Hitting time for doubly-weighted graphs with application to robotic surveillance

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***Abstract*— This article provides analysis results for the weighted hitting time and the pairwise weighted hitting time of a Markov chain. This concepts are useful in many applications from robotics to patrolling, environment monitoring and resources optimization. The aforementioned quantities are a performance indexes which measure the expected time taken by a random walker to travel from an arbitrary node of a graph to a second randomly selected node and the expected time to travel between two chosen nodes. These metrics can be used as objective function in optimization problems. In fact we propose a numerical example applied to robotic surveillance where the weighted hitting time is our cost to minimize with respect to the transition matrix of the agent.**

I. Introduction

In this work we introduce an alternative formulation of the *weighted hitting time* and a closed form for the *pairwise weighted hitting time*. This problem is of a general mathematical and engineering interest in the study of Markov Chains and random walks. In particular the weighted hitting time is a metric by which the performance of a random walker can be measured. The problem is also of interest in a number of robotic multi-agent applications; some examples include the monitoring of oil spills [1], the detection of forest fires [2], the tracking of border changes [3], the periodic patrolling of an environment [4], [5], the minimization of the emergency vehicle response times [6] and the servicing task in robotic warehouse management [7].

The hitting time of a random walk governed by a Markov chain, is the expected time taken by a random walker to travel between any two nodes in a network. For a single finite discrete-time Markov chain this quantity is known with many names: hitting time, Kemeny constant, mean first passage time, first hitting time and also eigentime. The weighted hitting time is an extension of the aforementioned concept, with the difference that a set of weight is associated to edge set of the underlying network. This generalization is particularly useful in many real application, like the ones modeling distance/time traveled or service costs/times.

The hitting time of a Markov chain first appeared in [8], however it has been studied by many researchers, e.g. [9], [10], [11]. In [12], [13] the authors provide bounds on the hitting time for various graph topology, and in [14] an alternative formulation is proposed. The authors of [15] found a formulation for the weighted hitting time, but not for the pairwise weighted hitting time.

The mean first passage time is closely related to other well-known metrics for graphs and Markov chains. One example is the *effective graph resistance* [16], [17], which is a metric quantifying the distance between pairs of vertices in an electric network. Then the relationship between electric networks and random walks is extensively studied in [18]. Another example is given by the *mixing rate* of an irreducible Markov chain, which is the rate of convergence to the stationary distribution [19]. Recently, [11] refers to the hitting time as the "expected time to mixing" and relates it to the mixing rate.

To achieve our results, we utilize the notion of *Kronecker graphs*. The first results for undirected Kronecker graph had been discovered in [20], where were found conditions to guarantee connectivity to graph generated by the Kronecker product of two graphs. In [21], [22] the authors introduce the concept of "stochastic" Kronecker graphs however this notion is not used in our work. In fact we are not attempting to generate new networks models, instead we are exploiting novel aspects of Kronecker products to discover areas not been already explored. Our approach is similar to the one exploited in [23] where the authors study the *group hitting time*, however in their analysis they do not consider doubly weighted graphs.

As main contributions of this work, we present an alternative formulation of the weighted hitting time and provide a closed form equation to compute the pairwise weighted hitting time. To the best of our knowledge a formulation for the former quantity has never been discovered before. We make no assumptions in our work, the random walker can move according to an arbitrary random walk with any kind of stationary distribution. From a practical point of view the first metric can be used when we want to optimize or discover the performances of the entire transition matrix, while the second is very useful when we are interested in only checking few connections of the graph. The last approach is also valuable when the problem at the hand has too many nodes/edges, in this scenario compute the weighted hitting time can be hard. However it is still possible to optimize or discover the performances of the crucial nodes of the graph using the pairwise weighted hitting time.

The paper is organized as follows. In Section II we introduce notation that will be used in the paper and we review some concepts about Markov chains and Kronecker products. In Section III we introduce the concept of doubly

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weighted graphs and mean first passage time. In Section IV we state our main results providing a new formulation for the weighted hitting time and the pairwise weighted hitting time. In Section V we provide insight into optimal weighted hitting time thorough an exampled applied to robotic surveillance, and finally in Section VI we present our conclusions and future research directions.

II. Mathematical preliminaries

In this section we report some useful definitions and notation. In the first subsection we recall some standard results on Markov chains. In the second subsection we introduce some concepts about tensors and conclude with a brief summary of the Kronecker products and some of its proprieties.

1) Markov Chains: A *Markov chain* is a sequence of random variables taking value in the *finite* set $\{1,\dots,n\}$ with the Markov property, namely that, the future state depends only on the present state. Let $X_k \in \{1,\dots,n\}$ denote the location of a random walker at time $k \in \{0,1,2,\dots\}$, then a Markov chain is *time-homogeneous* if $\mathbb{P}[X_{n+1}=j|X_n=i]=\mathbb{P}[X_n=j|X_{n-1}=i]=p_{i,j}$, where $P\in\mathbb{R}^{n\times n}$ is the transition matrix of the Markov chain. The transition matrix P is a row stochastic matrix which means that each of its rows is exactly summing to one. The vector $\boldsymbol{\pi}\in\mathbb{R}^{n\times 1}$ is a *stationary distribution* of P if $\sum_{i=1}^n \boldsymbol{\pi}_i = 1$, $0\le\boldsymbol{\pi}_i\le 1$ for all $i\in\{1,\dots,n\}$ and $\boldsymbol{\pi}^T P = \boldsymbol{\pi}^T$.

We will use of the following well-known results on Markov chains. A Markov chain is *irreducible* if, for each $i,j\in\{1,\dots,n\}$ there is a $t\in N$ such that $(P^t)_{i,j}>0$. If the Markov chain is *irreducible*, then there is a unique stationary distribution $\boldsymbol{\pi}$, and the corresponding eigenvalues of the transition matrix, λ_i for $i\in\{1,\dots,n\}$, are such that $\lambda_1=1$, $|\lambda_i|\le 1$ and $\lambda_i\neq 1$ for $i\in\{2,\dots,n\}$.

We let $\rho\,[P]$ denote the spectral radius of the matrix P, which is the supremum among the absolute values of the elements in its spectrum. Given a generic set A, with $|A|$ we denote its cardinality.

In this paper we consider finite irreducible time-homogeneous Markov chains. For more details on Markov chains or irreducible matrices see [8] or [24, Chapter 8], respectively.

2) Tensor Notation: Unless otherwise mentioned, vectors will be denoted by bold-faced letters (i.e., $\boldsymbol{a}$). We use the notation $\mathrm{diag}[\boldsymbol{a}]$ to indicate the diagonal matrix generated by vector $\boldsymbol{a}$ and $\mathrm{vec}[A]$ to indicate the vectorization of a matrix $A\in\mathbb{R}^{n\times m}$ where

$$\mathrm{vec}[A]=[A(1,1),\dots,A(n,1),\dots,A(m,1),\dots,A(n,m)]^T.$$

We also define the special matrix $[\mathcal{I}_{i,j}^{h,k}]$ as the matrix whose entries are all zero except for a single entry at (h,k) which has a value of 1, where h,k can only take values within the range of values that i,j take. With this matrix definition, it is easy to verify for $A=[a_{i,j}]$ that $a_{h,k}=\mathrm{vec}[[\mathcal{I}_{i,j}^{h,k}]]^T\,\mathrm{vec}[A]$. This enables us to go back and forth between the vectorized notation to the individual matrix elements. We denote $I_n\in\mathbb{R}^{n\times n}$ as the identity matrix of size n, $\mathbb{1}_n$ as the vector of ones of size n and $\mathbb{0}_{n\times n}$ as the matrix zeros of size $n\times n$. We define the Kronecker delta function $\delta_{i,j}$, by

$$\delta_{i,j}=\begin{cases}1, & \text{if there exists } i=j,\\ 0, & \text{otherwise.}\end{cases}$$

Then, with a slight abuse of notation, $A_d=[\delta_{i,j}a_{i,j}]$ represents the diagonal matrix generated by the elements of A. We are now ready to review some useful facts about Kronecker products. The Kronecker product, represented by the symbol $\otimes$, of two matrices $A\in\mathbb{R}^{n\times m}$ and $B\in\mathbb{R}^{q\times r}$ is a $nq\times mr$ matrix given by

$$A\otimes B=\begin{bmatrix} a_{1,1}B & \dots & a_{1,m}B\\ \vdots & \ddots & \vdots\\ a_{n,1}B & \ddots & a_{n,m}B\end{bmatrix}.$$

To build some intuition, notice for $A\in\mathbb{R}^{n\times n}$, that $I_n\otimes A$ is the block diagonal matrix with n copies of A on the diagonal:

$$I_n\otimes A=\begin{bmatrix} A & \mathbb{0}_{n\times n} & \dots & \mathbb{0}_{n\times n}\\ \mathbb{0}_{n\times n} & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0_{n\times n}\\ \mathbb{0}_{n\times n} & \dots & \mathbb{0}_{n\times n} & A\end{bmatrix}. \quad (1)$$

This implies for $A=I_n$, that $I_n\otimes A=I_n\otimes I_n=I_{n^2}$. The Kronecker product is bilinear and has many useful properties, two of which are summarized in the following lemma; see [25] for more information.

Lemma 1 (Properties of Kronercker product): Given the matrices A,B,C and D, the following relations hold for the Kronecker product.

(i) $(A\otimes B)(C\otimes D)=(AC)\otimes(BD)$,
(ii) $(B^T\otimes A)\,\mathrm{vec}[C]=\mathrm{vec}[ACB]$,

where it is assumed that the matrices are of appropriate dimension when matrix multiplication or addition occurs. In addition, for matrices $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{m\times m}$ with respective eigenvalues λ_i^A, $i\in\{1,\dots,n\}$ and λ_j^B, $j\in\{1,\dots,m\}$,

(iii) the eigenvalues of $A\otimes B$ are $\lambda_i^A\lambda_j^B$ for $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,m\}$.

III. Doubly weighted graphs and Mean First Passage Time

In most practical applications, distance/time traveled and service cost/times are important factors in the model of the system. To take into account also this factors we introduce an additional set of weighted edges in our graph which describe the distance/time (or service cost/time) to pass through the edge. We begin by first reviewing properties of the hitting time of random walker in a doubly weighted graph. Then, in the next section, we illustrate an alternative formulation of the hitting time and a way to compute the pairwise weighted hitting time between two nodes.

Consider a connected doubly-weighted graph $\mathcal{G}=(V,\mathcal{E},P,W)$, with node set $V:=\{1,\dots,n\}$, edge set $\mathcal{E}\subset V\times V$, and weight matrix $P=[p_{i,j}]$ with the property that $p_{i,j} \geq 0$ if $(i,j) \in \mathcal{E}$ and $p_{i,j} = 0$ otherwise. Let $P = [p_{i,j}]$ be the transition matrix associated with $\mathcal{G}$ with the property that $p_{i,j} \geq 0$ if $(i,j) \in \mathcal{E}$ and $p_{i,j} = 0$ otherwise. Therefore, P is a transition matrix of a Markov chain, where the element $p_{i,j}$ in the matrix represents the probability with which the random walk visits node j from node i. The matrix $W = [\omega_{i,j}]$ is the weight matrix with the properties that: if $(i,i) \in \mathcal{E}$, then $\omega_{i,i} \geq 0$; if $(i,j) \in \mathcal{E}$, $i \neq j$, then $\omega_{i,j} > 0$; and if $(i,j) \notin \mathcal{E}$, then $\omega_{i,j} = 0$.

Let $X_t \in \{1, \ldots, n\}$ denote the location of a random walker at time $t \in \{0, 1, 2, \ldots\}$. For any two nodes $i, j \in \{1, \ldots, n\}$, the *first passage time from* i *to* j, denoted by $T_{i,j}$, is the first time that the random walker starting at node i at time 0 reaches node j, that is,

$$T_{i,j}(W) = \min \left\{ \sum_{n=0}^{k-1} w_{X_n, X_{n+1}}, \text{ for } k \geq 1 | \right. \\ \left. X_k = j \text{ given that } X_0 = i \right\},$$

It is convenient to define the shorthand $m_{i,j} = \mathbb{E}[T_{i,j}]$ which represents the mean first passage time from i to j and the *mean first passage time matrix* M as the matrix whose (i,j)th entries are given by $m_{i,j}$. The *mean first passage time from start node* i, denoted by $\boldsymbol{h}_i$, is the expected first passage time from node i to any other node in the graph. The mean first passage time from node i, for a random walker described by a Markov chain with transition matrix P and stationary distribution $\boldsymbol{\pi}$, is given by

$$\boldsymbol{h}_i = \sum_{j=1}^{n} m_{i,j} \boldsymbol{\pi}_j.$$

By definition, the first passage time from i to j satisfies the following recursive equation

$$T_{i,j} = \begin{cases} w_{ij}, & \text{with probability } p_{i,j}, \\ T_{k,j} + w_{ik}, & \text{with probability } p_{i,k}, k \neq j. \end{cases}$$

Then, taking the expectation of the $T_{i,j}$ we get

$$\mathbb{E}\left[T_{i,j}\right] = w_{ij} p_{ij} + \sum_{k \neq j} p_{ik} \left(\mathbb{E}\left[T_{k,j}\right] + w_{ik} \right)$$

which also is equal to

$$m_{i,j} = w_{ij} p_{ij} + \sum_{k \neq j} p_{ik} \left(m_{k,j} + w_{ik} \right)$$

or in matrix notation

$$(I - P)M = (P \circ W) \mathbb{1}_n \mathbb{1}_n^T - P M_d, \tag{2}$$

where $(P \circ W)$ is the element-wise (Hadamard) product between P and W and $M_d = [\delta_{ij} m_{ij}]$. From [15] the weighted hitting time can be related to the hitting time in the following way.

Theorem 2 (Weighted hitting time): Given the doubly-weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, P, W)$ with transition matrix P and weight matrix W, the weighted hitting time H_W is given by

$$H_W(P, W) = \pi^T \left(P \circ W \right) H,$$

where H is the non-weighted hitting time.

IV. Weighted Hitting Time

In this section we provide an alternative way to compute the weighted hitting time which does not rely on the hitting time and that easily allows to compute the pairwise weighted hitting time.

Given the definition of the hitting time, one quickly sees that it can also be determined using the matrix M as follows,

$$H_W(P, W) = \boldsymbol{\pi}^T M \boldsymbol{\pi} = (\boldsymbol{\pi} \otimes \boldsymbol{\pi})^T \operatorname{vec}[M] \tag{3}$$

where $\boldsymbol{\pi}^T M \boldsymbol{\pi} = (\boldsymbol{\pi} \otimes \boldsymbol{\pi})^T \operatorname{vec}[M]$ is thanks to identity (ii) of Lemma 1. Applying again identity (ii) of Lemma 1 to (2) we get

$$(I - P)M = (P \circ W) \mathbb{1}_n \mathbb{1}_n^T - P M_{\mathrm{d}}. \tag{4}$$

The following result will be useful in constructing the alternate formulation for $H(P)$.

Lemma 3 (Eigenvalue shifting for stochastic matrices): Let $P \in \mathbb{R}^{n \times n}$ be an irreducible row-stochastic matrix, and let E be a diagonal matrix with diagonal elements $E_{ii} \in \{0, 1\}$, with at least one diagonal element which is zero. Then $\rho(PR) < 1$.

Proof: The stochastic matrix P is non-negative and, therefore, so is PE. Since P is irreducible, then $0 < PE < P$ and $\rho[PE] < \rho[P] = 1$ [26, Chapter 1]. ■

Before presenting the alternate representation of the weighted hitting time we introduce a useful equality:

$$\operatorname{vec}[M_d] = E \operatorname{vec}[M],$$

when E is defined by $E = \operatorname{diag}[[\delta_{i,j}]]$. We leave the proof of this simple book-keeping fact to the reader and present the main result of this section.

Theorem 4: (Hitting time and pairwise hitting times of a doubly-weighted graph): Given the doubly-weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, P, W)$ with transition matrix P and weight matrix W, the following properties hold:

(i) the weighted hitting time $H_W(P, W)$ of the Markov chain is given by

$$H_W(P, W) = (\boldsymbol{\pi} \otimes \boldsymbol{\pi})^T \operatorname{vec}[\mathcal{M}], \tag{5}$$

where

$$\operatorname{vec}[\mathcal{M}] = (I_{n^2} - (I_n \otimes P)(I_{n^2} - E))^{-1} \operatorname{vec}[(P \circ W) \mathbb{1}_n^T \mathbb{1}_n],$$

(ii) the pairwise weighted hitting time between nodes h and k, denoted by $m_{h,k}(W)$, is given by

$$m_{h,k}(W) = \operatorname{vec}[[\mathcal{I}_{i,j}^{h,k}]]^T \operatorname{vec}[\mathcal{M}].$$

Proof: First notice that by rearranging (4) and substituting $E \operatorname{vec}[M]$ for $\operatorname{vec}[M_d]$ gives that

$$(I_{n^2} - (I_n \otimes P)(I_{n^2} - E)) \operatorname{vec}[M] = \operatorname{vec}[(P \circ W) \mathbb{1}_n^T \mathbb{1}_n]. \tag{6}$$

From (3) we know that $H(P) = (\boldsymbol{\pi} \otimes \boldsymbol{\pi}) \operatorname{vec}[M]$, therefore it only remains to show that $(I_{n^2} - I_n \otimes P(I_{n^2} - E))$ is in fact invertible. First, recall from (1) that $I_n \otimes P$ results in the block diagonal matrix, whose diagonal blocks consist of copies of P. Second, notice that $(I_{n^2} - E)$ is simply the identity matrix with some diagonal entries set to zero. It can be easily verified that $(I_n \otimes P)(I_{n^2} - E)$ results in the block diagonal matrix where each block contains the matrix P with one column set to zero. For example, for $P \in \mathbb{R}^{3\times 3}$ we have that

$$(I_n \otimes P)(I_{n^2} - E) =$$
$$\begin{bmatrix} 0 & p_{12} & p_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{22} & p_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{32} & p_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{11} & 0 & p_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{21} & 0 & p_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{31} & 0 & p_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_{11} & p_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_{21} & p_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_{31} & p_{32} & 0 \end{bmatrix}.$$

Notice that each diagonal block will have at least one column set to zero. Hence, using Lemma 3, we have that maximum eigenvalue of each block is strictly less than one in magnitude, and thus $\rho[(I_n \otimes P)(I_{n^2} - E)] < 1$. Let λ_i denote the eigenvalues of $(I_n \otimes P)(I_{n^2} - E)$, then since the eigenvalues of $(I_{n^2} - (I_n \otimes P)(I_{n^2} - E))$ are simply $1 - \lambda_i$ and $|\lambda_i| < 1$ for all $i \in \{1, \ldots, n^2\}$, this implies the matrix $(I_{n^2} - (I_n \otimes P)(I_{n^2} - E))$ is invertible and $\mathrm{vec}[M]$ is the unique solution to (6). ∎

V. Application to Robotic Surveillance

The results on the weighted hitting time presented in this work can potentially be applied to many fields, here we focus on a surveillance problem. In particular, we define a doubly weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, P, W)$, whose vertices represent the watchtower and the main crossroads of the Italian medieval city of Marostica. We refer to this graph as the *Marostica roadmap*. The whole graph $\mathcal{G}$ has 42 vertices and 56 edges. The graph is divided in four partially-overlapping subgraphs, $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i, P_i, W_i)$, $i \in \{1, 2, 3, 4\}$ with $\mathcal{V}_i \subset \mathcal{V}$, where four agents can perform their surveillance action, see Figure 1.

In the following we study the transition matrices that arise from the numerical optimization of the weighted hitting time for the Marostica roadmap and its covering with four subgraphs. In particular, we look at the minimization of the cost function (5) described in Problem 5 below.

Problem 5: Given the stationary distributions $\boldsymbol{\pi}_i$ and the graphs $\mathcal{G}_i$ with $i \in \{1, 2, 3, 4\}$, find the transition matrices P_i solving:

$$\begin{aligned} &\inf\ H_W(P_i, W_i) \\ &\quad P_i \mathbb{1}_{|\mathcal{V}_i|} = \mathbb{1}_{|\mathcal{V}_i|}, \ \text{ for each } i \in \{1, 2, 3, 4\} \\ &\quad (\boldsymbol{\pi}_i)^T P_i = (\boldsymbol{\pi}_i)^T, \ \text{ for each } i \in \{1, 2, 3, 4\} \\ &\quad 0 \le p_i\,(h, k) \le 1, \ \text{ for each } (h, k) \in \mathcal{E}_i \text{ and } i \in \{1, 2, 3, 4\} \\ &\quad p_i\,(h, k) = 0, \ \text{ for each } (h, k) \notin \mathcal{E}_i \text{ and } i \in \{1, 2, 3, 4\} \end{aligned}$$

For simplicity, we assume that the stationary distributions $\pi_1, \ldots, \pi_4$ are uniform. The problem is numerically solved using the KNITRO solver as implemented by the TOMLAB package for MATLAB; see [27]. On an 2.7Ghz Intel Core i5 processor with 16 GB of memory, the computation time is 129 seconds. In Table I are reported the values of the optimized weighted hitting time and the result of the optimization is illustrated in Figure 2.

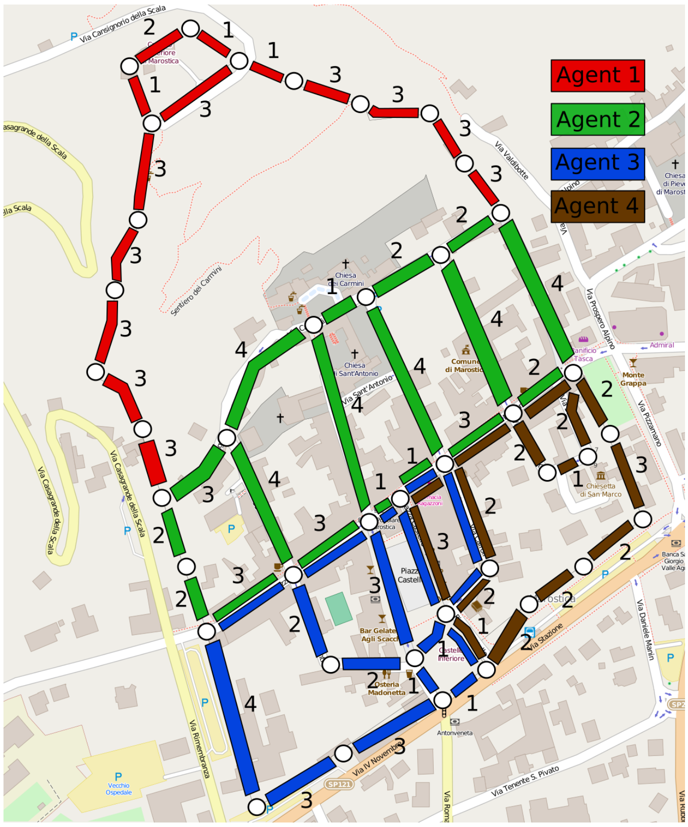

Fig. 1. Marostica roadmap with the agents subgraphs. The numbers on the edges represent the weights.

Random Walkers	$H_W(P_i, W_i)$
Agent 1	115.5
Agent 2	16.5
Agent 3	1.6
Agent 4	5.6

TABLE I

Weighted hitting time for random walkers in the four subgraphs of the Marostica roadmap.

VI. Conclusions

We have studied the problem of how to compute the weighted hitting time and the pairwise weighted hitting time of a Markov chain. We have presented two closed form solution to the two problems and we show an application to robotic surveillance. The work leaves open various directions for future research. One of them is to extend the results when multiple agents are involved, similarly to what has been done in [23].

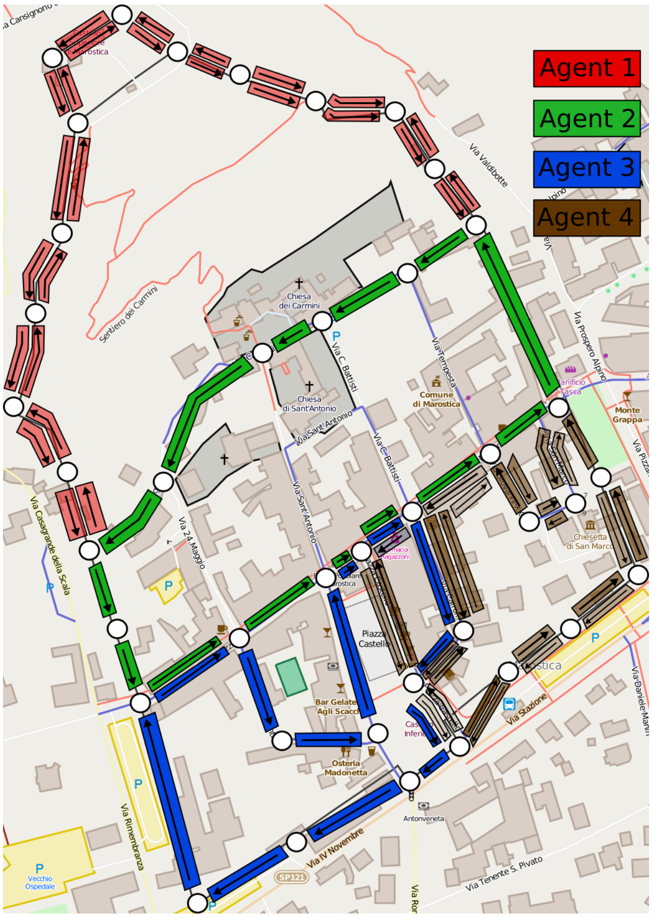

Fig. 2. Marostica roadmap with the optimized transition matrices. The transparency of the edges reflect the probability of the transition matrix. Less transparent is the edge higher is the probability and viceversa.

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