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Sequential Decision Aggregation with Social Pressure

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Abstract This paper proposes and characterizes a sequential decision aggregation system consisting of agents performing binary sequential hypothesis testing, and a fusion center which collects the individual decisions and reaches the global decision according to some threshold rule. Individual decision makers' behaviors in the system are influenced by other decision makers, through a model for social pressure; our notion of social pressure is proportional to the ratio of individual decision makers who have already made the decisions. For our proposed model, we obtain the following results: First, we derive a recursive expression for the probabilities of making the correct and wrong global decisions as a function of time, system size, and the global decision threshold. The expression is based on the individual decision makers' decision probabilities and does not rely on the specific individual decision making policy. Second, we discuss two specific threshold rules: the fastest rule and the majority rule. By means of a mean-field analysis, we relate the asymptotic performance of the fusion center, as the system size tends to infinity, to the individual decision makers' decision probability sequence. In addition to theoretical analysis, simulation work is conducted to discuss the speed/accuracy tradeoffs for different threshold rules.

Keywords sequential decision aggregation · decision accuracy · expected decision time · fusion center · fastest rule · majority rule

1 Introduction

1.1 Motivation and problem set-up

Decision making has been a classic research topic in the areas of industrial engineering as well as social science. In a centralized decision making model, all the signals are available to one decision maker, based on which the decision maker makes a choice among some candidate hypotheses according to some prescribed decision making policy. Numerous centralized decision making policies have been proposed. However, an isolated decision maker is always limited in decision accuracy and reliability. Moreover, in the context of sociological psychology, if we consider the decision maker as an individual in a social network, the individual is not likely to have access to all the disseminated information and make decisions independently. Instead, individuals have their private information sets and their decision making behaviors are influenced by others in the network. Therefore, it is of great research interest to study the group decision making problem. Recent years have seen much research on this topic with a focus on two objectives. The first is to establish the optimal group decision making policy. The second aspect is

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to build models to describe and understand the observed sociological phenomena. This paper aims to understand how grouping individual decision makers and their mutual interactions affect the accuracy and speed with which these individuals reach a collective decision.

In this paper, we consider a system consisting of a group of *sequential decision makers* (SDMs) and a fusion center. The SDMs are doing the sequential hypothesis test between two candidate hypotheses. The fusion center collects individual decisions and makes the global final decision. In our model, the individual SDMs make individual decisions based on both their private observations and the decisions of other SDMs. The latter amounts to a form of *social pressure*. We aim to relate the fusion center's global accuracy and expected decision time to the individuals' accuracy and expected decision time.

1.2 Literature Review

Group decision making has been extensively studied by numerous literature in both the engineering community [1–10], and the area of sociological psychology [20–25]. In engineering areas, such as control system and signal processing, two problems on group decision making are emphasized: 1) the communication between the individual decision makers and the fusion center; 2) the optimal decision making policy either in the individual level, or in the global level, to maximize the system's performance. In sociological psychology, researchers aim to investigate individuals' cognitive behavior in presence of social pressure and interactions, and the factors which influence individual or group decision making performance. Our model is closest to the work by Dandach et al. [8], of which the key feature is that, different from models in [1–6], the fusion center in [8] does not need to wait for all the SDMs' decisions. Our model generalizes [8] by allowing mutual interactions among SDMs.

The process, with which a decision maker updates its posterior belief, or likelihood function, according to the Bayesian formula and based on its private information set, is sometimes collectively referred to as *Bayesian learning*, e.g. [26, 27]. Bayesian learning has been used to model the individuals' rational behavior. As long as the signal-generation mechanism and the decision policy are given, the individual's decision making probabilities at any given time can be predicted. In this paper, we do not specify the signal structure and decision policy for an individual SDM, but assume that, when isolated, the SDM is adopting some Bayesian learning policy and its decision probabilities at each time step are given. On the other hand, *non-Bayesian learning* is a wording usually adopted to denote irrational decisions due to influence of other individuals in the system, or any other rule of thumb [29]. In our model, the non-Bayesian learning is characterized by the influence of social pressure. Therefore, our model can be considered as the combination of Bayesian learning [28, 7, 11] and non-Bayesian learning processes. Examples of the combination of Bayesian and non-Bayesian learning, either discrete-time or continuous-time, can be seen in [12, 10, 30], whereby individuals do not make any final decision but just update their posterior belief based on accumulated private information set (Bayesian), and combine it with the belief of their neighbors in the network (non-Bayesian).

In our model, the way that the decisions on either hypothesis propagates in the group through social pressure is similar to the *independent cascade model* [13–18] used in the computer science community to model the network contagion process. However, in the independent cascade model, the individuals are infected passively via the activated edges while in our model the decision makers proactively pick the other decision makers and follow the picked individuals' decisions with some probability.

1.3 Contribution

As the first contribution of this paper, we propose an algorithm to compute the fusion center's decision probabilities at each time step, based on the individual SDMs' decision probabilities. By introducing the concept of system state, we simplify our model, which is an exponential-dimension Markov chain, to a lumped polynomial-dimension Markov chain. The computation complexity of the iterative algorithm to compute the fusion center's decision probabilities is also polynomial. In addition, the algorithm does not rely on the specific decision making policies of the individual SDMs.

As the second contribution, we analyze the asymptotic accuracy and expected decision time of the fusion center as the system size n tends to infinity. We focus on two specific group decision making rules: the fastest rule and the majority rule. We give the exact expressions for the asymptotic accuracy and expected decision time in these two cases. Our model under the fastest rule has the same asymptotic performance as the model under the fastest rule in [8]. The analysis of the majority rule is based on the result on the mean-field convergence analysis proposed by Le Boudec et al. [19]. The asymptotic performance of the majority rule in our model is distinct from the model in [8] in that our model achieves faster decision speed, while at the cost of less accuracy, with the same individual SDMs. In addition, in our

Table 1 Notations frequently used in this paper

$D_i(t)$	decision of SDM i after time step t. $D_i(t) \in \{H_1, H_0.H_{\rm nd}\}$
$p_r(t)$	isolated SDM's probability of deciding H_r, for $r \in \{1, 0\}$, at time step t, on condition that it has not decided H_1 or H_0 before time t
$p_{\rm nd}(t)$	isolated SDM's probability of not deciding H_1 or H_0 at time step t, on condition that it has not decided H_1 or H_0 before time t
$f_r(t \,\vert\, N_1, N_0)$	SDM's probability of deciding H_r, $r \in \{1, 0\}$, after time step t, on condition that it has not decide H_1 or H_0, and N_1 (N_0 resp.) SDMs have already decided H_1 (H_0 resp.) before time t
$f_{\rm nd}(t \,\vert\, N_1, N_0)$	SDM's probability of not deciding H_1 or H_0 after time step t, on condition that it has not decide H_1 or H_0, and N_1 (N_0 resp.) SDMs have already decided H_1 (H_0 resp.) before time t
$N_1(t)$ ($N_0(t)$ resp.)	the number of SDMs who have decided H_1 (H_0 resp.) up to time step t
$p_r(t; n, q)$	the probability that the fusion center, running the q-out-of-n rule, decides H_r, $r \in \{1, 0\}$, right at time step t
$T_{\rm fc}$	decision time of the fusion center, which is a random variable
$p_c(n, q)$	the probability that the fusion center, running the q-out-of-n rule, makes the correct global decision, i.e., the accuracy of the fusion center
$\mathbb{E}[T_{\rm fc} \,\vert\, n, q]$	the expected decision time for the fusion center running the q-out-of-n rule

model under the majority rule, the decision speed and the global accuracy can simultaneously be better than the isolated SDM, which is not achieved by the model [8] without social pressure. Besides, leading order of a model parameter, which characterizes the individual SDMs' tendency of being influenced by the social pressure, is analyzed for the mean-field approximation of our system.

In addition, we present simulation work to validate the theoretical results and show how the accuracy and decision speed of our system vary with the system size, the group decision policies and the inclination of the decision makers to be influenced by the social pressure. We discuss how to adjust the model parameters to trade off between the system's accuracy and expected decision time.

1.4 Organization

The rest of this paper is organized as follows. Section 2 is the model description and problem statement. Section 3 provides the algorithm of computing the fusion center's decision probabilities for finite system sizes. Section 4 is the discussion of the asymptotic behavior as the system size tends to infinity. Some further simulation is provided in Section 5. Section 6 is the conclusion and discussion.

2 Notations, Model Description, and Problem Statement

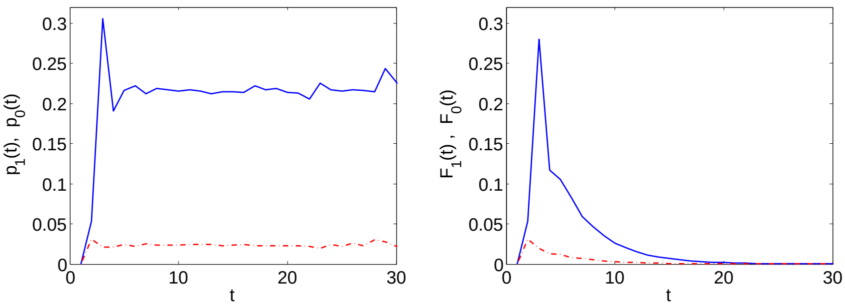
The group decision making system discussed in this paper consists of a fusion center and n identical individual decision makers indexed by $i \in V = \{1, 2, \ldots, n\}$. The individual decision makers are taking sequential hypothesis test between two hypotheses, H_1 and H_0, and are thus referred to as the *sequential decision makers* (SDMs). The SDMs make individual decisions based on both their private signals and communication with other SDMs in the system. The fusion center collects individual decisions and reach a global decision according to the *q-out-of-n* aggregation rule. Before the model description, we present in Table 1 all the notations frequently used in this paper.

2.1 Behavior of an isolated SDM

Our model of the isolated SDM is the same as that studied by Dandach et al. [8]. Suppose H_1 and H_0 are the candidate hypotheses and $H_{\rm nd}(t)$ corresponds to the state of "not deciding either H_1 or H_0". Without loss of generality, we always assume H_1 to be the correct hypothesis. Denote by $D_i(t)$ the decision state of SDM i at any time t, thereby $D_i(t) \in \{H_1, H_0, H_{\rm nd}\}$, and assume that the decision on H_1 or H_0 is irreversible. We assume that, when isolated from other SDMs, an SDM adopts some prescribed Bayesian learning and decision policy. We do not specify what the policy is, but just assume that the decision probabilities at each time, which can be predicted by the signal structure and the learning and decision policy, are given as the *individual decision probability sequence* (IDPS) $\{p_1(t), p_0(t), p_{\rm nd}(t)\}_{t\in\mathbb{N}}$, where

$$\begin{aligned} p_r(t) &= \mathbb{P}\big[D_i(t) = H_r \,\vert\, D_i(t-1) = H_{\rm nd}\big] \quad \text{for any } r \in \{1, 0\}, \text{ and} \\ p_{\rm nd}(t) &= \mathbb{P}\big[D_i(t) = H_{\rm nd} \vert D_i(t-1) = H_{\rm nd}\big]. \end{aligned} \tag{1}$$

Example: The *Sequential Probability Ratio Test* (SPRT) is a type of discrete-time Bayesian learning and decision policy, which achieves the minimum expected decision time for any prescribed error rate [26].



(a) $p_1(t)$ and $p_0(t)$ (b) $F_1(t)$ and $F_0(t)$

Fig. 1 IDPS for an SDM implementing an SPRT. In Figure 1(a), the blue solid curve represents $p_1(t)$ while the red dash curve represent $p_0(t)$. In Figure 1(b), the blue solid curve represent $F_1(t) = \mathbb{P}[D_i(t) = H_1,\ D_i(t-1) = H_{\mathrm{nd}}]$, i.e., the probability of deciding H_1 right at time step t. The red dash curve represents $F_0(t) = \mathbb{P}[D_i(t) = H_0,\ D_i(t-1) = H_{\mathrm{nd}}]$.

For an SDM running the SPRT, a signal S_t is received at each time step t, and, based on the accumulated information set $I_t = \{s_1, s_2, \ldots, s_t\}$, the SDM calculate the *log-likelihood function*

$$\Lambda(t) = \log\left(\frac{\mathbb{P}[S_1 = s_1, S_2 = s_2, \ldots, S_t = s_t \,|\, \theta = H_1]}{\mathbb{P}[S_1 = s_1, S_2 = s_2, \ldots, S_t = s_t \,|\, \theta = H_0]}\right),$$

according to the Bayesian formula, where θ denotes the underlying hypothesis. Prescribed thresholds $\eta_1 > 0$ and $\eta_0 < 0$ are used to manipulate the trade-off between decision accuracy and speed. Whenever $\Lambda(t) > \eta_1$ ($\Lambda(t) < \eta_0$ resp.), the SDM decides H_1 (H_0 resp.) at time step t. Given the signal structure, i.e., $f_{S|\theta=H_1}(s)$ and $f_{S|\theta=H_0}(s)$, and the thresholds η_1 and η_0, the IDPS, i.e., the probabilities of deciding H_1 or H_0 at each time step, can be predicted before the SPRT process occurs. We refer the computation algorithm to Appendix B in [8]. Figure 1 is an example of the IDPS for an SDM running the SPRT with $\eta_1 = 2.94$ and $\eta_0 = -2.94$. In this case the false-alarm and mis-detection probabilities are both 0.05.

In our model the IDPS of an isolated SDM are assumed to have the following property.

Assumption 1 (Isolated SDMs' almost-sure decision and finite expected decision time) *The isolated SDM, with the IDPS $\{p_1(t), p_0(t), p_{\mathrm{nd}}(t)\}_{t\in\mathbb{N}}$, makes the final individual decision almost surely, that is, $\prod_{t=1}^{\infty} p_{\mathrm{nd}}(t) = 0$. Moreover, the isolated SDM has finite expected decision time, i.e.,*

$$p_1(1) + p_0(1) + \sum_{t=2}^{\infty} t\Big(\big(p_1(t) + p_0(t)\big)\prod_{\tau=1}^{t-1} p_{\mathrm{nd}}(\tau)\Big) < \infty.$$

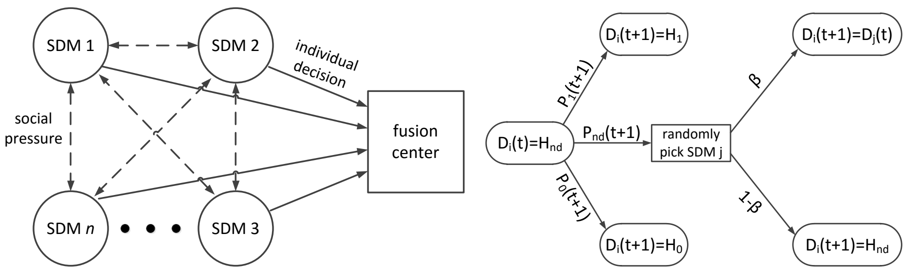
2.2 The n-SDM system

By n-SDM system we mean the system consisting of one fusion center and n identical and interacting SDMs. The behavior of the individual SDMs is described by the following assumption.

Assumption 2 (Individual decision making behavior in a n-SDM system) *In the n-SDM system, at each time step t, the following process occurs independently for any SDM $i \in V$ who has not made the final decision between H_1 and H_0:*

(i. SDM i first runs the sequential hypothesis test as an isolated SDM, i.e., SDM i decides H_1 (resp. H_0) with the probability $p_1(t)$ (resp. $p_0(t)$);
(ii. If no final decision is made in Step (i, SDM i will randomly pick one SDM j (can be SDM i itself) in the system and follow SDM j's previous decision state, i.e., $D_j(t-1)$, with some probability β.

In our model the more SDMs who have already made the decision H_1 (resp. H_0), the higher probability that the remaining SDMs decide H_1 (resp. H_0) at the current time step, that is, those SDMs who have made the final decision form the *social pressure*, which pushes other SDMs towards the final decisions. The probability β characterizes the inclination of the SDMs to be influenced by the social pressure. The model proposed by Dandach et. al. [8] is a special case when $\beta = 0$. Denote by $f_r(t \,|\, N_1, N_0)$, $r \in \{1, 0\}$, the probability that an SDM in the n-SDM system decides H_r at time step t, on condition that it has not made the final decision up to time $t-1$ and N_1 (resp. N_0) numbers of SDMs have decided H_1 (resp. H_0) before time t. Denote by $f_{\mathrm{nd}}(t \,|\, N_1, N_0)$ the probability that an SDM does not make the final decision



(a) The n-SDM system (b) Individual SDM's behavior at step $t+1$

Fig. 2 The first diagram shows the structure of the n-SDM system. The connections between the SDMs are bilateral with self loops. Therefore any SDM can be picked by any other SDM or itself. Once an individual final decision is made, the decision is sent to the fusion center. The second diagram describes how an SDM in the n-SDM system makes the individual decision at time step $t+1$.

at time step t, on condition that it has not made the final decision up to time $t-1$ and N_1 (resp. N_0) numbers of SDMs have decided H_1 (resp. H_0) before time t. According to Assumption 2,

$$
\begin{aligned}
f_r(t|N_1,N_0) &= p_r(t) + \beta p_{\mathrm{nd}}(t)\frac{N_r}{n} \quad \text{for } r\in\{1,0\}, \text{ and} \\
f_{\mathrm{nd}}(t|N_1,N_0) &= p_{\mathrm{nd}}(t)\left(\frac{n-N_1-N_0}{n} + (1-\beta)\frac{N_1+N_0}{n}\right).
\end{aligned} \tag{2}
$$

One can easily check that $f_1(t|N_1,N_0)+f_0(t|N_1,N_0)+f_{\mathrm{nd}}(t|N_1,N_0)=1$ for any t, N_1 and N_0.

Denote by $N_1(t)$ (resp. $N_0(t)$) the numbers of SDMs who have decided H_1 (resp. H_0) up to time step t. The fusion center receives each final individual decision from the SDMs and records $N_1(t)$ and $N_0(t)$. The global decision is made based on $N_1(t)$ and $N_0(t)$, according to the q-out-of-n rule defined below.

Definition 1 (The q-out-of-n rule) In an n-SDM sequential decision aggregation system, the fusion center running a q-out-of-n rule decides H_1 at time step t whenever $N_1(t) > N_0(t)$ and $N_1(t)\geq q$, where q is a prescribed threshold. The global decision H_0 is made if $N_0(t) > N_1(t)$ and $N_0(t)\geq q$.

Figure 2 gives a visual depiction of the n-SDM system structure and the individual SDMs' behavior.

2.3 Problem Statement

With the n-SDM system described in Section 2.1 and 2.2, we aim to solve the following problems.

Problem 1 (Finite-system behavior) For the fusion center running the q-out-of-n rule in a system with finite SDMs, given the IDPS $\{p_1(t),p_0(t),p_{\mathrm{nd}}(t)\}_{t\in\mathbb{N}}$, compute the probabilities $p_1(t;n,q)$, $p_0(t;n,q)$, $p_c(n,q)$, and the expected decision time $\mathbb{E}[T_{\mathrm{fc}}|n,q]$, as defined in Table 1.

Problem 2 (Asymptotic behavior) For the fusion center running the q-out-of-n rule in a n-SDM system, given the IDPS, compute the limit of the fusion center's accuracy and expected decision time as n tends to infinity, especially in the cases when $q=1$ or $q=\lceil n/2\rceil$.

3 The Behavior of the Fusion Center in a Finite n-SDM System

In this section we solve Problem 1, i.e., the fusion center's behavior in a system with finite SDMs. Firstly, we state a proposition on the almost-sure decision and finite expected decision time for the fusion center.

Proposition 1 (Almost-sure decision and finite expected decision time) *Consider an n-SDM system, assume that for the isolated SDM, there exists some $\tilde{t}\in\mathbb{N}$ such that $p_1(\tilde{t})\neq 0$, $p_0(\tilde{t})\neq 0$ and $\prod_{\tau=1}^{\tilde{t}-1} p_{\mathrm{nd}}(\tau)\neq 0$, then the fusion center has the almost-sure decision property if and only if*

(i. the isolated SDMs have almost-sure decision property;
(ii. the system size n is an odd number;
(iii. the threshold q satisfies $1\leq q\leq\lceil n/2\rceil$.

Moreover, in addition to the conditions (ii and (iii, if the isolated SDMs have finite expected decision time, then the fusion center also has finite expected decision time.

Proof: We first prove the contrapositive of the statement that the almost-sure decision of the fusion center leads to the conditions (i, (ii, and (iii.

(1) If the individual SDMs do not have the almost-sure decision property, i.e., $p_{\text{nd}} = \prod_{t=1} p_{\text{nd}}(t) \neq 0$, then the probability that none of the SDMs makes any final decision in the n-SDM system is equal to p_{nd}^n. Therefore, the probability that the fusion center does not make any global decision at all is no less than $p_{\text{nd}}^n > 0$.

(2) If n is even, the event "no SDM has made any final decision after time $\tilde{t}-1$, at time $\tilde{t}$, $n/2$ SDMs decide H_1 while $n/2$ SDMs decide H_0" has probability

$$\left(\prod_{\tau=1}^{\tilde{t}-1} p_{\text{nd}}(\tau)\right)^n \binom{n}{n/2} p_1(\tilde{t})^{n/2} p_0(\tilde{t})^{n/2} > 0.$$

If this event occurs, then the fusion center will never make a global decision.

(3) If $q > \lceil n/2 \rceil$, then consider the following event: "No SDM has decided up to $\tilde{t}-1$. At $\tilde{t}$, $\lceil n/2 \rceil$ SDMs decide H_1 while $\lfloor n/2 \rfloor$ SDMs decide H_0." This event has the probability

$$\left(\prod_{\tau=1}^{\tilde{t}-1} p_{\text{nd}}(\tau)\right)^n \binom{n}{\lceil n/2 \rceil} p_1(\tilde{t})^{\lceil n/2 \rceil} p_0(\tilde{t})^{\lfloor n/2 \rfloor} > 0.$$

In this case, neither N_1 nor N_0 has a chance to exceed the threshold, therefore the fusion center has a non-zero probability of making no global decision. Combining (1), (2) and (3) we conclude that the fusion center having the almost-sure decision property implies conditions (i, (ii, and (iii.

Next, we prove that conditions (i, (ii, and (iii lead to the almost-sure decision of the fusion center. Before the argument, we introduce some notations used in this proof. Define the random variable T_i as the decision time of SDM i when it is isolated, and define $T_i^{(n)}$ as the decision time of SDM i in an n-SDM system. Define $T_{\max}^{(n)}$ as $\max_i T_i^{(n)}$, i.e., the time instant when the last SDM makes the final individual decision. By definition, the fusion center's decision time must be prior or equal to $T_{\max}^{(n)}$. Let $\boldsymbol{T}_{-i}^{(n)} = (T_1^{(n)}, \dots, T_{i-1}^{(n)}, T_{i+1}^{(n)}, \dots, T_n^{(n)})$, i.e., the $(n-1)$-tuple of the decision time instants of all the SDMs except SDM i. Denote by ω one possible "trajectory" of the n-SDM system, i.e., a sequence of 2-tuples $\{(n_1(t), n_0(t))\}_{t\in\mathbb{N}}$, where $n_1(t)$, $n_0(t) \in \mathbb{N}$ and $n_1(t) + n_0(t) \leq n$ for any $t \in \mathbb{N}$. For simplicity, let $f_\alpha(t \,|\, \omega) = f_\alpha\big(t \,|\, n_1(t-1), n_0(t-1)\big)$ with the right-hand side of the equation defined by equations (2) for $\alpha = 1$, or 0, or "nd". Denote by Ω the set of all the possible trajectories, i.e., $\omega \in \Omega$.

Due to equations (2), $f_1(t \,|\, \omega) \geq p_1(t)$, $f_0(t \,|\, \omega) \geq p_0(t)$ and $f_{\text{nd}}(t \,|\, \omega) \leq p_{\text{nd}}(t)$ for any $\omega \in \Omega$. Since

$$\begin{aligned}\mathbb{P}[T_i^{(n)} < \infty \,|\, \boldsymbol{T}_{-i}^{(n)} < \infty] &= \sum_{\omega\in\Omega} \mathbb{P}[T_i^{(n)} < \infty \,|\, \omega, \boldsymbol{T}_{-i}^{(n)} < \infty] \mathbb{P}[\omega \,|\, \boldsymbol{T}_{-i}^{(n)} < \infty] \\ &= \sum_{\omega\in\Omega} \big(1 - \prod_{t=1}^{\infty} f_{\text{nd}}(t \,|\, \omega)\big) \mathbb{P}[\omega \,|\, \boldsymbol{T}_{-i}^{(n)} < \infty] \\ &\geq \sum_{\omega\in\Omega} \big(1 - \prod_{t=1}^{\infty} p_{\text{nd}}(t)\big) \mathbb{P}[\omega \,|\, \boldsymbol{T}_{-i}^{(n)} < \infty] = \mathbb{P}[T_i < \infty] = 1,\end{aligned}$$

we have

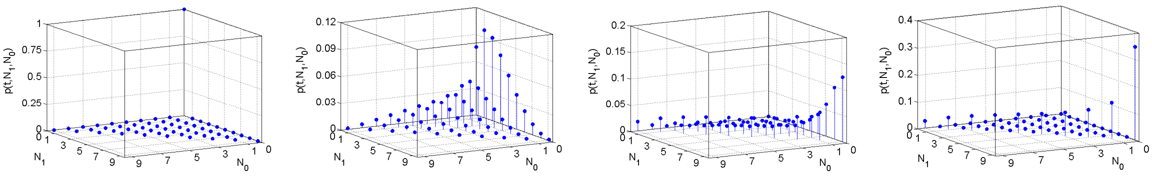
$$\mathbb{P}[T_{\max}^{(n)} < \infty] = \mathbb{P}[T_1^{(n)} < \infty, T_2^{(n)} < \infty, \dots, T_n^{(n)} < \infty] \geq \prod_{i=1}^{n} \mathbb{P}[T_i < \infty] = 1.$$

Therefore, $\mathbb{P}[T_{\max}^{(n)} < \infty] = 1$. Due to conditions (ii and (iii, the q-out-of-n rule must have been triggered no later than $T_{\max}^{(n)}$. Therefore, the fusion center makes the global decision almost surely.

We now prove the finite expected decision time for the fusion center. Conditions (ii and (iii lead to the inequality $T_{\text{fc}} \leq T_{\max}^{(n)} \leq T_1^{(n)} + T_2^{(n)} + \dots + T_n^{(n)}$ for any $\omega \in \Omega$. Moreover,

$$\begin{aligned}\mathbb{E}[T_i^{(n)}] &= \sum_{t=1}^{\infty} \mathbb{P}[T_i^{(n)} \geq t] = \sum_{t=1}^{\infty} \sum_{\omega\in\Omega} \mathbb{P}[T_i^{(n)} \geq t \,|\, \omega] \mathbb{P}[\omega] = 1 + \sum_{t=2}^{\infty} \sum_{\omega\in\Omega} \prod_{\tau=1}^{t-1} f_{\text{nd}}(t \,|\, \omega) \mathbb{P}[\omega] \\ &\leq 1 + \sum_{t=2}^{\infty} \sum_{\omega\in\Omega} \prod_{\tau=1}^{t-1} p_{\text{nd}}(t) \mathbb{P}[\omega] = 1 + \sum_{t=2}^{\infty} \mathbb{P}[T_i \geq t] = \mathbb{E}[T_i].\end{aligned}$$

Therefore, $\mathbb{E}[T_{\text{fc}} \,|\, n, q] \leq n\mathbb{E}[T_i] < \infty$ for any $1 \leq q \leq \lceil n/2 \rceil$. This concludes the proof. □



(a) $t = 1$ (b) $t = 21$ (c) $t = 41$ (d) $t = 101$

Fig. 3 The time-evolution of the state probability distribution of a 9-SDM system with social pressure. The IDPS for individual SDMs are as shown in Figure 1.

In the rest of this section, we quantitatively analyze the behavior of the fusion center in an n-SDM system, given the IDPS of the isolated SDM. We compute the probabilities of deciding either H_1 or H_0 at each time step, the accuracy, and the expected decision time of the fusion center.

1) The n-SDM system as a lumped Markov chain: The n-SDM sequential decision aggregation system is a 3^n-state Markov chain, since $D_i(t) \in \{H_1, H_0, H_{\text{nd}}\}$ for any $i \in V$ and at any time step the decision of any SDM only depends on the states of all the SDMs after the previous time step as well as the IDPS. Instead of focusing on any individual SDM's decision state, we discuss the time evolution of $N_1(t)$ and $N_0(t)$. Then the system is reduced to to $\frac{(n+1)(n+2)}{2}$-state Markov chain.

Definition 2 Consider the n-SDM sequential aggregation system. Define the system state after time step t by $\boldsymbol{N}(t) = (N_1(t), N_0(t))^T$ and define $p(t, N_1, N_0)$ as the probability distribution of the system state after time t. Define $\Gamma(t, \Delta N_1, \Delta N_0 \,|\, N_1, N_0)$ as the *state transition function*, which correspond to the probability of the following event: "on condition that N_1 SDMs have decided H_1 and N_0 SDMs have decided H_0 after time step $t-1$, ΔN_1 SDMs decide H_1 and ΔN_0 SDMs decide H_0 at time t."

The computation algorithm of the system's state probability distribution at any time t is given by the following proposition. The proof is a straightforward application of probability theory and thus omitted.

Proposition 2 (System state probability distribution) *The probability distribution of the n-SDM system state is given by the formulas below:*

(i. For $t = 1$, $p(1, N_1, N_0) = \Gamma(1, N_1, N_0 \,|\, 0, 0)$;
(ii. For $t \geq 2$, the probability distribution of the system state is computed from the distribution at last time step as

$$p(t, N_1, N_0) = \sum_{l=0}^{N_1} \sum_{k=0}^{N_0} p(t-1, l, k) \Gamma(t, N_1 - l, N_0 - k \,|\, l, k).$$

Here, the state transition function $\Gamma(t, \Delta N_1, \Delta N_0 \,|\, N_1, N_0)$ is computed by

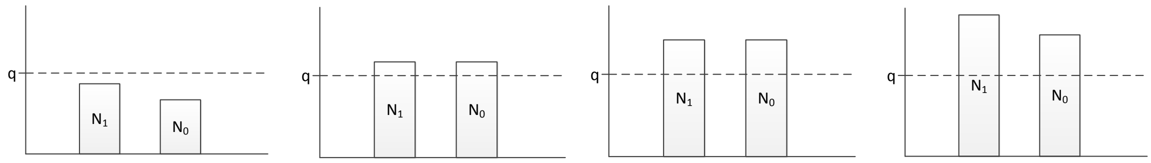
$$\Gamma(t, \Delta N_1, \Delta N_0 \,|\, N_1, N_0) = \binom{n - N_1 - N_0}{\Delta N_1} \binom{n - N_1 - N_0 - \Delta N_1}{\Delta N_0} \\ \times f_1^{\Delta N_1}(t|N_1, N_0) f_0^{\Delta N_0}(t|N_1, N_0) f_{\text{nd}}^{n - N_1 - N_0 - \Delta N_1 - \Delta N_0}(t|N_1, N_0),$$

where $t \in \mathbb{N}$ and $0 \leq \Delta N_1 + \Delta N_0 + N_1 + N_0 \leq n$.

Figure 3 illustrates the evolution of the probability distribution of the system state for a group of 9 SDMs in which all the SDMs are running the SPRT as shown in Figure 1. Initially, $(N_1, N_0) = (0, 0)$ is the only state with non-zero probability and then the states with non-zero probability spread out and finally aggregate on the diagonal line $N_1 + N_0 = 9$.

2) Computation of $p_1(t; n, q)$ and $p_0(t; , n, q)$: With the n-SDM system's state probability distribution at any time t, i..e, $p(t, N_1, N_0)$, we can compute $P_1(t; n, q)$ and $p_0(t; n, q)$ defined in Problem 1, that is, the probabilities that the fusion center running the q-out-of-n rule makes the global decision H_1 and H_0 respectively right at time step t. Notice that, in the sequential decision aggregation process for $1 \leq q \leq \lfloor n/2 \rfloor$, the *cancel-out* case may occur. The cancel-out case in which the fusion center finally decides H_1 corresponds to the intersection of the following three events:

(i. $N_1(\tau^* - 1) < q$ and $N_1(\tau^*) \geq q$ for some $\tau^* < t$;
(ii. For $\tau \in \{\tau^*, \tau^* + 1, \ldots, t-1\}$, $N_1(\tau) = N_0(\tau) \geq q$;
(iii. After time step t, $N_1(t) > N_0(t) \geq q$.



(a) $\tau = \tau^* - 1$ (b) $\tau = \tau^*$ (c) $\tau = t - 1$ (d) $\tau = t$

Fig. 4 The cancel-out case in which the number of votes for H_1 and H_0 both exceed the threshold q at time τ^* and remain equal till $t-1$. At time t, the vote for H_1 outnumbers H_0 and the fusion center decides H_1 at time t.

If the notations $N_1(t)$ and $N_0(t)$ are exchanged, the intersection of events (i), (ii) and (iii) corresponds to the cancel-out case in which the fusion center decides H_0. An example of the cancel-out case is illustrated by Figure 4. Based on whether the cancel-out case may occur, we discuss the computation of $p_1(t;n,q)$ and $p_0(t;n,q)$ in two cases, Case 1: $1 \leq q \leq \lfloor n/2 \rfloor$ and Case 2: $\lceil n/2 \rceil \leq q \leq n$.

Proposition 3 (Computation of $p_1(t;n,q)$ in Case 1) *Consider the n-SDM sequential decision aggregation system with the fusion center running the q-out-of-n rule and the individual SDMs with the IDPS $\{p_1(t), p_0(t), p_{\mathrm{nd}}(t)\}_{t\in\mathbb{N}}$. For $1 \leq q \leq \lfloor n/2 \rfloor$, the probability $p_1(t;n,q)$ defined in Problem 1 is computed by the following formulas:*

(i. For $t = 1$,

$$p_1(1;n,q) = \sum_{N_1=q}^{n} \sum_{N_0=0}^{\tilde{m}} p(1, N_1, N_0); \tag{3}$$

(ii. For $t \geq 2$,

$$\begin{aligned} p_1(t;n,q) = &\sum_{l=0}^{q-1}\sum_{k=0}^{q-1} p(t-1,l,k) \sum_{\Delta N_1=q-l}^{n-l-k} \sum_{\Delta N_0=0}^{\bar{m}} \Gamma(t, \Delta N_1, \Delta N_0 \,|\, l,k) \\ &+ \sum_{s=q}^{\lfloor n/2 \rfloor} p_{\mathrm{even}}(t-1,s) \sum_{\Delta N_1=1}^{n-2s} \sum_{\Delta N_0=0}^{m^*} \Gamma(t, \Delta N_1, \Delta N_0 \,|\, s,s), \end{aligned} \tag{4}$$

where $\tilde{m} = \min\{N_1 - 1, n - N_1\}$, $\bar{m} = \min\{\Delta N_1 + l - k - 1, n - l - k - \Delta N_1\}$ and $m^* = \min\{\Delta N_1 - 1, n - 2s - \Delta N_1\}$. The probability $p(t-1,l,k)$ for any $t \in \mathbb{N}$ and $0 \leq l + k \leq n$ is computed by Proposition 2 and the function $p_{\mathrm{even}}(t,s)$ for any $t \in \mathbb{N}$ and $q \leq s \leq \lfloor n/2 \rfloor$ is given by the following iteration formulas:

(i. For $t = 1$, $p_{\mathrm{even}}(t,s) = p(1,s,s)$;
(ii. For $t \geq 2$,

$$p_{\mathrm{even}}(t,s) = \sum_{l=0}^{q-1}\sum_{k=0}^{q-1} p(t-1,l,k)\Gamma(t, s-l, s-k \,|\, l,k) + \sum_{h=q}^{s} p_{\mathrm{even}}(t-1,h)\Gamma(t, s-h, s-h \,|\, h,h). \tag{5}$$

Proof: First we define $p_{\mathrm{even}}(t,s)$ as the probability of the intersection of the following tree events:

(i. $N_1(\tilde{\tau}) < q$ and $N_0(\tilde{\tau}) < q$ for some $\tilde{\tau} < t$;
(ii. For $\tau \in \{\tilde{\tau}, \tilde{\tau}+1, \ldots, t\}$, $N_1(\tau) = N_0(\tau)$;
(iii. After time step t, $N_1(t) = N_0(t) = s \geq q$.

Then equation (5) is a straightforward application of the total probability formula. For $t = 1$, $p_{\mathrm{even}}(1,s)$ is equal to $p(1,s,s)$ by definition. For the case $t \geq 2$, the first term of the right-hand side of equation (5) corresponds to the probability that both $N_1(t-1)$ and $N_0(t-1)$ are under the threshold q and $N_1(t) = N_0(t) = s \geq q$. The second term is the probability that, for any $\tau \leq t-1$, $N_1(\tau)$ and $N_0(\tau)$ remain equal if either of them exceeds the threshold q, and $N_1(t) = N_0(t) = s \geq q$.

With the computation algorithm of $p_{\mathrm{even}}(t,s)$, now we derive the formula for $p_1(t;,n,q)$. If the fusion center decides H_1 at $t = 1$, then $N_1(1) \geq q$ and $N_1(1) > N_0(1)$. Since all the system states $(N_1(1), N_0(1))$ are mutually exclusive, the probability that the fusion center decides H_1 at $t = 1$ is the sum of all the $p(1, N_1, N_0)$ satisfying $N_1 > N_0$ and $N_1 \geq q$. This concludes the proof of equation (3).

For $t \geq 2$, first we consider the case when the cancel-out case does not occur. The probability of the intersection of the following two events:

(i. At time $t-1$, both $N_1(t-1)$ and $N_0(t-1)$ are below the threshold. The probability of this event is $\sum_{l=0}^{q-1}\sum_{k=0}^{q-1} p(t-1,l,k)$;

(ii. On condition that after time $t-1$, the system is in some state (l,k) below the threshold, i.e., $l<q$ and $k<q$, the votes for H_1 outnumbers the votes for H_0 and exceeds the threshold at time step t,

is equal to

$$\sum_{\Delta N_1=q-l}^{n-l-k}\sum_{\Delta N_0=0}^{\bar{m}} \Gamma(t,\Delta N_1,\Delta N_0\,|\,l,k).$$

Applying the total probability formula we obtain the probability that the fusion center decides H_1 at t when the cancel-out case does not occur, which is the first term of the right-hand side of equation (4).

In the cancel-out case, the q-out-of-n condition is not triggered before t. After time step $t-1$, both $N_1(t-1)$ and $N_0(t-1)$ must have exceeded the threshold q and they are equal to s with probability $p_{\text{even}}(t-1,s)$ for any $s\in\{q,q+1,\dots,\lfloor n/2\rfloor\}$. On condition that $N_1(t-1)=N_0(t-1)=s\geq q$, the probability that $N_1(t)>N_0(t)\geq q$ is equal to $\sum_{\Delta N_1=1}^{n-2s}\sum_{\Delta N_0=0}^{m^*}\Gamma(t,\Delta N_1,\Delta N_0\,|\,s,s)$. According to the total probability formula, we obtain the second term of the right hand side of equation (4). This concludes the proof. □

The computation of $p_1(t;n,q)$ in the case $\lceil n/2\rceil$, in which there is no cancel-out case, is given by the proposition below. The proof is a straightforward application of the total probability formula.

Proposition 4 (Computation of $p_1(t;n,q)$ in Case 2) *Consider the n-SDM sequential decision aggregation process with the fusion center running the q-out-of-n rule. For $\lceil n/2\rceil\leq q\leq n$, the probability $p_1(t;n,q)$ is computed by the following formulas:*

(i. For $t=1$,

$$p_1(t;n,q)=\sum_{N_1=q}^{n}\sum_{N_0=0}^{n-N_1} p(1,N_1,N_0); \tag{6}$$

(ii. For $t\geq 2$,

$$p_1(t;n,q)=\sum_{l=0}^{q-1}\sum_{k=0}^{n-q} p(t-1,l,k)\sum_{\Delta N_0=q-l}^{n-l-k}\sum_{\Delta N_0=0}^{\bar{m}}\Gamma(t,\Delta N_1,\Delta N_0\,|\,l,k), \tag{7}$$

where $\bar{m}=n-l-k-\Delta N_1$.

To compute $p_0(t;n,q)$ we just need to switch all the indexes corresponding to H_1 and H_0 in equations (3), (4), (6), and (7).

3) Accuracy and expected decision time of the fusion center and the overall computation complexity: With the algorithm of computing $p_1(t;n,q)$ and $p_0(t;n,q)$, the fusion center's accuracy and expected decision time is given by the following equations:

$$p_c(n,q)=\sum_{t=1}^{\infty}p_1(t;n,q), \tag{8}$$

and

$$\mathbb{E}[T_{\text{fc}}\,|\,n,q]=\sum_{t=1}^{\infty}t\big(p_1(t;n,q)+p_0(t;n,q)\big). \tag{9}$$

The state transition function $\Gamma(t,\Delta N_1,\Delta N_0\,|\,N_1,N_0)$ is given by a closed form with the computation complexity $O(1)$. According to Proposition 2, the computation complxity for $p(t,N)1,N_0)$ is $O(1)$ for $t=1$ and $O(n^2)$ for $t\geq 2$. Knowing $p(t-,N_1,N_0)$ for any $0\leq N_1\leq n$, $0\leq N_0\leq n$ and $0\leq N_1+N_0\leq n$, the algorithm of computing $p_{\text{even}}(t,s)$ has the complexity $O(n^2)$. Therefore, according to Proposition 3 and Proposition 4 we know that the computation complexity for $p_1(t;n,q)$ is $O(n^5)$ when $1\leq q\leq\lfloor n/2\rfloor$ and is $O(n^4)$ when $\lceil n/2\rceil\leq q\leq n$.

4 Asymptotic Behaviors of the q-out-of-n Decision Aggregation System

By asymptotic behavior we mean the behavior of the fusion center in the n-SDM system as n tends to infinity. In this section, firstly we relate the accuracy and the expected decision time of the fusion center to the IDPS of the isolated SDMs, particularly for two special q-out-of-n rules: the fastest rule with $q=1$ and the majority rule with $q=\lceil n/2\rceil$. Then we discuss the influence of the parameter β on the sequential decision aggregation system as $n\to\infty$.

4.1 The fastest rule

According to Proposition IV.1 in the paper by Dandach et. al. [8], which is a n-SDM system with $\beta = 0$, the asymptotic accuracy and expected decision time of the fusion center running the fastest rule only depends on the first time instance when either $p_1(t) \neq 0$ or $p_0(t) \neq 0$. The following theorem states that the n-SDM system under the fastest rule leads to the same result for any $0 \leq \beta \leq 1$.

Theorem 1 (Asymptotic behavior for the fastest rule) *Consider the sequential decision aggregation system in which the fusion center is running the fastest rule. Define the earliest possible decision time $\bar{t}$ as*

$$\bar{t} = \min\{t \in \mathbb{N} \,|\, p_1(t) \neq 0 \text{ or } p_0(t) \neq 0\}.$$

Then the asymptotic accuracy of the fusion center satisfies

$$\lim_{n\to\infty} p_c(n,1) = \begin{cases} 1, & \text{if } p_1(\bar{t}) > p_0(\bar{t}),\\ 0, & \text{if } p_1(\bar{t}) < p_0(\bar{t}),\\ 1/2, & \text{if } p_1(\bar{t}) = p_0(\bar{t}), \end{cases} \tag{10}$$

and the asymptotic expected decision time satisfies

$$\lim_{n\to\infty} \mathbb{E}[T_{\text{fc}}|n,1] = \bar{t}. \tag{11}$$

Proof: In this proof it is convenient to modify our notation as follows: several systems with different IDPS are indexed by subscripts. Denote by $S_r^{(n)}$ the n-SDM system with index r and the IDPS $\{p_1^r(t), p_0^r(t), p_{\text{nd}}^r(t)\}_{t\in\mathbb{N}}$. Notice that here r is the system index rather than the power. The accuracy and expected decision time for the fusion center are denoted by $p_c(S_r^{(n)}, q)$ and $\mathbb{E}[T_{\text{fc}}|S_r^{(n)}, q]$ respectively.

We introduce three different n-SDM systems. Define

(i. $S_1^{(n)}$ as the n-SDM system with IDPS $\{p_1^1(t), p_0^1(t), p_{\text{nd}}^1(t)\}_{t\in\mathbb{N}}$, for which the earliest possible decision time $\bar{t}$ is defined by $\bar{t} = \min\{t \in \mathbb{N} \,|\, p_1^1(t) \neq 0 \text{ or } p_0^1(t) \neq 0\}$;

(ii. $S_2^{(n)}$ as the n-SDM system with $\beta = 0$, i.e., no social pressure, and the corresponding IDPS satisfying

$$\begin{cases} p_1^2(t) = p_1^1(t) \text{ and } p_0^2(t) = p_0^1(t), \text{ for } \forall t \leq \bar{t},\\ p_1^2(\bar{t}+1) = 1 \text{ and } p_0^2(\bar{t}+1) = 0,\\ p_1^2(t) = p_0^2(t) = 0 \text{ for } \forall t > \bar{t}+1; \end{cases}$$

(iii. $S_3^{(n)}$ as the n-SDM system with $\beta = 0$ and the IDPS satisfying

$$\begin{cases} p_1^3(t) = p_1^1(t) \text{ and } p_0^3(t) = p_0^1(t), \text{ for } \forall t \leq \bar{t},\\ p_1^3(\bar{t}+1) = 0 \text{ and } p_0^3(\bar{t}+1) = 1,\\ p_1^3(t) = p_0^3(t) = 0 \text{ for } \forall t > \bar{t}+1. \end{cases}$$

First we compare the accuracy of $S_1^{(n)}$ and the accuracy of $S_2^{(n)}$ when both are running the fastest rule. The systems $S_1^{(n)}$ and $S_2^{(n)}$ are identical for $t \leq \bar{t}$ since the social pressure terms $\beta p_{\text{nd}}(t)N_1(t)/n$ and $\beta p_{\text{nd}}(t)N_0(t)/n$ remain zero. For system $S_2^{(n)}$, at time step $\bar{t}+1$, all the SDMs who have not made final individual decisions will decide H_1. Therefore, $p_c(S_1^{(n)}, 1) \leq p_c(S_2^{(n)}, 1)$. Applying the same argument we have $p_c(S_3^{(n)}, 1) \leq p_c(S_1^{(n)}, 1)$. Moreover, according to Proposition IV.1 in [8], as n tends to infinity,

$$\begin{aligned} \lim_{n\to\infty} p_c(S_2^{(n)}, 1) &= \lim_{n\to\infty} p_c(S_3^{(n)}, 1)\\ &= \begin{cases} 1, & \text{if } p_1^1(\bar{t}) > p_0^1(\bar{t}),\\ 0, & \text{if } p_1^1(\bar{t}) < p_0^1(\bar{t}),\\ \frac{1}{2}, & \text{if } p_1^1(\bar{t}) = p_0^1(\bar{t}). \end{cases} \end{aligned} \tag{12}$$

This leads to equation (10).

Now we discuss the asymptotic expected decision time. If $p_1^1(\bar{t}) + p_0^1(\bar{t}) = 1$, obviously the fusion center's expected decision time would be $\bar{t}$ for any n. Suppose $0 < p_1^1(\bar{t}) + p_0^1(\bar{t}) < 1$. Define another system $S_4^{(n)}$ with the IDPS $\{p_1^4(t), p_0^4(t), p_{\text{nd}}^4(t)\}_{t\in\mathbb{N}}$ satisfies

$$\begin{aligned} &p_1^4(\bar{t}) = p_0^4(\bar{t}) = 0,\ p_{\text{nd}}^4(\bar{t}) = 1, \text{ and}\\ &p_1^4(t) = p_1^1(t),\ p_0^4(t) = p_0^1(t),\ p_{\text{nd}}^4(t) = p_{\text{nd}}^1(t) \text{ for any } t \neq \bar{t}, \end{aligned}$$

and the fusion center in system $S_4^{(n)}$ makes the global decision after the SDMs have decided H_1 or H_0. As long as $p_1^1(\bar{t}) + p_0^1(\bar{t}) < 1$, the isolated SDMs with the IDPS $\{p_1^4(t), p_0^4(t), p_{\mathrm{nd}}^4(t)\}_{t\in\mathbb{N}}$ still have almost-sure decision and finite expected decision time.

For system $S_1^{(n)}$,

$$\mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1] = \bar{t}\mathbb{P}[T_{\mathrm{fc}} = \bar{t} \,|\, S_1^{(n)}, q=1] + \mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1, T_{\mathrm{fc}} > \bar{t}]\mathbb{P}[T_{\mathrm{fc}} > \bar{t} \,|\, S_1^{(n)}, q=1].$$

By definition and according to the proof of Proposition 1,

$$\mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1, T_{\mathrm{fc}} > \bar{t}] \leq \mathbb{E}[T_{\max}^{(n)} \,|\, S_4^{(n)}] \leq n\mathbb{E}[T_i | S_4^{(n)}].$$

Moreover, according to the proof of Proposition IV.1 in [8], the term $\mathbb{P}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1]$ is in order $O(\epsilon)$ for some $0 < \epsilon < 1$ and $\lim_{n\to\infty} \mathbb{P}[T_{\mathrm{fc}} = \bar{t} \,|\, S_1^{(n)}, q=1] = 1$. Therefore,

$$\lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1, T_{\mathrm{fc}} > \bar{t}]\mathbb{P}[T_{\mathrm{fc}} > \bar{t} \,|\, S_1^{(n)}, q=1] = 0, \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, S_1^{(n)}, q=1] = \bar{t}.$$

□

4.2 The majority rule

Before analyzing the accuracy and expected decision time of the fusion center under the majority rule, we introduce a main result in the paper [19] on the mean-field convergence for systems with interacting objects, which can be applied to our model.

Consider a discrete-time Markov chain with n individuals. Denote by $X_i(t)$ the state of individual i after time step t. The individual states set is identical for all the individuals and is denoted by $\Theta = \{1, 2, \dots, S\}$, i.e., $X_i(t) \in \Theta$ for any $i \in \{1, 2, \dots, n\}$ and $t \in \mathbb{N}$.

Define the *occupancy measure* $\boldsymbol{M}^{(n)}(t) \in \mathbb{R}^{1\times S}$ by $M_r^{(n)}(t) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{X_i(t)=r\}}$ for any $r \in \Theta$. Define the *memory* $\boldsymbol{R}^{(n)}(t)$ as some d-dimension row vector, which is updated according to some continuous function $g : \mathbb{R}^{1\times d} \times \mathbb{R}^{1\times S} \to \mathbb{R}^{1\times d}$, that is, $\boldsymbol{R}^{(n)}(t+1) = g(\boldsymbol{R}^{(n)}(t), \boldsymbol{M}^{(n)}(t))$. Denote the individual state transition matrix by $K^{(n)}(t) = \big(K_{rm}^{(n)}(t)\big)_{S\times S}$, that is,

$$K_{rm}^{(n)}(t) = \mathbb{P}[X_i^{(n)}(t+1) = m \,|\, X_i^{(n)}(t) = r],$$

and $K_{rm}^{(n)}$ is an explicit function of $\boldsymbol{R}^{(n)}(t)$, i.e., $K^{(n)}(t) = \big(K_{rm}^{(n)}(\boldsymbol{R}^{(n)}(t))\big)_{S\times S}$. We rewrite [19, Theorem 4.1] as follows.

Lemma 1 (Mean-field convergence) *Consider the discrete-time Markov chain described above. Assume that,*

(i. For any $r, m \in \Theta$, as $n \to \infty$, $K_{rm}^{(n)}(\boldsymbol{r})$ converges uniformly in $\boldsymbol{r} \in \mathbb{R}^{1\times d}$ to some $K_{rm}(\boldsymbol{r})$, which is a continuous function of $\boldsymbol{r}$;

(ii. The vectors $\boldsymbol{M}^{(n)}(0)$ and $\boldsymbol{R}^{(n)}(0)$ converge almost surely to some deterministic limits $\boldsymbol{\mu}(t)$ and $\boldsymbol{\rho}(0)$.

Then for any fixed t, almost surely,

$$\lim_{n\to\infty} \boldsymbol{M}^{(n)}(t) = \boldsymbol{\mu}(t), \quad \text{and} \quad \lim_{n\to\infty} \boldsymbol{R}^{(n)}(t) = \boldsymbol{\rho}(t),$$

where $\boldsymbol{\mu}(t)$ and $\boldsymbol{\rho}(t)$ are defined by the following iteration formulas:

$$\boldsymbol{\mu}(t+1) = \boldsymbol{\mu}(t)K\big(\boldsymbol{\rho}(t)\big), \quad \text{and} \quad \boldsymbol{\rho}(t+1) = g\big(\boldsymbol{\rho}(t), \boldsymbol{\mu}(t+1)\big).$$

In the lemma above, the deterministic vector $\boldsymbol{\mu}(t)$ is referred to as the *mean-field limit* of $\boldsymbol{M}^{(n)}(t)$ as $n \to \infty$. Now we apply this lemma to our model. Define the occupancy measure $\boldsymbol{M}^{(n)}(t)$ by

$$\boldsymbol{M}^{(n)}(t) = \left(\frac{N_1(t)}{n}, \frac{N_0(t)}{n}, \frac{n - N_1(t) - N_0(t)}{n}\right), \tag{13}$$

and define the vector sequence $\{\boldsymbol{\mu}(t)\}_{t\in\mathbb{N}}$ by

$$\begin{aligned}
\boldsymbol{\mu}(0) &= (0, 0, 1), \\
\mu_1(t+1) &= \mu_1(t) + \mu_3(t)\big(p_1(t+1) + \beta p_{\mathrm{nd}}(t+1)\mu_1(t)\big), \\
\mu_2(t+1) &= \mu_2(t) + \mu_3(t)\big(p_0(t+1) + \beta p_{\mathrm{nd}}(t+1)\mu_2(t)\big), \\
\mu_3(t+1) &= 1 - \mu_1(t+1) - \mu_2(t+1).
\end{aligned} \tag{14}$$

The following proposition states that, as n tends to infinity, the occupancy measure $\boldsymbol{M}^{(n)}(t)$ in our model converges almost surely to the mean-field limit $\boldsymbol{\mu}(t)$.

Proposition 5 (Mean-field convergence in the n-SDM system) *Consider the n-SDM sequential decision aggregation system. For any $t \in \mathbb{N}$, as the system size n tends to infinity, the occupancy measure $\boldsymbol{M}^{(n)}(t)$, defined by equation (13), satisfies*

$$\lim_{n\to\infty} \boldsymbol{M}^{(n)}(t) = \boldsymbol{\mu}(t) \quad \textit{almost surely}, \tag{15}$$

where $\boldsymbol{\mu}(t)$ is defined by equation (14).

Proof: Define the memory vector by

$$\boldsymbol{R}^{(n)}(t) = \big(t, M_1^{(n)}(t), M_2^{(n)}(t)\big) = \left(t, \frac{N_1(t)}{n}, \frac{N_0(t)}{n}\right).$$

Therefore the function $g = (g_1, g_2, g_3)$ becomes:

$$\begin{aligned}
g_1\big(\boldsymbol{R}^{(n)}(t), \boldsymbol{M}^{(n)}(t+1)\big) &= R_1^{(n)}(t) + 1 = t + 1,\\
g_2\big(\boldsymbol{R}^{(n)}(t), \boldsymbol{M}^{(n)}(t+1)\big) &= M_1^{(n)}(t+1) = \frac{N_1(t+1)}{n},\\
g_3\big(\boldsymbol{R}^{(n)}(t), \boldsymbol{M}^{(n)}(t+1)\big) &= M_2^{(n)}(t+1) = \frac{N_0(t+1)}{n}.
\end{aligned}$$

Let the individual states set be $\Theta = \{1, 2, 3\}$, where the indexes 1, 2 and 3 correspond to H_1, H_0 and H_{nd} respectively. Define the matrix $K(\boldsymbol{r})$ by

$$\begin{aligned}
&K_{11}(\boldsymbol{r}) = 1, \quad K_{12}(\boldsymbol{r}) = 0, \quad K_{13}(\boldsymbol{r}) = 0; \quad K_{21}(\boldsymbol{r}) = 0, \quad K_{22}(\boldsymbol{r}) = 1, \quad K_{23}(\boldsymbol{r}) = 0;\\
&K_{31}(\boldsymbol{r}) = p_1(r_1+1) + \beta p_{\mathrm{nd}}(r_1+1) r_2, \quad K_{32}(\boldsymbol{r}) = p_0(r_1+1) + \beta p_{\mathrm{nd}}(r_1+1) r_3,\\
&K_{33}(\boldsymbol{r}) = 1 - K_{31}(\boldsymbol{r}) - K_{32}(\boldsymbol{r}).
\end{aligned}$$

Based on Assumption 2 and equations (1) and (2), in our model, the individual state transition matrix with any memory $\boldsymbol{r}$ satisfies $K^{(n)}(\boldsymbol{r}) = K(\boldsymbol{r})$, for any $n \in \mathbb{Z}_+$. Moreover, initially $\boldsymbol{M}^{(n)}(0) = \boldsymbol{\mu}(0)$ and $\boldsymbol{R}^{(n)}(0) = \boldsymbol{\rho}(0)$. According to Lemma 1, we obtain equation (15). □

Having completed all preparations, we now present the theorem on the asymptotic accuracy and expected decision time of the fusion center running the majority rule.

Theorem 2 (Asymptotic behavior for the majority rule) *Consider the n-SDM sequential decision aggregation system with the IDPS $\{p_1(t), p_0(t), p_{\mathrm{nd}}(t)\}_{t\in\mathbb{N}}$ known. Define the vector sequence $\{\boldsymbol{\mu}(t)\}_{t\in\mathbb{N}}$ by equation (14). As the system size n tends to infinity, the accuracy of the fusion center satisfies:*

$$\lim_{n\to\infty} p_c(n, \lceil n/2 \rceil) = \begin{cases} 1, & \textit{if } \lim_{t\to\infty} \mu_1(t) > 1/2,\\ 0, & \textit{if } \lim_{t\to\infty} \mu_2(t) > 1/2,\\ 1/2, & \textit{if } \exists T \in \mathbb{N},\ \textit{s.t. } \mu_1(T) = \mu_2(T) = 1/2. \end{cases} \tag{16}$$

As for the asymptotic expected decision time,

(i. if $\lim_{t\to\infty} \mu_1(t) > 1/2$ or $\lim_{t\to\infty} \mu_2(t) > 1/2$, then

$$t_{<\frac{1}{2}} + 1 \leq \lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, n, \lceil n/2 \rceil] \leq t_{>\frac{1}{2}},$$

where $t_{>\frac{1}{2}} = \min\{t \in \mathbb{N} \,|\, \max\big(\mu_1(t), \mu_2(t)\big) > 1/2\}$ and $t_{<\frac{1}{2}} = \max\{t \in \mathbb{N} \,|\, \max\big(\mu_1(t), \mu_2(t)\big) < 1/2\}$. Particularly, if there does not exists any $T \in \mathbb{N}$ such that $\mu_1(T) = 1/2$ or $\mu_2(T) = 1/2$, then $\lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, n, \lceil n/2 \rceil] = t_{>\frac{1}{2}}$;

(ii. if there exists $T \in \mathbb{N}$ such that $\mu_1(T) = \mu_2(T) = 1/2$, then

$$\lim_{n\to\infty} \mathbb{E}[T_{\mathrm{fc}} \,|\, n, \lceil n/2 \rceil] = t_{\frac{1}{2}},$$

where $t_{\frac{1}{2}} = \min\{t \in \mathbb{N} \,|\, \mu_1(t) = \mu_2(t) = 1/2\}$;

(iii. if for any $t \in \mathbb{N}$, $\mu_1(t) < 1/2$ and $\mu_2(t) < 1/2$, while $\lim_{t\to\infty} \mu_1(t) = \lim_{t\to\infty} \mu_2(t) = 1/2$, then the fusion center's expected decision time tends to infinity as $n \to \infty$ almost surely.

Proof: First we discuss the asymptotic accuracy. If $\lim_{t\to\infty}\mu_1(t) > 1/2$, there exists $\tilde{t}\in\mathbb{N}$ such that $\mu_1(\tilde{t}) > 1/2$. Since $\boldsymbol{M}^{(n)}(t)$ converges to $\boldsymbol{\mu}(t)$ almost surely, $M_1^{(n)}(\tilde{t}) = \frac{N_1(\tilde{t})}{n} > 1/2$ almost surely as $n\to\infty$. According to the majority rule,

$$\lim_{n\to\infty}\mathbb{P}[\text{The fusion center decides } H_1 \text{ no later than } \tilde{t}\,|\,n,\lceil n/2\rceil] = 1,$$

that is, $p_c(n,\lceil n/2\rceil)\to 1$ as $n\to\infty$. Following the same argument we have $p_c(n,\lceil n/2\rceil)\to 0$ when $\lim_{t\to\infty}\mu_2(t) > 1/2$.

Now consider the case when there exists $T\in\mathbb{N}$ such that $\mu_1(T)=\mu_2(T)=1/2$. Define $\bar{t} = \min\{t\,|\,\mu_1(t)=\mu_2(t)=1/2\}$. According to equation (14), for any $t<\bar{t}$, $\mu_1(t)<1/2$ and $\mu_2(t)<1/2$, which implies $N_1(t)/n<1/2$ and $N_0(t)/n<1/2$ almost surely as $n\to\infty$. Therefore, no global decision is made before $\bar{t}$ and after time step $\bar{t}$ the fusion center decides H_1 with probability 1/2 due to the symmetry.

Now we prove the results on the asymptotic expected decision time. First, we discuss the case when $\lim_{t\to\infty}\mu_1(t) > \frac{1}{2}$. The case $\lim_{t\to\infty}\mu_2(t) > \frac{1}{2}$ follows the same line of argument. For any $t\le t_{<\frac{1}{2}}$, $\mu_1(t)<\frac{1}{2}$, $\mu_2(t)<\frac{1}{2}$, and therefore

$$\mathbb{P}\Big[\lim_{n\to\infty}\frac{N_1(t)}{n} = \mu_1(t) < \frac{1}{2}\Big] = 1.$$

The fusion center makes no decision before $t_{<\frac{1}{2}}+1$, almost surely. For $t=t_{>\frac{1}{2}}$, $\mu_1(t)>\frac{1}{2}$, $\mu_2(t)<\frac{1}{2}$. We have

$$\mathbb{P}\Big[\lim_{n\to\infty}\frac{N_1(t_{>\frac{1}{2}})}{n} = \mu_1(t_{>\frac{1}{2}}) > \frac{1}{2}\Big] = 1.$$

Therefore, almost surely, $t_{<\frac{1}{2}}+1 \le T_{\text{fc}} \le t_{>\frac{1}{2}}$. Particularly, if there does not exist any T such that $\mu_1(T)=1/2$, then $t_{<\frac{1}{2}}+1 = t_{>\frac{1}{2}}$. This concludes the proof for Case (i.

In Case (ii, when $\mu_1(t_{\frac{1}{2}}) = \mu_2(t_{\frac{1}{2}}) = \frac{1}{2}$ for any $t<t_{\frac{1}{2}}$, we have $\mu_1(t)<\frac{1}{2}$ and $\mu_2(t)<\frac{1}{2}$. Therefore, as n tends to infinity, the fusion center makes the global decision at $t_{\frac{1}{2}}$ almost surely. The asymptotic expected decision time is $t_{\frac{1}{2}}$.

In Case (iii, since $\mathbb{P}[\lim_{n\to\infty}N_1(t)/n = \mu_1(t) < 1/2] = \mathbb{P}[\lim_{n\to\infty}N_0(t)/n = \mu_2(t) < 1/2] = 1$ for any $t\in\mathbb{N}$, the fusion center almost surely makes no global decision at any time. Therefore, $\lim_{n\to\infty}\mathbb{E}[T_{\text{fc}}\,|\,n,q] = \infty$. □

4.3 Analysis of the influence of parameter β

According to Proposition 5, $\mu_1(t)$ ($\mu_2(t)$, $\mu_3(t)$ resp.) is a mean-field approximation of $N_1(t)/n$ ($N_0(t)/n$, $\big(n-N_1(t)-N_0(t)\big)/n$ resp.) for large n. The parameter β plays an important role in the iteration of $\boldsymbol{\mu}(t)$. In this subsection we discuss the dynamical behavior of $\boldsymbol{\mu}(t)$ as a function of the parameter β.

1) β=0: The case $\beta=0$ corresponds to the system without social pressure. In this scenario the n-SDM system is degenerated to the model discussed in [8]. Denote by $\boldsymbol{\nu}(t) = \big(\nu_1(t),\nu_2(t),\nu_3(t)\big)$ the solution to equation (14) with $\beta=0$. Then we have

$$\boldsymbol{\nu}(t+1) = \boldsymbol{\nu}(t)A(t+1), \quad \text{with } A(t+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_1(t+1) & p_0(t+1) & p_{\text{nd}}(t+1) \end{bmatrix}, \tag{17}$$

and $\boldsymbol{\nu}(0) = (0,0,1)$. It is straightforward to check that the closed form of $\boldsymbol{\nu}(t)$ is given by

$$\begin{aligned} \nu_1(t) &= \begin{cases} p_1(1), & \text{for } t=1,\\ p_1(1) + \displaystyle\sum_{s=1}^{t-1} p_1(s+1)\prod_{\tau=1}^{s} p_{\text{nd}}(\tau), & \text{for } t\ge 2,\end{cases}\\ \nu_2(t) &= \begin{cases} p_0(1), & \text{for } t=1,\\ p_0(1) + \displaystyle\sum_{s=1}^{t-1} p_0(s+1)\prod_{\tau=1}^{s} p_{\text{nd}}(\tau), & \text{for } t\ge 2,\end{cases}\\ \nu_3(t) &= \prod_{\tau=1}^{t} p_{\text{nd}}(\tau). \end{aligned} \tag{18}$$

According to Assumption 1, $\lim_{t\to\infty}\nu_3(t)=0$. According to the iteration equations (17), $\nu_1(t)$ and $\nu_2(t)$ is non-decreasing with t and are both upper bounded by 1. Therefore, $\lim_{t\to\infty}\nu_1(t)$ and $\lim_{\to\infty}\nu_2(t)$ both exist. Moreover, with the closed-form of $\boldsymbol{\nu}(t)$, one can check that Theorem 2 for the case $\beta=0$ coincide with Proposition IV.3 and IV.4 in [8].

2) β=1: Denote by $\hat{\boldsymbol{\nu}}(t)$ the solution to equation (14) in the other extreme case when $\beta=1$. The iteration equation for $\hat{\boldsymbol{\nu}}(t)$ is nonlinear and written as

$$\begin{aligned}\hat{\nu}_1(t+1)&=\hat{\nu}_1(t)+\hat{\nu}_3(t)\big(p_1(t+1)+\beta p_{\mathrm{nd}}(t+1)\hat{\nu}_1(t)\big),\\ \hat{\nu}_2(t+1)&=\hat{\nu}_2(t)+\hat{\nu}_3(t)\big(p_0(t+1)+\beta p_{\mathrm{nd}}(t+1)\hat{\nu}_2(t)\big),\\ \hat{\nu}_3(t+1)&=p_{\mathrm{nd}}(t+1)\hat{\nu}_3(t)^2.\end{aligned}\tag{19}$$

One can deduce, from the third equation above, the closed form of $\hat{\nu}_3(t)$:

$$\hat{\nu}_3(t)=\prod_{\tau=1}^{t}p_{\mathrm{nd}}(\tau)^{2^{t-\tau}}.$$

Similar to the case when $\beta=0$, we conclude that the limit of $\hat{\boldsymbol{\nu}}(t)$ exists, as t tends to infinity. Moreover, with the same IDPS, $\hat{\nu}_3(t)$ decays to zero faster than $\nu_3(t)$, that is, in the system with large n and $\beta=1$, the expected decision time for the individual SDMs is no larger than in the case when $\beta=0$.

3) Small β: We conduct the leading order analysis in β, for the expression of $\boldsymbol{\mu}(t)$, when β is very small. The following proposition is stated without proof.

Proposition 6 (Leading order analysis for small β) *Consider the iteration equation* (14) *for* $\boldsymbol{\mu}(t)$ *with* β *positive but close to 0. Let* $\mu_r(t)=\nu_r(t)+g_r(t)\beta+O(\beta^2)$ *for any* $r\in\{1,2,3\}$*, where* $g_r(t)$ *is the coefficient of the leading order in* β *and* $\boldsymbol{\nu}(t)=\big(\nu_1(t),\nu_2(t),\nu_3(t)\big)$ *is given by equation* (18)*. Then,*

(i. for any $r\in\{1,2,3\}$, $g_r(t)$ *satisfies the following iteration formula:*

$$\begin{aligned}g_1(t+1)&=g_1(t)+p_1(t+1)g_3(t)+\nu_1(t)\nu_3(t)p_{\mathrm{nd}}(t+1),\\ g_2(t+1)&=g_2(t)+p_0(t+1)g_3(t)+\nu_2(t)\nu_3(t)p_{\mathrm{nd}}(t+1),\\ g_3(t+1)&=p_{\mathrm{nd}}(t+1)g_3(t)-p_{\mathrm{nd}}(t+1)\nu_3(t)\big(1-\nu_3(t)\big),\end{aligned}$$

and $g_1(t)+g_2(t)+g_3(t)=0$ *for any* $t\in\mathbb{N}$*;*

(ii. the closed form of $g_r(t)$ *is given by* $g_1(1)=g_2(1)=g_3(1)=0$, $g_1(2)=p_1(1)p_{\mathrm{nd}}(1)p_{\mathrm{nd}}(2)$, $g_2(2)=p_0(1)p_{\mathrm{nd}}(1)p_{\mathrm{nd}}(2)$, $g_3(2)=-p_{\mathrm{nd}}(1)p_{\mathrm{nd}}(2)\big(p_1(1)+p_0(1)\big)$*, and, for any* $t\geq 3$,

$$\begin{aligned}g_1(t)&=g_1(2)+\sum_{l=3}^{t}p_{\mathrm{nd}}(l)\nu_1(l-1)\nu_3(l-1)-\sum_{l=3}^{t}p_1(l)\sum_{s=2}^{l-1}\sum_{\tau=s}^{l-1}p_{\mathrm{nd}}(\tau)\nu_3(s-1)\big(1-\nu_3(s-1)\big),\\ g_2(t)&=g_2(2)+\sum_{l=3}^{t}p_{\mathrm{nd}}(l)\nu_2(l-1)\nu_3(l-1)-\sum_{l=3}^{t}p_0(l)\sum_{s=2}^{l-1}\sum_{\tau=s}^{l-1}p_{\mathrm{nd}}(\tau)\nu_3(s-1)\big(1-\nu_3(s-1)\big),\\ g_3(t)&=-\sum_{s=2}^{t}\prod_{\tau=s}^{t}p_{\mathrm{nd}}(\tau)\nu_3(s-1)\big(1-\nu_3(s-1)\big);\end{aligned}$$

(iii. for any $t\in\mathbb{N}$, $g_3(t)\leq 0$*, and therefore* $\mu_3(t)$ *is non-increasing with* β*;*

(iv. for any $t\in\mathbb{N}$, $g_1(t)$ *(*$g_2(t)$ *resp.) is non-decreasing with* $p_{\mathrm{nd}}(t)$ *and non-increasing with* $p_1(t)$ *(*$p_0(t)$ *resp.), and* $|g_3(t)|$ *is non-decreasing with* $p_{\mathrm{nd}}(t)$*.*

4) β close to 1: We present the following proposition on the leading order in $\delta=1-\beta$ for small δ.

Proposition 7 (Leading order analysis for β close to 1) *Consider equation* (14) *for* $\boldsymbol{\mu}(t)$ *with* β *close to but less than 1. Let* $\delta=1-\beta$ *and* $\mu_r(t)=\hat{\nu}_r(t)+\delta h_r(t)+O(\delta^2)$ *for* $r\in\{1,2,3\}$*, where* $H_r(t)$ *is the coefficient of the leading order in* δ *and* $\hat{\boldsymbol{\nu}}(t)$ *is given by equation* (19)*. Then we have:*

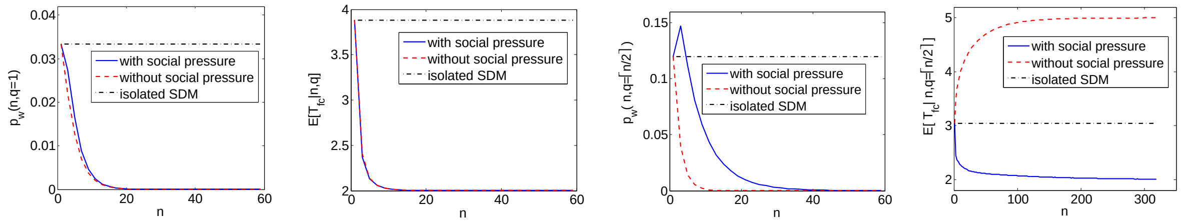
(i. for $r\in\{1,2,3\}$, $h_r(t)$ *satisfies the following iteration formula:*

$$\begin{aligned}h_1(t+1)&=\big(1+\hat{\nu}_3(t)p_{\mathrm{nd}}(t+1)\big)h_1(t)+p_1(t+1)h_3(t)+p_{\mathrm{nd}}(t+1)\hat{\nu}_1(t)\big(h_3(t)-\hat{\nu}_3(t)\big),\\ h_2(t+1)&=\big(1+\hat{\nu}_3(t)p_{\mathrm{nd}}(t+1)\big)h_2(t)+p_0(t+1)h_3(t)+p_{\mathrm{nd}}(t+1)\hat{\nu}_2(t)\big(h_3(t)-\hat{\nu}_3(t)\big),\\ h_3(t+1)&=p_{\mathrm{nd}}(t+1)\hat{\nu}_3(t)\big(2h_3(t)+\hat{\nu}_1(t)+\hat{\nu}_2(t)\big),\end{aligned}$$

and $h_1(t)+h_2(t)+h_3(t)=0$ *for any* $t\in\mathbb{N}$*;*

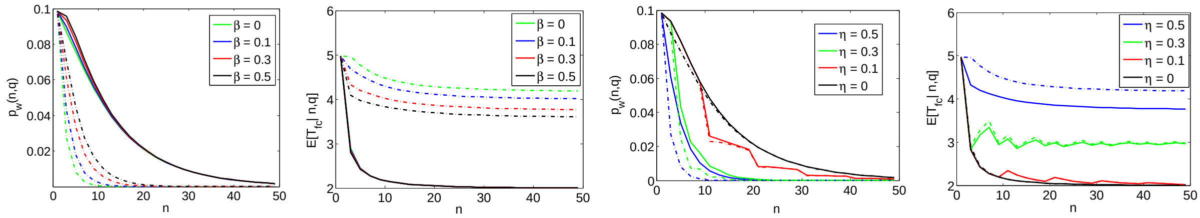
(ii. for any $t\in\mathbb{N}$, $h_3(t)\geq 0$*, and therefore* $\mu_3(t)$ *is non-decreasing with* β*;*

(iii. for any $t\in\mathbb{N}$, $h_1(t)$ *(*$h_2(t)$ *resp.) are non-decreasing with* $p_1(t)$ *(*$p_0(t)$ *resp.), and* $h_3(t)$ *is non-decreasing with* $p_{\mathrm{nd}}(t)$*.*



(a) error rate, fastest rule (b) expected decision time, fastest rule (c) error rate, majority rule (d) expected decision time, majority rule

Fig. 5 The probability of making wrong global decision, and the expected decision time, for the fusion center in n-SDM systems with the fastest rule and the majority rule. The blue curves correspond to the n-SDM systems with $\beta = 1$. The red dash-dot curves represent the n-SDM systems with $\beta = 0$ and the black dotted lines correspond to the isolated SDM.



(a) error rate, varying β (b) expected decision time, varying β (c) error rate, varying η (d) expected decision time, varying η

Fig. 6 The probability of making wrong global decision and the expected decision time, as functions of the system size n respectively, for different values of the parameter β and different q-out-of-n rules. In Figure (a) and (b), the solid lines correspond to the fastest rules while the dash lines correspond to the majority rules. In Figure (c) and (d), the solid lines correspond to the systems with $\beta = 0.3$ while the dash lines correspond to the system with $\beta = 0$.

5 Further Simulation

1) Validation of the asymptotic performance: Simulation work has been conducted to validate the results of Theorems 1 and 2. In Figure 5(a) and 5(b), the IDPS has $\bar{t} = 2$ and $p_1(\bar{t}) > p_0(\bar{t})$. The simulation result indicates that, as n increases, the fusion center's accuracy, i.e., $1 - p_w(n, 1)$ gets close to 1 and the expected decision time converges to $\bar{t}$. In Figure 5(c) and 5(d), the IDPS satisfies $\mu_1(\infty) > 1/2 > \mu_2(\infty)$ and $t_{>\frac{1}{2}} = 2$ for $\beta = 1$; $\mu_1(\infty) > 1/2 > \mu_2(\infty)$ and $t_{>\frac{1}{2}} = 5$ for $\beta = 0$. The simulation result indicates that, as n tends to infinity, the probability of making wrong global decision under the majority rule, i.e., the probability $p_w(n, \lceil n/2 \rceil)$, converges to 0 and the expected decision time converges to $t_{>\frac{1}{2}}$, as indicated by Theorem 2. Moreover, Figure 5(d) shows that, with the presence of social pressure, the expected decision time of the fusion center running the majority rule can be even less than the expected decision time of a single isolated SDM, while the expected decision time of the model without social pressure, as the red dash line in Figure 5(d) indicates, is much larger than the single isolated SDM's.

2) Comparison among different values of β: Simulation work has been conducted to compare the performances of systems with different values of the model parameter β. The IDPS shown in Figure 1 are used in the simulation work illustrated by Figure 6. Figure 6(a) and 6(b) are comparisons between the fastest rule and the majority rule with varying values of β. We can see that, for any fixed n and β, the fastest rule has less accuracy while faster decision speed than the majority. Moreover, the performance of the fastest rule is not sensitive to the value of β while, for the majority rule with fixed system size n, the probability of wrong global decision gets larger as β increases but the expected decision time decreases as β increases.

3) Comparison among different q-out-of-n rules: Refer to the *η-total rule* as the $\lceil \eta n \rceil$-out-of-n rule. The case $\eta = 0$ corresponds to the fastest rule while $\eta = 0.5$ is the majority rule. Figure 6(c) and 6(d) reveal that the system performance gets more sensitive to β as η increases. Moreover, for fixed n and β, the system's accuracy increases with the increase of η, at the cost of the higher expected decision time.

6 Conclusion and Discussion

This paper proposes a sequential decision aggregation model that does not rely on the specific individual decision making policy and incorporates social pressure. Individuals in our model are sequential decision makers (SDMs) influenced by the decisions of other individuals. We present an algorithm to compute the system's decision probabilities, accuracy and expected decision time. Two specific group decision rules, the fastest rule and the majority rule, are analyzed in detail. We then focus on the case when the system size tends to infinity and, via a mean-field analysis, provide the exact expression of the asymptotic accuracy and expected decision time for both the fastest rule and the majority rule. These results relate the group's decision making behavior to the isolated SDM's. In addition to the theoretical analysis, we provide some simulation work to present the performance of our group decision making model and compare it to the sequential decision aggregation model without social pressure, first proposed in [8]. Within our model, we also compared the performance of different q-out-of-n aggregation rules.

This model could be extended to a generalized problem, in which the SDMs' IDPS are heterogeneous. Moreover, the connections between the SDMs might not necessarily be all-to-all. If both the heterogeneous SDMs and the network structure are taken into consideration, the group decision making policy becomes more complicated. The generalized model would help to explain how a group of decision makers with different information sources and confidence levels collaborate together and the optimization of the group decision making performance will be related to the network topology.

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