Competitive Propagation: Models, Asymptotic Behavior and Quality-Seeding Games

Wenjun Mei Francesco Bullo

***Abstract*—In this paper we propose a class of propagation models for multiple competing products over a social network. We consider two propagation mechanisms: social conversion and self conversion, corresponding, respectively, to endogenous and exogenous factors. A novel concept, the product-conversion graph, is proposed to characterize the interplay among competing products. According to the chronological order of social and self conversions, we develop two Markov-chain models and, based on the independence approximation, we approximate them with two corresponding difference equations systems. Our theoretical analysis on these two approximated models reveals the dependency of their asymptotic behavior on the structures of both the product-conversion graph and the social network, as well as the initial condition. In addition to the theoretical work, we investigate via numerical analysis the accuracy of the independence approximation and the asymptotic behavior of the Markov-chain model, for the case where social conversion occurs before self conversion. Finally, we propose two classes of games based on the competitive propagation model: the repeated one-shot game and the dynamic infinite-horizon game. We characterize the quality-seeding trade-off for the first game and the Nash equilibrium in both games.**

***Index Terms*—competitive propagation, independence approximation, network structure, stability analysis, multi-stage uncooperative game, seeding, product quality**

I. Introduction

a) Motivation and problem description: It is of great scientific interest to model some sociological phenomenon as dynamics on networks, such as consensus, polarization, synchronization and propagation. Indeed, the past fifteen years have witnessed a flourishing of research on propagation of diseases, opinions, commercial products etc, collectively referred to as memes, on social networks. Much progress has been made both on obtaining and analyzing empirical data [2]–[5], and mathematical modeling [6]–[9]. In a more recent set of extensions, scientists have begun studying the simultaneous propagation of multiple memes, in which not only the interaction between nodes (or equivalently referred to as individuals) in the network, but also the interplay of multiple memes, plays an important role in determining the system's dynamical behaviors. These two forms of interactions together add complexity and research value to the multi-meme propagation model.

This paper proposed a series of mathematical models on the propagation of competing products. Three key elements: the interpersonal network, the individuals and the competing products, are modeled respectively as a graph with fixed topology, the nodes on the graph, and the states of nodes. Our models are based on the characterization of individuals' decision making behaviors under the social pressure. Two factors determine individuals' choices on which product to adopt: the endogenous factor and the exogenous factor. The endogenous factor is the social contact between nodes via social links, which forms a tendency of imitation, referred to as social pressure in this paper. The exogenous factor is what is unrelated to the network, e.g., the products' quality.

In the microscopic level, we model the endogenous and exogenous factors respectively as two types of product-adoption processes: the social conversion and the self conversion. In social conversion, any node randomly picks one of its neighbors and follows that neighbor's state with some given probability characterizing how open-minded the node is. In the self conversion, each node independently converts from one product to another with some given probability depending on the two products involved. Although individuals exhibit subjective preferences when they are choosing the products, statistics on a large scale of different individuals' behaviors often reveal that the relative qualities of the competing products are objective. For example, although some people may have special affections on feature phones, the fact that more people have converted from feature phones to smart phones, rather than the other way around, indicates that the latter is relatively better. We assume that the transit probabilities between the competing products are determined by their relative

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qualities and thus homogeneous among the individuals.

b) Literature review: Various models have been proposed to describe the propagation on networks, such as the percolation model on random graphs [10], [11], the independent cascade model [12]–[14], the linear threshold model [15]–[17] and the epidemic-like mean-field model [18]–[20].

As extensions to the propagation of a single meme, some recent papers have discussed the propagation of multiple memes, e.g., see [21]–[32]. Some of these papers adopt a Susceptible-Infected-Susceptible (SIS) epidemic-like model and discuss the long-term coexistence of multiple memes in single/multiple-layer networks, e.g., see [25]–[27]. Some papers focus instead on the strategy of initial seeding to maximize or prevent the propagation of one specific meme in the presence of adversaries [29]–[32]. Among all these papers mentioned in this paragraph, our model is most closely related to the work by Stanoev et. al. [28] but the social contagion process in [28] is different from our model and theoretical analysis on the general model is not included.

c) Contribution: Firstly we propose a generalized and novel model for the competitive propagation on social networks. By taking into account both the endogenous and exogenous factors and considering the individual variance as well as the interplay of the competing products, our model is general enough to describe a large class of multi-meme propagation processes. Moreover, many existing models have difficulty in dealing with the simultaneous contagions of multiple memes, and have to avoid the problem by adding an additional assumption of the infinitesimal step length that only allows the occurrence of a single contagion at every step. Differently from these models, the problem of multiple contagions does not occur in our model since we model the contagion process as the individual's initiative choice under the social pressure, which is more suitable for the product-adoption process. In addition, compared with the independent cascade model, in which individuals' choices are irreversible, our models adopt a more realistic assumption that conversions from one product to another are reversible and occur persistently.

Secondly, we propose a new concept, the product-conversion graph, to characterize the interplay between the products. There are two graphs in our model: the social network describing the interpersonal connections, and the product-conversion graph defining the transitions between the products in self conversion, which in turn reflect the products' relative quality.

Thirdly, starting from the description of individuals' behavior, we develop two Markov-chain competitive propagation models different in the chronological order of the social conversion and the self conversion processes. Applying the independence approximation, we propose two corresponding network competitive propagation models, which are difference equations systems, such that the dimension of our problem is reduced and some theorems in the area of dynamical systems can be applied to the analysis of the approximation models.

Fourthly, both theoretical analysis and simulation results are presented on the dynamical properties of the network competitive propagation models. We discuss the existence, uniqueness and stability of the fixed point, as well as how the systems' asymptotic state probability distribution is determined by the social network structure, the individuals' open-mindedness, the initial condition and, most importantly, the structure of the product-conversion graph. We find that, if the product-conversion graph contains only one absorbing strongly connected component, then the self conversion dominates the system's asymptotic behavior; With multiple absorbing strongly connected components in the product-conversion graph, the system's asymptotic state probability distribution also depends on the initial condition, the network topology and the individual open-mindedness. In addition, simulation results are presented to show the high accuracy of the independence approximation and reveal that the original Markov-chain model also exhibits the same asymptotic behavior.

At last, based on the network competitive propagation model, we propose two classes of non-cooperative games. In both games the players are the competing companies with bounded investment budgets on seeding, e.g., advertisement and promotion, and improving their products' quality. The first model is a infinitely repeated one-shot game, in which the players myopically maximize their next-step pay-off. We investigate the unique Nash equilibrium at each stage. Theoretical analysis also reveals some strategic and realistic insights on the seeding-quality trade-off and the allocation of seeding resources among the individuals. The second model is a dynamic game with infinite horizon, in which the players aim to maximize their discounted accumulated pay-offs. The existence of Nash equilibrium for the two-player case is proved and numerical analysis is given on the comparison with the one-shot game.

d) Organization: The rest of this paper is organized as follows. Section II give the assumptions for two Markov-chain propagation models. Section III and IV discuss the approximation of these two models respectively. In Section V, we discuss the two classes of games. Section VI is the conclusion.

II. Model Description and Notations

a) Social network as a graph: In this model, a social network is considered as an undirected, unweighted, fixed-topology graph $G = (V, E)$ with n nodes. The nodes are indexed by $i \in V = \{1, 2, \dots, n\}$. The

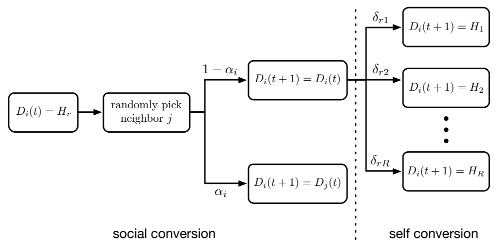


Fig. 1. Diagram illustration for the social-self conversion model

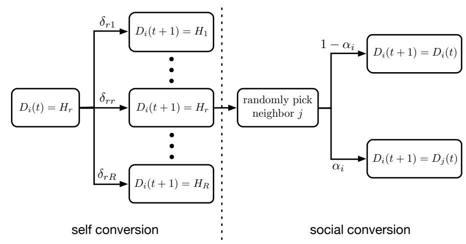


Fig. 2. Diagram illustration for the self-social conversion model

adjacency matrix is denoted by $A = (a_{ij})_{n\times n}$ with $a_{ij} = 1$ if $(i,j) \in E$ and $a_{ij} = 0$ if $(i,j) \notin E$.

The row-normalized adjacency matrix is denoted by $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, where $\tilde{a}_{ij} = \frac{1}{N_i} a_{ij}$ with $N_i = \sum_{j=1}^{n} a_{ij}$. The graph $G = (V,E)$ is always assumed connected and there is no self loop, i.e., $\tilde{a}_{ii} = 0$ for any $i \in V$.

b) Competing products and the states of nodes: Suppose there are R competing products, denoted by $H_1, H_2, \dots, H_R$, propagating in the network. We consider a discrete-time model, i.e., $t \in \mathbb{N}$, and assume the products are mutually exclusive. We do not specify the state of adopting no product and collectively refer to all the states as "products". Denote by $D_i(t)$ the state of node i after time step t. For any $t \in \mathbb{N}$, $D_i(t) \in \{H_1, H_2, \dots, H_R\}$. For simplicity let $\Theta = \{1, 2, \dots, R\}$, i.e., the set of the product indexes.

c) Nodes' production adoption behavior: Two mechanisms define the individuals' behavior: the social conversion and the self conversion. The following two assumptions propose respectively two models different in the chronological order of the social and self conversions.

Assumption 1 (Social-self conversion model): Consider the competitive propagation of R products in the network $G = (V,E)$. At time step $t+1$ for any $t \in \mathbb{N}$, suppose the previous state of any node i is $D_i(t) = H_r$. Node i first randomly pick one of its neighbor j and following j's previous state, i.e., $D_i(t+1) = D_j(t)$, with probability α_i. If node i does not follow j's state in the social conversion, with probability $1-\alpha_i$, then node i converts to product H_s with probability δ_{rs} for any $s \neq r$, or stay in H_r with probability δ_{rr}.

Assumption 2 (Self-social conversion model): At any time step $t+1$, any node i with $D_i(t) = H_r$ converts to H_s with probability δ_{rs} for any $s \neq r$, or stay in the state H_r with probability δ_{rr}. If node i stays in H_r in the process above, then node i randomly picks a neighbor j and following $D_j(t)$ with probability α_i, or still stay in H_r with probability $1-\alpha_i$.

Assumptions 1 and 2 are illustrated by Figure 1 and Figure 2 respectively. By introducing the parameters δ_{rs} we define a directed and weighted graph with the adjacency matrix $\Delta = (\delta_{rs})_{R\times R}$, referred to as the *product-conversion graph*. Figure 3 gives an example of the product-conversion graph for different smart phone operation systems. Based on either of the two assumptions, Δ is row-stochastic. In this paper we discuss several types of structures of the product-conversion graph, e.g., the case when it is strongly connected, or consists of a transient subgraph and some isolated absorbing subgraphs. The parameter α_i characterizes node i's inclination to be influenced by social pressure. Define $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)^\top$ as the individual open-mindedness vector. Assume $0 < \alpha_i < 1$ for any $i \in V$.

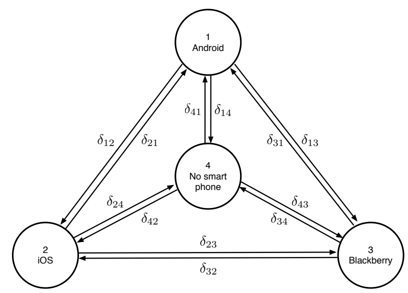


Fig. 3. An example of the product-conversion graph for different smart phone operation systems.

d) Problem description: According to either Assumption 1 or Assumption 2, at any time step $t+1$, the probability distribution of any node's states depends on its own state as well as the states of all its neighbors at time t. Therefore, the collective evolution of nodes' states is a R^n-state discrete-time Markov chain. Define $p_{ir}(t)$ as the probability that node i is in state H_r after time step t, i.e., $p_{ir}(t) = \mathbb{P}[D_i(t) = H_r]$. We aim to understand the dynamics of $p_{ir}(t)$. Since the Markov chain models have exponential dimensions and are difficult to analyze, we approximate it with lower-dimension difference equations systems and analyze instead the dynamical properties of the approximation systems.

e) Notations: Before proceeding to the next section, we introduce some frequently used notations in Table I.

TABLE I
NOTATIONS FREQUENTLY USED IN THIS PAPER

$\succeq$ ($\preceq$)	entry-wise no less(greater) than
$\succ$ ($\prec$)	entry-wise strictly greater(less) than
$\mathbb{1}_n$, $\mathbb{0}_n$	$(1,1,\dots,1)^\top \in \mathbb{R}^{n\times 1}$, $(0,0,\dots,0)^\top \in \mathbb{R}^{n\times 1}$
$\mathbb{0}_{n\times m}$	$(0)_{n\times m}$
V	set of individuals. $V=\{1,\dots,n\}$
Θ	set of products. $\Theta=\{1,\dots,R\}$
$S_{nm}(\boldsymbol{a})$	the set $\{\boldsymbol{X}\in\mathbb{R}^{n\times m}\,\vert\, X\succeq \mathbb{0}_{n\times m}, X\mathbb{1}_m=\boldsymbol{a}\}$ for any $\boldsymbol{a}\in\mathbb{R}^n$
$\tilde{S}_{nm}(\boldsymbol{a})$	the set $\{\boldsymbol{X}\in\mathbb{R}^{n\times m}\,\vert\, X\succeq \mathbb{0}_{n\times m}, X\mathbb{1}_m\preceq\boldsymbol{a}\}$ for any $\boldsymbol{a}\in\mathbb{R}^n$
$\boldsymbol{w}(M)$	the normalized dominant left eigenvector for matrix $M\in\mathbb{R}^{l\times l}$ if it has one
$\boldsymbol{x}^r$	the r-th column vector of the matrix $X\in\mathbb{R}^{n\times m}$
$\boldsymbol{x}^{(i)}$	the i-th row vector of the matrix $X\in\mathbb{R}^{n\times m}$
$\boldsymbol{x}^{(-i)}$	the i-th row vector of the matrix $\tilde{A}X\in\mathbb{R}^{n\times m}$, i.e., $\boldsymbol{x}^{(-i)}=(x_{-i1},x_{-i2},\dots,x_{-im})$ where $x_{-ir}=\sum_{j=1}^n \tilde{a}_{ij}x_{jr}$
$G(A)$	the graph with the adjacency matrix A

III. Network Competitive Propagation Model with Social-self conversion

This section is based on Assumption 1. We first derive an approximation model for the time evolution of $p_{ir}(t)$, referred to as the *social-self conversion network competitive propagation model* (social-self NCPM), and then analyze the asymptotic behavior of the approximation model and its relation to the social network topology, the product-conversion graph, the initial condition and the individuals open-mindedness. Further simulation work is presented in the end of this section.

A. Derivation of the social-self NCPM

Some notations are used in this section.

Notation 3: For the competitive propagation of products $\{H_1,H_2,\dots,H_R\}$ on the network $G=(V,E)$,

(1) define the random variable $X_i^r(t)$ by $X_i^r(t)=1$ if $D_i(t)=H_r$; $X_i^r(t)=0$ if $D_i(t)\neq H_r$. Due to the mutual exclusiveness of the products, for any $i\in V$, if $X_i^r(t)=1$, then $X_i^s(t)=0$ for any $s\neq r$;

(2) Define the $n-1$ tuple $\boldsymbol{D}_{-i}(t) = (D_1(t),\dots,D_{i-1}(t),D_{i+1}(t),\dots,D_n(t))$, i.e., the states of all the nodes except node i after time step t;

(3) Define the following notations for simplicity:

$$P_{ij}^{rs}(t)=\mathbb{P}[X_i^r(t)=1\,\vert\, X_j^s(t)=1],$$
$$P_i^r(t;-i)=\mathbb{P}[X_i^r(t)=1\,\vert\,\boldsymbol{D}_{-i}(t)],$$
$$\Gamma_i^r(t;s,-i)=\mathbb{P}[X_i^r(t+1)=1\,\vert\, X_i^s(t)=1,\boldsymbol{D}_{-i}(t)].$$

In the derivation of the network competitive propagation model, the following approximation is adopted:

Approximation 4 (Independence Approximation): For the competitive propagation of R products on the network $G=(V,E)$, approximate the conditional probability $P_{ij}^{ms}(t)$ by its corresponding total probability $p_{im}(t)$ for any $m,s\in\Theta$ and any $i,j\in V$.

With the *independence approximation*, the social-self NCPM is presented in the theorem below.

Theorem 5 (Social-self NCPM): Consider the competitive propagation based on Assumption 1, with the social network and the product-conversion graph represented by their adjacency matrices $\tilde{A}=(\tilde{a}_{ij})_{n\times n}$ and $\Delta=(\delta_{rs})_{R\times R}$ respectively. The probability $p_{ir}(t)$ satisfies

$$\begin{aligned} &p_{ir}(t+1)-p_{ir}(t)\\ &\quad=\sum_{s\neq r}\alpha_i\sum_{j=1}^n\tilde{a}_{ij}\big(P_{ij}^{sr}(t)p_{jr}(t)-P_{ij}^{rs}(t)p_{js}(t)\big)\\ &\qquad+\sum_{s\neq r}(1-\alpha_i)(\delta_{sr}p_{is}(t)-\delta_{rs}p_{ir}(t)), \end{aligned} \tag{1}$$

for any $i\in V$ and $r\in\Theta$. Applying the independence approximation, the approximation model for equation (1), i.e., the social-self NCPM, is

$$\begin{aligned} &p_{ir}(t+1)\\ &\quad=\alpha_i\sum_{j=1}^n\tilde{a}_{ij}p_{jr}(t)+(1-\alpha_i)\sum_{s=1}^R\delta_{sr}p_{is}(t). \end{aligned} \tag{2}$$

Proof: By definition,

$$p_{ir}(t+1)-p_{ir}(t)=\mathbb{E}\big[\mathbb{E}[X_i^r(t+1)-X_i^r(t)\,\vert\,\boldsymbol{D}_{-i}(t)]\big],$$

where the conditional expectation is given by

$$\begin{aligned} &\mathbb{E}[X_i^r(t+1)-X_i^r(t)\,\vert\,\boldsymbol{D}_{-i}(t)]\\ &\quad=\sum_{s\neq r}\big(\Gamma_i^r(t;s,-i)P_i^s(t;-i)-\Gamma_i^s(t;r,-i)P_i^r(t;-i)\big). \end{aligned}$$

According to Assumption 1,

$$\begin{aligned} &\Gamma_i^r(t;s,-i)P_i^s(t;-i)\\ &\quad=\alpha_i\sum_j\tilde{a}_{ij}X_j^r(t)P_i^s(t;-i)+(1-\alpha_i)\delta_{sr}P_i^s(t;-i). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[\Gamma_i^r(t;s,-i)P_i^s(t;-i)]&=\alpha_i\sum_j\tilde{a}_{ij}\mathbb{E}[X_j^r(t)P_i^s(t;-i)]\\ &\quad+(1-\alpha_i)\delta_{sr}\mathbb{E}[P_i^s(t;-i)]. \end{aligned}$$

One the right-hand side of the equation above, $\mathbb{E}[P_i^s(t;-i)]=p_{is}(t)$. Moreover,

$$\begin{aligned} &\mathbb{E}[X_j^r(t)P_i^s(t;-i)]\\ &\quad=\sum_{\boldsymbol{d}_{-i-j}}\mathbb{P}[X_i^s(t)=1,X_j^r(t)=1,\boldsymbol{D}_{-i-j}(t)=\boldsymbol{d}_{-i-j}]\\ &\quad=P_{ij}^{sr}(t)p_{jr}(t). \end{aligned}$$

Apply the same computation to $\mathbb{E}[\Gamma_i^s(t;r,-i)P_i^r(t;-i)]$ and then we obtain equation (1). Replace $P_{ij}^{sr}(t)$ and $P_{ij}^{rs}(t)$ by $p_{is}(t)$ and $p_{ir}(t)$ respectively and according to the equations $\sum_{s\neq r} p_{is}(t) = 1 - p_{ir}(t)$ and $\sum_{s\neq r} \delta_{rs} = 1 - \delta_{rr}$, we obtain equation (2). ■

The derivation of Theorem 5 is equivalent to the widely adopted mean-field approximation in the modeling of the network epidemic spreading [19], [33], [34]. Notice that the independence approximation neither neglects the correlation between any two nodes' states, nor destroys the network topology, since $p_{jr}(t)$, $p_{js}(t)$ and $\tilde{a}_{ij}$ all appear in the dynamics of $p_{ir}(t)$.

B. Asymptotic behavior of the social-self NCPM

Define the map $f : \mathbb{R}^{n\times R} \to \mathbb{R}^{n\times R}$ by

$$f(X) = \mathrm{diag}(\boldsymbol{\alpha})\tilde{A}X + (I - \mathrm{diag}(\boldsymbol{\alpha}))X\Delta. \quad (3)$$

According to equation (2), the matrix form of the social-self NCPM is written as

$$P(t+1) = f\big(P(t)\big), \quad (4)$$

where $P(t) = (p_{ir}(t))_{n\times R}$. We analyze how the asymptotic behavior of system (4), i.e., the existence, uniqueness and stability of the fixed point of the map f, is determined by the two graphs introduced in our model: the social network with the adjacency matrix $\tilde{A}$, and the product-conversion graph with the adjacency matrix Δ.

1) Structures of the social network and the product-conversion graph: Assume that the social network $G(\tilde{A})$ has a globally reachable node. As for the product-conversion graph, we consider the more general case. Suppose that the product-conversion graph $G(\Delta)$ has m absorbing strongly connected components (absorbing SCCs) and a transient subgraph. Re-index the products such that the product index set for any l-th absorbing SCCs is given by $\Theta_1 = \{1, 2, \dots, k_1\}$, and

$$\Theta_l = \Big\{ \sum_{u=1}^{l-1} k_u + 1, \sum_{u=1}^{l-1} k_u + 2, \dots, \sum_{u=1}^{l} k_u \Big\},$$

for any $l \in \{2, 3, \dots, m\}$, and the index set for the transient subgraph is $\Lambda = \{\sum_{l=1}^{m} k_l + 1, \dots, \sum_{l=1}^{m} k_l + 2, \dots, R\}$. then the adjacency matrix Δ of the product-conversion graph takes the following form:

$$\Delta = \begin{bmatrix} \bar{\Delta} & \mathbb{0}_{(R-k_0)\times k_0} \\ B_{k_0\times(R-k_0)} & \Delta_0 \end{bmatrix}, \quad (5)$$

where $\bar{\Delta} = \mathrm{diag}[\Delta_1, \Delta_2, \dots, \Delta_m]$ and $B = [B_1, B_2, \dots, B_m]$, with $B_l \in \mathbb{R}^{k_0\times k_1}$ for any $l \in \{1, 2, \dots, m\}$, is nonzero and entry-wise non-negative. Matrix $\Delta_l = (\delta_{rs}^{\Theta_l})_{k_l\times k_l}$, with $\delta_{rs}^{\Theta_1} = \delta_{rs}$ and $\delta_{rs}^{\Theta_l} = \delta_{\sum_{u=1}^{l-1} k_u + r, \sum_{u=1}^{l-1} k_u + s}$ for any $l \in \{2, 3, \dots, m\}$, is the adjacency matrix of the l-th absorbing SCC, and is thus irreducible and row-stochastic. The following definition classifies four types of structures of $G(\Delta)$.

Definition 6 (Four sets of product-conversion graphs): Based on whether the product-conversion graph $G(\Delta)$ has a transient subgraph and a single or multiple absorbing SCCs, we classify the adjacency matrix Δ into the following four cases:

(i) Case 1 (single SCC): The graph $G(\Delta)$ is strongly connected, i.e., $\Delta = \Delta_1$, with $k_1 = R$;
(ii) Case 2 (single SCC + transient subgraph): The graph $G(\Delta)$ contains one absorbing SCC and a transient subgraph, i.e., $\bar{\Delta} = \Delta_1$ and $k_0 \geq 1$;
(iii) Case 3 (multi-SCC): The graph $G(\Delta)$ contains m absorbing SCCs, i.e., $\Delta = \mathrm{diag}[\Delta_1, \Delta_2, \dots, \Delta_m]$, with $\sum_{l=1}^{m} k_l = R$;
(iv) Case 4 (multi-SCC + transient subgraph): The graph $G(\Delta)$ contains m absorbing SCCs and a transient subgraph, with Δ given by equation (5).

2) Stability analysis of the social-self NCPM: The following theorem states the distinct asymptotic behaviors of the social-self NCPM, with different structures of the product-conversion graph.

Theorem 7 (Asymptotic behavior for social-self NCPM): Consider the social-self NCPM on a strongly connected social network $G(\tilde{A})$, with the product-conversion graph $G(\Delta)$. Assume that

(i) Each absorbing SCC $G(\Delta_l)$ of $G(\Delta)$ is aperiodic;
(ii) For any Δ_l, $l \in \{1, 2, \dots, m\}$, as least one column of Δ_l is entry-wise strictly positive;
(iii) For any $r \in \Lambda$, $\sum_{s\in\Lambda} \delta_{rs} < 1$, i.e., $\Delta_0 \mathbb{1}_{k_0} \prec \mathbb{1}_{k_0}$.

Then, for any $P(0) \in S_{nR}(\mathbb{1}_n)$, the solution $P(t)$ to equation (4) has the following properties, depending upon the structure of Δ:

(i) in Case 1, $P(t)$ converges to $P^* = \mathbb{1}_n \boldsymbol{\Delta}^\top$ exponentially fast, where P^* is the unique fixed point in $S_{nR}(\mathbb{1}_n)$ for the map f defined by equation (3). Moreover, the convergence rate is $\epsilon(\Delta) = \alpha_{\max} + (1 - \alpha_{\max})\zeta(\Delta)$, where $\alpha_{\max} = \max_i \alpha_i$ and $\zeta(\Delta) = 1 - \sum_{r=1}^{R} \min_s \delta_{sr}$;
(ii) in Case 2, for any $i \in V$,
$$\lim_{t\to\infty} p_{ir}(t) = \begin{cases} 0, & \text{for any } r \in \Lambda, \\ w_r(\Delta_1), & \text{for any } r \in \Theta_1; \end{cases}$$
(iii) in Case 3, for any $l \in \{1, 2, \dots, m\}$ and $i \in V$,
$$\lim_{t\to\infty} \boldsymbol{p}^{\Theta_l(i)}(t) = \big(\boldsymbol{w}^\top(M) P^{\Theta_l}(0) \mathbb{1}_{k_l}\big) \boldsymbol{w}^\top(\Delta_l),$$
where $M = \mathrm{diag}(\boldsymbol{\alpha})\tilde{A} + I - \mathrm{diag}\,\boldsymbol{\alpha}$ and $P^{\Theta_l}(t) = \big(p_{ir}^{\Theta_l}(t)\big)_{n\times k_l}$, with $p_{ir}^{\Theta_l}(t) = p_{i,\sum_{u=1}^{l-1} k_u + r}(t)$ and $\boldsymbol{p}^{\Theta_l(i)}(t)$ being the i-th row of $P^{\Theta_l}(t)$;
(iv) in Case 4, for any $l \in \{1, 2, \dots, m\}$ and $i \in V$,
$$\lim_{t\to\infty} p_{ir}(t) = \begin{cases} 0, & \text{for any } r \in \Lambda, \\ \gamma_l w_r(\Delta_l), & \text{for any } r \in \Theta_l, \end{cases}$$
where γ_l depends on $\tilde{A}$, B_l, $P^{\Theta_l}(0)$, $P^{\Lambda}(0)$ and satisfies $\sum_{l=1}^{m} \gamma_l = 1$.

Before proving the theorem above, a useful and well-known lemma is stated without the proof.

Lemma 8 (Row-stochastic matrices after pairwise-difference similarity transform): Let $M \in \mathbb{R}^{n\times n}$ be row-stochastic. Suppose the graph $G(M)$ is aperiodic and has a globally reachable node. Then the nonsingular matrix

$$Q = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ 1/n & \dots & 1/n & 1/n \end{bmatrix}$$

satisfies

$$QMQ^{-1} = \begin{bmatrix} M_{\text{red}} & \mathbb{0}_{n-1} \\ \boldsymbol{c}^\top & 1 \end{bmatrix}$$

for some $\boldsymbol{c} \in \mathbb{R}^{n-1}$ and $M_{\text{red}} \in \mathbb{R}^{(n-1)\times(n-1)}$. Moreover, M_{red} is discrete-time exponentially stable.

Proof of Theorem 7: (1) Case 1:

Since matrix Δ is row-stochastic, irreducible and aperiodic, according to the Perron-Frobenius theorem, $\boldsymbol{w}(\Delta) \in \mathbb{R}^R$ is well-defined. By substituting P^*, defined by $\boldsymbol{p}^{*(i)} = \boldsymbol{w}(\Delta)^\top$ for any $i \in V$, into equation (3), we verify that P^* is a fixed point of f.

For any X and $Y \in \mathbb{R}^{n\times R}$, define the distance $d(\cdot,\cdot)$ by $d(X,Y) = \|X - Y\|_\infty$. Then $(S_{nR}(\mathbb{1}_n), d)$ is a complete metric space. For any $X \in S_{nR}(\mathbb{1}_n)$, it is easy to check that $f(X) \succeq \mathbb{0}_{n\times R}$ and

$$f(X)\mathbb{1}_R = \operatorname{diag}(\boldsymbol{\alpha})\tilde{A}X\mathbb{1}_R + (I - \operatorname{diag}(\boldsymbol{\alpha}))X\mathbb{1}_R = \mathbb{1}_n.$$

Therefore, f maps $S_{nR}(\mathbb{1}_n)$ to $S_{nR}(\mathbb{1}_n)$.

For any $X \in S_{nR}(\mathbb{1}_n)$, according to equation (3),

$$\begin{aligned} \|f(X)^{(i)} - f(P^*)^{(i)}\|_1 & \\ \le \alpha_i \|\boldsymbol{x}^{(-i)} - \boldsymbol{p}^{*(-i)}\|_1 & \\ + (1-\alpha_i)\|(\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)})\Delta\|_1. & \end{aligned} \tag{6}$$

The first term of the right-hand side of (6) satisfies

$$\begin{aligned} \|\boldsymbol{x}^{(-i)} - \boldsymbol{p}^{*(-i)}\|_1 &\le \sum_{r=1}^R \sum_{j=1}^n \tilde{a}_{ij}|x_{jr} - w_r(\Delta)| \\ &\le \|X - P^*\|_\infty. \end{aligned}$$

The second term of the right-hand side of (6) satisfies

$$\|(\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)})\Delta\|_1 = \sum_{r=1}^R \Big|\sum_{s=1}^R \big(x_{is} - w_s(\Delta)\big)\delta_{sr}\Big|.$$

If $\boldsymbol{x}^{(i)} = \boldsymbol{p}^{*(i)}$, then $\|f(X)^{(i)} - f(P^*)^{(i)}\|_1 \le \alpha_i\|X - P^*\|_\infty$. If $\boldsymbol{x}^{(i)} \ne \boldsymbol{p}^{*(i)}$, since $\boldsymbol{x}^{(i)}\mathbb{1}_R = \boldsymbol{p}^{*(i)}\mathbb{1}_R = 1$, both the set $\theta_1 = \{s \,|\, x_{is} \ge w_s(\Delta)\}$ and the set $\theta_2 = \{s \,|\, x_{is} < w_s(\Delta)\}$ are nonempty and

$$\begin{aligned} \sum_{s\in\theta_1}\big(x_{is} - w_s(\Delta)\big) &= \sum_{s\in\theta_2}\big(w_s(\Delta) - x_{is}\big) \\ &= \frac{1}{2}\sum_{s=1}^R |x_{is} - w_s(\Delta)|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|(\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)})\Delta\|_1 \\ &\quad= \sum_{r=1}^R \sum_{s=1}^R |x_{is} - w_s(\Delta)|\delta_{sr} \\ &\qquad - 2\sum_{r=1}^R \min\{\sum_{s\in\theta_1}(x_{is} - w_s(\Delta))\delta_{sr}, \\ &\qquad\qquad \sum_{s\in\theta_2}(w_s(\Delta) - x_{is})\delta_{sr}\}, \text{ where} \end{aligned} \tag{7}$$

$$\begin{aligned} &\min\{\sum_{s\in\theta_1}(x_{is} - w_s(\Delta))\delta_{sr}, \sum_{s\in\theta_2}(w_s(\Delta) - x_{is})\delta_{sr}\} \\ &\qquad \ge \frac{1}{2}\min_s \delta_{sr}\|\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)}\|_1. \end{aligned}$$

Substituting the inequality above into (7), we obtain

$$\|(\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)})\Delta\|_1 \le \Big(1 - \sum_{r=1}^R \min_s \delta_{sr}\Big)\|\boldsymbol{x}^{(i)} - \boldsymbol{p}^{*(i)}\|_1.$$

Since $\sum_{r=1}^R \delta_{sr} = 1$ for any s, $\sum_{r=1}^R \min_s \delta_{sr}$ is no larger than 1. In addition, since at least one column of Δ is strictly positive, $\sum_{r=1}^R \min_s \delta_{sr} > 0$. Therefore, $0 \le \zeta(\Delta) = 1 - \sum_{r=1}^R \min_s \delta_{sr} < 1$, and

$$\|f(X)^{(i)} - \boldsymbol{p}^{*(i)}\|_1 \le \big(\alpha_i + (1-\alpha_i)\zeta(\Delta)\big)|X - P^*\|_\infty.$$

This leads to

$$\|f(X) - f(P^*)\|_\infty \le \epsilon(\Delta)\|X - P^*\|_\infty,$$

for any $X \in S_{nR}(\mathbb{1}_n)$ and $0 < \epsilon(\Delta) < 1$. This concludes the proof for Case 1.

(2) Case 2:

For the transient subset Λ, define $P^\Lambda(t) = \big(p_{ir}^\Lambda(t)\big)_{n\times k_0}$, with $p_{ir}^\Lambda(t) = p_{i,r+k_1}(t)$, for any $i \in V$ and $r \in \{1, 2, \dots, k_0\}$. Then,

$$P^\Lambda(t+1) = \operatorname{diag}(\boldsymbol{\alpha})\tilde{A}P^\Lambda(t) + (I - \operatorname{diag}(\boldsymbol{\alpha}))P^\Lambda(t)\Delta_0.$$

According to Assumption (iii) of Theorem 7,

$$c = \max_{r\in\{1,2,\dots,k_0\}} \sum_{s=1}^{k_0} \delta_{rs}^\Lambda < 1, \quad \text{and} \quad \Delta_0\mathbb{1}_{k_0} \le c\mathbb{1}_{k_0}.$$

Therefore,

$$\begin{aligned} &P^\Lambda(t+1)\mathbb{1}_{k_0} \\ &\qquad \preceq \Big(\operatorname{diag}(\boldsymbol{\alpha})\tilde{A} + c\big(I - \operatorname{diag}(\boldsymbol{\alpha})\big)\Big)P^\Lambda(t)\mathbb{1}_{k_0}. \end{aligned}$$

Since $\rho\Big(\operatorname{diag}(\boldsymbol{\alpha})\tilde{A} + c\big(I - \operatorname{diag}(\boldsymbol{\alpha})\big)\Big) < 1$, for any $P^\Lambda(0) \in \tilde{S}_{nk_0}(\mathbb{1}_n)$, $P^\Lambda(t) \to \mathbb{0}_{n\times k_0}$ exponentially fast.

Define $P^{\Theta_1}(t) = (p_{ir}(t))_{n\times k_1}$. then we have

$$\begin{aligned} &P^{\Theta_1}(t+1) \\ &\quad = \operatorname{diag}(\boldsymbol{\alpha})\tilde{A}P^{\Theta_1}(t) + \big(I - \operatorname{diag}(\boldsymbol{\alpha})\big)P^{\Theta_1}(t)\Delta_1 \\ &\qquad + \big(I - \operatorname{diag}(\boldsymbol{\alpha})\big)P^\Lambda(t)B. \end{aligned}$$

Since $P^{\Lambda}(t)$ converges to $\mathbb{0}_{n\times k_0}$ exponentially fast, we have: 1) there exists $C>0$ and $0<\xi<1$ such that

$$\|\big(I-\operatorname{diag}(\boldsymbol{\alpha})P^{\Lambda}(t)B\big)\|_\infty \le C\xi^t;$$

2) $\|P^{\Theta_1}(t)\mathbb{1}_{k_1}-\mathbb{1}_{k_1}\|_\infty \to 0$ exponentially fast, which implies $d\big(P^{\Theta_1}(t), S_{nk_1}(\mathbb{1}_n)\big)\to 0$ exponentially fast.

For any $X\in \tilde{S}_{nk_1}(\mathbb{1}_n)$, define map $\tilde{f}$ by

$$\tilde{f}(X)=\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}X+\big(I-\operatorname{diag}(\boldsymbol{\alpha})\big)X\Delta_1.$$

According to the proof for Case 1, there exists a unique fixed point $\tilde{P}^*$ for the map $\tilde{f}$ in $S_{nk_1}(\mathbb{1}_n)$, given by $\tilde{p}^*_{ir}=w_r(\Delta_1)$. Moreover, there exists $0<\epsilon<1$ such that, for any $X\in S_{nk_1}(\mathbb{1}_n)$,

$$\|\tilde{f}(X)-\tilde{P}^*\|_\infty \le \epsilon\|X-\tilde{P}^*\|_\infty.$$

Since the function $\frac{\|\tilde{f}(X)-\tilde{P}^*\|_\infty}{\|X-\tilde{P}^*\|_\infty}$ is continuous in $\tilde{S}_{nk_1}(\mathbb{1}_n)\setminus\tilde{P}^*$ and $d\big(P^{\Theta_1}(t), S_{nk_1}(\mathbb{1}_n)\big)\to 0$, there exists $T>0$ and $0<\eta<1$ such that, for any $t>T$,

$$\|\tilde{f}\big(P^{\Theta_1}(t)\big)-\tilde{P}^*\|_\infty \le \eta\|P^{\Theta_1}(t)-\tilde{P}^*\|_\infty.$$

For $t\in\mathbb{N}$ much larger than T,

$$\begin{aligned}&\|P^{\Theta_1}(t)-\tilde{P}^*\|_\infty\\ &\quad\le \eta^{t-T}\|P^{\Theta_1}(T)-\tilde{P}^*\|_\infty + C\frac{\xi^t-\eta^{t-T}\xi^T}{\eta/\xi}.\end{aligned}$$

Since $0<\eta<1$, $0<\xi<1$ as $t\to\infty$, $\|P^{\Theta_1}(t)-\tilde{P}^*\|_\infty\to 0$. This concludes the proof for Case 2.

(3) Case 3:

For any $l\in\{1,2,\dots,m\}$,

$$\begin{aligned}&P^{\Theta_l}(t+1)\\ &\quad=\hat{f}\big(P^{\Theta_l}(t)\big)\\ &\quad=\big(I-\operatorname{diag}(\boldsymbol{\alpha})\big)P^{\Theta_l}(t)\Delta_l+\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}P^{\Theta_l}(t),\end{aligned}$$

where $\Delta_l\mathbb{1}_{k_l}=\mathbb{1}_{k_l}$ since Θ_l is absorbing and strongly connected. Therefore,

$$P^{\Theta_l}(t+1)\mathbb{1}_{k_l}=MP^{\Theta_l}(t)\mathbb{1}_{k_l},$$

where $M=I-\operatorname{diag}(\boldsymbol{\alpha})+\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}$ is row-stochastic and aperiodic. Moreover, the graph $G(M)$ has a globally reachable node and therefore the matrix M has a normalized dominant left eigenvector $\boldsymbol{w}(M)$. Applying the Perron-Frobenius theorem,

$$\lim_{t\to\infty}P^{\Theta_l}(t)\mathbb{1}_{k_l}=\big(\boldsymbol{w}^\top(M)P^{\Theta_l}(0)\mathbb{1}_{k_l}\big)\mathbb{1}_n.$$

Let $c_l=\boldsymbol{w}^\top(M)P^{\Theta_l}(0)\mathbb{1}_{k_l}$. Following the same line of argument in the proof for Case 2, $\hat{f}$ maps $S_{nk_l}(c_l\mathbb{1}_n)$ to $S_{nk_l}(c_l\mathbb{1}_n)$, and maps $\tilde{S}_{nk_l}(c_l\mathbb{1}_n)$ to $\tilde{S}_{nk_l}(c_l\mathbb{1}_n)$. Moreover, $\hat{P}^*\in\mathbb{R}^{n\times k_l}$ with $\hat{\boldsymbol{p}}^{*(i)}=c_l\boldsymbol{w}^\top(\Delta_l)$, for any $i\in V$, is the unique fixed point of the map $\hat{f}$ in $S_{nk_l}(c_l\mathbb{1}_n)$. In addition, there exists $0<\epsilon<1$ such that for any $X\in S_{nk_l}(c_l\mathbb{1}_n)$,

$$\|\hat{f}(X)-\hat{P}^*\|_\infty\le\epsilon\|X-\hat{P}^*\|_\infty.$$

The function $\hat{h}(X)=\frac{\|\hat{f}(X)-\hat{P}^*\|_\infty}{\|X-\hat{P}^*\|_\infty}$ is continuous in $\tilde{S}_{nk_l}(c_l\mathbb{1}_n)\setminus\hat{P}^*$. Since for any $P^{\Theta_l}(0)\in\tilde{S}_{nk_l}(c_l\mathbb{1}_n)\setminus\hat{P}^*$, we have $P^{\Theta_l}(t)\mathbb{1}_{k_l}\to c_l\mathbb{1}_{k_l}$, which implies $d\big(P^{\Theta_l}(t), S_{nk_l}(c_l\mathbb{1}_{k_l})\big)\to 0$ as $t\to 0$. Therefore, there exists $0<\eta<1$ and $T>0$ such that for any $t>T$,

$$\|\hat{f}\big(P^{\Theta_l}(t)\big)-\hat{P}^*\|_\infty\le\eta\|P^{\Theta_l}(t)-\hat{P}^*\|_\infty.$$

Therefore, $P^{\Theta_l}(t)\to\hat{P}^*$ as $t\to\infty$.

(4) Case 4:

$$\begin{aligned}P^{\Theta_l}(t+1)&=\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}P^{\Theta_l}(t)\\ &\quad+\big(I-\operatorname{diag}(\boldsymbol{\alpha})\big)P^{\Theta_l}(t)\Delta_l\\ &\quad+\big(I-\operatorname{diag}(\boldsymbol{\alpha})\big)P^{\Lambda}(t)B_l.\end{aligned}$$

for any $l\in\{1,2,\dots,m\}$. Therefore,

$$P^{\Theta_l}(t+1)\mathbb{1}_{k_l}=MP^{\Theta_l}(t)\mathbb{1}_{k_l}+\boldsymbol{\phi}(t),\qquad(8)$$

where $M=\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}+I-\operatorname{diag}(\boldsymbol{\alpha})$ is row-stochastic and primitive. The vector $\boldsymbol{\phi}(t)$ is a vanishing perturbation according to the proof for Case 2.

Let $\boldsymbol{x}(t)=P^{\Theta_l}(t)\mathbb{1}_{k_l}$ and $\boldsymbol{y}(t)=Q\boldsymbol{x}(t)$ with Q defined in Lemma 8. Let $\boldsymbol{y}_{\text{err}}(t)=(y_1(t),y_2(t),\dots,y_{n-1}(t))^\top$, where $y_i(t)=x_{i+1}(t)-x_i(t)$ for any $i=1,2,\dots,n-1$. Then we have

$$\boldsymbol{y}(t+1)=QMQ^{-1}\boldsymbol{y}(t)+Q\boldsymbol{\phi}(t).$$

Let $\boldsymbol{\varphi}(t)=\big(\varphi_1(t),\varphi_2(t),\dots,\varphi_{n-1}(t)\big)^\top$ with $\varphi_i(t)=\sum_j Q_{ij}\phi_j(t)$. $\boldsymbol{\varphi}(t)$ is also a vanishing perturbation and

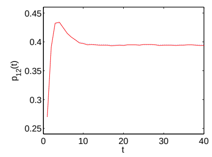
$$\boldsymbol{y}_{\text{err}}(t+1)=M_{\text{red}}\boldsymbol{y}_{\text{err}}(t)+\boldsymbol{\varphi}(t).$$

The equation above is an exponentially stable linear system with a vanishing perturbation. Since $\rho(M_{\text{red}})<1$, $\boldsymbol{y}_{\text{err}}\to\mathbb{0}_{n-1}$ as $t\to\infty$, which implies that $P^{\Theta_l}(t)\mathbb{1}_{k_l}\to\gamma\mathbb{1}_n$ and γ_l depends on M, B_l, $P^{\Theta_l}(0)$ and $P^{\Lambda}(0)$. Moreover, $\sum_l\gamma_l=1$ since $P(t)\mathbb{1}_R=\mathbb{1}_n$. Following the same argument in the proof for Case 3, we obtain

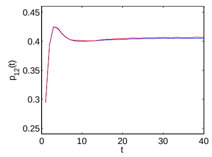
$$\lim_{t\to\infty}\boldsymbol{p}^{\Theta_l(i)}(t)=\gamma_l\boldsymbol{w}^\top(\Delta_l).$$

■

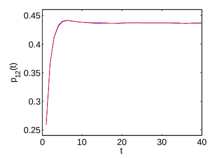
3) Interpretations of Theorem 7: Analysis on Case 1 to 4 leads to the following conclusions: 1) The probability of adopting any product in the transient subgraph eventually decays to zero; 2) For the product-conversion graph with only one absorbing SCC $G(\Delta_1)$, the system's asymptotic product-adoption probability distribution only depends on $\boldsymbol{w}(\Delta_1)$. In this case, the self conversion dominates the competitive propagation



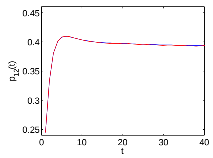
(a) $n = 5$, $p = 1$



(b) $n = 10$, $p = 0.5$



(c) $n = 50$, $p = 0.1$



(d) $n = 50$, $p = 1$

Fig. 4. Difference between the solutions to the social-self NCPM (blue dash) and the original Markov-chain model (red) in complete graphs or Erdős-Rényi graphs.

process; 3) With multiple absorbing SCCs in the product-conversion graph, the initial condition $P(t)$ and the structure of the social network $G(\tilde{A})$ together determine the fraction each absorbing SCC eventually takes in the total probability 1; 4) In each absorbing SCC $G(\Delta_l)$, the asymptotic adoption probability for each product is proportional to its corresponding entry of Δ_l.

C. Further simulation work

a) Accuracy of the social-self NCPM solution: Simulation results have been presented to compare the solution to the social-self NCPM with the solution to the original Markov chain model defined by Assumption 1. Let the matrix Δ take the following form

$$\Delta = \begin{bmatrix} \Delta_1 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \Delta_2 & \mathbb{0} \\ B_1 & B_2 & \Delta_0 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.8 & 0 & 0.2 \end{bmatrix}. \quad (9)$$

The Markov-chin solution is computed by the Monte Carlo method. In each sampling, A, $\boldsymbol{\alpha}$ and $P(0)$ are randomly generated and set identical for the Markov chain and the NCPM. The probability $p_{12}(t)$ is plotted for both models on different types of social networks, such as the complete graph, the Erdős-Rényi graph, the power-law graph and the star graph. As shown in Figure 4 and Figure 5, the solution to the social-self NCPM nearly overlaps with the Markov-chain solution in every plot, due to the i.i.d self conversion process.

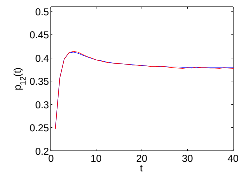
b) Asymptotic behavior of the Markov chain model In Figure 6 and Figure 7, all the trajectories $p_{ir}(t)$, for the Markov-chain model on an Erdős-Rényi graph with $n = 5, p = 0.4$ and randomly generated $\boldsymbol{\alpha}$, are computed by the Monte Carlo method. Figure 6(a) corresponds to the structure of the product-conversion graph defined by Case 4 in Definition 6 with

$$\Delta_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \Delta_2 = 1, \Delta_0 = 0.2, B = [0\ 0.8\ 0].$$

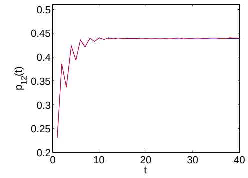
The transient subgraph is only connected to SCC Θ_1 and the intial adoption probability for H_3 is 0. Figure 6(b) corresponds to the structure of the product-conversion graph defined by Case 3 in Defintion 6 with

$$\Delta = \begin{bmatrix} \Delta_1 & \mathbb{0} \\ \mathbb{0} & \Delta_2 \end{bmatrix}, \Delta_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix}.$$

The simulation results shows that, in these two cases the Markov-chain solutions converge exactly to the values indicated by the social-self NCPM, regardless of the initial condition. The matrix Δ used in Figure 7 is given by equation (9). As illustrated by Figure 7, the asymptotic adoption probabilities vary with the initial condition in the Markov-chain model, in consistence with the results of Theorem 7.

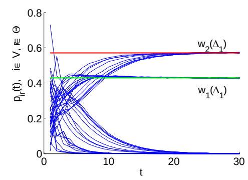


(a) power-law graph

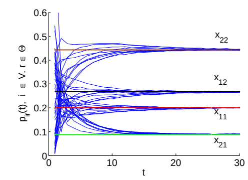


(b) star graph

Fig. 5. Difference between the solutions to the social-self NCPM (blue dash) and the original Markov-chain model (red) in the power-law graph and the star graph. The power-law graph has 100 nodes, with the degree distribution $p(k) = 1010k^{-2.87}$. The star graph consists of 10 nodes with node 1 as the center.



(a) multi-SCC + transient subgraph

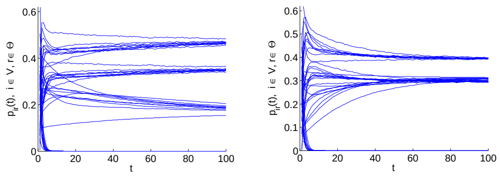


(b) multi-SCC

Fig. 6. Asymptotic behavior of the Markov chain model with the production-conversion graphs defined by Case 3 or Case 4 in Definition 6. Every curve in this plot is a trajectory $p_{ir}(t)$ for $i \in V$ and $r \in \Theta$. Here $x_{lr} = \boldsymbol{w}^\top(M)P^{\Theta_l}(0)\mathbb{1}_{k_l}w_r(\Delta_l)$.

IV. Analysis on the Self-social Network Competitive Propagation Model

In this section we discuss the network competitive propagation model based on Assumption 2, i.e, the case in which self conversion occurs before social conversion at each time step. Firstly we propose an approximation model, referred to as the *self-social network competitive*



(a) initial condition 1 (b) initial condition 2

Fig. 7. Asymptotic behavior of the Markov chain model with the production-conversion graph consisting of multiple SCCs and a transient subgraph. Every curve in this plot is a trajectory $p_{ir}(t)$ for $i \in V$ and $r \in \Theta$.

propagation model (self-social NCPM), and then analyze the dynamical properties of this approximation model.

Theorem 9 (Self-social NCPM): Consider the competitive propagation model based on Assumption 2, with the social network and the product-conversion graph represented by their adjacency matrices $\tilde{A}$ and Δ respectively. The probability $p_{ir}(t)$ satisfies

$$\begin{aligned} p_{ir}(t+1) - p_{ir}(t) = &\sum_{s\neq r} \big(\delta_{sr} p_{is}(t) - \delta_{rs} p_{ir}(t)\big) \\ &+ \sum_{s\neq r} \delta_{ss}\alpha_i \sum_{j=1}^{n} \tilde{a}_{ij} p_{is}(t) P_{ji}^{rs}(t) \\ &- \sum_{s\neq r} \delta_{rr}\alpha_i \sum_{j=1}^{n} \tilde{a}_{ij} p_{ir}(t) P_{ji}^{sr}(t), \end{aligned}$$

for any $i \in V$ and $r \in \Theta$. Applying the independence assumption, the matrix form of the self-social NCPM is

$$\begin{aligned} &P(t+1) \\ &\quad = P(t)\Delta + \operatorname{diag}(\boldsymbol{\alpha}) \operatorname{diag}\big(P(t)\boldsymbol{\delta}\big)\tilde{A}P(t) \qquad (10) \\ &\quad\quad - \operatorname{diag}(\boldsymbol{\alpha}) P(t) \operatorname{diag}(\boldsymbol{\delta}), \end{aligned}$$

with $P(t) = (p_{ir}(t))_{n\times R}$ and $\boldsymbol{\delta} = (\delta_{11}, \delta_{22}, \dots, \delta_{RR})^\top$.

It is straightforward to check that, for any $P(t) \in S_{nR}(\mathbb{1}_n)$, $P(t+1)$ is still in $S_{nR}(\mathbb{1}_n)$. According to the Brower fixed point theorem, there exists at least one fixed point for the system (10) in $S_{nR}(\mathbb{1}_n)$. Since the nonlinearity of equation (10) add much difficulty to the analysis of it, in the remaining part of this section we discuss the special case when $R = 2$.

For simplicity, in this section, let $\boldsymbol{p}(t) = \boldsymbol{p}_2(t) = \big(p_{12}(t), p_{22}(t), \dots, p_{n2}(t)\big)^\top$. Without loss of generality, assume $\delta_{22} \geq \delta_{11}$. Define the map $h : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{aligned} h(\boldsymbol{x}) = &\,\delta_{12}\mathbb{1}_n + (1 - \delta_{12} - \delta_{21})\boldsymbol{x} \\ &+ \delta_{11} \operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{x} - \delta_{22} \operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{x} \qquad (11) \\ &+ (\delta_{22} - \delta_{11}) \operatorname{diag}(\boldsymbol{\alpha}) \operatorname{diag}(\boldsymbol{x})\tilde{A}\boldsymbol{x}. \end{aligned}$$

Then the self-social NCPM for $R = 2$ is written as

$$\boldsymbol{p}(t+1) = h(\boldsymbol{p}(t)), \qquad (12)$$

and $\boldsymbol{p}_1(t)$ is computed by $\boldsymbol{p}_1(t) = \mathbb{1}_n - \boldsymbol{p}(t)$.

We present below the main theorem of this section.

Theorem 10 (Dynamical behavior of self-social NCPM with $R = 2$): Consider the two-product self-social NCPM, given by equations (11) and (12), with the parameters $\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \alpha_1, \dots, \alpha_n$ all in the interval $(0, 1)$, and $\delta_{22} \geq \delta_{11}$. We conclude that,

(i) system (12) has a unique fixed point $\boldsymbol{p}^* \in [0,1]^n$;

(ii) the unique fixed point $\boldsymbol{p}^*$ satisfies

$$\frac{1}{2}\mathbb{1}_n \preceq \boldsymbol{p}^* \preceq \frac{\delta_{12}}{\delta_{12}+\delta_{21}}\mathbb{1}_n, \quad \text{and} \qquad (13)$$

$$p_i^* - p_{-i}^* \leq \frac{1-\frac{1}{2}\alpha_i}{\alpha_i}\frac{\delta_{22}-\delta_{11}}{\delta_{22}+\delta_{11}}; \qquad (14)$$

(iii) if $\delta_{22} = \delta_{11}$, the unique fixed point $\boldsymbol{p}^*$ for system (12) is globally exponentially stable; (By "globally" we mean "for any $\boldsymbol{p}(0) \in [0,1]^n$.")

(iv) if $\delta_{22} > \delta_{11}$, and

$$\alpha_i < \frac{8\delta_{11}\delta_{22}}{(\delta_{22}-\delta_{11})^2 + 8\delta_{11}\delta_{22}} \text{ for any } i \in V, \qquad (15)$$

then $\boldsymbol{p}^*$ is locally asymptotically stable;

(v) if $\delta_{22} > \delta_{11}$, and

$$\alpha_i < \frac{\delta_{22}+\delta_{11}}{3\delta_{22}-\delta_{11}} \text{ for any } i \in V, \qquad (16)$$

then $\boldsymbol{p}^*$ is globally exponentially stable. Moreover, the convergence rate is upper bounded by $\max_i\big(\max\{\epsilon_i, K_i\epsilon_i + K_i - 1\}\big)$, where ϵ_i and K_i are defined as $\epsilon_i = (2\delta_{22} - \delta_{11})\alpha_i/K_i$ and $K_i = \delta_{12} + \delta_{21} + \delta_{22}\alpha_i$, respectively.

Proof: We start the proof by establishing that h is a continuous map from $[0,1]^n$ to $[0,1]^n$ itself. Firstly, since

$$\begin{aligned} h(\boldsymbol{x}) = &\,\delta_{12}(\mathbb{1}_n - \boldsymbol{x}) + \delta_{11} \operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{x} \\ &+ (1-\delta_{21})\boldsymbol{x} - \delta_{22} \operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{x} \\ &+ (\delta_{22}-\delta_{11}) \operatorname{diag}(\boldsymbol{\alpha}) \operatorname{diag}(\boldsymbol{x})\tilde{A}\boldsymbol{x}, \quad \text{and} \end{aligned}$$

$$(1-\delta_{21})\boldsymbol{x} - \delta_{22} \operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{x} \succeq (1-\delta_{21}-\delta_{22})\boldsymbol{x} = \mathbb{0}_n,$$

the right-hand side of the expression of h is non-negative. Therefore, for any $\boldsymbol{x} \in [0,1]^n$, $h(\boldsymbol{x}) \succeq \mathbb{0}_n$. Secondly, recall that $x_{-i} = (\tilde{A}\boldsymbol{x})_i = \sum_j \tilde{a}_{ij}x_j$. That is, x_{-i} is the weighted average of all the x_j's except x_i and the value of x_{-i} does not depend on x_i since $\tilde{a}_{ii} = 0$. Moreover, since $\sum_j \tilde{a}_{ij} = 1$ for any $i \in V$, x_{-i} is also in the interval $[0,1]$. According to equation (11), rewrite the i-th entry of $h(\boldsymbol{x})$ as

$$h(\boldsymbol{x})_i = \delta_{12} + \delta_{11}\alpha_i x_{-i} + \eta_i x_i,$$

where $\eta_i = 1 - \delta_{12} - \delta_{21} - \delta_{22}\alpha_i + (\delta_{22} - \delta_{11})\alpha_i x_{-i}$. The maximum value of η_i is $1 - \delta_{12} - \delta_{21} - \delta_{11}\alpha_i$, obtained when $x_{-i} = 1$. Therefore,

$$\eta_i x_i \leq \max(1 - \delta_{12} - \delta_{21} - \delta_{11}\alpha_i,\ 0).$$

Then we have

$$\begin{aligned}h(\boldsymbol{x})_i &\leq \delta_{12} + \delta_{11}\alpha_i + \max(1-\delta_{12}-\delta_{21}-\delta_{11}\alpha_i,\ 0)\\ &= \max(\delta_{22},\ \delta_{12}+\delta_{11}\alpha_i) < 1.\end{aligned}$$

The inequality above leads to $h(\boldsymbol{x}) \preceq \mathbb{1}_n$ for any $\boldsymbol{x} \in [0,1]^n$. Since h maps $[0,1]^n$ to $[0,1]^n$ itself, according to the Brower fixed point theorem, there exists $\boldsymbol{p}^*$ such that $h(\boldsymbol{p}^*) = \boldsymbol{p}^*$. This concludes the proof of the existence of a fixed point.

Any fixed point of h should satisfy $h(\boldsymbol{p}^*) = \boldsymbol{p}^*$, i.e.,

$$\begin{aligned}\mathbb{0}_n = {} & \delta_{12}\mathbb{1}_n + \delta_{11}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{p}^*\\ & + (\delta_{22}-\delta_{11})\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{p}^*)\tilde{A}\boldsymbol{p}^*\\ & - (\delta_{12}+\delta_{21})\boldsymbol{p}^* - \delta_{22}\operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{p}^*.\end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned}\boldsymbol{p}^* = {} & \delta_{12}K^{-1}\mathbb{1}_n + \delta_{11}K^{-1}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{p}^*\\ & + (\delta_{22}-\delta_{11})K^{-1}\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{p}^*)\tilde{A}\boldsymbol{p}^*,\end{aligned}$$

where $K = (\delta_{12}+\delta_{21})I + \delta_{22}\operatorname{diag}(\boldsymbol{\alpha})$ is a positive diagonal matrix. Define a map $T: \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{aligned}T(\boldsymbol{x}) = {} & \delta_{12}K^{-1}\mathbb{1}_n + \delta_{11}K^{-1}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A}\boldsymbol{x}\\ & + (\delta_{22}-\delta_{11})K^{-1}\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{x})\tilde{A}\boldsymbol{x}.\end{aligned} \tag{18}$$

We have that map h has a unique fixed point if and only if map T has a unique fixed point. For any $\boldsymbol{x}$ and $\boldsymbol{y} \in [0,1]^n$, define the distance $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|_\infty$. Then $([0,1]^n, d)$ is a complete metric space. According to equation (18), since K^{-1}, $\operatorname{diag}(\boldsymbol{\alpha})$, $\tilde{A}$, $\delta_{22}-\delta_{11}$ and $\operatorname{diag}(\boldsymbol{x})$ are all nonnegative, for any $\boldsymbol{x}$, $\boldsymbol{y} \in [0,1]^n$ and $\boldsymbol{x} \preceq \boldsymbol{y}$, we have $T(\boldsymbol{x}) \preceq T(\boldsymbol{y})$. Moreover,

$$T(\mathbb{0}_n) = \delta_{12}K^{-1}\mathbb{1}_n \succ \mathbb{0}_n, \quad \text{and}$$

$$\begin{aligned}T(\mathbb{1}_n) &= \delta_{12}K^{-1}\mathbb{1}_n + \delta_{11}K^{-1}\boldsymbol{\alpha} + (\delta_{22}-\delta_{11})K^{-1}\boldsymbol{\alpha}\\ &= \delta_{12}K^{-1}\mathbb{1}_n + \delta_{22}K^{-1}\boldsymbol{\alpha}.\end{aligned}$$

Since

$$T(\mathbb{1}_n)_i = \frac{\delta_{12}+\delta_{22}\alpha_i}{\delta_{12}+\delta_{21}+\delta_{22}\alpha_i} < 1,$$

we have $T(\mathbb{1}_n) \prec \mathbb{1}_n$. Therefore, T maps $[0,1]^n$ to $[0,1]^n$. For any $\boldsymbol{x}$, $\boldsymbol{y} \in [0,1]^n$,

$$\begin{aligned}T(\boldsymbol{x})_i - T(\boldsymbol{y})_i = {} & \frac{\delta_{11}\alpha_i}{K_i}(x_{-i}-y_{-i})\\ & + \frac{(\delta_{22}-\delta_{11})\alpha_i}{K_i}(x_i x_{-i} - y_i y_{-i}).\end{aligned}$$

Moreover,

$$|x_{-i}-y_{-i}| \leq (\sum_{j=1}^n \tilde{a}_{ij}) \max_j |x_j - y_j| = \|\boldsymbol{x}-\boldsymbol{y}\|_\infty,$$

and

$$\begin{aligned}&|x_i x_{-i} - y_i y_{-i}|\\ &\quad \leq \max\big(\max_i y_i^2 - \min_i x_i^2,\ \max_i x_i^2 - \min_i y_i^2\big)\\ &\quad \leq 2\|\boldsymbol{x}-\boldsymbol{y}\|_\infty.\end{aligned}$$

Therefore,

$$|T(\boldsymbol{x})_i - T(\boldsymbol{y})_i| \leq \epsilon_i \|\boldsymbol{x}-\boldsymbol{y}\|_\infty,$$

where $\epsilon_i = \frac{(2\delta_{22}-\delta_{11})\alpha_i}{\delta_{12}+\delta_{21}+\delta_{22}\alpha_i}$. One can check that $\epsilon_i < 1$ for any $i \in V$ and ϵ_i does not depend on the $\boldsymbol{x}$ and $\boldsymbol{y}$. Let $\epsilon = \max_i \epsilon_i$. Then for any $\boldsymbol{x}$, $\boldsymbol{y} \in [0,1]^n$,

$$\|T(\boldsymbol{x}) - T(\boldsymbol{y})\|_\infty \leq \epsilon \|\boldsymbol{x}-\boldsymbol{y}\|_\infty \quad \text{with } \epsilon < 1.$$

Applying the Banach fixed point theorem, we know that the map T possesses a unique fixed point $\boldsymbol{p}^*$ in $[0,1]^n$. In addition, for any $\boldsymbol{p}(0)$, the sequence $\{\boldsymbol{p}(t)\}_{t\in\mathbb{N}}$ defined by $\boldsymbol{p}(t+1) = T\big(\boldsymbol{p}(t)\big)$ satisfies $\lim_{t\to\infty}\boldsymbol{p}(t) = \boldsymbol{p}^*$. This concludes the proof of statement (i).

For statement (ii), one can check that T maps $S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \frac{1}{2}\mathbb{1}_n \preceq \boldsymbol{x} \preceq \frac{\delta_{12}}{\delta_{12}+\delta_{21}}\mathbb{1}_n\}$ to S itself. Since T is a contraction map, the unique fixed point $\boldsymbol{p}^*$ is in S. The concludes the proof for equation (13). According to equation (17), we have $C_i p_i^* - C_{-i} p_{-i}^* = \delta_{12} - \delta_{12} p_i^*$, where $C_i = \delta_{21} + \delta_{22}\alpha_i$ and $C_{-i} = \delta_{11}\alpha_i + (\delta_{22} - \delta_{11})\alpha_i p_i^*$. Firstly we point out that $C_i > C_{-i}$, since $C_i - C_{-i} = \delta_{21} + \alpha_i(\delta_{22}-\delta_{11})(1-p_i^*) > 0$. Moreover,

$$\begin{aligned}&p_i^* - p_{-i}^*\\ &\quad = \frac{\delta_{12} - \big(\delta_{12}+\delta_{21}+\alpha_i(\delta_{22}-\delta_{11})(1-p_i^*)\big)p_i^*}{\delta_{11}\alpha_i + (\delta_{22}-\delta_{11})\alpha_i p_i^*}.\end{aligned}$$

The right-hand side of the equation above with $\frac{1}{2} \leq p_i^* \leq \frac{\delta_{12}}{\delta_{12}+\delta_{21}}$ achieves its maximum value $\frac{1-\frac{1}{2}\alpha_i}{\alpha_i}\frac{\delta_{22}-\delta_{11}}{\delta_{22}+\delta_{11}}$ at $p_i^* = \frac{1}{2}$. This concludes the proof for equation (14).

Now we prove statement (iii). With $\delta_{11} = \delta_{22}$,

$$h\big(\boldsymbol{x}\big) = \boldsymbol{x} + \delta_{12}\mathbb{1}_n - 2\delta_{12}\boldsymbol{x} + \delta_{11}\operatorname{diag}(\boldsymbol{\alpha})\big(\tilde{A}\boldsymbol{x} - \boldsymbol{x}\big).$$

One can check that $\boldsymbol{p}^* = \frac{1}{2}\mathbb{1}_n$ is a fixed point. According to statement (i), the fixed point is unique. Let $\boldsymbol{p}(t) = \boldsymbol{y}(t) + \frac{1}{2}\mathbb{1}_n$. Then the two-product self-social NCPM becomes $\boldsymbol{y}(t+1) = M\boldsymbol{y}(t)$, where $M = (1-2\delta_{12})I + \delta_{11}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A} - \delta_{11}\operatorname{diag}(\boldsymbol{\alpha})$. For any $i \in V$, if $1 - 2\delta_{12} - \delta_{11}\alpha_i \geq 0$, then

$$\sum_{j=1}^n |M_{ij}| = 1 - 2\delta_{12} - \delta_{11}\alpha_i + \delta_{11}\alpha_i = 1 - 2\delta_{12} < 1;$$

and, if $1 - 2\delta_{12} - \delta_{11}\alpha_i < 0$, then

$$\sum_{j=1}^n |M_{ij}| = 2\delta_{12} + \delta_{11}\alpha_i + \delta_{11}\alpha_i - 1 < 1.$$

Since $\rho(M) \leq \|M\|_\infty = \max_i \sum_{j=1}^n |M_{ij}|$, the spectral radius of M is strictly less than 1. Therefore, the fixed point $\boldsymbol{p}^* = \frac{1}{2}\mathbb{1}_n$ is globally exponentially stable.

Now consider the case when $\delta_{22} > \delta_{11}$. Let $\boldsymbol{p}(t) = \boldsymbol{y}(t) + \boldsymbol{p}^*$. Then system (12) becomes

$$\boldsymbol{y}(t+1) = M\boldsymbol{y} + (\delta_{22}-\delta_{11})\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{y}(t))\tilde{A}\boldsymbol{y}(t).$$

The right-hand side of the equation above is a linear term $M\boldsymbol{y}(t)$ with a constant matrix M, plus a quadratic term. The matrix M can be decomposed as $M = \tilde{M} - \delta_{12} I$ and $\tilde{M} = \tilde{M}^{(1)} + \tilde{M}^{(2)}$ is further decomposed as a diagonal matrix $\tilde{M}^{(1)}$ plus a matrix $\tilde{M}^{(2)}$ in which all the diagonal entries are 0. Since

$$\begin{aligned}\tilde{M}^{(1)} &= (1-\delta_{12})I - \delta_{22}\operatorname{diag}(\boldsymbol{\alpha}) \\ &\quad + (\delta_{22}-\delta_{11})\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\tilde{A}\boldsymbol{p}^*)\end{aligned}$$

is a positive diagonal matrix, and

$$\tilde{M}^{(2)} = \delta_{11}\operatorname{diag}(\boldsymbol{\alpha})\tilde{A} + (\delta_{22}-\delta_{11})\operatorname{diag}(\boldsymbol{\alpha})\operatorname{diag}(\boldsymbol{p}^*)\tilde{A}$$

is a matrix with all the diagonal entries being zero and all the off-diagonal entries being nonnegative. The matrix $\tilde{M} = \tilde{M}^{(1)} + \tilde{M}^{(2)}$ is nonnegative.

Since $\tilde{A} = \operatorname{diag}(\frac{1}{N_1}, \frac{1}{N_2}, \dots, \frac{1}{N_n})A$, the matrix $\tilde{M}$ can be written in the form $DA + E$, where A is symmetric and D, E are positive diagonal matrix. One can easily prove that all the eigenvalues of any matrix in the form $\tilde{M} = DA + E$ are real since $\tilde{M}$ is similar to the symmetric matrix $D^{\frac{1}{2}}(A + D^{-1}E)D^{\frac{1}{2}}$.

The local stability of $\boldsymbol{p}^*$ is equivalent to the inequality $\rho(M) < 1$, which is in turn equivalent to the intersection of the following two conditions: $\lambda_{\max}(\tilde{M}) < 1 + \delta_{12}$ and $\lambda_{\min}(\tilde{M}) > -1 + \delta_{12}$. First we prove $\lambda_{\max}(\tilde{M}) < 1 + \delta_{12}$. Since A is irreducible and $\boldsymbol{\alpha} \succ \mathbb{0}_n$, $\boldsymbol{p}^* \succ \mathbb{0}_n$, we have $\tilde{M}_{ij} > 0$ if and only if $a_{ij} > 0$ for any $i \neq j$. In addition, $\tilde{M}_{ii} > 0$ for any $i \in V$. Therefore, $\tilde{M}$ is irreducible, aperiodic and thus primitive. According to the Perron-Frobenius theorem, $\lambda_{\max}(\tilde{M}) = \rho(\tilde{M})$. We have $\rho(\tilde{M}) \leq \|\tilde{M}\|_\infty$ and for any $i \in V$,

$$\sum_j |\tilde{M}_{ij}| = 1 - \delta_{21} + (\delta_{22}-\delta_{11})\big(\alpha_i(p^*_{-i} + p^*_i) - \alpha_i\big).$$

According to equation (13), for any $i \in V$,

$$1-\delta_{21} \leq \sum_j |\tilde{M}_{ij}| \leq 1-\delta_{21} + \frac{(\delta_{12}-\delta_{21})^2}{\delta_{12}+\delta_{21}}\alpha_i < 1+\delta_{12}.$$

Therefore,

$$\lambda_{\max}(\tilde{M}) \leq 1 - \delta_{21} + \frac{(\delta_{12}-\delta_{21})^2}{\delta_{12}+\delta_{21}}\alpha_i < 1 + \delta_{12}.$$

Now we prove $\lambda_{\min}(\tilde{M}) > -1 + \delta_{12}$. According to the Gershgorin circle theorem,

$$\lambda_{\min}(\tilde{M}) \geq \min_i (\tilde{M}_{ii} - \sum_{j\neq i} |\tilde{M}_{ij}|).$$

For any $i \in V$,

$$\begin{aligned}\tilde{M}_{ii} - \sum_{j\neq i}|\tilde{M}_{ij}| &= 1 - \delta_{21} - \alpha_i(\delta_{22}+\delta_{11}) \\ &\quad - \alpha_i(\delta_{22}-\delta_{11})(p^*_i - p^*_{-i}).\end{aligned}$$

According to equation (14),

$$p^*_i - p^*_{-i} \leq \frac{1-\frac{1}{2}\alpha_i}{\alpha_i}\frac{\delta_{22}-\delta_{11}}{\delta_{22}+\delta_{11}}.$$

Moreover, inequality (15) is necessary and sufficient to

$$\frac{1-\frac{1}{2}\alpha_i}{\alpha_i}\frac{\delta_{22}-\delta_{11}}{\delta_{22}+\delta_{11}} < \frac{1-\alpha_i}{\alpha_i}\frac{\delta_{22}+\delta_{11}}{\delta_{22}-\delta_{11}}.$$

Therefore,

$$\begin{aligned}\tilde{M}_{ii} - &\sum_{j\neq i}|\tilde{M}_{ij}| \\ &> 1 - \delta_{21} - \alpha_i(\delta_{22}+\delta_{11}) - (1-\alpha_i)(\delta_{22}+\delta_{11}) \\ &= -1 + \delta_{12},\end{aligned}$$

for any $i \in V$. That is to say, the inequality (15) is sufficient for $\rho(M) < 1$, i.e., the local stability of $\boldsymbol{p}^*$. This concludes the proof for statement (iv).

For statement (v), observe that the maps h and T satisfy the following relation:

$$h(\boldsymbol{x}) = KT(\boldsymbol{x}) + (I-K)\boldsymbol{x},$$

for any $\boldsymbol{x} \in [0,1]^n$, where $K = (\delta_{12}+\delta_{21})I + \delta_{22}\operatorname{diag}(\boldsymbol{\alpha})$. For any $\boldsymbol{x}$, $\boldsymbol{y} \in [0,1]^n$,

$$\begin{aligned}|h(\boldsymbol{x})_i - &h(\boldsymbol{y})_i| \\ &= |K_i\big(T(\boldsymbol{x})_i - T(\boldsymbol{y})_i\big) + (1-K_i)(x_i - y_i)|.\end{aligned}$$

We estimate the upper bound of $|h(\boldsymbol{x})_i - h(\boldsymbol{y})_i|$ in terms of $\|\boldsymbol{x}-\boldsymbol{y}\|_\infty$ in two cases.

Case 1: $\delta_{12} + \delta_{21} + \delta_{22}\alpha_i < 1$ for any i. Firstly,

$$\frac{\delta_{11}}{\delta_{22}} + 1 - \frac{1}{\delta_{22}} < \frac{\delta_{11}+\delta_{22}}{3\delta_{22}-\delta_{11}}$$

always holds as long as $\delta_{11} < \delta_{22}$. Then recall that, for any $\boldsymbol{x}$, $\boldsymbol{y} \in [0,1]^n$,

$$|T(\boldsymbol{x})_i - T(\boldsymbol{y})_i| \leq \epsilon_i \|\boldsymbol{x}-\boldsymbol{y}\|_\infty,$$

where $\epsilon_i = \frac{(2\delta_{22}-\delta_{11})\alpha_i}{K_i} < 1$. Therefore,

$$|h(\boldsymbol{x})_i - h(\boldsymbol{y})_i| \leq (K_i\epsilon_i + 1 - K_i)\|\boldsymbol{x}-\boldsymbol{y}\|_\infty,$$

for any $i \in V$. The coefficient $K_i\epsilon_i + 1 - K_i$ is always strictly less than 1 because it is a convex combination of $\epsilon_i < 1$ and 1. Therefore, h is a contraction map.

Case 2: There exists some i such that $\delta_{12} + \delta_{21} + \delta_{22}\alpha_i \geq 1$. In this case, for any such i,

$$|h(\boldsymbol{x})_i - h(\boldsymbol{y})_i| \leq (K_i\epsilon_i + K_i - 1)\|\boldsymbol{x}-\boldsymbol{y}\|_\infty.$$

If $\alpha_i < \frac{\delta_{11}+\delta_{22}}{3\delta_{22}-\delta_{11}}$, then we have

$$\begin{aligned}K_i\epsilon_i + K_i - 1 &= (3\delta_{22}-\delta_{11})\alpha_i + \delta_{12} + \delta_{21} - 1 \\ &< \delta_{11} + \delta_{22} + \delta_{12} + \delta_{21} - 1 = 1.\end{aligned}$$

Therefore, h is also a contraction map.

Combining Case 1 and Case 2 we conclude that if $\alpha_i < \frac{\delta_{11}+\delta_{22}}{3\delta_{22}-\delta_{11}}$ for any $i \in V$, then h is a contraction

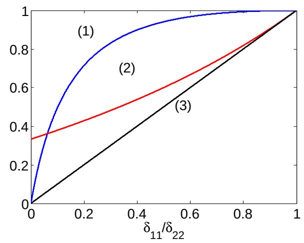


Fig. 8. This figure illustrates how the conditions for the local stability and global stability change with the ratio δ_{11}/δ_{22}. Curve (1) is $8\delta_{11}\delta_{22}/((\delta_{22})-\delta_{11})^2+8\delta_{11}\delta_{22})$, i.e, corresponding to the condition for local stability. Curve (2) is $(\delta_{22}+\delta_{11})/(3\delta_{22}-\delta_{11})$, corresponding to the condition for global stability. Curve (3) is δ_{11}/δ_{22}.

map. According to the proof for statement (i), h maps $[0,1]^n$ to $[0,1]^n$. Therefore, according to the Banach fixed point theorem, for any initial condition $\boldsymbol{p}(0) \in [0,1]^n$, the solution $\boldsymbol{p}(t)$ converges to $\boldsymbol{p}^*$ exponentially fast and the convergence rate is upper bounded by $\max_i \big(\max(\epsilon_i, K_i\epsilon_i + K_i - 1)\big)$. ■

The rest of this section are some remarks of Theorem 10. Firstly, equation (13) has a meaningful interpretation: The condition $\delta_{22} \geq \delta_{11}$ implies that product H_2 is advantageous to H_1, in the sense that the nodes in state H_1 have a higher or equal tendency of converting to H_2 than the other way around. As the result, the fixed point is in favor of H_2, i.e., $\boldsymbol{p}^* \geq \frac{1}{2}\mathbb{1}_n$.

From the proof of statement (iv), we know that, around the unique fixed point, the linearized system is $\boldsymbol{y}(t+1) = M\boldsymbol{y}(t)$, where M is a Metzler matrix and is Hurwitz stable. Usually the Metzler matrices are presented in continuous-time network dynamics models, e.g., the epidemic spreading model [35], [36]. In the proof of Theorem 10 (iv), we provide an example of the Metzler matrix in a stable discrete-time system.

Figure 8 plots the right-hand sides of inequalities (15) and (16), respectively, as functions of the ratio $\frac{\delta_{11}}{\delta_{22}}$, for the case when $0 < \frac{\delta_{11}}{\delta_{22}} < 1$. One can observe that, for a large range of $\frac{\delta_{11}}{\delta_{22}}$, the sufficient condition we propose for the global stability is more conservative than the sufficient condition for the local stability.

One major difference between the self-social and the social-self NCPM in the asymptotic property is that, in the self-social NCPM, every individual's state probability distribution is not necessarily identical. Moreover, distinct from the social-self NCPM, for any of the four cases of $G(\Delta)$ defined in Definition 6, the asymptotic behavior of the self-social NCPM depends on not only the structure of $G(\Delta)$, but also the structure of the social network $G(\tilde{A})$ and the individual open-mindedness $\boldsymbol{\alpha}$.

V. Non-cooperative Quality-Seeding Games

Based on the social-self NCPM given by equation (4), we propose two non-cooperative multi-player games distinct in the pay-off functions, and analyze their Nash equilibria. These two games share the common idea that, companies benefit from the adoption of their products, and thereby invest on both improving their products' quality, and seeding, e.g., advertisement and promotion, to maximize their products' adoption probabilities. All the notations in Table I and the previous sections still apply, and, in Table II, we introduce some additional notations and functions exclusively for this section.

TABLE II
Notations and functions used in Section V

$X(t)$	seeding matrix at time t. $X(t) = \big(x_{ir}(t)\big)_{n\times R}$, where $x_{ir}(t) \geq 0$ is company r's investment on seeding for individual i. $\boldsymbol{x}_r(t)$ is the r-th column of $X(t)$ and $\boldsymbol{x}^{(i)}(t)$ is the i-th column of $X(t)$
$\boldsymbol{w}(t)$	the quality investment vector at time t. $\boldsymbol{w}(t) \in \mathbb{R}^{R\times 1}$, and each entry $w_r(t) \geq 0$ is company r's investment at time t on product H_r's quality
$Y(t)$	action matrix at time t. $Y(t) = \big(X(t)^\top, \boldsymbol{w}(t)\big)^\top$, in which any $\boldsymbol{y}_r(t) = (\boldsymbol{x}_r(t)^\top, w_r(t))^\top$ is Player r's action at t.
$\boldsymbol{c}$	the budget vector. $\boldsymbol{c} \in \mathbb{R}^{R\times 1}$ and $\boldsymbol{c} \succ \mathbb{0}_R$. entry c_r is the budget limit for company r
Ω_r	player r's action set. $\Omega_r = \{\boldsymbol{y} \in \mathbb{R}^{n+1}_{\geq 0} \,\|\, \mathbb{1}^\top_{n+1}\boldsymbol{y} \leq c_r\}$
$\psi_r(\boldsymbol{x}^{(i)};\gamma)$	$\psi_r : \mathbb{R}^{1\times R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $\psi_r(\boldsymbol{x}^{(i)};\gamma) = x_{ir}/(\boldsymbol{x}^{(i)}\mathbb{1}_R + \gamma)$, with model parameter $\gamma > 0$
$g_r(\boldsymbol{w};\boldsymbol{\varsigma})$	$g_r : \mathbb{R}^{R\times 1}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $g_r(\boldsymbol{w};\boldsymbol{\varsigma}) = (w_r + \varsigma_r)/\mathbb{1}^\top_R(\boldsymbol{w}+\boldsymbol{\varsigma})$, where $\boldsymbol{\varsigma} \in \mathbb{R}^R_{\geq 0}$
$\boldsymbol{\beta}_r(t)$	$\boldsymbol{\beta}_r(t) = \big(\beta_{1r}(t),\dots,\beta_{nr}(t)\big)^\top = \tilde{A}\boldsymbol{p}_r(t)$
$u_r(P)$	single-stage reward for player r with system state P. $u_r(P) = \mathbb{1}^\top_n \boldsymbol{p}_r$

A. Repeated one-shot quality-seeding game

1) Game set-up and analysis: In this subsection we consider the scenario in which the companies allocate their investments aiming to maximize their instant payoffs. The game is referred to as the *repeated one-shot quality-seeding game*, and is formalized as follows.

(a) Players: The players are the R companies. Each company r has a product H_r competing on the network.

(b) Players' actions: At each stage (or time step equivalently) t, each company r has two types of investments. The investment on seeding, i.e., $\boldsymbol{x}_r(t)$, and the investment on quality, i.e., $w_r(t)$. The total investment is bounded by a fixed budget c_r, i.e., $\mathbb{1}^\top_n x_r(t) + w_r(t) \leq c_r$.

(c) Rules: The investment on seeding changes the individuals' product-adoption probability in the social conversion process. For any individual $i \in V$, each company r's investment $x_{ir}(t)$ creates a "virtual node"

in the network, who is always adopting the product H_r. In the social conversion process, the probability that individual i picks company r's virtual node is $\psi_r\big(\boldsymbol{x}^{(i)}(t);\gamma\big)$ for any $i \in V$ and $r \in \Theta$. The probability that individual i picks individual j in the social conversion process is then given by $\Big(1-\sum_{s=1}^R \psi_s\big(\boldsymbol{x}^{(i)};\gamma\big)\Big)\tilde{a}_{ij}$. The investment on quality, i.e., $w_r(t)$, influences the product-conversion graph. We assume that the product-conversion graph is associated with a rank-one adjacency matrix $[\delta_1 \mathbb{1}_n, \delta_2 \mathbb{1}_n, \dots, \delta_R \mathbb{1}_n]$ and $\delta_r = g_r(\boldsymbol{w}(t);\boldsymbol{\varsigma})$ is determined by all the companies' investments on product quality and the products' preset qualities $\boldsymbol{\varsigma} = (\varsigma_1,\dots,\varsigma_R)^\top \succ \mathbb{0}_R$. With each company r's action $\boldsymbol{y}_r(t) = \big(\boldsymbol{x}_r(t)^\top, w_r(t)\big)^\top$ at time t, the dynamics of the product-adoption probabilities $P(t) \in \mathbb{R}^{n\times R}_{\geq 0}$ is given by

$$P(t+1) = H\big(P(t), \boldsymbol{y}_1(t), \dots, \boldsymbol{y}_R(t)\big), \tag{19}$$

where the map H is defined by

$$\begin{aligned} H\big(P,&\boldsymbol{y}_1(t),\dots,\boldsymbol{y}_R(t)\big)_{ir} \\ &= \alpha_i \frac{\gamma}{\boldsymbol{x}^{(i)}(t)\mathbb{1}_R + \gamma}\sum_{k=1}^n \tilde{a}_{ik} p_{kr} \\ &\quad + \alpha_i \psi_r\big(\boldsymbol{x}^{(i)}(t);\gamma\big) + (1-\alpha_i) g_r\big(\boldsymbol{w}(t);\boldsymbol{\varsigma}\big), \end{aligned}$$

for any $P \in S_{nR}(\mathbb{1}_n)$, $i \in V$, and $r \in \Theta$.

(d) Pay-offs and goals: At each stage t, each player r chooses its action $\boldsymbol{y}_r(t)$, in order to maximize the pay-off $u_r(P(t+1)) = \mathbb{1}_n^\top \boldsymbol{p}_r(t+1)$, i.e., the total adoption probability of product H_r at the next stage.

The following theorem gives a closed-form expression of the Nash equilibrium at each stage and the system's asymptotic behavior when every player is adopting the policy at the Nash equilibrium.

Theorem 11 (Repeated one-shot quality-seeding game): Consider the R-player quality-seeding game described in this subsection. Further assume that the budget limit c_r for any company r satisfies

$$c_r \geq \max\Big\{ \big(\frac{n}{\min_i \alpha_i} - 1\big)\gamma - \varsigma_r, \frac{\mathbb{1}_n^\top \boldsymbol{\alpha}}{n - \mathbb{1}_n^\top \boldsymbol{\alpha}}\varsigma_r \Big\}. \tag{20}$$

Then we have the following conclusions:

i) for each t, there exists a unique pure-strategy Nash equilibrium $Y^*(t) = \big(X^*(t)^\top, \boldsymbol{w}^*(t)\big)^\top$, given by

$$x^*_{ir}(t) = \frac{\alpha_i}{n}c_r + \frac{\alpha_i\gamma}{n}\mathbb{1}_n^\top \boldsymbol{\beta}_r(t) + \frac{\alpha_i}{n}\varsigma_r - \beta_{ir}(t)\gamma, \tag{21}$$

$$w^*_r(t) = \big(1 - \frac{\mathbb{1}_n^\top\boldsymbol{\alpha}}{n}\big)\big(c_r + \mathbb{1}_n^\top \boldsymbol{\beta}_r(t)\gamma\big) - \frac{\mathbb{1}_n^\top\boldsymbol{\alpha}}{n}\varsigma_r, \tag{22}$$

and $x^*_{ir}(t) \geq 0$, $w^*_r(t) \geq 0$ for any $i \in V, r \in \Theta$;

ii) if $\big(X(t), \boldsymbol{w}(t)\big) = \big(X^*(t), \boldsymbol{w}^*(t)\big)$ for any $t \in \mathbb{N}$ and $P(0) \in S_{nR}(\mathbb{1}_n)$, then $P(t)$ obeys the following iteration equations:

$$\boldsymbol{p}_r(t+1) = \frac{c_r + \varsigma_r + \mathbb{1}_n^\top \tilde{A}\boldsymbol{p}_r(t)\gamma}{\mathbb{1}_R^\top \boldsymbol{c} + \mathbb{1}_R^\top \boldsymbol{\varsigma} + n\gamma}\mathbb{1}_n, \tag{23}$$

for any $r \in \Theta$, $t \in \mathbb{N}$. As the result, $\boldsymbol{p}_r(t)$ converges to $(c_r + \varsigma_r)\big/\big(\mathbb{1}_R^\top(\boldsymbol{c}+\boldsymbol{\varsigma})\big)$ exponentially fast with the rate $n\gamma\big/\big(\mathbb{1}_R^\top(\boldsymbol{c}+\boldsymbol{\varsigma}) + n\gamma\big)$.

Proof: Since we only discuss the actions at stage t in this proof, for simplicity of notations and without causing any confusion, we use x_{ir} (w_r, x^*_{ir}, w^*_r resp.) for $x_{ir}(t)$ ($w_r(t)$, $x^*_{ir}(t)$, $w^*_r(t)$ resp.).

If company r knows the actions of all the other companies at time step t, i.e., $\boldsymbol{y}_s$, for any $s \neq r$, the optimal response for company r is the solution to the following optimization problem:

$$\begin{aligned} &\underset{(\boldsymbol{x},w)\in\Omega_r}{\text{minimize}} && -\mathbb{1}_n^\top \boldsymbol{p}_r(t+1) \\ &\text{subject to} && \mathbb{1}_n^\top \boldsymbol{x} + w - c_r \leq 0. \end{aligned} \tag{24}$$

Let $\tilde{x}_{ir} = x_{ir} + \beta_{ir}(t)\gamma$, $\tilde{w}_r = w_r + \varsigma_r$, and $L_r(\boldsymbol{x}_r, w_r, \mu_r) = -\mathbb{1}_n^\top \boldsymbol{p}_r(t+1) + \mu_r \mathbb{1}_n^\top \boldsymbol{x}_r + \mu_r w_r - \mu_r c_r$, for any $i \in V$ and $r \in \Theta$. The solution to the optimization problem (24) satisfies

$$\frac{\partial L_r}{\partial x_{ir}} = -\alpha_i \frac{\sum_{s\neq r}\tilde{x}_{is}}{(\sum_{s=1}^R \tilde{x}_{is})^2} + \mu_r = 0, \tag{25}$$

$$\frac{\partial L_r}{\partial w_r} = -\mathbb{1}_n^\top(\mathbb{1}_n - \boldsymbol{\alpha})\frac{\sum_{s\neq r}\tilde{w}_s}{(\mathbb{1}_R^\top \tilde{\boldsymbol{w}})^2} + \mu_r = 0, \tag{26}$$

$$\frac{\partial L_r}{\partial \mu_r} = \mathbb{1}_n^\top \boldsymbol{x}_r + w_r - c_r = 0. \tag{27}$$

According to the definition of Nash equilibrium, $(\boldsymbol{x}^*_r, w^*_r)$ solves the optimization problem (24) with $(\boldsymbol{x}_s, w_s) = (\boldsymbol{x}^*_s, w^*_s)$ for any $s \neq r$. One immediate result is that $\mathbb{1}_n^\top \boldsymbol{x}^*_r + w^*_r - c_r = 0$ for any $r \in \Theta$. Moreover, equation (25) leads to:

$$\frac{1}{\sqrt{\mu_r}} = \frac{1}{\sum_{k=1}^n \sqrt{\alpha_k \sum\limits_{s\neq r} \tilde{x}^*_{ks}}}\sum_{s=1}^R \big(c_s - w^*_s + \mathbb{1}_n^\top \boldsymbol{\beta}_s(t)\gamma\big),$$

and therefore,

$$\frac{\sqrt{\alpha_i \sum_{s\neq r}\tilde{x}^*_{is}}}{\sum\limits_{k=1}^n \sqrt{\alpha_k \sum\limits_{s\neq r}\tilde{x}^*_{ks}}} = \frac{\sum\limits_{s=1}^R \tilde{x}^*_{is}}{\sum\limits_{s=1}^R \big(c_s - w^*_s + \mathbb{1}_n^\top \boldsymbol{\beta}_s(t)\gamma\big)}. \tag{28}$$

The right-hand side of the equation above does not depend on the product index r. Therefore,

$$\frac{\sum_{s\neq r}\tilde{x}^*_{is}}{\sum_{s\neq\tau}\tilde{x}^*_{is}} = \left(\frac{\sum_{k=1}^n \sqrt{\alpha_k \sum_{s\neq r}\tilde{x}^*_{ks}}}{\sum_{k=1}^n \sqrt{\alpha_k \sum_{s\neq\tau}\tilde{x}^*_{ks}}}\right)^2,$$

for any $r, \tau \in \Theta$. Since the right-hand side of the equation above does not depend on i, we have

$$\frac{\sum_{s\neq r}\tilde{x}^*_{is}}{\sum_{s\neq r}\tilde{x}^*_{js}} = \frac{\sum_{s\neq\tau}\tilde{x}^*_{is}}{\sum_{s\neq\tau}\tilde{x}^*_{js}} = \frac{\sum_{s=1}^R \tilde{x}^*_{is}}{\sum_{s=1}^R \tilde{x}^*_{js}} = \frac{\tilde{x}^*_{ir}}{\tilde{x}^*_{jr}},$$

for any $r, \tau \in \Theta$. Combine the equation above with equation (28) and then we obtain

$$\frac{\sum_{s=1}^{R}\tilde{x}_{is}^{*}}{\sum_{s=1}^{R}\tilde{x}_{js}^{*}}=\sqrt{\frac{\alpha_i}{\alpha_j}}\sqrt{\frac{\sum_{s\neq r}\tilde{x}_{is}^{*}}{\sum_{s\neq r}\tilde{x}_{js}^{*}}}\quad\Rightarrow\quad\frac{\tilde{x}_{ir}^{*}}{\tilde{x}_{jr}^{*}}=\frac{\alpha_i}{\alpha_j},$$

for any $r \in \Theta$. Therefore,

$$\tilde{x}_{ir}^{*}=\frac{\alpha_i}{\mathbb{1}_n^{\top}\boldsymbol{\alpha}}\big(c_r-w_r^{*}+\mathbb{1}_n^{\top}\boldsymbol{\beta}_r(t)\gamma\big). \tag{29}$$

Combining equation (29) and (26), we obtain

$$\frac{c_r-w_r^{*}+\mathbb{1}_n^{\top}\boldsymbol{\beta}_{\boldsymbol{r}}(t)\gamma}{\tilde{w}_r^{*}}=\frac{c_\tau-w_\tau^{*}+\mathbb{1}_n^{\top}\boldsymbol{\beta}_\tau(t)\gamma}{\tilde{w}_\tau^{*}}=\eta,$$

for any $r, \tau \in \Theta$ and some constant η. Substitute the equation above into equation (26), we solve that $\eta = \mathbb{1}_n^{\top}\boldsymbol{\alpha}/\mathbb{1}_n^{\top}(\mathbb{1}_n - \boldsymbol{\alpha})$. Therefore, we obtain equation (22), and by substituting equation (22) into equation (29) we obtain equation (21). The uniqueness of the pure-strategy Nash equilibrium $({X^*}^{\top}, \boldsymbol{w})^{\top}$ is implied from the computation. Moreover, equation (20) guarantees $\tilde{x}_{ir}^{*} \geq 0$ and $w_r^{*} \geq 0$ for any $i \in V$ and $r \in \Theta$.

Substituting euqation (21) and (22) into the dynamical system (19), after simplification, we obtain equation (23) and thereby all the results in Conclusion ii). ■

2) Interpretations and Remarks: : The basic idea of seeding-quality trade-off in the competitive seeding-quality game is similar to the work by Fazeli et. al. [32], but, in our model, players take actions at every step, instead of just at the beginning of the game. Moreover, our model is based on a different propagation model.

Theorem 11 reveals the behavior of the competitive propagation dynamics under the players' rational but myopic actions, and provides some strategic insights on the investment decisions and the seeding-quality trade-off for short-term reward maximization.

(a) Interpretation of $\beta_{ir}(t)$: By definition, $\beta_{ir}(t)$ is the average probability, among all the neighbors of individual i, of adopting product H_r at time step t. The larger $\beta_{ir}(t)$, the more individual i is inclined to adopt H_r via social conversion. Therefore, $\beta_{ir}(t)$ characterizes the current "social attraction" of H_r for individual i, and $\mathbb{1}_n^{\top}\boldsymbol{\beta}_r(t)/n$ characterizes the current overall social attraction of product H_r in the network.

(b) Seeding-quality trade-off: According to equation (22), at the Nash equilibrium, the investment on H_r's product quality monotonically decreases with $\mathbb{1}_n^{\top}\boldsymbol{\alpha}/n$, and increases with $\mathbb{1}_n^{\top}\boldsymbol{\beta}_r$. This observation implies that: 1) in a society with relatively low open-mindedness, the competing companies should relatively emphasize more on improving their products' quality, rather than seeding, and vice versa; 2) for products which do not have much social attraction, seeding is more efficient than improving the product's quality.

(c) Allocation of seeding resources among the individuals: According to equation (21), for any company r, at the Nash equilibrium at each time step t, the investment on seeding for any individual i, i.e., $x_{ir}(t)$, increases with individual i's open-mindedness, since it is easier for a more open-minded individual to be influence by seeding. Moreover, by rewriting equation (21), one would observe that $x_{ir}^{*}(t)$ monotonically decreases with $\beta_{ir}(t)$. A possible interpretation is that, seeding is relatively not efficient for products with strong social attraction. Moreover, one can also observe that $x_{ir}^{*}(t)$ increases with $\sum_{l=1}^{n}\tilde{a}_{li}p_{ir}(t)$, in which $\sum_{l=1}^{n}\tilde{a}_{li}$ is individual i's in-degree, reflecting i's potential of influencing the others, and $\sum_{l=1}^{n}\tilde{a}_{li}p_{ir}(t)$ characterizes individual i's potential of converting other individuals to product H_r.

(d) Nash equilibrium on the boundary: Without equation (20), the right-hand sides of equation (21) and (22) could be non-positive. In this case, the Nash equilibrium would be on the boundary of the feasible action set, i.e., some of the $x_{ir}^{*}(t)$ or $w_r^{*}(t)$ might be 0.

B. Dynamic quality-seeding game with infinite-horizon

In this subsection we introduce a multi-stage game among more farsighted players than in the previous subsection. The players aim to maximize the accumulated pay-offs of all the stages. We refer to this game as the *dynamic quality-seeding game*. The model set-up is the same with the game defined in the previous subsection, except for the following two modifications:

(a) Players' policies Denote by $\mathcal{Y}_r$ the set of functions mapping $S_{nR}(\mathbb{1}_n)$ to Ω_r. Each player r's policy is a sequence of maps, denoted by $\mathscr{Y}_r = \{\mathscr{Y}_{r,t}\}_{t\in\mathbb{N}}$, where $\mathscr{Y}_{r,t} \in \mathcal{Y}_r$ for any t. Player r's action at each stage t is thus given by $\boldsymbol{y}_t = \mathscr{Y}_{r,t}\big(P(t)\big)$. We refer to $\mathscr{Y}_r = \{\mathscr{Y}_{r,t}\}_{t\in\mathbb{N}}$ as stationary policy if $\mathscr{Y}_{r,t} = \mathscr{Y}_{r,\tau}$ for any $t \neq \tau$, and simply use $\mathscr{Y}_r$ for the map at each stage.

(b) Pay-offs and goals: Denote by $v_r(P; \mathscr{Y}_1, \dots, \mathscr{Y}_R)$ the pay-off of Player r, with initial condition $P(0) = P$ and each Player s adopting the policy $\mathscr{Y}_s$. The pay-off $v_r(P; \mathscr{Y}_1, \dots, \mathscr{Y}_R)$ is given by the accumulated step pay-offs with discount, that is,

$$v_r(P;\mathscr{Y}_1,\dots,\mathscr{Y}_R)=\sum_{t=0}^{\infty}\varepsilon^t u_r(P(t)),$$

where $P(0) = P$ and $P(t+1) = H\big(P(t); \mathscr{Y}_1(P(t)), \dots, \mathscr{Y}_R(P(t))\big)$ for any $t \in \mathbb{N}$.

This model set-up defines a multi-stage non-cooperative dynamic game with infinite horizon. One interpretation of the discounted accumulated pay-off is that, people tend to value the immediate profit more than the future profit. An alternative explanation is that, the discount factor ε characterizes the interest rate $1/\varepsilon - 1$ when the players deposit their current pay-off to the banks, or use them for some other investments.

The R-tuple $(\mathscr{Y}_1^*, \dots, \mathscr{Y}_R^*)$ is a Nash equilibrium if, for any $P \in S_{nR}(\mathbb{1}_n)$ and $r \in \Theta$, $v_r(P; \mathscr{Y}_1^*, \dots, \mathscr{Y}_R^*) \geq v_r(P; \mathscr{Y}_1^*, \dots, \mathscr{Y}_{r-1}^*, \mathscr{Y}_r, \mathscr{Y}_{r+1}^*, \dots, \mathscr{Y}_R^*)$, for any $\mathscr{Y}_r \in \mathcal{Y}_r^\infty = \mathcal{Y}_r \times \mathcal{Y}_r \times \dots$. In this subsection, we limit our discussion to the case of two players. The following theorem presents some results on the stationary Nash equilibrium and the equilibrium pay-off function for this dynamic quality-seeding game.

Theorem 12 (Two-player infinite-horizon dynamic game): Consider the dynamic quality-seeding game defined in this subsection, with $R = 2$. Define the subset of continuously differentiable functions $\mathcal{V} = \{v : [0,1]^n \to \mathbb{R} \,|\, v \text{ satisfies properties } \mathcal{P}_1 \text{ and } \mathcal{P}_2\}$, where

$$\mathcal{P}_1 : \boldsymbol{p} \preceq \hat{\boldsymbol{p}} \Rightarrow v(\boldsymbol{p}) \leq v(\hat{\boldsymbol{p}}) \text{ for any } \boldsymbol{p}, \hat{\boldsymbol{p}} \in [0,1]^n,$$
$$\mathcal{P}_2 : v(\boldsymbol{p}) \text{ is convex in } \boldsymbol{p}.$$

We conclude that:

(i) There exists a Nash equilibrium $(\mathscr{Y}_1^*, \mathscr{Y}_2^*)$, where $\mathscr{Y}_1^*$ and $\mathscr{Y}_2^*$ are both stationary policies;

(ii) The total pay-off for Player 2 at this Nash equilibrium is given by $v_2(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*) = v^*(P\boldsymbol{e}_2)$, where $\boldsymbol{e}_2$ is the second standard basis vector of $\mathbb{R}^2$, and v^* is the unique fixed point of the map $\mathcal{T} : \mathcal{V} \to \mathcal{V}$, defined by
$$\mathcal{T}v(\boldsymbol{p}) = \mathbb{1}_n^\top \boldsymbol{p} + \varepsilon \sup_{\boldsymbol{y}_2 \in \Omega_2} \inf_{\boldsymbol{y}_1 \in \Omega_1} v\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big),$$
where $P = [\mathbb{1}_n - \boldsymbol{p}, \boldsymbol{p}] \in \mathbb{R}^{n \times 2}$. As a result, $v_1(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*) = n/(1-\varepsilon) - v_2(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*)$;

(iii) The stationary Nash policies $\mathscr{Y}_1^*$, $\mathscr{Y}_2^*$ are given by
$$\mathscr{Y}_1^*(P) = \operatorname*{argmin}_{\boldsymbol{y}_1 \in \Omega_1} \sup_{\boldsymbol{y}_2 \in \Omega_2} v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big),$$
$$\mathscr{Y}_2^*(P) = \operatorname*{argmax}_{\boldsymbol{y}_2 \in \Omega_2} \inf_{\boldsymbol{y}_1 \in \Omega_1} v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big).$$

Before proving the theorem above, we summarize Theorem 4.4 and Property 4.1 in [37], on the two-player zero-sum continuous games, into the following lemma.

Lemma 13 (Pure-strategy Nash equilibrium): Consider the two-player zero-sum continuous game with Player 1 as the minimizer and Player 2 as the maximizer. Suppose the action sets of Player 1 and 2, denoted by Ω_1 and Ω_2 respectively, are both compact and convex subsets of finite-dimension Euclidean spaces. If the cost function $v(\boldsymbol{y}_1, \boldsymbol{y}_2) : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is continuously differentiable, convex in $\boldsymbol{y}_1$, and concave in $\boldsymbol{y}_2$, then: (1) the game admits at least one saddle-point Nash equilibrium in pure strategies; (2) if there are multiple saddle points, the saddle points satisfy the ordered interchangeability property. That is, if $(\boldsymbol{y}_1^*, \boldsymbol{y}_2^*)$ and $(\tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2)$ are saddle points, so are $(\boldsymbol{y}_1^*, \tilde{\boldsymbol{y}}_2)$ and $(\tilde{\boldsymbol{y}}_1, \boldsymbol{y}_2^*)$.

Proof of Theorem 12: In this proof, for simplicity, denote by $\boldsymbol{p}$ the second column of the matrix P, i.e., $P = [\mathbb{1}_n - \boldsymbol{p}, \boldsymbol{p}]$, and correspondingly, $\hat{P} = [\mathbb{1}_n - \hat{\boldsymbol{p}}, \hat{\boldsymbol{p}}]$. Since Ω_1 and Ω_2 are compact subsets of $\mathbb{R}^{n+1}$, for any $v \in \mathcal{V}$, there exists $(\boldsymbol{y}_1, \boldsymbol{y}_2)$ such that $\mathcal{T}v(\boldsymbol{p}) = \mathbb{1}_n^\top \boldsymbol{p} + \varepsilon v\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big)$. Moreover, from the expression of map H, one can deduce that $H(P, \boldsymbol{y}_1, \boldsymbol{y}_2)$ satisfies
$$\boldsymbol{p} \preceq \hat{\boldsymbol{p}} \Rightarrow H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2 \preceq H(\hat{P}; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2,$$
for any $(\boldsymbol{y}_1, \boldsymbol{y}_2) \in \Omega_1 \times \Omega_2$ and $\boldsymbol{p}, \hat{\boldsymbol{p}} \in [0,1]^n$. This leads to the conclusion that $\mathcal{T}v$ also satisfies property $\mathcal{P}_1$. Moreover, by definition, $H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)$ is linear in P. Since $v(\boldsymbol{p})$ is convex in $\boldsymbol{p}$, one can check that $\mathcal{T}v(\boldsymbol{p})$ is also convex in $\boldsymbol{p}$. Therefore, $\mathcal{T}$ satisfies property $\mathcal{P}_2$ and maps $\mathcal{V}$ to $\mathcal{V}$ itself. Now we prove that $\mathcal{T}$ is a contraction map. Define the function norm $\|\cdot\|$ for any $v \in \mathcal{V}$ as $\|v\| = \sup_{\boldsymbol{p} \in [0,1]^n} |v(\boldsymbol{p})|$. For any $v, \hat{v} \in \mathcal{V}$, we have
$$\|\mathcal{T}v - \mathcal{T}\hat{v}\| = \varepsilon \sup_{\boldsymbol{p} \in [0,1]^n} |\mathcal{T}v(\boldsymbol{p}) - \mathcal{T}\hat{v}(\boldsymbol{p})|$$
$$\leq \varepsilon \sup_{\boldsymbol{p} \in [0,1]^n} \sup_{\boldsymbol{y}_2 \in \Omega_2} \sup_{\boldsymbol{y}_1 \in \Omega_1} |v(\boldsymbol{p}) - \hat{v}(\boldsymbol{p})| \leq \varepsilon \|v - \hat{v}\|.$$
According to the Banach fixed-point theorem, there exists a unique $v^* \in \mathcal{V}$ satisfying
$$v^*(P\boldsymbol{e}_2) = \mathbb{1}_n^\top P\boldsymbol{e}_2 + \varepsilon \sup_{\boldsymbol{y}_2 \in \Omega_2} \inf_{\boldsymbol{y}_1 \in \Omega_1} v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big).$$
According to the expression of the map $H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)$, one can check that, for any $\eta \in [0,1]$, $P \in S_{nR}(\mathbb{1}_n)$, and $\boldsymbol{y}_1, \hat{\boldsymbol{y}}_1 \in \Omega_1$,
$$H\big(P; \eta\boldsymbol{y}_1 + (1-\eta)\hat{\boldsymbol{y}}_1, \boldsymbol{y}_2\big)\boldsymbol{e}_2$$
$$\preceq \eta H(P; \boldsymbol{y}_1, \boldsymbol{y}_2) + (1-\eta) H(P; \hat{\boldsymbol{y}}_1, \boldsymbol{y}_2).$$
Since $v^*(\boldsymbol{p})$ satisfies properties $\mathcal{P}_1$ and $\mathcal{P}_2$,
$$v^*\big(H(P; \eta\boldsymbol{y}_1 + (1-\eta)\hat{\boldsymbol{y}}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big)$$
$$\leq \eta v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big) + (1-\eta) v^*\big(H(P; \hat{\boldsymbol{y}}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big).$$
That is, $v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big)$ is convex in $\boldsymbol{y}_1$. Similarly, we have $v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big)$ is concave in $\boldsymbol{y}_2$.

According to Lemma 13, for any $P \in S_{nR}(\mathbb{1}_n)$ and the two-player zeros-sum game with cost function $v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big)$, there exists a saddle-point Nash equilibrium $(\boldsymbol{y}_1^*, \boldsymbol{y}_2^*) \in \Omega_1 \times \Omega_2$ such that
$$v^*\big(H(P; \boldsymbol{y}_1^*, \boldsymbol{y}_2^*)\boldsymbol{e}_2\big) = \sup_{\boldsymbol{y}_2 \in \Omega_2} \inf_{\boldsymbol{y}_1 \in \Omega_1} v^*\big(H(P; \boldsymbol{y}_1^*, \boldsymbol{y}_2^*)\boldsymbol{e}_2\big)$$
$$= \inf_{\boldsymbol{y}_1 \in \Omega_1} \sup_{\boldsymbol{y}_2 \in \Omega_2} v^*\big(H(P; \boldsymbol{y}_1, \boldsymbol{y}_2)\boldsymbol{e}_2\big).$$
Therefore, there exists functions $\mathscr{Y}_1$, $\mathscr{Y}_2$ such that $\boldsymbol{y}_1^* = \mathscr{Y}_1^*(P)$ and $\boldsymbol{y}_2^* = \mathscr{Y}_2^*(P)$ satisfy the equation above, for any $P \in S_{nR}(\mathbb{1}_n)$. Moreover, since
$$v^*(P\boldsymbol{e}_2) - v_2(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*)$$
$$= \varepsilon\Big(v^*\big(H(P; \mathscr{Y}_1^*(P), \mathscr{Y}_2^*(P))\boldsymbol{e}_2\big)$$
$$- v_2\big(H(P; \mathscr{Y}_1^*(P), \mathscr{Y}_2^*(P)); \mathscr{Y}_1^*, \mathscr{Y}_2^*\big)\Big),$$

for any $P \in S_{nR}(\mathbb{1}_n)$, and functions v and v_2 are bounded, we conclude that $v^*(P\boldsymbol{e}_2) = v_2(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*)$. Therefore, for any $\tau \in \mathbb{N}$, we have

$$v_2(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*) \geq \sum_{t=0}^{\tau-1} \varepsilon^t u_2(P(t)) + \varepsilon^\tau v_2\Big(H\big(P(\tau); \mathscr{Y}_1^*(P(\tau)), \boldsymbol{y}_2\big); \mathscr{Y}_1^*, \mathscr{Y}_2^*\big)\Big),$$

for any $\boldsymbol{y}_2 \in \Omega_2$, and, due to the fact that $v_1(P; \mathscr{Y}_1, \mathscr{Y}_2) = n/(1-\varepsilon) - v_2(P; \mathscr{Y}_1, \mathscr{Y}_2)$ for any $(\mathscr{Y}_1, \mathscr{Y}_2)$, we have

$$v_1(P; \mathscr{Y}_1^*, \mathscr{Y}_2^*) \geq \sum_{t=0}^{\tau-1} \varepsilon^t u_1(P(t)) + \varepsilon^\tau v_1\Big(H\big(P(\tau); \boldsymbol{y}_1, \mathscr{Y}_2^*(P(\tau))\big); \mathscr{Y}_1^*, \mathscr{Y}_2^*\big)\Big),$$

for any $\boldsymbol{y}_1 \in \Omega_1$. Since both $v_1(P; \mathscr{Y}_1, \mathscr{Y}_2)$ and $v_2(P; \mathscr{Y}_1, \mathscr{Y}_2)$ satisfy the property of continuity at infinity, according to the one-stage deviation principle, $(\mathscr{Y}_1^*, \mathscr{Y}_2^*)$ is a Nash equilibrium of the dynamics game. This concludes the proof. ■

Theorem 12 provides an iteration algorithm to compute the stationary Nash policy $(\mathscr{Y}_1^*, \mathscr{Y}_2^*)$, and the players' respective pay-offs at the Nash equilibrium. A comparison by simulation is given in Figure 9, between the Nash policies for the dynamic game discussed in this subsection, and the repeated one-shot game in the previous subsection. The model parameters are set as $n = 3$, $\boldsymbol{\alpha} = (0.51, 0.87, 0.77)^\top$, $\gamma = 5$, $\varsigma_1 = \varsigma_2 = 1$, $c_1 = 30$, $c_2 = 60$, $\varepsilon = 0.8$, and $\tilde{A}$ such that $\tilde{a}_{13} = \tilde{a}_{23} = 1$, $\tilde{a}_{31} = \tilde{a}_{32} = 0.5$, and $\tilde{a}_{ij} = 0$ otherwise. Simulation results show that, with the same initial condition, for the two types of games, the players' total pay-offs at the respective Nash equilibria are very close to each other. Moreover, from Figure 9 we can observe that, for each of the two games, the players' pay-offs are almost linear to the initial average probability of adopting H_2.

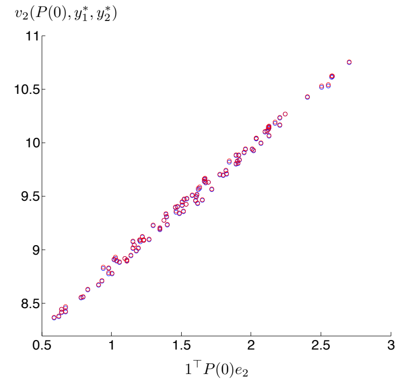


Fig. 9. Comparison between the Nash policies for the dynamic game discussed in Subsection V.B (blue dots), and the repeated one-shot game in Subsection V.A (red dots).

VI. Conclusion

This paper discusses a class of competitive propagation models based on two product-adoption mechanisms: the social conversion and the self conversion. Applying the independence approximation we propose two difference equations systems, referred to as the social-self NCPM and the self-social NCPM respectively. Theoretical analysis reveals that the structure of the product-conversion graph plays an important role in determining the nodes' asymptotic state probability distributions. Simulation results reveal the high accuracy of the independence approximation and the asymptotic behavior of the original social-self Markov chain model. Based on the social-self NCPM, we propose two-types of competitive propagation games and discuss their Nash equilibria, as well as the trade-off between seeding and quality for the repeated one-shot game. One possible future work is the deliberative investigation on the Nash equilibrium on the boundary. It is also of research value to explore the extension of the analysis in Section V.B to the case of multiple-player dynamic games. Another open problem is the stability analysis of the self-social NCPM with $R > 2$. Simulation results support the claim that, for the case when $R > 2$, there also exists a unique fixed point P^* and, for any initial condition $P(0) \in S_{nR}(\mathbb{1}_n)$, the solution $P(t)$ to equation (10) converges to P^*. We leave this statement as a conjecture.

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