A Solvability Condition for Reactive Power Flow

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***Abstract*— A central question in the analysis and operation of power networks is the feasibility of a unique high-voltage solution to the power flow equations satisfying operational constraints. For planning, monitoring, and contingency analysis in transmission networks, the high-voltage solution of these nonlinear equations can be constructed only numerically or roughly approximated using a linear DC power flow. In this work we analytically study the solvability of the nonlinear decoupled reactive power flow equations, and present a solvability condition relating the existence of a unique high-voltage solution to the spatial distribution of loading and the effective impedances between load buses. We validate the accuracy and applicability of our results through standard power network test cases.**

I. Introduction

The interactions between the nodal phasor voltages and complex power injections of a synchronous AC power network are described by a set of nonlinear algebraic equations called the power flow equations. These coupled equations are the foundation of nearly all power systems analysis, planning and control. Most notable are their extensive use in optimal power flow, contingency analysis, model reduction, VAR allocation, transient stability studies, wide-area frequency stability and control, and voltage security assessment [1].

When the equations are taken in their most general form, the corresponding power flow solution space admits a rich and complex phenomenology [2]. The number of solutions is known to scale exponentially with the system size [3], there can be several realistic solutions [4], and the system is known to exhibit multidimensional bifurcation phenomena [5]. In heavily loaded networks, distinct stable solutions can exist so close to one another in voltage-space that distinguishing them in real time may prove difficult [6], while a further increase in load can cause these neighboring solutions to undergo saddle-node bifurcation and vanish entirely [5]. Analytic results are typically unavailable, and despite early ideas to the contrary [7], for a given network the feasible set in the space of power injections (parameter-space) is generally non-convex [2].

A. Solutions, Loading Margins & Stability Indices

The most fundamental question regarding the power flow equations is as follows: under what conditions on the givens do they admit a high-voltage solution satisfying operational constraints? This difficult problem in essence attempts to quantify the physical limits of the network [8]–[15]. While good numerical methods are available to calculate these transfer capability limits, theoretical understanding of network limits remains somewhat lacking. Known solvability conditions are either theoretically conservative [12], or restrict the loading profile and offer no uniqueness guarantees [8], [9]. Moreover, as noted in [11], in transmission networks the reactive power security problem is considerably more challenging than the corresponding problem for active power flow, due to the presence of non-negligible reactive power losses; see [15] for an analysis of solutions to the active power flow equations in transmission networks. See also the approximations techniques in [16], [17], and results for distribution networks in [18].

A related but distinct set of literature instead seeks to determine *loading margins and voltage proximity indices*, which are, respectively, parameter-space and voltage-space distance-to-bifurcation indicators. These quantities serve as safety margins for online assessment of network stability. Analytic results are available only in the simplest cases, while proximity indices for practical networks require numerical computation. In [19] Venikov *et al.* proposed using the determinant of the power flow Jacobian as a measure of proximity to voltage collapse. This observation generated a massive effort in the power systems literature, evolving into spectral and singular value methods [3], [5], [20]–[25], continuation methods [2], [26], [27], optimization techniques [14], [28], and later energy approaches [29]. These indices typically only provide a state-space metric of voltage stability, and struggle to be explicitly related to parametric loading margins without additional support from numerical approaches. Theoretical guarantees regarding proximity indices are typically unavailable.

B. Contributions

In the present work we provide a novel condition guaranteeing the existence of a high-voltage solution to the decoupled reactive power flow equations. The algebraic inequality condition we present generalizes the classic single-line existence condition by relating the network topology, impedances and load demands, and can be roughly stated as "network coupling should be larger than loading." The degree to which our condition is satisfied corresponds in a natural way to the deviations of load bus voltages from generator voltage setpoints. Moreover, we show that under our condition the decoupled reactive power flow Jacobian is nonsingular at the corresponding solution, indicating that the solution is robust under perturbations. This non-singularity

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property is crucial for reliability of numerical proximity index techniques, stability of associated load dynamics [25], [30]–[34], robustness of solutions under perturbations [5], [14], and droop control in microgrids [35]. We interpret the main result before presenting several interesting corollaries, highlighting how our condition provides a convex approximation of the feasible injection region in parameter-space, and how it naturally implies an upper bound on the reactive power dissipated in the network.

Overall, the distinguishing feature of this work is the analytic approach, which takes a preliminary attempt at bridging the gap between the theory of power flow and established numerical approaches. To focus on the reactive power problem, throughout we work under several simplifying assumptions such as power flow decoupling, the absence of shunt elements and phase shifting transformers, and uniform generator voltages. Relaxations of some of these assumptions, technical proofs, and extensive numerical testing are deferred to a journal article to follow.

The remainder of this section recalls some mostly-standard notation that is used in the sequel. In Section II we formulate our model of the reactive power flow equations. In Section III we review some basic results on reactive power flow. In Section IV we present our main results, and show how our novel parametric condition leads to the results described above. In Section V we present numerical evidence showing our results are accurate in practical power networks, before offering concluding remarks in Section VI.

C. Preliminaries and Notation

Sets, vectors and functions: Given a finite set $\mathcal{V}$, let $|\mathcal{V}|$ denote its cardinality. The set $\mathbb{S}_1$ is the unit circle, $\mathbb{T}^n = \mathbb{S}_1 \times \cdots \times \mathbb{S}_1$ is the n-torus, and $\mathbb{R}$ (resp. $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$) is the set of real (resp. nonnegative real, strictly positive real) numbers. For $x \in \mathbb{R}^n$, $[x] \in \mathbb{R}^{n\times n}$ is the associated diagonal matrix. Given $x, y \in \mathbb{R}^n_{\geq 0}$, we write $x \geq y$ if $x_i \geq y_i$ for each $i \in \{1, \ldots, n\}$. We let $\mathbb{1}_n$ and $\mathbb{0}_n$ n-dimensional vectors of unit and zero entries. The ∞-norm of $x \in \mathbb{R}^n$ is given by $\|x\|_\infty = \max_{i\in\{1,\ldots,n\}} |x_i|$.

II. Network Modeling and Decoupled Reactive Power Flow

Throughout this work we consider a connected, phase-balanced power network operating in sinusoidal steady-state. The network is modeled as a weighted graph $G(\mathcal{V}, \mathcal{E})$ with two distinct types of nodes (or *buses*) : loads $\mathcal{V}_L$ and generators $\mathcal{V}_G$, such that $\mathcal{V} = \mathcal{V}_L \cup \mathcal{V}_G$. For notational simplicity, we set $n = |\mathcal{V}_L|$, $m = |\mathcal{V}_G|$ and assume $n, m \geq 1$. Each branch $\{i, j\} \in \mathcal{E}$ is weighted by a transfer admittance $y_{ij} = g_{ij} + jb_{ij}$, where $g_{ij} \geq 0$ and $b_{ij} \leq 0$. We encode the weights and topology in the bus admittance matrix Y, with elements $Y_{ij} = -y_{ij}$ and $Y_{ii} = -\sum_{j=1}^{n+m} y_{ij}$. The conductance matrix G and susceptance matrix B are defined by $G = \mathrm{Re}(Y)$ and $B = \mathrm{Im}(Y)$. To each bus we associate a phasor voltage $U_i = V_i e^{j\theta_i} \in \mathbb{C}$ where $V_i \geq 0$ is the voltage magnitude and $\theta_i \in \mathbb{S}$ is the voltage angle. The complex power injection at bus $i \in \mathcal{V}$ is given by $S_i = P_i + jQ_i$.

First, for the high-voltage transmission networks we consider, the branches are dominantly inductive; the conductance matrix G is therefore negligible and we assume that $Y \simeq jB$For simplicity of presentation, we also assume that the network lacks any phase-shifting transformers, shunt elements, and line charging capacitors. Under these conditions, we have the following standard characterization of B [9].

Fact 1: **(Properties of Susceptance Matrix).** The (negative) susceptance matrix $-B$ is the Laplacian matrix of a weighted, undirected and connected graph. In particular,

(i) ***Sign Structure:*** $B_{ii} < 0$, $B_{ij} \geq 0$ for all $i, j \in \mathcal{V}$;
(ii) ***Symmetry:*** $B_{ij} = B_{ji} \geq 0$ for all $i, j \in \mathcal{V}$, with strict inequality if $\{i, j\} \in \mathcal{E}$;
(iii) ***Zero Row-Sums:*** $\sum_{j=1}^{n+m} B_{ij} = 0$ for all $i \in \mathcal{V}$.

The *power flow functions* $g_i, h_i : \mathbb{R}^{n+m} \times \mathbb{T}^{n+m} \to \mathbb{R}$ relate the voltages and power injections at each bus $i \in \mathcal{V}$ via

$$g_i(V, \theta) \triangleq \sum_{j=1}^{n+m} V_i V_j B_{ij} \sin(\theta_i - \theta_j) \,, \tag{1a}$$

$$h_i(V, \theta) \triangleq -\sum_{j=1}^{n+m} V_i V_j B_{ij} \cos(\theta_i - \theta_j) \,. \tag{1b}$$

Physically, g_i and h_i are the *active power* and *reactive power* injected at node $i \in \mathcal{V}$.

As is standard in power flow analysis, the load buses $\mathcal{V}_L$ are modeled as PQ buses, at which the active and reactive power injections $P_i, Q_i \in \mathbb{R}$ are specified. Generator buses $\mathcal{V}_G$ are modeled as PV buses, at which active power injections $P_i \in \mathbb{R}$ and voltage magnitudes $V_i \geq 0$ are regulated to constant values. Although the reactive power injections Q_i are unknown at the generator buses $\mathcal{V}_G$, they are determined uniquely by (1b) after solving the remainder of the problem for the remaining unknowns θ_i $(i \in \mathcal{V})$ and V_i $(i \in \mathcal{V}_L)$. Extensions to include generator reactive power limits are deferred to a future journal publication.

It follows that we may discard the m additional reactive power flow equations for the generators $\mathcal{V}_G$. Given this PQ and PV modeling of loads and generators, the injections given by (1a)–(1b) are constrained at each node to meet the demands, yielding the algebraic power flow equations

$$P_i = g_i(V, \theta) \,, \quad i \in \mathcal{V} \,, \tag{2a}$$

$$Q_i = h_i(V, \theta) \,, \quad i \in \mathcal{V}_L \,, \tag{2b}$$

where $V = (V_1, \ldots, V_n) \in \mathbb{R}^n_{>0}$ is the vector of load voltage magnitudes. Typically $Q_i < 0$, corresponding to an inductive load. This work will focus on the n *reactive power flow equations* (2b); see [15], [34] and the references therein for a detailed analysis of the active power flow (2a).

Remark 1: **(Load Modeling).** There are myriad of static or dynamic load models which could be used in place of the PQ modeling we adopt herein to formulate the power flow equations (2a)–(2b). It has been shown that for security analysis problems, models can without loss of generality be assumed static [36]. Throughout this work *we model all loads* *as PQ loads, i.e., stiff constant power demands*. The literature has firmly established that this is the most challenging case for steady-state analysis, and the one most relevant from the perspective of both classic [1] and modern [37], [38] power system operation. For constant-impedance loads, constant-current loads, or their combination, (1b) becomes linear [11], [13]. See [13], [25], [35], [39] for additional information and analysis. □

In practice, realistic power flow solutions have the property that $|\theta_i - \theta_j| \simeq 0$. Indeed, under normal operating conditions, a typical angular difference is roughly $5°$, for which $\cos(5°) \simeq 0.99$. It is therefore common to study (2b) under a *decoupling* assumption, in which the power angles are assumed to be known and constant [11], [12], or even negligible [8], [9]. We therefore make the following technical assumption (see [9] and Ref. 14 of [11] for analysis on the error introduced by decoupling).

Assumption 1: **(Decoupling).** The power angles θ satisfy $|\theta_i - \theta_j| = 0$ for all branches $\{i, j\} \in \mathcal{E}$ of the network. □

Assumption 1 can easily be relaxed to constant (but fixed) angular differences, but the special case of zero differences introduces some additional structure and allows for some elegant graph-theoretic interpretations; we comment more on this assumption in Section VI. To even further simplify the presentation, we assume that the voltages V_i all generators $i \in \mathcal{V}_G$ are regulated to the same value, which in a preferred system of units can be taken to be equal to one. Extensions to non-uniform generator voltages are straightforward.

With these assumptions, the decoupled power balance equations we will study are

$$Q_i = -\sum_{j\in\mathcal{V}_L} V_i V_j B_{ij} - \sum_{j\in\mathcal{V}_G} V_i B_{ij}\,, \quad i \in \mathcal{V}_L\,. \tag{3}$$

Our goal is to find a sufficient condition guaranteeing that (3) is solvable for the voltages $V = (V_1, \ldots, V_n)$.

III. Single Load Problem & Necessary Solvability Condition

In this section we build intuition regarding the general properties of the decoupled reactive power flow (3) by first recalling the results for a single load. We then generalize the insights from the single load case into a necessary solvability condition which can be applied to any network.

For $n = 1$, the reactive power flow (3) is a single quadratic equation, and the necessary and sufficient condition for the existence of a solution follows immediately by studying its discriminant [25, Section 2.2.3].

Proposition 3.1: **(Necessary and Sufficient Condition for One Load).** For a single load $|\mathcal{V}_L| = 1$, consider the decoupled reactive power flow (3) which reduces to

$$0 = q + bv^2 - bv\,, \tag{4}$$

where $b < 0$. The following statements are equivalent:

(i) ***Unique High-Voltage Solution:*** There exists a unique high-voltage solution $v^+ > 1/2$ to (4);

(ii) ***Load Limit:*** $4q/b < 1$.

Moreover, if either of the above statements are true, then

$$v^+ = \frac{1}{2}\left(1 + \sqrt{1 - \frac{q}{b/4}}\right)\,.$$

The exact solution in Proposition 3.1 shows that the solution exists only for loads which are less inductive than the critical load $b/4$. When $q = 0$, $v = 1$, that is, the load voltage aligns with the generator voltage. Moreover, one easily verifies that when $q = b/4$, the Jacobian of (4) is singular at the corresponding solution $v = 1/2$, which is the maximum power transfer point at which the high and low-voltage solutions coalesce [5]. This bifurcation point at $(v, q) = (1/2, b/4)$ motivates the following necessary condition for solvability, which applies to (3) defined over any transmission network. The proof is not reported here, but follows easily from change of coordinates and definiteness arguments. See also [37, Lemma 1] for a related result.

Proposition 3.2: **(Necessary Condition for Solvability).** Consider the decoupled reactive power flow equation (3). If a solution to (3) exists, then

$$\frac{4\sum_{i=1}^{n} Q_i}{\sum_{i,j=1}^{n} B_{ij}} < 1\,. \tag{5}$$

Physically, Proposition 3.2 restricts the sum of load in the network from being overly inductive. From the definition of the susceptance matrix, the sum $\sum_{i,j=1}^{n} B_{ij}$ in the denominator of (5) is equal to $-\sum_{i\in\mathcal{V}_L}\sum_{j\in\mathcal{V}_G} B_{ij}$, which is the parallel combination of all susceptances connecting generators to loads. Similarly, $\sum_{i=1}^{n} Q_i$ is the total reactive power loading in the network. Comparing this to the single load condition in Proposition 3.1, the necessary condition (5) can be interpreted as aggregating all loads down to a single load, and connecting all generators to that load through a single effective susceptance.

The gap between this necessary condition and the sufficient condition developed in the next section is filled by the grid topology. That is, while the aggregated condition in Proposition 3.2 must always be fulfilled, the relative locations of load and generation and the effective impedances between them must be included. We summarize the following key observations from Propositions 3.1 and 3.2:

- The high-voltage solutions should deviate minimally from the generator voltage which is fixed at unity;
- The reactive load should not be overly inductive, and should be compared to a measure of total susceptance or total reactance;
- The voltage level $1/2$ is a "trouble spot" for (3);
- The relative locations of loading should be included in the feasibility condition;

IV. Main Results

We preface our main results with a useful notion of effective impedance, which will help us state a circuit-theoretic version of our result. Given a power network satisfying our previous assumptions, let v_i be the voltage measured at load i when a unit reactive current is extracted at load j with all other current injections zero. We then call $X_{ij}^{\text{eff}} = v_i - 1$ the *differential effective reactance* between loads $i, j \in \mathcal{V}_L$. In other words, X_{ij}^{eff} is the proportionality coefficient from current injections at load j to voltage deviations at load i. The differential effective reactance provides a useful measure of how electrically distant loads i and j are from one another in the network. It is easily seen that $X_{ij}^{\text{eff}} = X_{ji}^{\text{eff}}$, and can be shown that $X_{ii}^{\text{eff}} \geq X_{ij}^{\text{eff}}$ for all i, j.

The following is our main result, a sufficient condition under which we are assured the existence of a solution to (3) along with some invertibility properties of the Jacobian matrix at the solution. The proof is not reported here.

Theorem 4.1: **(Solution Existence).** Consider the decoupled reactive power flow equation (3), and assume that

$$\mathsf{M} \triangleq \max_{i\in\mathcal{V}_L} \sum_{j\in\mathcal{V}_L} 4X_{ij}^{\text{eff}} Q_j < 1\,. \tag{6}$$

The following statements hold:

1) ***Existence:*** There exists a unique high-voltage solution $V^+ \in \mathbb{R}^n_{>0}$ of (3) satisfying $V^+ > \frac{1}{2}\mathbb{1}_n$;
2) ***Bounding:*** The solution V^+ satisfies the bound $\|V^+ - \mathbb{1}_n\|_\infty \leq \epsilon$, where $\epsilon = \mathsf{M}/2$;
3) ***Jacobian Invertibility:*** The Jacobian matrix of (3) evaluated at V^+ is Hurwitz.

Moreover, the condition (6) is tight; the marginal case $\mathsf{M} = 1$ is achieved by the extremal load profile $(Q_1, \ldots, Q_n)$ defined component-wise by $Q_i = \sum_{j=1}^n B_{ij}$. In this case the exact solution to (3) is $V^+ = \frac{1}{2}\mathbb{1}_n$.

Remark 2: **(Interpretation of Theorem 4.1).** Theorem 4.1 captures the intuition regarding the compromise between voltage profile cohesiveness (i.e., homogeneity) and severity of loading. The parametric condition (6) compactly expresses the interplay between the load demands Q_i, which tend to distort the voltage profile, and the electrical distances in the network as expressed in terms of the differential effective reactances X_{ij}^{eff}. As the margin M approaches zero, ϵ approaches zero and the voltage profile becomes flat. As the margin M approaches unity, ϵ approaches 1/2 and the bifurcation point may be reached. □

Let us first compare the result of Theorem 4.1 to the two-node case of Proposition 3.1. In this case, $|\mathcal{V}_L| = 1$, $X_{11}^{\text{eff}} = 1/b$, and the condition (6) therefore reduces to the necessary and sufficient condition $4q/b < 1$. While the necessary condition of (3.2) restricts an aggregated load $Q_{\text{agg}} = \sum_{i=1}^n Q_i$ compared to an aggregated reactance $X_{\text{agg}} = 1/\sum_{i,j=1}^n B_{ij}$, the sufficient condition takes into account the topology of the graph connecting the loads through the effective reactances X_{ij}^{eff}. Surprisingly, the final statement in Theorem 4.1 states that the necessary condition and the sufficient condition coincide for a specific load distribution. In Section V we show that in practical networks, the condition (6) provides good estimates on the locations of solutions.

We state two interesting corollaries of Theorem 4.1. First, the condition (6) furnishes an inner-convex estimate of the so-called "injection region" — the set of power injections $Q_L \in \mathbb{R}^n$ such that a unique high-voltage solution exists.

TABLE I

ACCURACY OF $\|V^+ - \mathbb{1}_n\|_\infty \leq \epsilon$ BOUND IN TEST CASES

Network	Error η (%)
IEEE 14 Bus	0.22
IEEE 30 Bus	0.59
IEEE 57	0.31
IEEE 118 Bus	0.19
Polish (W'99-'00)	1.16

Corollary 4.2: **(Convex Estimate of Injection Region).** The set $\{(Q_1, \ldots, Q_n) \in \mathbb{R}^n \;:\; \mathsf{M} < 1\}$ is a convex subset of the injection region.

This convex region can be regarded as a "safe" region of parameter space — load demands that are guaranteed to lead to high-voltage solutions. From (6), one sees that the estimate is simply the intersection of various half-spaces. Second, in [12] it was noted that "...*the reactive power problem is characterized by large reactive losses relative to the transmitted reactive power*...". A simple corollary of Theorem 4.1 makes this intuition precise by upper bounding the reactive power absorbed by the network.

Corollary 4.3: **(Reactive Power Losses).** Assume the condition of Theorem 4.1 holds, and let

$$\mathcal{L}(V^+) \triangleq -\widetilde{V}^T B \widetilde{V} \in \mathbb{R}_{\geq 0}$$

denote the reactive power absorbed/dissipated by network at the solution V^+, where $\widetilde{V} = (V^+, \mathbb{1}_m)$ is the vector of load voltages and generator voltages. Then $\mathcal{L}(V^+)$ satisfies the bound

$$\frac{\mathcal{L}(V^+)}{\sum_{i,j=1}^n B_{ij}} \leq \epsilon\,.$$

Note that as the reactive loading in the network goes to zero, ϵ goes to zero and so does the reactive power absorbed, since the network voltage profile becomes flat.

V. SIMULATION STUDY

In this section we demonstrate that the bounds developed in Theorem 4.1 provide accurate results in several standard test cases. We computed [40] the exact operating point V^{exact}, and then computed the deviation via $\epsilon^{\text{exact}} = \|V^{\text{exact}} - \mathbb{1}_n\|_\infty$. This was then compared to the bound ϵ developed in Theorem 4.1 via the relative error $\eta \triangleq (\epsilon - \epsilon^{\text{exact}})/\epsilon^{\text{exact}}$. The results of these computations are reported in Table I. All tests generated errors of less than 1%, aside from the large (2383 bus) Polish test system. Hence, the bound is fairly accurate in these test networks, suggesting its accuracy in even larger practical networks may be acceptable.

VI. CONCLUSIONS

This paper presented an analytical technique to assess to solvability of the decoupled reactive power flow equations. We have presented circuit-theoretic conditions on the effective impedances and load demands to ensure the existence of a high-voltage solution. While in this article we have included many assumptions in our analysis, our approach appears to give good results in standard test cases. A forthcoming journal publication will relax many of the assumptions listed throughout this work including decoupling and additional load models, will provide proofs of the main results, and will provide more extensive numerical testing of our approach. We foresee that the results herein and the subsequent extensions will be useful for control applications, where simple, tractable models are needed for linear feedback design, and as constraints or objective functions for optimization.

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