VEHICLE ROUTING ALGORITHMS FOR RADIALLY ESCAPING TARGETS *

PUSHKARINI AGHARKAR †, SHAUNAK D. BOPARDIKAR‡, AND FRANCESCO BULLO

Abstract. We introduce a novel dynamic vehicle routing problem termed the *Radially Escaping Targets (RET)* problem in which mobile targets appear uniformly randomly on a disk according to a stochastic process and move radially outward to escape the disk in a minimum amount of time. A single vehicle is assigned the task of intercepting the targets before they escape. We first obtain two fundamental upper bounds on the fraction of targets intercepted by the vehicle in the steady state - termed the *capture fraction* - for the RET problem. We then propose three policies to maximize the capture fraction for the RET problem and identify parameter regimes in which they are suitable. All three policies are within a constant factor of the optimal in specific parameter regimes. For the asymptotic regime of low arrival rate this factor is equal to one. For the asymptotic regime of high arrival rate, the factor is equal to 2.52 when the disk radius is greater than or equal to one. For the moderate speed regimes, this factor is dependent on the target speed. We verify performance of the policies with numerical simulations.

Key words. vehicle routing problems, dynamic vehicle routing, moving target routing problems

1. Introduction.

The subject of this paper is a dynamic vehicle routing (DVR) problem involving moving targets: targets appear throughout an environment and move with a constant speed in order to escape its boundary in the least amount of time. A single vehicle is assigned the task of intercepting as many targets as it can before they escape the environment. One application of this problem setup is in robotic patrolling where it is necessary to stop malicious agents from leaving a region so as to protect the surroundings.

1.1. Related work.

The classical Traveling Salesman Problem (TSP) and its extensions to other DVR problems have been explored extensively [4, 21]. Due to a recent surge of activity in the area of motion planning for autonomous robots, a lot of variants of DVR have been addressed over the last decade. An extensive list of such problems can be found in [8]. A variant, like the problem studied in this paper, is the classic static vehicle routing problem with time windows [21], which is known to be NP-hard. Modified versions of the problem have been studied in [5, 16]. Another variant is the vehicle routing problem with moving targets.

Several researchers have worked on dynamic vehicle routing problems (VRPs) involving moving targets in the past. The approximation complexity of Moving-Target TSP was studied in [10], where it was shown that Moving-Target TSP with n targets cannot be approximated better than by a factor of $2^{O(\sqrt{n})}$ times optimal within polynomial time unless $P = NP$. The authors in the same work also showed that if targets have the same velocities, then there is a polynomial time approximation for the Moving-Target TSP. Authors in [11] give a $2 + \epsilon$ approximation algorithm for instances of the Moving-Target TSP in which $O(\frac{\log n}{\log \log n})$ of the n points are moving with arbitrary velocity. Authors in [6] study a variant of the Moving-Target VRP in which targets appear on a segment and move with the same velocity. They prove that

*This work was supported in part by NSF grant CNS 1035917 and ARO grant W911NF-11-1-0092.

†Pushkarini Agharkar and Francesco Bullo are with the Mechanical Engineering Department and the Center for Control, Dynamical Systems and Computation, University of California at Santa Barbara, USA, agharkar@umail.ucsb.edu, bullo@engineering.ucsb.edu

‡Shaunak D. Bopardikar is with United Technologies Research Center Inc., Berkeley, CA 94705, USA, bshaunak@gmail.com

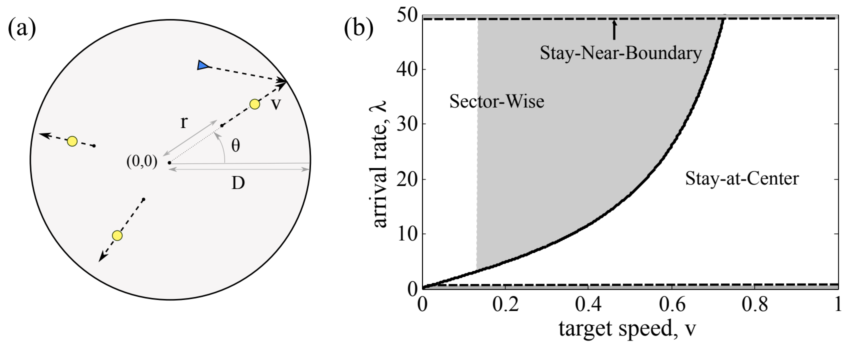

FIG. 1. *(a) Schematic of the Radially escaping targets (RET) problem. (b) The parameter regimes where the Stay-at-Center (SAC), Sector-Wise(SW) and Stay-Near-Boundary(SNB) policies are designed are shown for D=1. The gray shaded regions indicate the parameter regimes in which the policies are constant factor optimal.*

a first come first serve policy minimizes the expected time to service a target when the target arrival rate is very high as well as when the target speed is close to the vehicle speed. Authors in [9] study a kinetic variant of the k-delivery TSP where all targets move with the same velocity and a robotic arm moving with a finite capacity must intercept them. They provide constant-factor approximation for the problem. Authors in [2] study a grasp and delivery problem motivated by robot navigation and propose a 2-factor approximation algorithm. In [5], the moving targets have to be serviced within a time-window and a policy based on repeated computation of longest paths through the available set of targets is proposed to this end. In [16], authors also consider stochastic time windows within which static targets are required to be serviced.

More recent results on the subject of routing problems involving the task of target interception consider more general models for target behavior [3, 15, 14]. In [3], the authors propose a partitioning strategy for a multiple vehicle multiple target problem in which the targets can apply an evading strategy in response to the actions of the service vehicle. In this work, a single target maintains the same velocity throughout with the intention of escaping the environment as quickly as possible. The problem setup in this paper is significantly different from earlier setups in the following ways: The moving targets have different velocities depending on their angular location, as opposed to having same velocities as assumed in many problem setups looked at in literature [6, 10, 19]. They also have different deadlines depending on their radial location as opposed to having the same deadline or time window before which they should be serviced [5, 6].

1.2. Contribution. The contributions of this paper can be summarized as follows. We introduce a novel dynamic vehicle routing problem termed the Radially Escaping Targets (RET) problem. The RET problem has three parameters: the target arrival rate λ, the target speed $v < 1$ and the environment radius D. We first determine two policy independent upper bounds on the fraction of targets that can be captured for the RET problem. In the process, we derive a novel method to establish upper and lower bounds on the path through radially escaping targets. Next, we formulate three policies: Stay-at-Center (SAC), Sector-wise (SW) and Stay-Near-Boundary (SNB) policy. The SAC policy is designed for low arrival rates while the SW policy is formulated for moderate arrival rates. The SNB policy is designed for

Table 1
Performance of policies for the RET problem

Design Regime	Algorithm	Regime of constant factor optimality	Factor of optimality
Light load	Stay-At-Center	$\lambda \to 0^+$	1
Moderate load	Sector-wise	$\lambda > \dfrac{7\pi v}{(1-v^2)^{3/2}D}, v > \dfrac{1}{4\sqrt{2}}$	$\dfrac{1}{\alpha(v)}$
Fixed speed, heavy load	Stay-Near-Boundary	$\lambda \to +\infty$, $D > 1$	$\dfrac{7\beta}{2}$

high arrival rates. Lower bounds on the fraction of targets captured using the SAC, SW and SNB policies are obtained. In Table 1, we summarize these lower bounds and also present the factor of optimality (defined as the ratio of the fundamental upper bound for the RET problem to the capture fraction of a policy). The symbol $\beta \approx 0.7120 \pm 0.0002$ and

$$\alpha(v) = \frac{\sqrt{v}}{\pi^2}\left(\left(\frac{v}{(1-v^2)^{3/2}}\right)^{1/2} + \frac{10}{3}\frac{(1-v^2)^{1/2}}{v}\right)^{-1/2}.$$

In Fig. 1(b), the design regimes for the SAC, SW and SNB policies are shown. The gray shaded regions indicate the regimes where the policies are constant factor optimal. The SAC and SNB policy are constant factor optimal in the asymptotic regimes of $\lambda \to 0^+$ and $\lambda \to +\infty$ respectively. The gray shaded regions separated by dashed lines are representative of these asymptotic regimes. For fixed target speed, the SW policy is within a constant factor of the optimal in the gray shaded region in the middle. We present numerical simulations which empirically verify our results.

The set-up of the RET problem can be viewed as a dynamical system where targets are generated via a stochastic process. The dynamical system needs to be controlled using a control law or policy in order to stop the targets from escaping the environment. The performance metric to evaluate the policy is the capture fraction of the targets which needs to be maximized. Fundamental upper bounds and achievable lower bounds on the capture fraction is the topic of the paper. We study the gap between them as well.

1.3. Organization. The paper is organized as follows. In Section 2, we formally introduce the RET problem and its parameters and state the problem statement. In Section 3 we establish some preliminary results which will be used to evaluate performance of algorithms as well as obtain two fundamental upper bounds on the performance of any algorithm for the RET problem. In Section 4, we propose three policies for the RET problem and provide provable guarantees on their performance. Simulation results are presented in Section 5 and conclusions and future directions are discussed in Section 6. Finally, in order to facilitate the reading, the proofs of all the theoretical results on the policies are in the appendices of the paper.

2. Problem Formulation. We start with introducing a DVR problem in which the environment $\mathcal{E} = \{(r,\theta) : 0 \le r \le D \text{ for all } \theta \in [0, 2\pi)\}$ is a disk of radius D. Targets appear independently and uniformly distributed in $\mathcal{E}$ with uniform spatial density.

Their arrival times are modeled using a Poisson process with rate λ [18]. Uniform spatial distribution of the targets is realized through probability density functions $f(r) = 2r/D^2$ and $e(\theta) = 1/2\pi$ where r and θ are random variables describing the location of appearing targets in radial coordinates. Once the targets appear, they move radially outwards with a constant speed $v < 1$ and eventually reach the boundary of the environment. A vehicle with speed of 1 and confined to move in $\mathcal{E}$ intercepts the targets and captures them before they escape the environment. We refer to this problem as the Radially Escaping Targets (RET) problem for convenience and a schematic of the problem is shown in Fig. 1(a). The parameters of the RET problem are the target speed v, arrival rate λ and disk radius D.

Let $\mathcal{Q}(t) \subset \mathcal{E}$ denote the set of positions of all targets that have appeared but have not been serviced or have escaped before time t. Let $p(t) \in \mathcal{E}$ be the position of the vehicle at time t. A policy for the vehicle is a map $P : \mathcal{E} \times \mathbb{F}(\mathcal{E}) \to \mathbb{R}^2$, where $\mathbb{F}(\mathcal{E})$ is the set of finite subsets of $\mathcal{E}$, assigning a velocity to the service vehicle as a function of the current state of the system: $\dot{p}(t) = P(p(t), \mathcal{Q}(t))$. Let $m_{\text{cap}}(t)$ be the number of targets that have appeared and have been captured before time t and $m_{\text{miss}}(t)$ be the number of targets that have escaped and $m_{\text{tot}}(t) = m_{\text{miss}}(t) + m_{\text{cap}}(t)$, then the goal of this problem can be stated as follows:

Problem Statement. Find policies P that maximize the fraction of targets that are serviced $\mathbb{F}_{\text{cap}}(P)$, termed as the *capture fraction*. Formally, for a policy P, we define the steady state average capture fraction as

$$\mathbb{F}_{\text{cap}}(P) := \limsup_{t \to +\infty} \mathbb{E}\left[\frac{m_{\text{cap}}(t)}{m_{\text{cap}}(t) + m_{\text{miss}}(t)}\right]$$

where the expectation is with respect to the stochastic process that generates the targets.

Each target has a deadline depending on when and where it appears in the environment. We propose policies for the service vehicle suitable for specific target speeds and arrival rates with provable guarantees on their performance. We first present some preliminary results which will be used to analyze policies for the RET problem.

3. Preliminary results.

We start with reviewing some established results to intercept moving targets in shortest time as well as propose methods to obtain bounds on paths through a set of moving targets.

3.1. Time to capture a single target.

The optimal strategy (i.e., taking minimum time) for a vehicle to capture a target moving at a speed less than that of the vehicle is to move in a straight line with maximum speed to intercept the target based on the constant bearing principle [12]. In the following definition, this result is stated in terms of radial coordinates.

DEFINITION 3.1. (Constant bearing principle) *The time taken by the vehicle starting from $p = (x, 0)$ and moving with unit speed to capture a target located at $q = (r, \theta)$ and moving radially outward with constant speed $v < 1$ is*

$$T(p, q) = \frac{-v(x\cos\theta - r) + (v^2(x\cos\theta - r)^2 - (1 - v^2)(2rx\cos\theta - x^2 - r^2))^{1/2}}{1 - v^2}.$$

The next result gives a relation of the distance between the vehicle and target location to the time required to capture the moving target.

LEMMA 3.2. (Time to capture) *The time $T(p, q)$ required by the vehicle starting from $p = (x, 0)$ and moving with unit speed to capture a target at $q = (r, \theta)$ moving* *radially outward with speed v satisfies the following inequality*

$$T(p,q) \leq \left(\frac{2v}{1-v^2} + \frac{1}{\sqrt{1-v^2}}\right) d(p,q),$$

where $d(p,q) = \sqrt{x^2+r^2-2xr\cos\theta}$ is the Euclidean distance between p and q. If $r \leq x\cos\theta$, then

$$T(p,q) \leq \left(\frac{1}{\sqrt{1-v^2}}\right) d(p,q)$$

Proof. We start with providing an upper bound on the positive root y^+ of a quadratic equation. For the quadratic equation $ay^2+by+c=0$, if $a>0$ and $c<0$, then there are two possibilities: $b \geq 0$ or $b<0$.

$$y^+ = \frac{-b+\sqrt{b^2-4ac}}{2a} = \begin{cases} \dfrac{-b+\sqrt{b^2+4a\,|c|}}{2a} \leq \dfrac{-b+b+2\sqrt{a\,|c|}}{2a} = \sqrt{\dfrac{|c|}{a}}, & b \geq 0, \\ \dfrac{-b+\sqrt{|b|^2+4a\,|c|}}{2a} \leq \dfrac{|b|}{a} + \sqrt{\dfrac{|c|}{a}}, & b<0. \end{cases}$$

Since the time taken $T := T(p,q)$ to capture a target at q starting from p satisfies the following quadratic equation,

$$T^2(1-v^2) + 2vT(x\cos\theta - r) - (x^2+r^2-2xr\cos\theta) = 0,$$

the result follows. ∎

3.2. Optimal placement of vehicle. By optimal placement, we mean the location at which the vehicle should be placed in order for it to have the highest probability of capturing a target. To determine optimal placement, we start by defining the capturable set of a vehicle location.

DEFINITION 3.3. (Capturable set) *A vehicle located at $(x,0)$ and moving with unit speed can only reach targets located in the capturable set*

$$C(x,v,D) := \{(r,\theta) \in \mathcal{E} : r < r_c \ \ \textit{for all}\ \theta \in [0,2\pi)\}$$

using the constant bearing principle, where

$$r_c(x,v,D,\theta) = \max\left(0, D - v\sqrt{D^2+x^2-2xD\cos\theta}\right).$$

These are the locations for which $r+vT \leq D$. The expression for r_c is obtained by setting $r_c+vT=D$. The radial location r_c corresponds to the locations of targets that the vehicle can capture just before they escape the disk. The probability that a target is in the capturable set of a particular vehicle location $(x,0)$ is given by

$$\rho(x,v,D) := \frac{\int_0^{2\pi}\int_0^D \mathbb{P}\left[(r,\theta) \in C(x,v,D)\right] f(r)e(\theta)drd\theta}{\int_0^{2\pi}\int_0^D \mathbb{P}\left[(r,\theta) \in \mathcal{E}\right] f(r)e(\theta)drd\theta} = \frac{\int_0^{2\pi}\int_0^{r_c} f(r)e(\theta)drd\theta}{\pi D^2}.$$

When the vehicle is at location $p^* = (x^*(v,D),0)$ where

$$x^*(v,D) := \arg\max_{0\leq x\leq D} \rho(x,v,D), \tag{3.1}$$

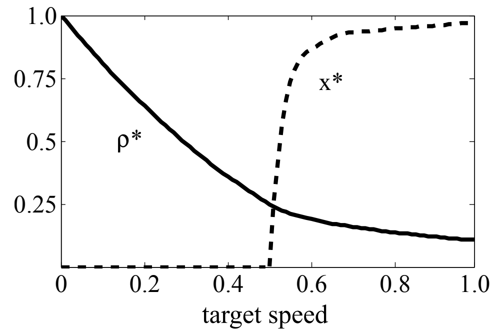

FIG. 2. *Optimal vehicle location x^* and the maximum probability ρ^* of capturing an escaping target starting from $(x^*, 0)$ as a function of target speed v for the RET problem with $D = 1$.*

the probability of it capturing a target is maximum. The vehicle location $x^*(v, D)$ is referred to as the *optimal location.* Let $\rho^*(v, D) := \rho(x^*, v, D)$. Closed form expressions for x^* and ρ^* do not appear to be possible for all $v \in (0, 1)$. However, from numerical calculations it is known that $x^* = 0$ for $v \in (0, 0.5]$ irrespective of the value of the parameter D. The numerically computed variation of $x^*(v, D)$ and $\rho^*(v, D)$ for $D = 1$ is shown in Fig. 2. For target speed $v \leq 0.5$, $x^* = 0$ and the vehicle location $p^* = (0, 0)$ maximizes the probability of the vehicle being able to capture a target before it escapes. For higher speeds, this location is closer to the boundary. There is a qualitative difference between these two cases. For the former case, $p^* = (0, 0)$ is the unique vehicle location which maximizes ρ whereas for the later case, the set of corresponding optimal locations is all points with radial coordinate equal to x^*.

THEOREM 3.4. (Capture fraction upper bound) *For every policy P for the RET(v, λ) problem, $\mathbb{F}_{\text{cap}}(P) \leq \rho^*(v, D)$.*

Proof. Let the vehicle start from x_1 and service target at p_1. The probability of the vehicle capturing this target is maximum when $x_1 = x^*$. The best case scenario is that no new target appears while the vehicle services it and repositions itself at x_2 so as to increase the probability of capturing a new target at p_2. This can be realized for a suitably low value of arrival rate λ. In order to maximize the probability of capturing the new target, $x_2 = x^*$ as well. Thus, to maximize the probability of capturing every new target, the vehicle returns to x^* and waits for a target to appear. With this strategy, the vehicle can still only capture targets which appear within $C(x^*, v, D)$. The fraction of targets which satisfy this criterion is $\rho^*(v, D)$. Thus, the vehicle can capture no more than $\rho^*(v, D)$ fraction of targets. □

3.3. Quantification of targets inside the environment. In this subsection we quantify the number of targets in an unserviced region in the environment. We distinguish between targets *originating* and *accumulating* in a certain region. Targets are said to have accumulated in a region when after appearing, they spend time in the region, in the course of their trajectories.

DEFINITION 3.5. (Annular section) *The annular section $A(a, b, \theta_1, \theta_2) \subset \mathbb{R}^2$ is the set $A(a, b, \theta_1, \theta_2) := \{(r, \theta) | a \leq r \leq b, \theta \in [\theta_1, \theta_2]\}$.*

LEMMA 3.6. (Accumulated targets in an annular section) *For $0 < a < b < D$, let n_A be the number of targets accumulated at steady state in an unserviced annular section $A(a, b, 0, 2\pi)$ and $f_a(x)$ be the distribution of the accumulating targets w.r.t the radial location $x \in [0, D]$. Then,*

(i) $\mathbb{E}[n_A] = (b^3 - a^3)\lambda/3vD^2$,

(ii) $\mathrm{Var}[n_A] = (b^3 - a^3)\lambda/3vD^2$ *and*

(iii) $f_a(x) = \lambda x^2/vD^2$.

Proof. Firstly, steady state is assumed, meaning that the initial transient has already passed, hence the time at which the snapshot is taken is $t \geq D/v$. Also, by unserviced, we mean that the vehicle has not serviced targets in the region under consideration for at least time D/v before the time instant under consideration. Let us examine the number of targets accumulating in the annulus $A_r := A(r, r + \Delta r, 0, 2\pi)$ due to targets appearing in the annulus $R_1 := A(p_1, p_1 + \Delta p_1, 0, 2\pi)$. Let us also assume that Δr and Δp are infinitesimal. The intensity of the Poisson process on R_1 is directly proportional to its area and is equal to $2\pi p_1 \Delta p_1 \lambda/\pi D^2 = 2p_1\Delta p_1\lambda/D^2$.

$$\begin{aligned}
&\mathbb{P}[A_r \text{ contains } n \text{ targets originating from } R_1] \\
&\qquad = \mathbb{P}\left[n \text{ targets originated from } R_1 \text{ in time interval } \left[t, t + \frac{\Delta r}{v}\right]\right] \\
&\qquad = \mathbb{P}\left[n \text{ targets originated from } R_1 \text{ in time interval of length} \frac{\Delta r}{v}\right] \\
&\qquad = \exp\left(-\frac{2p_1\Delta p_1 \lambda \Delta r}{D^2 v}\right) \frac{\left(\frac{2p_1\Delta p_1\lambda\Delta r}{D^2 v}\right)^n}{n!},
\end{aligned}$$

where $t = (r - p_1)/v$. Thus, the process of targets accumulating in A_r due to targets originating in R_1 is spatially Poisson with intensity $\mathrm{area}(R_1)/(\pi D^2)\lambda/v = 2p_1\Delta p_1\lambda/D^2 v$.

Next, let us examine the process of accumulation of targets in A_r due to two annuli $R_1 := A(p_1, p_1 + \Delta p_1, 0, 2\pi)$ and $R_2 := A(p_2, p_2 + \Delta p_2, 0, 2\pi)$.

$$\begin{aligned}
\mathbb{P}[A_r \text{ contains } n \text{ targets from } R_1 \cup R_2] &= \sum_{i=0}^{n} \left[\exp\left(\frac{-2p_1\Delta p_1\lambda\Delta r}{D^2 v}\right) \frac{\left(\frac{2p_1\Delta p_1\lambda\Delta r}{D^2 v}\right)^i}{i!}\right. \\
&\quad \left.\times \exp\left(\frac{-2p_2\Delta p_2\lambda\Delta r}{D^2 v}\right) \frac{\left(\frac{2p_2\Delta p_2\lambda\Delta r}{D^2 v}\right)^{(n-i)}}{(n-i)!}\right] \\
&= \exp\left(\frac{-2(p_1\Delta p_1 + p_2\Delta p_2)\lambda\Delta r}{D^2 v}\right) \\
&\quad \times \frac{\left(\frac{2(p_1\Delta p_1 + p_2\Delta p_2)\lambda\Delta r}{D^2 v}\right)^n}{n!}
\end{aligned} \tag{3.2}$$

Thus the process of targets accumulating in A_r due to targets originating in $R_1 \cup R_2$ is also spatially Poisson and the intensity of this process, given by $(2p_1\Delta p_1 + 2p_2\Delta p_2)\lambda/D^2 v = (\mathrm{area}(R_1) + \mathrm{area}(R_2))/(\pi D^2)\lambda/v$, is the sum of the intensities due to R_1 and R_2. This can be extended to all the rings of radii $p \in [0, r]$. So arrival process of all targets accumulating in A_r is also spatially Poisson and has intensity $\mathrm{area}(A(0, r, 0, 2\pi))/(\pi D^2)\lambda/v = (r^2\lambda/vD^2)$. Thus the expected number as well as the variance of targets accumulating in the unserviced annulus A_r is $r^2\lambda\Delta r/vD^2$.

Next, consider an annular section $A(a, b, 0, 2\pi)$. Since Poisson processes are additive, the arrival process of targets accumulating in $A(a, b, 0, 2\pi)$ is Poisson and is

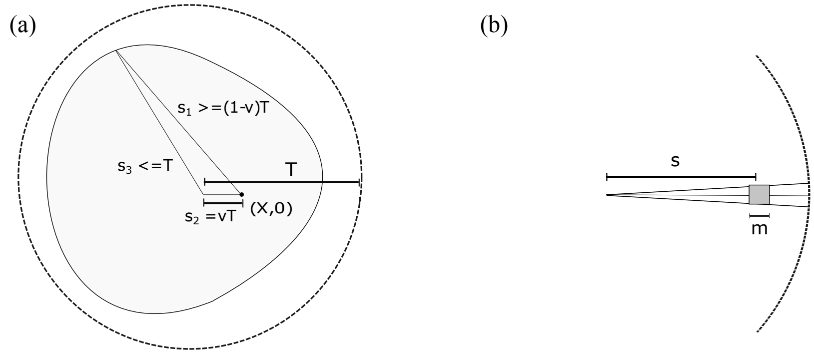

FIG. 3. *(a) The set S_T for the RET problem is shown by the gray shaded region. The dashed circle is the boundary of $\bar{S_T}$ which is a circle of radius T centered at $(X - vT, 0)$. (b) The area element ζ of length and width m in $\bar{S_T}$.*

the sum of processes of targets accumulating in disjoint annuli like A_r with $r \in [a, b]$. Hence the expected number and variance of targets accumulating in $A(a, b, 0, 2\pi)$ is $\int_a^b (r^2\lambda/vD^2)dr = (b^3 - a^3)\lambda/3vD^2$. Let $f_a(x)$ be the distribution of the number of accumulating targets w.r.t the radial location x. Since $\int_0^s f_a(x)dx = s^3\lambda/3vD^2$, we get $f_a(x) = \lambda x^2/vD^2$.
□

LEMMA 3.7. (Travel time bound for RET problem) *Let targets arrive uniformly in $\mathcal{E}$ according to a Poisson process of rate λ and move radially outward with speed v. Let Q be the set of targets accumulated in $\mathcal{E}$ at time t and T_d be the random variable giving the minimum amount of time required to travel to a target in Q starting from vehicle position $(X, 0)$. Then,*

$$\mathbb{E}[T_d] \geq \sqrt{\frac{\pi v D}{2\lambda}}.$$

Proof. To get a bound on the travel time, we start with defining a set S_T shown in Fig. 3(a), such that any target in it can be reached from the vehicle position $(X, 0)$ in T time units or less. Mathematically,

$$S_T := \left\{(r, \theta) \in \mathcal{E} | X^2 + (r + vT)^2 - 2X(r + vT)\cos(\theta) \leq T^2\right\}.$$

Also, let $\bar{S_T} := \left\{(r, \theta) \in \mathcal{E} | (X - vT - r\cos\theta)^2 + (r\sin\theta)^2 \leq T^2\right\}$. Since the relative velocity of any target with respect to the vehicle is more than or equal to $(1 - v)$, the distance s_1 of any point on the boundary of S_T from $(X, 0)$ is greater than or equal to $T(1 - v)$. Using the triangle inequality, the distance s_2 of that point from $(X - vT, 0)$ is less than or equal to T. Then, $S_T \subseteq \bar{S_T}$.

If T_d is the random variable giving the minimum amount of time to go from vehicle location $(X, 0)$ to a target, then $T_d > T$ if S_T is empty and $\mathbb{P}[T_d > T] = \mathbb{P}[|S_T| = 0]$. Here, the notation $|S_T|$ is used to denote the number of outstanding targets in the set S_T. Further,

$$\mathbb{P}[|\bar{S}_T| = 0] = \mathbb{P}[|S_T| = 0]\mathbb{P}[|\bar{S}_T \backslash S_T| = 0] \leq \mathbb{P}[|S_T| = 0]. \tag{3.3}$$

We now calculate the probability that an infinitesimal area element ζ of length m and width m centered at $(s, 0)$ shown in Fig. 3(b) is empty:

$$\begin{aligned}\mathbb{P}[|\zeta| = 0] &= \exp\left(-\lambda \frac{m}{v} \frac{1}{\pi D^2} \int_0^s r\theta dr\right) = \exp\left(-\lambda \frac{m}{v} \frac{1}{\pi D^2} \int_0^s r \frac{m}{s} dr\right) \\ &= \exp\left(\frac{-m^2}{v} \frac{\lambda s}{2\pi D^2}\right) \geq \exp\left(\frac{-m^2 \lambda}{2\pi v D}\right) = \exp\left(\frac{-\lambda}{2\pi v D} \operatorname{area}(\zeta)\right), \qquad (3.4)\end{aligned}$$

where the inequality follows from the fact that $s \in [0, D]$, and the exponential function has a minimum at $s = D$. The last equality is true since $\operatorname{area}(\zeta) = m^2$. Further, every compact set can be written as a countable union of non-overlapping rectangles. Thus, Eq. (3.4) holds for the compact measurable set $\bar{S}_T$ as well. Then, using the results from Eq. (3.3) and Eq. (3.4),

$$\mathbb{P}[|S_T| = 0] \geq \mathbb{P}[|\bar{S}_T| = 0] \geq \exp\left(\frac{-\lambda}{2\pi v D} \operatorname{area}(\bar{S}_T)\right) = \exp\left(\frac{-\lambda}{2\pi v D} \pi T^2\right),$$

and the expectation of T_d can be bounded as follows:

$$\begin{aligned}\mathbb{E}[T_d] = \int_0^{+\infty} \mathbb{P}[T_d > T] dT &= \int_0^{+\infty} \mathbb{P}[|S_T| = 0] \geq \int_0^{+\infty} \mathbb{P}[|\bar{S}_T| = 0] dT \\ &\geq \int_0^{+\infty} \exp\left(\frac{-T^2 \lambda}{2vD}\right) dT \geq \frac{\sqrt{\pi}}{2} \sqrt{\frac{2vD}{\lambda}} = \sqrt{\frac{\pi v D}{2\lambda}},\end{aligned}$$

so the result is obtained. ∎

THEOREM 3.8. (Policy Independent Upper Bound on Service Fraction) *An upper bound on the service fraction of any policy P for the RET problem satisfies*

$$\mathbb{F}_{\text{cap}}(P) \leq \sqrt{\frac{2}{\pi v \lambda D}}.$$

Proof. This follows from the fact that in order to service a fraction $c \in (0, 1]$ of targets, we require that the rate at which targets are serviced is more than the rate at which they arrive [13], i.e., $c\lambda \mathbb{E}[T] \leq 1$. Since $T > T_d$, the result now follows by using Lemma 3.7. ∎

3.4. Bounds on paths and tours through static and escaping targets. To distinguish static targets from moving targets, we introduce some terminology. A target moving radially outward is referred to as an escaping target. A target is said to have been 'captured' by the vehicle if the vehicle reaches the target before it escapes the environment. The following results are used to estimate and bound the length of the path through targets in the environment.

THEOREM 3.9. (Upper bound on path through escaping targets) *Let targets starting from (r_i, θ_i), $i \in \{1, \dots, N\}$ move radially outward with speed v. Let T be the length of the path through these escaping targets in some arbitrary order $\delta : \{1, \dots, N\} \to \{1, \dots, N\}$. Let T_s be the length of the path through static targets located at $(r_i + v\bar{T}, \theta_i)$, $i \in \{1, \dots, N\}$ processed in order δ and $\bar{T} \geq T$. Then,*

$$T \leq \frac{T_s}{1 - v}.$$

Proof. Without loss of generality, let the targets be labeled in the order in which they are processed. Let the vehicle take time T_j to service the $j-$th escaping target having serviced the $(j-1)-$th escaping target. Consider the $i-$th escaping target starting from (r_i, θ_i). The vehicle services this target at time $\sum_{j=1}^{i} T_j$. It then starts for the escaping target $i+1$ and reaches it in time T_{i+1}. Let $T^{'}_{i+1}$ be the distance between $(r_i + v\sum_{j=1}^{i+1} T_j, \theta_i)$ and $(r_{i+1} + v\sum_{j=1}^{i+1} T_j, \theta_{i+1})$. Also, let $T^{''}_{i+1}$ be the distance between $(r_i + vT, \theta_i)$ and $(r_{i+1} + vT, \theta_{i+1})$ while $T_{s,i+1}$ is the distance between $(r_i + v\bar{T}, \theta_i)$ and $(r_{i+1} + v\bar{T}, \theta_{i+1})$. Since the distance between two targets moving radially outward with the same speed is a non-decreasing function of time, $T^{'}_{i+1} \leq T^{''}_{i+1} \leq T_{s,i+1}$. Referring to Fig. 4, from the triangle inequality, $T^{'}_{i+1} + vT_{i+1} \geq T_{i+1}$, i.e., $T_{i+1} \leq (T^{'}_{i+1})/(1-v) \leq (T_{s,i+1})(1-v)$. Extending this to all the targets in the path,

$$T = \sum_{i=1}^{n} T_{i+1} \leq \sum_{i=1}^{n} \frac{T_{s,i+1}}{1-v} = \frac{T_s}{1-v}.$$

□

The upper bound on the length of the path through escaping targets can thus be expressed in terms of the length of the path through their static locations *in the future*.

THEOREM 3.10. (Lower bound on path through escaping targets) *Let targets starting from (r_i, θ_i), $i \in \{1, \ldots, N\}$ move radially outward with speed v. Let T be the length of the path through these escaping targets in some arbitrary order δ : $\{1, \ldots, N\} \to \{1, \ldots, N\}$. Let T_0 be the length of the path through static targets located at (r_i, θ_i), $i \in \{1, \ldots, N\}$ processed in order δ. Then,*

$$T \geq \frac{T_0}{1+v}.$$

Proof. The proof is similar to that of Theorem. 3.9 and is omitted for brevity. □

Given a set K of n points, the Euclidean traveling salesperson problem (ETSP) is to determine the shortest tour, i.e., a closed path that visits each point exactly once. We now state the classic result providing a limit for the length of the ETSP through large number of points. We will leverage this result in the next section to derive tighter analytic bounds on the performance of our policies in regimes of high target arrival rates.

THEOREM 3.11. (Asymptotic ETSP length,[20]) *If a set K of n points is distributed independently and identically in a compact set Q, then there exists a constant β such that*

$$\lim_{n \to +\infty} \frac{ETSP(K)}{\sqrt{n}} = \beta \int_Q \varphi(q)^{1/2} dq,$$

where φ is the density of the absolutely continuous part of the point distribution. The constant β has been estimated numerically as $\beta \approx 0.7120 \pm 0.0002$ [17].

4. Policies. In this section, we propose three policies for the RET problem. The SAC policy is designed for low target arrival rates while the SW policy is designed for moderate target arrival rates. Finally, the SNB policy is proposed for high arrival rates.

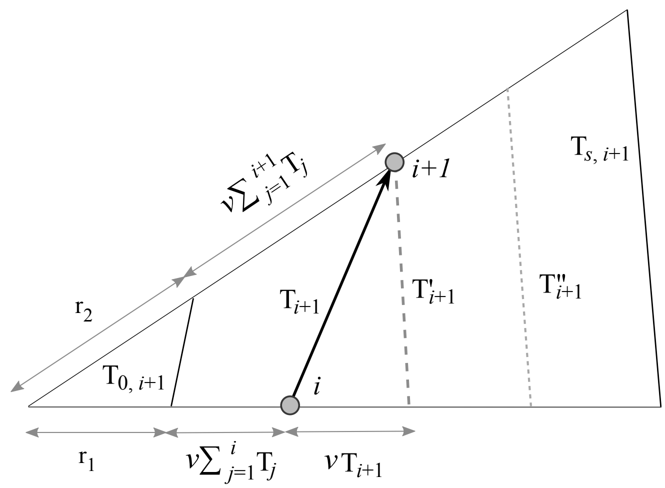

FIG. 4. *The thick line labeled T_{i+1} indicates the trajectory of the vehicle starting from the target i to service the target $i+1$. The gray circles indicate the locations at which the vehicle intercepts the targets.*

4.1. Stay at Center (SAC) Policy. According to this policy, the vehicle stays at the optimal location in the disk and waits for new targets to appear in its capturable set. For $v \in [0, 0.5]$, this location is the center. The SAC policy is suitable for low target arrival rates at which the optimal vehicle location takes prominence.

Given a vehicle location $(x, 0) \in \mathcal{E}$, recall that $C(x, v, D)$ denotes the capturable set for the vehicle. Let x^* and ρ^* be defined as per Eq. (3.1). The formal description of the SAC policy is given in Algorithm 1.

Algorithm 1: Stay At Center (SAC) policy

Assumes: v, D known and the vehicle placed at $(x^*, 0)$.
1 Intercept a target that appears inside $C(x^*, v, D)$;
2 Return back to $(x^*, 0)$;
3 Repeat from step 1.

Algorithm 1 has the following guarantee on capture fraction.

THEOREM 4.1. (SAC Policy Capture Fraction) *The capture fraction of the SAC policy satisfies*

$$\mathbb{F}_{\text{cap}}(\text{SAC}) \geq \frac{\rho^*(v, D)}{2\rho^*(v, D)\lambda D + 1}.$$

If $v \in [0, 0.5]$ so that the optimal vehicle location $x^* = 0$, then the above fraction becomes equal to

$$\mathbb{F}_{\text{cap}}(\text{SAC}) \geq \frac{(1 - v)^2}{2\lambda(1 - v)^2 D + 1}.$$

Proof. See Appendix A. ▯

REMARK 4.2. (Optimality in light load, i.e., $\lambda \to 0^+$) *In the light load regime of $\lambda \to 0^+$, the capture fraction achieved equals $\rho^*(v, D)$, which is exactly equal to the probability that a target falls within the capturable set C when the vehicle is located at the optimal location $(x^*, 0)$. Comparing with Theorem 3.4, we see that the SAC policy is optimal in this limiting regime.*

4.2. Sector-wise (SW) Policy. In the Sector-wise policy, the vehicle stays closer to the boundary and utilizes the high relative velocity of the outgoing targets. It starts every iteration at a radial location X and services the first target with the smallest counterclockwise angular separation in a specific subset associated with the iteration. One such subset J_1 which the vehicle encounters in the first iteration is shown by the shaded region in Fig. 5(a). It then proceeds to the nearest location with radial coordinate X in the disk and waits for a specified time to begin its next iteration. The formal description of the policy is given in Algorithm 2. While the SW policy is applicable in all parameter regimes of the RET problem, for a fixed speed v, it is constant factor optimal in moderate arrival regimes as established in Theorem 4.4.

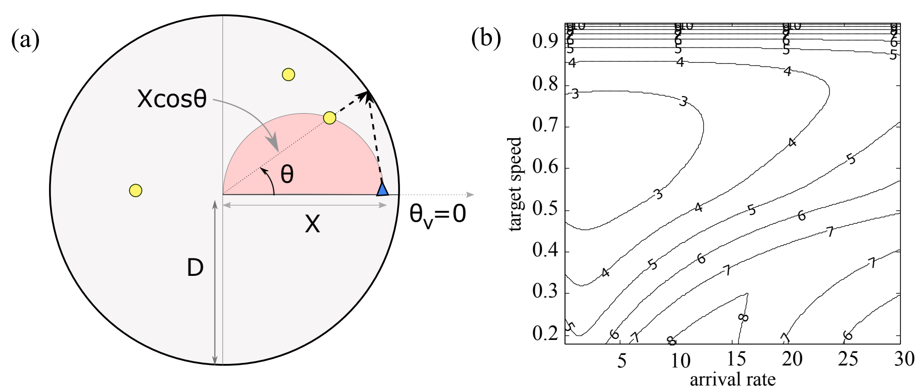

FIG. 5. *(a) In the first iteration, the vehicle located at $p = (X, 0)$ services outstanding targets in the set J_1 shown in the shaded region. (b) Factor of optimality of the SW policy in different parameter regimes v, λ when $D = 1$.*

Algorithm 2 has the following guarantee on capture fraction.

LEMMA 4.3. (SW Policy Capture Fraction) *The capture fraction of the SW policy satisfies*

$$\mathbb{F}_{\text{cap}}(\text{SW}) \geq \frac{1}{\lambda}\left(W + \frac{\pi D}{4}(3\eta_1(k) + \eta_2(k)) + \frac{8}{\lambda(1-v^2)}\eta_3(k)\right)^{-1},$$

where

$$\begin{aligned}
\eta_1(k) &= \left(L_{-1}(8k) - I_1(8k) - L_{-1}(20k) + I_1(20k)\right), \\
\eta_2(k) &= \left(I_0(8k) - L_0(8k) - I_0(20k) + L_0(20k)\right), \\
\eta_3(k) &= 1 - \pi/2\left(I_0(12k) - L_0(12k) - I_0(20k) + L_0(20k)\right),
\end{aligned} \tag{4.1}$$

$W = \max(0, D\sqrt{1-v^2}(1/4v - \sqrt{2}))$, $k = \lambda D(1-v^2)^{3/2}/72\pi v$, *and I_0 and I_1 are modified Bessel functions of the first kind and L_0 and L_{-1} are modified Struve functions [1].*

Algorithm 2: Sector-wise (SW) policy

```
   Assumes: v, D known and the vehicle placed at (X, 0).
 1 Set X = D√(1 − v²), W = max(0, X(1/4v − √2));
 2 repeat
 3     if there are targets in ℰ with counterclockwise angular separation θ < π/2
       such that their radial coordinate r satisfies r ≤ X cos θ then
 4         Service the target with smallest angular separation and move to
           nearest location in ℰ with radial coordinate X;
 5         Wait for time W and return to step 3.
 6     else
 7         Stay at current location.
 8     end
 9 until all targets are serviced or have escaped;
```

Proof. See Appendix B □

THEOREM 4.4. (Performance in moderate arrival rates) *For* $\lambda > \frac{7\pi v}{(1-v^2)^{3/2}D}$ *and* $v \in (1/4\sqrt{2}, 1)$*, the capture fraction of the SW policy satisfies*

$$\mathbb{F}_{\text{cap}}(\text{SW}) \geq \alpha(v)\sqrt{\frac{2}{\pi v \lambda D}},$$

where

$$\alpha(v) = \frac{\sqrt{v}}{\pi^2}\left(\left(\frac{v}{(1-v^2)^{3/2}}\right)^{1/2} + \frac{10}{3}\frac{(1-v^2)^{1/2}}{v}\right)^{-1/2}. \tag{4.2}$$

Proof. See Appendix C □

Thus, for moderate arrival rates and $v \in (1/4\sqrt{2}, 1)$, using the result from Theorem 4.4 and the fundamental bound obtained in Theorem 3.8, the SW policy is also a constant factor policy with the factor equal to $1/\alpha(v)$.

4.3. Stay-Near-Boundary (SNB) Policy. We now introduce the SNB policy for the high arrival regime. In this regime, the density of targets accumulating close to the boundary of the disk is high. Hence, the distance traversed by the vehicle (and the time taken) between capturing consecutive targets is small. Consequently, the distance by which the targets move between consecutive captures is also small. Hence, the vehicle can plan ahead and capture multiple targets in a single iteration. To determine the order of captures, it uses the solution to the Euclidean Minimum Hamiltonian Path (EMHP) problem which can be stated as follows:

> Given a set of n (stationary) points, determine the length of the shortest path which visits each point exactly once.

The SNB policy makes use of three parameters g, h and n_{cap}. At the beginning of every iteration, the vehicle computes an EMHP through the locations that the targets accumulated in $A(g, h, 0, 2\pi)$ will have after time $(D - h)$. This is done to levarage the result from Theorem 3.9. It uses the order obtained from the EMPH to service the first n_{cap} targets using constant bearing principle. A formal statement of the SNB policy is given in Algorithm 3.

The parameters g, h and n_{cap} are chosen in a way which ensures that the vehicle will service all the n_{cap} targets accumulated in $A(g, h, 0, 2\pi)$ at the beginning of the iteration before the last target escapes the environment. This can be achieved in the following way:

1. parameters g and h are solutions to variables a and b respectively in the following Optimization Problem:

$$\max_{a,b} \left(\frac{b^3 - a^3}{b^2 - a^2}\right)$$

subject to

$$\begin{aligned}
&\mu_A = \frac{\lambda(b^3 - a^3)}{3vD^2},\\
&\frac{\beta}{1-v}\sqrt{\frac{6\pi}{b^3 - a^3}}\left(\frac{b^2 - a^2}{2}\right)\sqrt{\mu_A(1+v)} \leq \frac{D-b}{v},\\
&\frac{\beta}{1+v}\sqrt{\frac{6\pi}{b^3 - a^3}}\left(\frac{b^2 - a^2}{2}\right)\sqrt{\mu_A(1-v)} \geq \frac{b-a}{v},\\
&0 \leq a < b < D.
\end{aligned}$$

2. parameter n_{cap} is set as follows:

$$n_{\text{cap}} := \frac{\lambda(1-v)}{3vD^2}(h^3 - g^3).$$

Algorithm 3: Stay-Near-Boundary (SNB) policy

Assumes: g, h and n_{cap} are known and the vehicle is at $(h, 0)$.

```
if A(g, h, 0, 2π) contains outstanding targets then
    s_1 := set of locations of outstanding targets in A(g, h, 0, 2π);
    s_2 := set of their locations if they move radially outward by distance (D − h);
    Ψ := order of the EMHP starting from (D, 0), visiting targets in s_2 and ending at (D, 0);
    service the first n_cap targets in s_1 in order from Ψ using constant bearing principle and return to (h, 0).
end
```

Algorithm 3 has the following guarantee on the capture fraction of the RET problem.

THEOREM 4.5. (SNB Policy Capture Fraction) *For any fixed $v \in (0, 1)$, in the limit as $\lambda \to +\infty$, the capture fraction of the SNB policy satisfies*

$$\mathbb{F}_{\text{cap}}(\text{SNB}) \geq \frac{2p}{7\beta}\sqrt{\frac{2}{\pi\lambda v D}}$$

with probability one, where

$$p := p(D) = \begin{cases} 1, & D > 1,\\ \dfrac{5\sqrt{D}}{6}, & \textit{otherwise}. \end{cases}$$

Proof. See Appendix D ∎

COROLLARY 4.6. (Performance of the SNB policy) *In the limit as $\lambda \to +\infty$ such that $\lambda > (1+v)^2/2\pi\beta^2 v(1-v)$, the SNB policy is within a factor $7\beta/2p$ of the optimal. For $D \geq 1$, this factor is ≈ 2.52.*

5. Simulations. The numerical performance of the SAC and SW policies for arrival rates of $\lambda = 2$ and $\lambda = 10$ respectively and all target speeds is shown in Fig. 6. The parameter $D = 1$ for these simulations. The mean of the capture fraction based on 1000 simulations is shown along with its standard deviation. It agrees well with the theoretical lower bounds. The theoretical bounds are still conservative. For the SAC policy, the conservativeness comes from the application of Jensen's inequality in Eq. (A.1). For the SW policy, the conservativeness of the bound is because of inequalities introduced in Eq. (B.1),(B.3) to bound integrals.

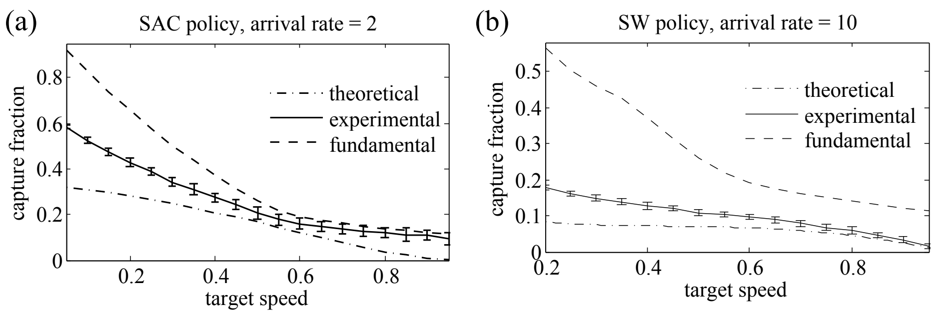

FIG. 6. *Performance of the (a) SAC and (b) SW policies for arrival rates $\lambda = 2$ and $\lambda = 10$ respectively for the RET problem with $D = 1$. The theoretical bounds are from Theorem 4.1 and Theorem 4.4 respectively.*

6. Conclusion and Future Directions. We have introduced a novel vehicle routing problem termed the RET problem in which targets move radially outward in a disk with the intention of escaping it quickly. We have established two policy independent upper bounds on the performance of any algorithm for the RET problem. We have also proposed three policies for different parameter regimes of the RET problem. In Table 1, we have summarized the lower bounds on the capture fraction achieved by these policies as well as their factor of optimality. The SAC policy is optimal for $\lambda \to 0^+$ while for moderate arrival rates, for a fixed target speed, the SW policy is within a constant factor of the optimal. The SNB policy is within a constant factor of the optimal for $\lambda \to +\infty$. When the disk radius is greater than or equal to one, this factor is equal to 2.52.

The current problem setup can be extended in various ways. We assume that the vehicle needs to intercept the target exactly in order to capture it. An interesting and realistically motivated modification of the problem is when the vehicle has a small capture radius. The SAC policy may be extended and applied relatively easily to that setup. On the other hand, the other policies would require extensive computation due to the history dependence which would be introduced because of the capture radius model. In the current setup, the targets move radially outward with the intention of escaping the environment in minimum time. Another modification of the setup is the case in which the targets modify their trajectories in order to evade the pursuing vehicle.

A variation of the RET problem is also the scenario in which the targets are moving radially inward towards an inner boundary instead of moving radially outward and the vehicle has to stop the targets from reaching the inner boundary. Generalizations of the RET problem, like for instance, when the distribution of targets in the environment is not uniform, or when the environment is an arbitrary closed curve and the targets have arbitrary velocities are also open to exploration.

Appendix. Proofs of theorems. In this section we present the proofs of the performance guarantees of the three policies for the RET problem.

Appendix A. Proof of Theorem 4.1. Notice that if $m_{\mathrm{cap}}(t) > 0$ for some $t > 0$, then

$$\begin{aligned}\limsup_{t\to+\infty}\mathbb{E}\left[\tfrac{m_{\mathrm{cap}}(t)}{m_{\mathrm{cap}}(t)+m_{\mathrm{miss}}(t)}\right] &= \limsup_{t\to+\infty}\mathbb{E}\left[\tfrac{1}{1+\frac{m_{\mathrm{miss}}(t)}{m_{\mathrm{cap}}(t)}}\right]\\ &\geq \left(1+\limsup_{t\to+\infty}\mathbb{E}\left[\tfrac{m_{\mathrm{miss}}(t)}{m_{\mathrm{cap}}(t)}\right]\right)^{-1},\end{aligned} \tag{A.1}$$

where the last step comes from an application of Jensen's inequality [7]. Thus, we can determine a lower bound on the capture fraction by studying the number of targets that escape per captured target. Consider a tagged target i which falls within $C(x^*, v, D)$. The time t_i taken by vehicle to intercept target i and return to the optimal location satisfies $t_i \leq 2D$. Therefore, the number of targets that escape because the vehicle intercepts the i-th target is equal to the sum of 1) the number of targets that arrive anywhere in the environment during the time interval of t_i and 2) the number of targets that are generated outside of $C(x^*, v, D)$ while the vehicle is waiting for the next capturable target. Since the target arrival process is a Poisson process, the expected number of targets in case 1 are given by $\lambda t_i \leq 2\lambda D$. The spatial distribution of the targets is uniform random. Further, $\mathrm{area}(C(x^*, v, D)) = \rho^*(v, D)\pi D^2$. Therefore, the targets missed in case 2, denoted by N_{miss} is a random variable distributed as follows.

$$N_{\mathrm{miss}} = \begin{cases} 0, \text{ with probability } \rho^*(v,D), \\ 1, \text{ with probability } \rho^*(v,D)\Big(1-\rho^*(v,D)\Big), \\ 2, \text{ with probability } \rho^*(v,D)\Big(1-\rho^*(v,D)\Big)^2, \\ \vdots \\ k, \text{ with probability } \rho^*(v,D)\Big(1-\rho^*(v,D)\Big)^k, \\ \vdots \end{cases}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[N_{\mathrm{miss}}\right] &= \sum_{k=1}^{\infty} k\rho^*(v,D)\Big(1-\rho^*(v,D)\Big)^k = \rho^*(v,D)\sum_{k=1}^{\infty}\Big(1-\rho^*(v,D)\Big)^k\\ &= \rho^*(v,D)\frac{1-\rho^*(v,D)}{(\rho^*(v,D))^2} = \frac{1}{\rho^*(v,D)} - 1.\end{aligned}$$

Substituting the upper bound for case 1 and the expression for case 2 in (A.1), we obtain

$$\mathbb{F}_{\text{cap}}(\text{SAC}) \geq \frac{1}{2\lambda D + \frac{1}{\rho^*(v,D)}} = \frac{\rho^*(v,D)}{2\rho^*(v,D)\lambda D + 1}.$$

Appendix B. Proof of Theorem 4.3. In the sector-wise policy, the vehicle starts every iteration at a distance $X = D\sqrt{1-v^2}$ from the center. If θ_v is the angular position of the vehicle in its i-th iteration and

$$J_i := \{(r,\theta) \mid 0 \leq r \leq X\cos(\theta-\theta_v), \theta-\theta_v \in [0,\pi/2]\},$$

then if there is an outstanding target in J_i, the vehicle services the target in J_i with the smallest angular separation from θ_v in the counterclockwise direction. The choice of X ensures that the vehicle always services any target in J_i before it escapes the disk.

We now calculate the expectation of the time required for a single iteration of the SW policy. Without loss of generality we assume that $i=1$, $\theta_v=0$ initially and the vehicle is at $(X,0)$. We also assume that the environment is unserviced. Let $K(\gamma_1,\gamma_2) := \{(r,\phi) | 0 \leq r \leq X\cos\phi, \phi \in [\gamma_1,\gamma_2]\}$ for $\gamma_1 < \gamma_2$. Also, for infinitesimal $\delta\theta$, let $\theta^+ = \theta + \delta\theta$. Then the probability of the first outstanding target in J_1 being at an angular location θ, i.e.

$$\begin{aligned}\mathbb{P}[\text{first target is in } K(\theta,\theta^+) |\ J_1 \text{ is not empty}] &= \mathbb{P}\left[|K(0,\theta)|=0\right]\mathbb{P}\left[|K(0,\theta^+)| \neq 0\right] \\ &= \exp\left(-k\left(9\sin\theta + \sin 3\theta\right)\right) \\ &\quad \times \left(1 - \exp\left(-k\left(9\sin\theta^+ + \sin 3\theta^+\right)\right)\right) \\ &\geq \exp(-8k\sin\theta)\left(1-\exp(-12k\sin\theta)\right), \end{aligned} \tag{B.1}$$

where $k = \frac{\lambda X^3}{72\pi v D^2}$. A number of results are used to obtain Eq. (B.1). The first result, which is derived in the same spirit as Eq. (3.4), is that for $\alpha \in [0,2\pi]$,

$$\mathbb{P}\left[|K(0,\alpha)|=0\right] = \exp\left(-k\left(9\sin\alpha + \sin 3\alpha\right)\right). \tag{B.2}$$

The second supporting result is the following empirically obtained inequality for $\alpha \in [0,\pi/2]$: $12\sin\alpha \geq 9\sin\alpha + \sin 3\alpha \geq 8\sin\alpha$.

Now that we have an expectation of the first target being at an angular location θ relative to the vehicle, we calculate the time taken to capture this target. Let T_θ be the random variable denoting the time taken to start from $(X,0)$, service a target at (r,θ) and go to (X,θ) to start the next iteration. We determine a bound on the expectation of T_θ. Once again, we assume that the environment is unserviced and note that the probability distribution of the outstanding targets is given by $f_a(r) = \lambda r^2/vD^2$ as obtained in Lemma 3.6. Since $r \leq X\cos\theta$, we use Lemma 3.2 to obtain a lower bound on the expectation of T_θ:

$$\mathbb{E}[T_\theta] \leq \frac{2\displaystyle\int_{r=0}^{X\cos\theta} \left(\frac{\lambda r^2}{vD^2}\right)\left(\frac{\sqrt{X^2+r^2-2Xr\cos\theta}}{\sqrt{1-v^2}}\right) dr}{\displaystyle\int_{r=0}^{X\cos\theta} \left(\frac{\lambda r^2}{vD^2}\right) dr}$$

$$= \frac{2\displaystyle\int_{s=0}^{\cos\theta} X\left(\frac{\lambda s^2}{vD^2}\right)\left(\frac{\sqrt{1+s^2-2s\cos\theta}}{\sqrt{1-v^2}}\right) ds}{\displaystyle\int_{s=0}^{\cos\theta} \left(\frac{\lambda s^2}{vD^2}\right) ds} \tag{B.3}$$

$$\leq \frac{6X}{\sqrt{1-v^2}}\left(\frac{\sin\theta}{4}+\frac{1}{12}\right) =: \Gamma(\theta). \tag{B.4}$$

The factor of two is required since the vehicle has to go to (X, θ) to start the next iteration and the time required for this is always less than or equal to the time required to service the target at (r, θ) starting from $(X, 0)$. If T is the random variable denoting the time required to start from $(X, 0)$, service the first target in J_1 and return to the radial location X, then using the result from Eq. (B.1),

$$\begin{aligned}
\mathbb{E}[T|J_1 \text{ is not empty}] &= \int_{\theta=0}^{\pi/2} \mathbb{E}[T_\theta]\ \mathbb{P}[\text{ first target in } K(\theta,\theta^+)]d\theta \\
&\leq \int_{\theta=0}^{\pi/2} \Gamma(\theta)\left(\exp(-8k\sin\theta)\left(1-\exp(-12k\sin\theta)\right)\right)d\theta \\
&= \left(\frac{6X}{\sqrt{1-v^2}}\right)\left(\frac{\pi}{8}\eta_1(k)+\frac{\pi}{24}\eta_2(k)\right). && \text{(B.5)}
\end{aligned}$$

(B.6)

where the functions η_1 and η_2 are as defined in Eq. (4.1). Further, using the result from Eq. (B.2),

$$\begin{aligned}
\mathbb{P}[\ J_1 \text{ is empty }] \leq& 1 - \int_0^{\pi/2} \exp(-12k\sin\theta)\left(1-\exp(-8k\sin\theta)\right)\ d\theta \\
=& 1 - \frac{\pi}{2}\left(I_0(12k) - L_0(12k) - I_0(20k) + L_0(20k)\right) \\
=& \eta_3(k), && \text{(B.7)}
\end{aligned}$$

and the expected time that the vehicle has to wait for a new target to appear in J_1 is less than $8/\lambda(1-v^2)$ since the area of J_1 is $(1-v^2)/8$ times the area of the disk. This is in addition to the time W that the vehicle waits at the beginning of the iteration. So, using Eq. (B.5) and Eq. (B.7), the time T taken to finish a single iteration of the SW policy has the following expectation:

$$
\begin{aligned}
\mathbb{E}[T] =& W + \mathbb{E}[T|\ J_1 \text{ is not empty }] + \mathbb{E}[T|\ J_1 \text{ is empty }] \\
=& W + \mathbb{E}[T|\ J_1 \text{ is not empty }] + \frac{8}{\lambda(1-v^2)}\mathbb{P}[\ J_1 \text{ is empty }] \\
\leq& W + \left(\frac{6X}{\sqrt{1-v^2}}\right)\left(\frac{\pi}{8}\eta_1(k) + \frac{\pi}{24}\eta_2(k)\right) + \left(\frac{8}{\lambda(1-v^2)}\right)\eta_3(k) \\
=& W + \frac{3\pi D}{4}\eta_1(k) + \frac{\pi D}{4}\eta_2(k) + \frac{8}{\lambda(1-v^2)}\eta_3(k).
\end{aligned}
$$

Then, $\mathbb{F}_{\text{cap}}(SW) \geq 1/\lambda\mathbb{E}[T]$. Finally, in the most favorable scenario for the vehicle, it intercepts each new target at the end of each quadrant at a radial location X, waits for time W and begins a new iteration. The time in which it returns to a quadrant in this manner is equal to $4\sqrt{2}X + 4W$. Since $X/v < 4\sqrt{2}X + 4W$ for all $v < 1$, the assumption that the vehicle always begins an iteration in an unserviced region holds true.

Appendix C. Proof of Theorem 4.4. From Lemma 4.3, we know that the capture fraction of the SW policy satisfies

$$
\mathbb{F}_{\text{cap}}(\text{SW}) \geq \frac{1}{\lambda}\left(W + \frac{\pi D}{4}(3\eta_1(k) + \eta_2(k)) + \frac{8}{\lambda(1-v^2)}\eta_3(k)\right)^{-1}. \tag{C.1}
$$

When $v > 1/4\sqrt{2}$, $W = 0$. Further, when $\lambda > \frac{7\pi v}{(1-v^2)^{3/2}D}$, $k = \frac{\lambda(1-v^2)^{3/2}}{72\pi v} > 0.1$. Using upper and lower bounds on Bessel and Struve functions, the following hold true for $k > 0.1$,

$$
\begin{aligned}
\frac{3}{4}\eta_1(k) + \frac{1}{4}\eta_2(k) = & \frac{3}{4}(L_{-1}(8k) - I_1(8k) - L_{-1}(20k) + I_1(20k)) \\
& + \frac{1}{4}\left(I_0(8k) - L_0(8k) - I_0(20k) + L_0(20k)\right) \leq \frac{1}{12\sqrt{k}},
\end{aligned} \tag{C.2}
$$

$$
\eta_3(k) = 1 - \frac{\pi}{2}\left(I_0(12k) - L_0(12k) - I_0(20k) + L_0(20k)\right) \leq \frac{5\sqrt{k}}{2}. \tag{C.3}
$$

Using Eq. C.2 and C.3, and the result in Eq. C.1, $\mathbb{F}_{\text{cap}}(\text{SW})$ can be bounded from below and the result is obtained.

Appendix D. Proof of Theorem 4.5. We start with calculating an upper bound on the length of the tour through all the targets in $A(g, h, 0, 2\pi)$. Let $Q := \{(r, \theta) \in A(g, h, 0, 2\pi)\}$ be the set of locations of targets accumulated in $A(g, h, 0, 2\pi)$ and $n = |Q|$. From Lemma 3.6, the normalized distribution of these targets w.r.t the radial location x is given by $f_n(x) = 3x^2/(h^3 - g^3)$ for $x \in [g, h]$. Let $\bar{Q}$ be the set of locations (s, ϕ) of these targets if they move outwards by distance d and occupy $A(g + d, h + d, 0, 2\pi)$. The normalized distribution functions of the random variables s and ϕ which denote locations of these targets are

$$
f_s(x) = \frac{3(x-d)^2}{h^3 - g^3} \quad \text{and} \quad e_\phi(y) = \frac{1}{2\pi}.
$$

Using Theorem 3.11 and assuming that $n \to \infty$ (which we will revisit later),

$$\lim_{n\to+\infty} \frac{ETSP(\bar{Q})}{\sqrt{n}} = \beta\sqrt{\frac{6\pi}{h^3-g^3}}\left(\frac{h^2-g^2}{2}\right)$$

with

$$\varphi(s,\phi) = f_s(x)e_\phi(y) = \frac{3(x-d)^2}{(h^3-g^3)}\frac{1}{2\pi}.$$

Using Chebyshev's inequality, if μ_A and σ_A are the mean and standard deviation of the random variable n, then for any fixed $v \in (0,1)$,

$$\mathbb{P}[n < \mu_A(1+v)] \geq 1 - \sigma_A^2/v^2\mu_A^2.$$

Then, the condition

$$t_u := \frac{\beta}{1-v}\sqrt{\frac{6\pi}{h^3-g^3}}\left(\frac{h^2-g^2}{2}\right)\sqrt{\mu_A(1+v)} \leq \frac{D-g}{v} \tag{D.1}$$

from the Optimization Problem 1 ensures that the n targets will be serviced before they escape the disk with at least a probability of $1 - \sigma_A^2/v^2\mu_A^2$. Similarly since $n_{\text{cap}} = \mu_A(1-v)$ and $v > 0$,

$$\mathbb{P}[n > \mu_A(1-v)] \geq 1 - \sigma_A^2/v^2\mu_A^2 \tag{D.2}$$

so that $n > n_{\text{cap}}$ and the vehicle services n_{cap} targets in an iteration with probability of at least $1 - \sigma_A^2/v^2\mu_A^2$. Further, the condition

$$\frac{\beta}{1+v}\sqrt{\frac{6\pi}{h^3-g^3}}\left(\frac{h^2-g^2}{2}\right)\sqrt{\mu_A(1-v)} \geq \frac{h-g}{v} \tag{D.3}$$

from the Optimization Problem 1 ensures that with probability of at least $1-\sigma_A^2/v^2\mu_A^2$, when the vehicle starts an iteration, $A(g,h,0,2\pi)$ is unserviced. In the above inequality, the left-hand side is the lower bound on the length of the tour through $\mu_A(1-v)$ targets in $A(g,h,0,2\pi)$ obtained by using Theorem 3.10. We also know that

$$\mu_A = \frac{\lambda(h^3-g^3)}{3vD^2}$$

and $\sigma_A^2 = \mu_A$ from Lemma 3.6. When $\lambda \to +\infty$ and $v \in (0,1)$, then $\mu_A \to +\infty$ and $n_{\text{cap}} \to +\infty$ so that Eq. (D.1),(D.2) and (D.3) hold true with probability one. Then, since $n_{\text{cap}} \to +\infty$ and $n > n_{\text{cap}}$ with probability one, our earlier assumption that $n \to +\infty$ is true as well.

If $k_{\text{tot}}(i)$ and $k_{\text{cap}}(i)$ are the number of targets that have appeared and have been serviced in the $i-$th iteration of the SNB policy, and $\mathbb{F}_{\text{i}}(\text{SNB}) = \mathbb{E}\left[k_{\text{cap}}(i)/k_{\text{tot}}(i)\right]$, then since at every iteration, $k_{\text{cap}}(i) \geq n_{\text{cap}}$,

$$\mathbb{F}_{\text{i}}(\text{SNB}) \geq \mathbb{E}\left[\frac{n_{\text{cap}}}{k_{\text{tot}}(i)}\right] = n_{\text{cap}}\mathbb{E}\left[\frac{1}{k_{\text{tot}}(i)}\right] \geq \frac{n_{\text{cap}}}{\mathbb{E}[k_{\text{tot}}(i)]}, \tag{D.4}$$

where the last inequality holds true using Jensen's inequality for convex function $1/k_{\text{tot}}(i)$.

Next, when $\lambda > (1+v)^2/2\pi\beta^2 v(1-v)$, the solution to the Optimization Problem 1 exists and the parameters g and h obtained by solving it satisfy

$$\frac{h^3 - g^3}{h^2 - g^2} \geq \frac{4Dp}{7} \frac{1-v}{\sqrt{1+v}}.$$

Now, when $\lambda \to +\infty$, for a fixed speed, the above condition on λ is met. Then, from Eq. D.4 and using the fact that $\mathbb{E}[k_{\mathrm{tot}}(i)] \leq \lambda t_u$,

$$\mathbb{F}_{\mathrm{i}}(\mathrm{SNB}) \geq \frac{2p(1-v)^2}{7(1+v)} \frac{1-v}{\beta} \sqrt{\frac{2}{\pi\lambda v}}.$$

Let the countably infinite set $Y := \{\mathbb{F}_{\mathrm{i}}(\mathrm{SNB}) \text{ for all } i \in \mathbb{N}\}$. Also, let the uncountable set $Z := \{\mathbb{E}\left[m_{\mathrm{cap}}(t)/m_{\mathrm{tot}}(t)\right] \text{ for all } t \in \mathbb{R}_{\geq 0}\}$. Since $Y \subseteq Z$,

$$\mathbb{F}_{\mathrm{cap}}(\mathrm{SNB}) = \limsup_{t\to\infty} \mathbb{E}\left[\frac{m_{\mathrm{cap}}(t)}{m_{\mathrm{tot}}(t)}\right] \geq \limsup_{i\to\infty} \mathbb{F}_{\mathrm{i}}(\mathrm{SNB}) \geq \frac{2p(1-v)^2}{7(1+v)} \frac{1-v}{\beta} \sqrt{\frac{2}{\pi\lambda v}},$$

the result is obtained.

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