k \in \{1, \ldots, K\}$ with $n_k \geq 2$, denote $\bar{x}_{kk_j}(s) = x_{kk_j}(s)/x_{kk_j}^*$ for all $j \in \{1, \ldots, n_k\}$, $\bar{x}_{kk_{\max}}(s) = \max\{\bar{x}_{kk_j}(s), j \in \{1, \ldots, n_k\}\}$, and $\bar{x}_{kk_{\min}}(s) = \min\{\bar{x}_{kk_j}(s), j \in \{1, \ldots, n_k\}\}$.

Define a Lyapunov function candidate $V_k(x_{kk}(s)) = \bar{x}_{kk_{\max}}(s)/\bar{x}_{kk_{\min}}(s)$ for each $k \in \{1, \ldots, K\}$. It is clear that 1) any sublevel set of V_k is compact and invariant, 2) V_k is strictly decreasing anywhere in $A_k := \{x \in \mathbb{R}^{n_k} \mid x \geq 0,\ \mathbb{1}_{n_k}^T x = \zeta_k^*\}$ except x_{kk}^*, which can be proved in the similar way as in Theorem 4 [28, Appendix F], (3) V_k and F are continuous. Therefore, every trajectory starting in A_k converges asymptotically to the equilibrium point x_{kk}^* by the LaSalle Invariance Principle as stated in [9, Theorem 1.19]. Moreover, since $x_{kk}(s) \in A_k$ for all $s \geq 2$ and for all initial states x, $\lim_{s \to \infty} x_{kk}(s) = x_{kk}^*$.

Regarding fact (iv), the results are derived based upon two facts that 1) $W(x(s))$, consistent with C, has K left eigenvectors associated eigenvalue 1 for $s \geq 1$, and 2) the dominant left eigenvectors of $W(x(s))$ can be described by (14) and $x(s+1)$ can be calculated by (15) for $s \geq 1$. ∎

Appendix F. Proof of Proposition 14.

Proof. Denote $\alpha^* = 1/(\sum_{j=1}^{n_k} \frac{c_{kk_j}}{1 - x_{kk_j}^*})$. Define ${c_{\text{thrshld}}}^k = 1 - \alpha^*$, or equivalently

$$\frac{1}{1 - {c_{\text{thrshld}}}^k} = \sum_{j=1}^{n_k} \frac{c_{kk_j}}{1 - x_{kk_j}^*},$$

which implies that $\min\{x_{kk_1}^*, \ldots, x_{kk_{n_k}}^*\} < {c_{\text{thrshld}}}^k < \max\{x_{kk_1}^*, \ldots, x_{kk_{n_k}}^*\}$. Moreover, since $F(x^*) = x^*$ with F defined in (9), for all $j \in \{1, \ldots, n_k\}$, from (10)

$$\frac{x^*_{kk_j}(1 - x^*_{kk_j})}{\zeta^*_k c_{kk_j}} = \alpha^* = \frac{c_{\mathrm{thrshld}}{}^k(1 - c_{\mathrm{thrshld}}{}^k)}{c_{\mathrm{thrshld}}{}^k}. \tag{18}$$

For $c_{\mathrm{thrshld}}{}^k < 0.5$: First, if $\zeta^*_k c_{kk_j} > c_{\mathrm{thrshld}}{}^k$, then $x^*_{kk_j}(1 - x^*_{kk_j}) > \zeta^*_k c_{kk_j}(1 - \zeta^*_k c_{kk_j})$. Since $\zeta^*_k c_{kk_j} < 0.5$, it is clear that $x^*_{kk_j} > \zeta^*_k c_{kk_j}$. Second, if $\zeta^*_k c_{kk_j} < c_{\mathrm{thrshld}}{}^k$, then $x^*_{kk_j}(1 - x^*_{kk_j}) < \zeta^*_k c_{kk_j}(1 - \zeta^*_k c_{kk_j})$, which implies $x^*_{kk_j} < \zeta^*_k c_{kk_j}$ or $x^*_{kk_j} > 1 - \zeta^*_k c_{kk_j} > 0.5$. Furthermore, since $c_{\mathrm{thrshld}}{}^k < 0.5$, we can show $c_{\mathrm{thrshld}}{}^k < \max\{\zeta^*_k c_{kk_1}, \ldots, \zeta^*_k c_{kk_{n_k}}\}$ (otherwise, if $0.5 > c_{\mathrm{thrshld}}{}^k \geq \max\{\zeta^*_k c_{kk_1}, \ldots, \zeta^*_k c_{kk_{n_k}}\}$, then by simple calculation we can show $c_{\mathrm{thrshld}}{}^k \geq \max\{x^*_{kk_1}, \ldots, x^*_{kk_{n_k}}\}$, which is a contradiction). Thus, there exists another individual i such that $c_{kk_i} > c_{kk_j}$, which by fact (ii.3) of Theorem 10 implies $x^*_{kk_i} > x^*_{kk_j}$. Therefore, $x^*_{kk_j} < \zeta^*_k c_{kk_j}$ for $\zeta^*_k c_{kk_j} < c_{\mathrm{thrshld}}{}^k$, otherwise, $x^*_{kk_i} > x^*_{kk_j} > 0.5$ contradicts the fact that $x^*_{kk_j} + x^*_{kk_i} < 1$. Third, if $\zeta^*_k c_{kk_j} = c_{\mathrm{thrshld}}{}^k$, then $x^*_{kk_j}(1 - x^*_{kk_j}) = \zeta^*_k c_{kk_j}(1 - \zeta^*_k c_{kk_j})$ from (18). Similarly, we can show $x^*_{kk_j} < 0.5$ and hence $x^*_{kk_j} = \zeta^*_k c_{kk_j}$.

For $c_{\mathrm{thrshld}}{}^k \geq 0.5$: Denote

$$x^*_{kk_{\max}} = \max\{x^*_{kk_1}, \ldots, x^*_{kk_{n_k}}\}, \quad \text{and} \quad c_{kk_{\max}} = \max\{c_{kk_1}, \ldots, c_{kk_{n_k}}\}.$$

By fact (ii.3) of Theorem 10 and the fact that $0.5 \leq c_{\mathrm{thrshld}}{}^k < x^*_{kk_{\max}}$, there exists only one individual denoted by $j_{\max}$ associated with $c_{kk_{\max}}$ and her equilibrium self-weight is $x^*_{kk_{\max}}$. Since $c_{\mathrm{thrshld}}{}^k < x^*_{kk_{j_{\max}}}$, equation (18) implies $\zeta^*_k c_{kk_{j_{\max}}} < x^*_{kk_{j_{\max}}}$. For any other individual $i \neq j_{\max}$, we have $\zeta^*_k c_{kk_i} < 0.5 \leq c_{\mathrm{thrshld}}{}^k$, which implies $x^*_{kk_i}(1 - x^*_{kk_i}) < c_{\mathrm{thrshld}}{}^k(1 - c_{\mathrm{thrshld}}{}^k)$ from (18). As $c_{\mathrm{thrshld}}{}^k + x^*_{kk_i} < x^*_{kk_{j_{\max}}} + x^*_{kk_i}$, we obtain $x^*_{kk_i} < 0.5 \leq c_{\mathrm{thrshld}}{}^k$ and hence $x^*_{kk_i} < \zeta^*_k c_{kk_i}$ from (18).

Regarding fact (iii), since $F(x^*) = x^*$ for F defined in (9), for any individuals $i, j \in \{1, \ldots, n_k\}$, we have $x^*_{kk_i}/x^*_{kk_j} = \big(c_{kk_i}/(1 - x^*_{kk_j})\big)/\big(c_{kk_j}/(1 - x^*_{kk_j})\big)$. By using the similar argument in the proof of Proposition 7 fact (iii), $c_{kk_i} > c_{kk_j}$ implies $x^*_{kk_i} > x^*_{kk_j}$ and then implies $x^*_{kk_i}/c_{kk_i} > x^*_{kk_j}/c_{kk_j}$. ∎