Modeling and Analysis of Competitive Propagation with Social Conversion

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Abstract— **In this paper we model a class of propagation processes for multiple competing products on a contact network and analyze the resulting dynamical behaviors. We assume three types of product-adoption processes for each individual: self adoption, social adoption and social conversion. On this basis, we build a Markov chain model of the competitive propagation process. Based on the independence approximation, a difference equations system, referred to as the network competitive propagation model, is derived to approximate the original Markov chain. Both simulation work and theoretical results are given to show the accuracy of the independence approximation. The network competitive propagation model does not exclude the long-term coexistence of the mutually exclusive competing products spreading in a single-layer network. The result on coexistence is contrary to some previous literature on the propagation of multiple memes. Moreover, we find that the probability distributions of nodes' states achieve asymptotic consensus, which indicates that our network competitive propagation model is a good example of network dynamics with both consensus and propagation behaviors.**

I. INTRODUCTION

a) Motivation and problem description: It is great scientific interest to model some aspects of human society as a large-scale complex network exhibiting dynamics such as consensus, polarization, synchronization and propagation. Indeed, the past fifteen years have witnessed a flourishing of research on propagation process on social networks. Much progress has been achieved both statistically [1], [2], [3] and theoretically [4], [5], [6], [7], [8]. Propagation processes over network can describe the spread of an infectious disease in a contact network, an innovative product in an economic network or a ideology in an influence network. Infections, products and ideologies are collectively referred to as "memes."

In a more recent set of extensions, scientists have began studying the simultaneous possibly-competitive propagation of multiple memes. Such phenomena have significant research value since there exist numerous real examples of competitive propagation processes in real society. Due to the interactions among both the nodes in the network and propagating memes, multi-meme propagation systems exhibit interesting dynamical behaviors deserving of scientific investigation.

This paper studies a class of propagation processes for competing innovative products. We focus on the kind of innovative products that pervade the social network and become daily necessities for every people. The popularization of cellphones is an appropriate example. Studying the competitive propagation of such kind of products helps to understand network dynamics with multiple types of interactions, and might provide some directions for business companies to evaluate and adjust their marketing and promotion strategies. In this paper, we build a mathematical model of such competitive propagation phenomena and analyze their dynamic behavior.

b) Literature review: Propagation on networks are usually modeled and analyzed in the context of epidemic spreading or innovation diffusion. Among various models of propagation processes on networks, three are widely adopted: the percolation model on random graphs [9], [5], the Markov chain on fixed topology with the mean-field approximation [10], [6], [11], [12] and the linear threshold model [7], [13], [14].

As extensions to the propagation of a single meme, some recent papers have focused on the propagation of multiple memes. The modeling of multiple-meme propagation has been studied from various perspectives, e.g., see [15], [16], [17], [18], [19], [20], [21]. Among these papers, our paper is closely related to [19], [20], [21], which present variants of the Susceptible-Infected-Susceptible (SIS) epidemic spreading model, i.e., the network SI_1SI_2S model. The model proposed in [19] assumes that the two diseases are mutually exclusive. Under such an assumption they proved that the long-term coexistence of the two diseases is impossible. Paper [20] relaxed the mutual exclusion assumption and showed that coexistence is possible if any node can be infected by two diseases at the same time. In this model, infection of one disease affects the infection probability of another disease. The mutual exclusion assumption was kept in [21], while the propagation of memes was assumed to occur on multilayer networks and each competing meme propagates on its own layer. On such conditions the coexistence of competitive memes is also possible.

c) Contribution: Our first contribution is a novel Markov chain model which is more suitable to the competitive propagation processes we aim to understand, compared with the epidemic-like models [19], [20], [21]. Similar to the linear threshold model, our paper assumes that it is the ratio, instead of the number, of neighbors in different states that influences an individual's behavior because adopting a product is an initiative selection process rather than a passive contagion. The main difference from the linear threshold model is that our model is stochastic and no predetermined threshold is assumed. Moreover, our model is a variant of SI model instead of the SI_1SI_2S model for that we are considering the propagation of innovative products, such as cellphones, which successfully pervade the social network and become daily necessities. Adopting such a product can be considered as an irreversible action.

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Secondly, starting from discussing individuals' product-adoption behaviors, we develop a difference equations system, referred to as the *network competitive propagation model*, to approximate the original Markov chain. The derivation is based on the *independence approximation*, which is mathematically equivalent to the widely-used mean-field mean-field approximation [6], [12], [22], [23]. A rigorous and self-contained derivation of the network competitive propagation model is given. Simulation work is done to show the accuracy of the independence approximation in the cases of Erdős-Rényi graphs and complete graphs.

Thirdly, by relaxing the independence approximation to the *independence assumption*, a theorem is given that in complete graphs the solution to the original Markov chain converges to the solution to the network competitive propagation model as the size of the network tends to infinity.

Our fourth contribution is a long-term prediction that, mutually exclusive competing products persist in a single-layer network. The prediction of coexistence is distinct from the works [19], [20], [21] but is consistent with numerous real world examples such as the competition between different auto insurances.

At last, the theoretical analysis of the network propagation model reveals some interesting dynamical properties. As an extension to the classic Bass diffusion model [24], the network competitive propagation model persists the almost-sure adoption/infection property. Moreover, an important property is found: as long as the network is connected, the probability distribution of every individual's state achieves consensus as time tends to infinity, i.e., our system achieves *social learning* in the sense of probability distribution. The consensus property makes our model a good example of network dynamics possessing both consensus and propagation.

d) Organization: The rest of this paper is organized as follows. Section II is the basic assumptions of the Markov chain competitive propagation model. Section III is the derivation of the network competitive propagation model. In Section IV we discuss the accuracy of the independence approximation and the qualitative properties of the network competitive propagation model. Section V gives conclusion.

We include some proofs of the theorems in this paper and refer the interested reader to a following journal submission for proofs of all the other theorems/propositions/lemmas.

II. Description of the Markov Chain Model

A. Social Network as a Graph

In this model, a social network is considered as an undirected, unweighted graph $G = (V, E)$ with fixed topology. The nodes set $V = \{1, 2, \dots, n\}$ refers to the set of individuals in the network. The set E of social links is represented by the *adjacency matrix* $A = (a_{ij})_{n\times n}$. Entry $a_{ij} = 1$ if the pair of nodes $(i, j) \in E$ and $a_{ij} = 0$ if $(i, j) \notin E$. The presence of self loops is not considered, i.e., $a_{ii} = 0$ for any $i \in V$. If $a_{ij} = 1$, we say that node i and j are neighbors. The cardinality of node i's neighbor set is denoted by N_i, which is the ith row sum of A, i.e., $N_i = \sum_{j=1}^{n} a_{ij}$.

B. States of Nodes

Suppose that there are R competitive products spreading on the graph. The set of the competing products is $\{H_1, H_2, \dots, H_R\}$. These products are mutually exclusive, that is, at any time, every individual in the network adopts at most one product. We consider the competitive propagation as a discrete-time stochastic process, i.e., $t \in \mathbb{Z}_{\geq 0}$. Denote the state of node i after time step t by $D_i(t)$, $i \in V$, then $D_i(t) \in \{H_0, H_1, H_2, \dots, H_R\}$, where H_0 corresponds to the state of *not adopting any product*.

C. Production-adoption Process

In this model, the product-adoption process is divided into three possible chronologically-ordered events: self adoption, social adoption and social conversion. The behavior of nodes is described by the following three assumptions, which apply to all sections in this paper.

Assumption 1 (Self adoption): For competitive propagation on social networks, at any time step t, for any node $i \in V$, if it has not adopted any product up to time $t-1$, i.e., $D_i(t-1) = H_0$, then node i will adopt product H_r with probability ϵ_r, where $r \in \{1, 2, \dots, R\}$ and $0 \leq \epsilon_1 + \epsilon_2 + \cdots + \epsilon_R \leq 1$. If some product H_r is adopted in the self adoption process, the adoption process for node i at time step t ends up with state $D_i(t) = H_r$.

Assumption 2 (Social adoption): For any node i, if it has not adopted any product up to time $t-1$ and not adopted any product in the self adoption process at time step t (with probability $\epsilon_0 = 1 - \epsilon_1 - \cdots - \epsilon_R$), this node will first uniformly likely pick one of its neighbors j, and then follow node j's state at the previous time step $t-1$, i.e., $D_i(t) = D_j(t-1)$, with probability β.

Assumption 3 (Social Conversion): For any node i, if it has already adopted some product H_r before time step t, then at time t, it will first uniformly likely pick one of its neighbors j and then follow node j's state at time $t-1$ with probability α.

The last two assumptions depict how a node is influence by the "social pressure" formed by its neighbors: the more neighbors have adopted some product H_s, the higher probability that this node will adopt or be converted to the product H_s. For simplicity, in this paper we assume homogeneous nodes, i.e., the parameters $\alpha, \beta, \epsilon_1, \dots, \epsilon_R$, which together defines nodes' adoption behavior, are identical for any node $i \in V$. The complete adoption process for node i at time step $t+1$ is illustrated as a diagram in Figure 1.

D. Competitive Propagation as a Markov Chain

According to the assumptions above, at any time step $t+1$, the probability distribution of any node's states depends on its own state at previous time step, $D_i(t)$, as well as the states of all its neighbors at time t, i.e., $D_j(t)$ for any j satisfying $a_{ij} = 1$. Since every node has $R+1$ possible states, the collective evolution of nodes' states is a $(R+1)^n$-state discrete-time Markov chain. The exponential dimension adds much difficulty to the analysis of this model. An approximation model is proposed in the next section to reduce the dimension and simply the analysis of the problem.

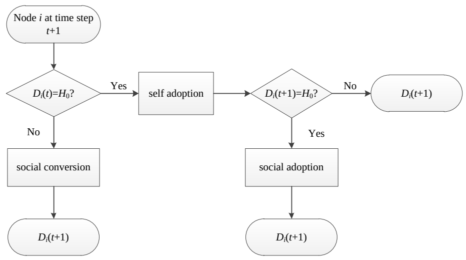


Fig. 1. This graph illustrates the self adoption, social adoption and social conversion process for node i, at time step $t+1$.

III. Derivation of the Network Competitive Propagation Model

In this section we derive the network competitive propagation model to approximate the Markov chain process defined in the previous section. The core idea is to figure out the dynamical evolution of the probability distributions of nodes' states.

Define $p_i^r(t)$, $i \in V$, $r \in \{0,1,2,\dots,R\}$ and $t \in \mathbb{Z}_{\geq 0}$, as the probability that node i is in state H_r at time step t, that is, $p_i^r(t) = \mathbb{P}[D_i(t) = H_r]$. Our goal is to find an expression of $p_i^r(t+1)$ as a function of $p_j^s(t)$, $j \in V$, $s \in \{0,1,2,\dots,R\}$, $t \in \mathbb{Z}_{\geq 0}$, with parameters α, β, ϵ_1, $\dots$, ϵ_R. With such an expression, given the initial condition, $p_i^r(0)$ for any i, r and the model parameters, the probability distribution $p_i^r(t)$, $r \in \{1,2,\dots,R\}$, for any node i at any time t can be computed iteratively.

In the derivation of the iteration function, a useful lemma and the independence approximation are used and stated below. The useful lemma can be proved using probability theory and Assumptions 1, 2 and 3.

Lemma 4: Define random variables $X_i^r(t)$, $i \in V, r \in \{0,1,\dots,R\}, t \in \mathbb{Z}_{\geq 0}$, as "whether node i has adopted product H_r at time step t", i.e.,

$$X_i^r(t) = \begin{cases} 1, & \text{if } D_i(t) = H_r, \\ 0, & \text{if } D_i(t) \neq H_r, \end{cases}$$

and define

$$\boldsymbol{D}_{-i}(t) = \big(D_1(t), \dots, D_{i-1}(t), D_{i+1}(t), \dots, D_n(t)\big)$$

as the states of all the nodes except node i at time t. For simplicity, let

$$\begin{aligned} P_{ij}^{rs}(t) &= \mathbb{P}[D_i(t) = H_r \,|\, D_j(t) = H_s], \\ P_i^r(t;-i) &= \mathbb{P}[D_i(t) = H_r \,|\, \boldsymbol{D}_{-i}(t)], \\ P_i^{rs}(t;-i) &= \mathbb{P}[D_i(t) = H_r, D_i(t+1) = H_s \,|\, \boldsymbol{D}_{-i}(t)] \end{aligned}$$

for any $i,j \in V, r,s \in \{0,1,\dots,R\}$ and $t \in \mathbb{Z}_{\geq 0}$. Then the following equations hold:

$$P_i^{0r}(t;-i) = \Big(\epsilon_r + \frac{\beta\epsilon_0}{N_i}\sum_j a_{ij} X_j^r(t)\Big) P_i^0(t;-i) \tag{1}$$

$$P_i^{sr}(t;-i) = \frac{\alpha}{N_i}\sum_j a_{ij} X_j^r(t) P_i^s(t;-i) \tag{2}$$

and

$$\mathbb{E}[X_j^s(t) P_i^r(t;-i)] = p_j^s(t) P_{ij}^{rs}(t). \tag{3}$$

The independence approximation is stated as follows. The core idea of the independence approximation is to take the state of any node as independent of any other node's state at the same time.

Approximation 5 (Independence Approximation): For the competitive propagation model on an arbitrary graph, approximate any conditional probability $\mathbb{P}[D_i(t) = H_r \,|\, D_j(t) = H_s]$ by its corresponding total probability $\mathbb{P}[D_i(t) = H_r]$, i.e.,

$$P_{ij}^{rs}(t) \simeq p_i^r(t)$$

for any $i,j \in V$, $r,s \in 0,1,\dots,R$ and $t \in \mathbb{Z}_{\geq 0}$.

With this approximation, the approximation model to the original Markov-chain competitive propagation is stated below.

Theorem 6 (Network Competitive Propagation Model): For the competitive propagation model with self adoption rates $\epsilon_1, \epsilon_2, \dots, \epsilon_R$, social adoption rate β and social conversion rate α, the probability that node i has adopted H_r at time t satisfies

$$\begin{aligned} &p_i^r(t+1) - p_i^r(t) \\ &\quad = \epsilon_r \phi_i(t) + \frac{\beta\epsilon_0}{N_i}\sum_j a_{ij} p_j^r(t) P_{ij}^{0r}(t) \\ &\qquad + \frac{\alpha}{N_i}\sum_{s\neq r}\sum_j a_{ij}\Big(p_j^r(t) P_{ij}^{sr}(t) - p_j^s(t) P_{ij}^{rs}(t)\Big). \end{aligned} \tag{4}$$

Based on Approximation 5, the discrete-time dynamics for the probability $p_i^r(t)$ is

$$\begin{aligned} &p_i^r(t+1) - p_i^r(t) \\ &\quad = \epsilon_r \phi_i(t) + \frac{\beta\epsilon_0}{N_i}\sum_j a_{ij} p_j^r(t) \phi_i(t) \\ &\qquad + \frac{\alpha}{N_i}\sum_{s\neq r}\sum_j a_{ij}\Big(p_j^r(t) p_i^s(t) - p_j^s(t) p_i^r(t)\Big), \end{aligned} \tag{5}$$

where $\phi_i(t) = 1 - p_i^1(t) - p_i^2(t) - \cdots - p_i^R(t)$ and $P_{ij}^{0r}(t), P_{ij}^{sr}(t), P_{ij}^{rs}(t)$ are defined in Lemma 4.

Proof: In order to compute the time evolution of $p_i^r(t)$, notice that

$$\begin{aligned} p_i^r(t+1) - p_i^r(t) &= \mathbb{E}[X_i^r(t+1) - X_i^r(t)] \\ &= \mathbb{E}\Big[\mathbb{E}[X_i^r(t+1) - X_i^r(t) \,|\, \boldsymbol{D}_{-i}(t)]\Big]. \end{aligned}$$

The conditional expectation can be expressed as

$$\begin{aligned} &\mathbb{E}[X_i^r(t+1) - X_i^r(t) \,|\, \boldsymbol{D}_{-i}(t)] \\ &\quad = 0 \cdot \mathbb{P}[X_i^r(t+1) - X_i^r(t) = 0 \,|\, \boldsymbol{D}_{-i}(t)] \\ &\qquad + 1 \cdot \mathbb{P}[X_i^r(t+1) - X_i^r(t) = 1 \,|\, \boldsymbol{D}_{-i}(t)] \\ &\qquad + (-1) \cdot \mathbb{P}[X_i^r(t+1) - X_i^r(t) = -1 \,|\, \boldsymbol{D}_{-i}(t)] \end{aligned} \tag{6}$$

According to Assumption 1, 2 and 3, we have the following equations:

$$\begin{aligned} &\mathbb{P}[X_i^r(t+1) - X_i^r(t) = 1 \,|\, \boldsymbol{D}_{-i}(t)] \\ &= P_i^{0r}(t;-i) + \sum_{s\neq r} P_i^{sr}(t;-i) \end{aligned} \tag{7}$$

and

$$\begin{aligned}&\mathbb{P}[X_i^r(t+1)-X_i^r(t)=-1\,|\,\boldsymbol{D}_{-i}(t)]\\&=\sum_{s\neq r}P_i^{rs}(t;-i).\end{aligned}\tag{8}$$

The expressions of $P_i^{0r}(t;-i)$, $P_i^{sr}(t;-i)$ and $P_i^{rs}(t;-i)$ can be computed according to equation (1) and (2) in Lemma 4. Substitute equation (7) and (8) into equation (6) we obtain

$$\begin{aligned}&\mathbb{E}[X_i^r(t+1)-X_i^r(t)\,|\,\boldsymbol{D}_{-i}(t)]\\&=\Big(\epsilon_r+\frac{\beta\epsilon_0}{N_i}\sum_j a_{ij}X_j^r(t)\Big)P_i^0(t;-i)\\&\quad+\frac{\alpha}{N_i}\sum_{s\neq r}\sum_j a_{ij}\Big(X_j^r(t)P_i^s(t;-i)-X_j^s(t)P_i^r(t;-i)\Big).\end{aligned}$$

Compute the expectations of both sides of the equation above and according to equation (3) in Lemma 4, we obtain the exact equation (4) for the time evolution of $p_i^r(t)$. Finally, applying Approximation 5 to equation (4) we obtain the network competitive propagation model (5). ■

Remark 7: Theorem 6 provides an algorithm to compute the approximated probability distributions of nodes' state at any time step, given all the model parameters and initial conditions. As simulation results indicate, it is not necessarily true that only one of $\lim_{t\to\infty}\Big(p_1^r(t)+p_2^r(t)+\cdots+p_n^r(t)\Big)$, $r\in\{1,2,\ldots,R\}$, is nonzero, which means that the network competitive propagation model does not exclude the persistent coexistence of multiple competing products.

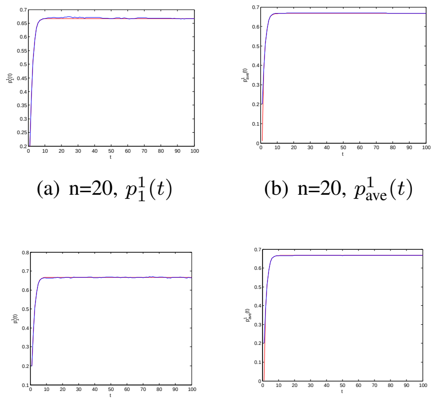
The derivation procedure of Theorem 6 was recently proposed by van Mieghem et. al. [23] and the independence approximation underlying this paper is equivalent to the widely adopted mean-field approximation [6], [22], [23]. However, the independence approximation has it own value: in the case of complete graphs, it can easily relaxed to the independence assumption, based on which we can prove that the solution to the original Markov chain converges to the solution to the network competitive propagation model as the system size tends to infinity. These concepts will be discussed in the next section.

IV. Network Competitive Propagation Model: Accuracy and Qualitative Properties

A. Accuracy of the Independence Approximation

Since the independence approximation is adopted, the solution to the network competitive propagation is an approximation to the original Markov chain model described by Assumptions 1, 2 and 3. In this subsection we discuss how the solution to the network model approximates the original Markov chain solution. Two kinds of graphs are considered: the Erdős-Rényi random graph and the complete graph.

1) Erdős-Rényi graphs: The Erdős-Rényi graph is a random graph in which any pair of two nodes forms a link with an uniform probability p. The probability p together with the number of nodes, n, determines the structure of a Erdős-Rényi graph. We consider two competing products propagating on Erdős-Rényi graphs with $n=20$, $p=0.5$ and $n=100$, $p=0.1$ respectively. The model parameters are set to be $\epsilon_1=0.2,\epsilon_2=0.1,\alpha=0.2,\beta=0.4$. The initial conditions are all zeroes. Equation (5) gives the approximation solution and the Markov chain solution is estimated by the Monte Carlo method.



(a) n=20, $p_1^1(t)$ (b) n=20, $p_{\text{ave}}^1(t)$ (c) n=100, $p_1^1(t)$ (d) n=100, $p_{\text{ave}}^1(t)$

Fig. 2. Difference between the solutions to the network competitive propagation model and the original Markov chain model in Erdős-Rényi graphs. The red curves are the solution to the network model and the blue curves correspond to the Markov chain solution.

In figure 2 we plot the probability that node 1 adopts product H_1, and the average probability of adopting H_1 of all the nodes, as functions of time step t, i.e., $p_1^1(t)$ and $p_{\text{ave}}^1(t)$. The simulation result indicates that both in the Erdős-Rényi graph with 20 nodes and 100 nodes, the approximated solutions are very close to the original Markov chain solutions.

2) Complete graphs: Now we consider the competitive propagation on complete graphs, i.e., on graphs where every node is linked to all the other nodes. Therefore a complete graph $G=(V,E)$ with the adjacency matrix A satisfies $N_i=n-1$ and $a_{ii}=0$, $a_{ij}=1$ for any $i,j\in V$, $i\neq j$. Figure 3 indicates that for competitive propagation on complete graphs, as the size of the graph increases, the difference between the solution (take $p_1^1(t)$ as an example) to the network model and the Markov chain model converges to zero at any time t.

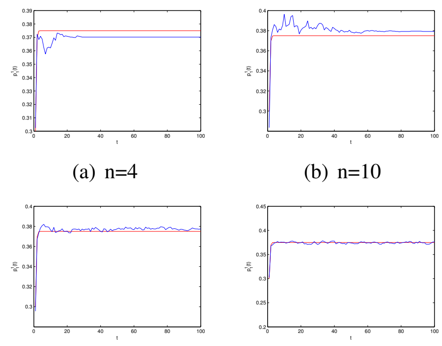
If we consider identical initial conditions, say $p_i^r(0)=0$ for any $i\in V$, the time evolution of state probability distribution in complete graphs will be identical for any node. Therefore we can omit the node index and then network competitive propagation model takes a very simple form:

$$p^r(t+1)-p^r(t)=\epsilon_r\phi(t)+\beta\epsilon_0p^r(t)\phi(t)\tag{9}$$

and the exact solution, i.e., the Markov chain solution satisfies

$$\begin{aligned}&p^r(t+1;n)-p^r(t;n)\\&\quad=\epsilon_r\phi(t;n)+\beta\epsilon_0p^r(t;n)P_{ij}^{0r}(t;n)\\&\qquad+\alpha\sum_{s\neq r}\Big(p^r(t;n)P_{ij}^{sr}(t;n)-p^s(t;n)P_{ij}^{rs}(t;n)\Big).\end{aligned}\tag{10}$$

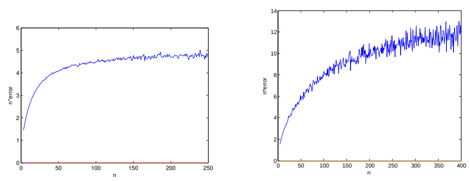
where $r\in\{1,2,\ldots,R\}$ and the system size n is added into the expressions in the exact equation (10) as an important parameter.



(a) n=4 (b) n=10

(c) n=30 (d) n=100

Fig. 3. Convergence of the original Markov chain to the network competitive propagation solution on complete graphs. The approximated solution are the red curves and the Markov chain solution are the blue curves. As the size of the system, n, increases, the original Markov chain solution gets closer to the approximated solution.



(a) t=5 (b) t=15

Fig. 4. The system size n multiplied by the error between the conditional probability $\mathbb{P}[D_2(t) = H_1 \,|\, D_1(t) = H_1]$ and the total probability $\mathbb{P}[D_2(t) = H_1]$, as a function of n. As n increases, the product converges to a constant value. This result validates that the error is in order $O\left(\frac{1}{n}\right)$.

The simulation result in Figure 3 can be stated as the following theorem.

Theorem 8 (Convergence for Complete Graphs): For the competitive propagation on complete graphs, suppose $p^r(t;n)$ satisfies the exact equation (10) while $p^r(t)$ satisfies the approximated equation (9), as the system size n tends to infinity, for any $t \in \mathbb{Z}_{\geq 0}$ and $r \in \{1, 2, \dots, R\}$, we have

$$\lim_{n\to\infty} p^r(t;n) - p^r(t) = 0.$$

This theorem is based on the following assumption, which is a relaxation of the independence approximation.

Assumption 9 (Asymptotic Independence Assumption): For the competitive propagation model described in Section 2, if the graph is complete and all the nodes have identical initial conditions, then the following equations hold:

$$P_{ij}^{rs}(t;n) - p_i^r(t;n) = O\left(\frac{1}{n}\right) \tag{11}$$

for any $s, r \in \{0, 1, \dots, R\}$ and any $i \neq j$, $t \in \mathbb{Z}_{\geq 0}$.

Equation (11) is validated by simulation work, see Figure 4.

B. Qualitative Properties of the Network Model

1) Almost-sure Adoption: The network competitive propagation model is a variant of the classic susceptible-infected (SI) epidemic model. A salient feature of SI model is that as time tends to infinity, the probability of being infected tends to 1 for any node as long as the graph is connected. Our model has a similar property named as *almost-sure adoption*, that is, if we consider "adopt some product" as one state, then the probability of being in such state tends to 1 for any node, as $t \to \infty$. This property is stated as the following proposition.

Proposition 10 (Almost-sure Adoption): For the network competitive propagation model given by Theorem 6, if at least one of the self adoption probabilities $\epsilon_1, \epsilon_2, \dots, \epsilon_R$ is non-zero, then for any initial states $p_i^r(0)$, $i \in V, r \in \{1, 2, \dots, R\}$, we have

$$\lim_{t\to\infty} \sum_{r=1}^{R} p_i^r(t) = 1. \tag{12}$$

The result of Proposition 10 is consistent with the original Markov chain model described by Assumptions 1, 2 and 3. In the Markov chain model, at any time step the probability of not adopting any product for any given node i is less than or equal to $\epsilon_0 = 1 - \epsilon_1 - \epsilon_2 - \dots - \epsilon_R$. Therefore, the probability of not adopting any product in t time steps is no larger than ϵ_0^t, which means the probability of not adopting any product before t vanishes for any node as $t \to \infty$. In addition, once a node has adopted some product, by our assumptions, it is not possible for this node to convert from the state *adopting some product* to *not adopting any product*. Therefore, that *every node has adopted some product* is a necessary condition which the fixed point of the original Markov chain must satisfy.

2) Consensus of Probability Distributions of Nodes' States: Consider $p_i^r(t)$, $r \in \{1, 2, \dots, R\}$, as the probability distribution of node i's states at time t. The network competitive propagation model has a very interesting property: as time tends to infinity, the state probability distribution for any node tends to be identical, i.e., the network competitive propagation model achieves consensus in the sense of state probability distribution $p_i^r(t)$. The consensus property is stated as the theorem below.

Theorem 11 (Consensus of Probability Distribution): For the network competitive propagation model on any connected graph, with initial condition $0 \leq \sum_{r=1}^{R} p_i^r(t) \leq 1$ for any $i \in V$, if at least one of the self adoption probabilities $\epsilon_1, \epsilon_2, \dots, \epsilon_R$ is non-zero, the state probability distribution for any node achieves consensus as time tends to infinity, that is, for any $r \in \{1, 2, \dots, R\}$,

$$\lim_{t\to\infty} \boldsymbol{p}^r(t) = c_r \mathbb{1}_n$$

where $\boldsymbol{p}^r(t) = \left(p_1^r(t), p_2^r(t), \dots, p_n^R(t)\right)^T$ and for any r, $c_r \in [0, 1]$ is a constant depending on the network topology, initial conditions and the model parameters. Moreover, $c_1 + c_2 +, \dots, +c_R = 1$.

Theorem 11, together with Proposition 10, indicates that the network competitive propagation model proposed in this paper is a combination of both propagation and consensus on social networks. Moreover, in the research of social dynamics, the time evolution of nodes' state probability distributions is usually interpreted as the social learning

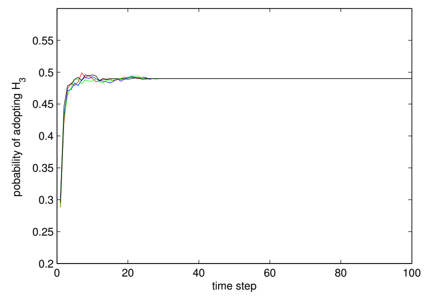


Fig. 5. The consensus of the probability $p_i^3(t)$ for the original Markov chain model. Different colors correspond to the curves for different nodes.

process [25]. If we elaborate the self-adoption process in our model as a sequential hypothesis testing [26], then the self-adoption process is a type of *Bayesian learning* and the social adoption and social conversion can be considered as *non-Bayesian learning* processes. From this perspective, our model can be extended to a mixture of both Bayesian and non-Bayesian learning, similar to the model proposed in [27].

The time evolution of $p_i^r(t)$ in the Markov chain competitive propagation model is simulated by Monte Carlo method. We consider the competitive propagation of three products, H_1, H_2 and H_3, on a graph with 4 nodes. Nodes 2, 3 and 4 are all linked to node 1 and there is no other link in the graph. The model parameters are set as $\epsilon_1 = 0.1$, $\epsilon_2 = 0.2$, $\epsilon_3 = 0.3$, $\alpha = 0.3$ and $\beta = 0.4$. Take the probability of adopting H_3 as an example. Probabilities $p_1^3(t)$, $p_2^3(t)$, $p_3^3(t)$ and $p_4^3(t)$ are estimated as functions of t. As Figure 5 indicates, the curves of $p_1^3(t)$, $p_2^3(t)$, $p_3^3(t)$ and $p_4^3(t)$ merge into one for large t. This simulation result shows that the consensus property still holds in the original Markov chain.

V. Conclusion

This paper discusses a class of competitive propagation processes. Starting from the description of individuals' behavior, we proposed a Markov chain model and approximated it with the network competitive propagation model based on the independence approximation. The network competitive propagation model reduces the dimension of problem from $(R+1)^n$ to nR and provides an algorithm to compute the approximated probability distributions of nodes' states. Simulation has been done to show how well the network model approximates the original Markov chain. In addition, we considered a specific case: competitive propagation on complete graphs. Based on the asymptotic independence assumption, we showed that the solution to the network competitive propagation model converges to the original Markov chain solution, as the system size n tends to infinity. Moreover, we have discussed some qualitative properties of the network competitive propagation model, such as almost-sure adoption and consensus of state probability distributions.

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