Centroidal Area-Constrained Partitioning for Robotic Networks

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We consider the problem of optimal coverage with area-constraints in a mobile multi-agent system. For a planar environment with an associated density function, this problem is equivalent to dividing the environment into optimal subregions such that each agent is responsible for the coverage of its own region. In this paper, we design a continuous-time distributed policy which allows a team of agents to achieve a convex area-constrained partition of a convex workspace. Our work is related to the classic Lloyd algorithm, and makes use of generalized Voronoi diagrams. We also discuss practical implementation for real mobile networks. Simulation methods are presented and discussed.

1 Introduction

Problem description and motivation The applications of multi-agent systems to accomplish complex tasks in a complex environment are vast. They include but are not limited to tasks such as search and rescue operations, ad-hoc mobile wireless networks, warehouse management, and environmental monitoring [1–4]. When working with large robotic networks, it is often desired to partition the environment amongst all agents in an optimal way so that the workload can be equalized across all agents. For example, in ocean surveillance the cost of travel is large so it is desirable to survey one contiguous region as opposed to traveling around multiple ones. A partitioning policy is one in which an environment $Q \subset \mathbb{R}^d$ is partitioned into n disjoint subregions $W_i \subset Q$, $i \in \{1,\dots,n\}$ whose union is Q. Given some measure ϕ, the partitioning policy is equitable if $\phi(W_i) = \phi(W_j)$ for all $i \neq j$. Placement of agents within such regions is also of importance. In the case of surveillance or warehouse management, it is desirable to be at the center of your region so tasks are more easily serviced. Placing agents optimally and defining optimal regions simultaneously can be complicated as the region of an agent and its position are related to one another. We provide methods to accomplish this simultaneous operation using generalized Voronoi partitions.

Literature review Partitioning and coverage control is discussed in detail in [5] along with the application of multicenter functions to robotic networks. Results on specific manifestations of generalized Voronoi partitions and partitioning can be found in [6]. Results on the existence of area-constrained power diagrams along with a method to determine them are presented in [7]. More detailed results on existence of generalized Voronoi partitions for arbitrary area constraints are presented in [8]. Linear programming is used to handle generalized Voronoi partitions in [9] for fixed agents, showing that generalized Voronoi partitions are optimal for a certain class of multicenter functions. Similar results are obtained in [8], together with a discrete-time algorithm to solve the problem of optimal deployment of agents, while satisfying constraints on the areas.

Contributions The contributions of this work are several. First, we design a provably correct, spatially-distributed continuous-time algorithm to compute area-constrained generalized Voronoi partitions of a convex environment. Our approach improves upon the work done in [7] for power diagrams both in generality and in numerical stability. Second, we build on work in [8], by introducing a continuous-time spatially-distributed algorithm to compute centroidal area-constrained generalized Voronoi partitions of a convex environment. Although an approximate method is proposed in [7] to achieve this for power diagrams, here we intro-

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duce a simpler generalized method which bridges the gap between [7] and [8] in that it is the continuous-time analogue to [8]. More precisely, the continuous-time algorithm presented in this paper and the discrete-time algorithm in [8] both converge to the set of centroidal area-constrained Voronoi partitions, which shall be formally defined below. Our proposed continuous-time algorithms are not only of academic interest in their own right, but also fill application gaps where a discrete-time algorithm may fall short. One such application is in mobile robotic networks, when there is a need to have continuous adaptive coverage and the understanding of the region's underlying density distribution changes frequently. For such cases, in the discrete time setting it is possible that, before the agents arrive at their optimal configuration, the optimal configuration has changed due to a new understanding of the density distribution. Moreover, the path that is taken may cause coverage cost to increase temporarily, since only the configuration is optimal not the path taken. In the continuous-time setting, the agents are always moving in a path that improves their coverage cost, and a change in the underlying distribution simply means a correction in their path. Third, we introduce a practical method for implementation of our algorithms, and show their performance in simulation. Finally, as theoretical contributions, we prove that the set of mappings that generate area-constrained partitions are unique up to translation and we compute several novel partial derivatives of relevant operators. Specifically, we generalize a classic result about multi-center optimization: the partial derivative of the multicenter function evaluated at area-constrained Voronoi partitions has the same direction as the vector connecting the robot locations to the centroids of their regions.

A preliminary short version of this article is to appear at the 2013 DSCC conference as [10] and is available at `http://motion.me.ucsb.edu`. In comparison, this article contains various addenda and updates not found in [10]. First, we present detailed proofs of theorems that were presented in [10]. Second, we introduce a new and better conditioned algorithm to determine area-constrained partitions. Third, a new policy to determine centroidal area-constrained partitions is presented. Finally, simulation results were extended and updated to demonstrate the performance of the new policies not present in [10].

Paper organization The paper is organized as follows. In section 2 we setup preliminary notation, introduce the concept of generalized Voronoi partitions and present our problem in technical detail. In section 3 we compute some useful properties of the objective functions of interest. In section 4 we state existence properties of area-constrained generalized Voronoi partitions and present algorithms to reach the set of area-constrained Voronoi partitions. In section 5 we state the main result of our paper on centroidal equitable generalized Voronoi partitions. In section 7 we discuss the application of our algorithm to real systems and provide numerical simulations. In the final section we present our conclusions and future directions for research.

2 Preliminaries and problem statement

Let us have a convex compact set $Q \subset \mathbb{R}^2$, endowed with a density function $\phi : Q \to \mathbb{R}_{\geq 0}$, so that the measure (or area) of a region $A \subset Q$ is defined as

$$\phi(A) = \int_A \phi(q)dq,$$

provided the set A is measurable in the sense of Lebesgue. Without loss of generality, we assume that Q has unit measure, that is, $\phi(Q) = \int_Q \phi(q)dq = 1$. Let $p_1, \ldots, p_n$ denote the positions of n robotic agents in Q. We assume that each agent is associated with a (measurable) sub-region $W_i \subset Q$, where $\{W_i\}_{i=1}^n$ partitions Q into sets whose interiors are pairwise disjoint. A vector can be defined to collect the measures of the regions of a partition, as $\phi(W) = [\phi(W_1), \ldots, \phi(W_n)]^T$. By our assumptions on Q, we have $\sum_{i=1}^n \phi(W_i) = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex, increasing, and differentiable function. Then, given n locations $p = (p_1, \ldots, p_n)$ and a partition $W = (W_1, \ldots, W_n)$, the *multicenter function* is defined by

$$\mathcal{H}(p,W) = \sum_{i=1}^n \int_{W_i} f(\|q - p_i\|)\phi(q)dq.$$

Our goal in this work is minimizing the function $\mathcal{H}$ under certain constraints, namely, that the areas of each region are fixed. Specifically, we consider constants $c_i > 0$ for each agent $i \in \{1, \ldots, n\}$ such that $\sum_{i=1}^n c_i = 1$ and we require $\phi(W_i) = c_i$ for every i. For brevity, we denote $S = \{c \in \mathbb{R}^n_{>0} \mid \sum_{i=1}^n c_i = 1\}$.

Problem 1 (Multicenter optimization with area constraint)**.** *Given $c \in S$, determine the locations of the agents $p = (p_1, \ldots, p_n)$ and the partition $W = (W_1, \ldots, W_n)$ solving:*

$$\begin{aligned} \min_{p,W} \quad & \mathcal{H}(p,W) \\ \text{subject to} \quad & \phi(W_i) = c_i, \;\; i \in \{1, \ldots, n-1\}. \end{aligned} \tag{1}$$

Note that the nth constraint $\phi(W_n) = c_n$ is omitted because it is redundant.

In order to solve this problem, we introduce a useful partitioning scheme. To begin, we define $\mathcal{D} := \{p \in Q^n \mid p_i \neq p_j, \;\; i \neq j\}$ as the set of disjoint positions in Q. Then, given the function f as above, n distinct locations $p \in \mathcal{D}$, and n scalar weights $w = (w_1, \ldots, w_n)$, the *generalized Voronoi partition* of Q is the collection of subsets $V^f(p,w) = (V_1^f(p,w), \ldots, V_n^f(p,w))$ of Q, defined by

$$\begin{aligned} V_i^f(p,w) = \{q \in Q \mid \; & f(\|q - p_i\|) - w_i \\ & \leq f(\|q - p_j\|) - w_j, \;\; \forall j \neq i\}. \end{aligned} \tag{2}$$

Generalized Voronoi partitions enjoy several important properties. First, any generalized Voronoi partition of Q is in fact a partition of Q. Second, the generalized partition generated by (p,w) is equal to the generalized partition generated by $(p, w+\alpha\mathbb{1}_n)$, for any $\alpha \in \mathbb{R}$ (Here, $\mathbb{1}_n$ is the vector in $\mathbb{R}^n$ whose entries are all equal to 1). Finally, as opposed to Voronoi partitions, for generalized Voronoi partitions we do not require that $p_i \in V_i^f(p,w)$. Thus, agents need not be located in their partition. From here onward, we will refer to generalized Voronoi partitions simply as *Voronoi partitions*. Two important special cases are described below.

Example 1 (Standard Voronoi Diagram)**.** *Given $p \in \mathcal{D}$ and n scalar weights $w = (w_1, \dots, w_n)$, the* Standard Voronoi Diagram *of Q is given by* (2) *with $w = 0$. The partition is given by*

$$V_i^{SD}(p,w) = \{q \in Q \mid f(\|q-p_i\|) \leq f(\|q-p_j\|)\},$$

regardless of the choice of $f(x)$. We call each V_i^{SD} a standard *Voronoi region. These regions are convex and have boundaries that are given by straight line segments; moreover, every generator p_i is contained in its respective region V_i^{SD}.*

Example 2 (Power Diagrams)**.** *Given $p \in \mathcal{D}$ and n scalar weights $w = (w_1, \dots, w_n)$, the* power diagram *of Q is given by* (2) *with $f(x) = x^2$. The partition is given by*

$$V_i^{PD}(p,w) = \{q \in Q \mid \|q-p_i\|^2 - w_i \leq \|q-p_j\|^2 - w_j\},$$

and we call each Voronoi region V_i^{PD} a power cell. Note that Standard Voronoi Diagrams are a special case of Power Diagrams, since $V_i^{PD}(p,0) = V^{SD}(p)$. These regions are convex and their boundaries that are line segments; however, it is possible that the generators p_i are not contained by their respective power cells V_i^{PD}.

We now define the dual graph of a partition $\{W_i\}_{i=1}^n$, which will be useful later: the node set is $\{1, \dots, n\}$, and there exists an edge $\{i, j\}$ if the boundary between agents i and j, denoted Δ_{ij}, has positive measure. In that case, we say that j is a neighbor of i and we write $j \in N_i$. The dual graph of the standard Voronoi Partition is known to be the classic Dirichlet triangulation, as illustrated in Figure 1.

Before we are ready to define a second problem of interest, we first introduce a useful definition. Given n distinct locations $p = (p_1, \dots, p_n)$, the set of weights $w = (w_1, \dots, w_n)$ such that every partition has non-zero measure is defined by $U = \{w \in \mathbb{R}^n \mid \phi(V_i^f(p,w)) > 0 \; \forall i\}$. Since $V^f(p,w) = V^f(p, w+\alpha\mathbb{1}_n)$ for every scalar α, this equivalence relation naturally defines equivalence classes in U: with a slight abuse of notation, in what follows we sometimes refer to such classes as the elements of U. The following problem is a simplified version of Problem 1.

Problem 2 (Multicenter Voronoi partition optimization with area constraints)**.** *Given $c \in S$, determine the locations of the agents $p \in \mathcal{D}$ and weights $w \in U$ solving:*

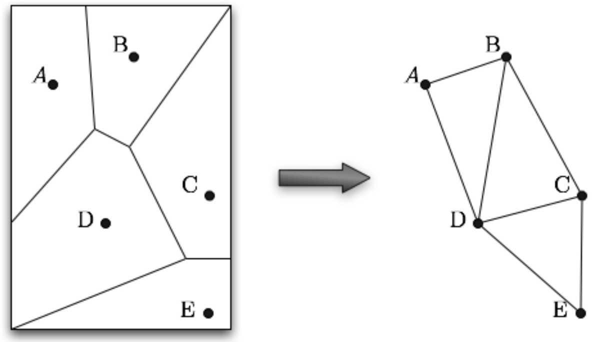

Fig. 1. The image on the left is the Standard Voronoi Partition generated by nodes A through E. The image on the right shows the dual graph for this partition.

$$\begin{aligned} \min_{p,w} \quad & \mathcal{H}(p, V^f(p,w)) \\ \text{subject to} \quad & \phi(V_i^f(p,w)) = c_i, \;\; i \in \{1, \dots, n-1\}. \end{aligned}$$

As a preliminary step, we should make sure that this problem has feasible solutions: this fact is shown in Section 4, which also provides a method to find a set of weights in U for every set of locations. Problems 1 and 2 are known to be equivalent in the following sense.

Proposition 1 (Proposition V.1 in [8])**.** *Let $p \in Q^n$ be the agent locations and $w \in U$ a weight assignment which satisfies the area constraint. Then, the Voronoi partition $V^f(p,w)$ optimizes $\mathcal{H}(p,W)$ among all partitions satisfying the area constraint.*

In order to derive a useful consequence of this fact, we consider the simpler case when there is only one agent in Q: then, the multicenter function becomes

$$p \mapsto \mathcal{H}_1(p,Q) := \int_Q f(\|q-p\|)\phi(q)dq. \qquad (3)$$

Since f is strictly convex, $\mathcal{H}_1$ is too, and the following holds: If Q is convex, then there is a unique minimizer of (3), which we denote by $\mathrm{Ce}(Q)$. Moreover, we have that $\frac{\partial \mathcal{H}_1(p,Q)}{\partial p} = 0$ at $p = \mathrm{Ce}(Q)$, where $\frac{\partial \mathcal{H}_1(p,Q)}{\partial p} = \int_Q \frac{\partial}{\partial p} f(\|q-p\|)\phi(q)dq$ from [5]. The Voronoi partition $V^f(p,w)$ generated by (p,w) is said to be *centroidal* if

$$\mathrm{Ce}[V_i^f(p,w)] = p_i,$$

for all $i \in \{1, \dots, n\}$. This notation allows us to state the following fact [8]: for every solution (p^*, W^*) of Problem 1, there exists a weight assignment $w^* \in U$ such that $W^* = V^f(p^*, w^*)$ and $p_i^* = \mathrm{Ce}(W_i^*)$ for all $i \in \{1, \dots, n\}$. Equivalently, the solutions to Problem 1 are centroidal Voronoi partitions whose regions have the prescribed areas.

In the rest of this paper, we will go beyond this abstract characterization of the optimal solutions and give an optimization algorithm which is amenable to practical implementation.

3 Relevant partial derivatives

This section is devoted to compute relevant partial derivatives of the multicenter function, which shall be used to solve Problem 2 in the subsequent sections.

Given $p \in \mathcal{D}$, and $w \in U$, we define the partition of Q by $V^f(p,w)$. It is convenient to define the *Voronoi multi-center function* as

$$(p,w) \mapsto \overline{\mathcal{H}}(p,w) = \sum_{i=1}^{n} \int_{V_i^f(p,w)} f(\|q-p_i\|)\phi(q)dq,$$

or equivalently $\overline{\mathcal{H}}(p,w) = \mathcal{H}(p, V^f(p,w))$. It is also convenient to define the *generators-to-areas function* as

$$(p,w) \mapsto \mathcal{M}(p,w) = \left[\int_{V_1^f(p,w)} \phi(q)dq, \ldots, \int_{V_n^f(p,w)} \phi(q)dq\right]^T,$$

or equivalently $\mathcal{M}(p,w) = \phi(V^f(p,w))$. In the rest of this section, we shall compute the gradients of the functions $\overline{\mathcal{H}}$ and $\mathcal{M}$. In order to state our results, we need some notation. Let $\Delta_{ij}(p,w)$ denote the boundary between the ith and jth Voronoi region and $\vec{n}_{ij}$ the normal to this boundary, pointing towards region W_j. Given locations $p \in \mathcal{D}$ and weights $w \in U$, let $L_a(p,w)$ and $L_{b_k}(p,w)$, $k \in \{1,2\}$, be the $n \times n$ matrices, whose entries a_{ij} and $b_{ij}^{(k)}$ are defined by

$$a_{ij}(p,w) = \begin{cases} -\int_{\Delta_{ij}} \phi(q)\Big(\frac{\partial q}{\partial w_i} \cdot \vec{n}_{ij}\Big)dq, & \text{if } i \neq j, \\ -\sum_{i=1}^{n} a_{ij}, & \text{otherwise}, \end{cases} \quad (4)$$

and

$$b_{ij}^{(k)}(p,w) = \begin{cases} -\int_{\Delta_{ij}} \phi(q)\big(\frac{\partial q}{\partial p_i^{(k)}} \cdot \vec{n}_{ij}\big)dq, & \text{if } i \neq j, \\ -\sum_{i=1}^{n} b_{ij}^{(k)}, & \text{otherwise}, \end{cases} \quad (5)$$

where $q \in \mathbb{R}^2$ has components $q^{(1)}$ and $q^{(2)}$. Clearly, entries a_{ij} and b_{ij} are zero if Δ_{ij} has zero measure.

We are now ready to state the two main results of this section, which imply that computing the gradients of $\overline{\mathcal{H}}(p,w)$ and the $\mathcal{M}(p,w)$ is spatially-distributed over the dual graph of the Voronoi partition.

Proposition 2 (Partial derivatives of the Voronoi multi-center function)**.** *Given $p \in \mathcal{D}$ and $w \in U$, let $p_i^{(k)}$, $k \in \{1,2\}$, denote the two components of $p_i \in \mathbb{R}^2$, for $i \in \{1,\ldots,n\}$, and define L_a and L_{b_k} as in equations* (4) *and* (5)*. Then, the partial derivatives of $\overline{\mathcal{H}}(p,w)$ are*

$$\frac{\partial \overline{\mathcal{H}}(p,w)}{\partial p^{(k)}} = \Big[\int_{V_1^f} \frac{\partial}{\partial p_1^{(k)}} f(\|q-p_1\|)\phi(q)dq, \ldots, \int_{V_n^f} \frac{\partial}{\partial p_n^{(k)}} f(\|q-p_n\|)\phi(q)dq\Big] + w^T L_{b_k}(p,w), \quad (6)$$

$$\frac{\partial \overline{\mathcal{H}}(p,w)}{\partial w} = w^T L_a(p,w). \quad (7)$$

Proof. Note that we write V for V^f throughout the proof and everything is done with respect to one component of p_i where $p_i \in \mathbb{R}^2$ (we drop the k from $p_i^{(k)}$ for clarity). Differentiating with respect to p_i we see that

$$\begin{aligned} \frac{\partial \overline{\mathcal{H}}}{\partial p_i} &= \int_{V_i} \frac{\partial}{\partial p_i} f(\|q-p_i\|)\phi(q)dq \\ &+ \int_{\partial V_i} f(\|q-p_i\|)\phi(q)\left(\frac{\partial q}{\partial p_i} \cdot \vec{n}_{ij}\right)dq \\ &+ \sum_{j \in N_i} \int_{\partial V_i \cap \partial V_j} f(\|q-p_j\|)\phi(q)\left(\frac{\partial q}{\partial p_i} \cdot \vec{n}_{ji}\right)dq, \end{aligned} \quad (8)$$

where (8) easily falls out from the conservation law [5, Proposition 2.23]. The second term on the RHS is defined as

$$\begin{aligned} &\int_{\partial V_i} f(\|q-p_i\|)\phi(q)\left(\frac{\partial q}{\partial p_i} \cdot n\right)dq \\ &= \sum_{j \in N_i} \int_{\Delta_{ij}} f(\|q-p_i\|)\phi(q)\left(\frac{\partial q}{\partial p_i} \cdot \vec{n}_{ij}\right)dq \\ &= \sum_{j \in N_i} \int_{\Delta_{ij}} (f(\|q-p_j\|) + w_i - w_j)\phi(q)\left(\frac{\partial q}{\partial p_i} \cdot \vec{n}_{ij}\right)dq, \end{aligned}$$

where $f(\|q-p_i\|) = f(\|q-p_j\|) + w_i - w_j$ holds true along the boundary Δ_{ij} and is given from the definition of the Voronoi partition. Therefore, combining the second and third terms on the RHS of (8) and noting that $\vec{n}_{ij} = -\vec{n}_{ji}$, we have

$$\begin{aligned} \frac{\partial \overline{\mathcal{H}}}{\partial p_i} &= \int_{V_i} \frac{\partial}{\partial p_i} f(\|q-p_i\|)\phi(q)dq \\ &+ \sum_{j \in N_i} (w_i - w_j) \int_{\Delta_{ij}} \phi(q)\left(\frac{\partial q}{\partial p_i} \cdot \vec{n}_{ij}\right)dq. \end{aligned} \quad (9)$$

Writing (9) in vector form gives $\frac{\partial \overline{\mathcal{H}}(p,w)}{\partial p}$ as defined in (6), with matrices whose entries are defined by (5). Similarly the derivative with respect to w_i is given as

$$\frac{\partial \overline{\mathcal{H}}}{\partial w_i} = \sum_{j \in N_i} (w_i - w_j) \int_{\Delta_{ij}} \phi(q)\left(\frac{\partial q}{\partial w_i} \cdot \vec{n}_{ij}\right)dq. \quad (10)$$

The vector form of this equation can easily be seen to be (7) with the Laplacian matrix L_a defined by (4). □

Proposition 3 (Partial derivatives of the generators-to-areas function)**.** *Given $p \in \mathcal{D}$ and $w \in U$, let $p_i^{(k)}$, $k \in \{1,2\}$, denote the two components of $p_i \in \mathbb{R}^2$, for $i \in \{1,\dots,n\}$, and define L_a and L_{b_k} as in equations* (4) *and* (5)*. Then, the partial derivatives of $\mathcal{M}(p,w)$ are*

$$\frac{\partial \mathcal{M}(p,w)}{\partial p^{(k)}} = L_{b_k}(p,w), \tag{11}$$

$$\frac{\partial \mathcal{M}(p,w)}{\partial w} = L_a(p,w). \tag{12}$$

Proof. For clarity, we write V for V^f throughout the proof. Looking at the i^{th} component of $\mathcal{M}(p,w)$ and differentiating with respect to w_i, by the conservation law [5, Proposition 2.23] we get

$$\begin{aligned}\frac{\partial \mathcal{M}_i}{\partial w_i} &= \int_{V_i} \frac{\partial}{\partial w_i}\phi(q)dq + \int_{\partial V_i} \phi(q)\left(\frac{\partial q}{\partial w_i}\cdot \vec{n}_{ij}\right)dq,\\ &= \sum_{j\in N_i}\int_{\Delta_{ij}} \phi(q)\left(\frac{\partial q}{\partial w_i}\cdot \vec{n}_{ij}\right)dq,\end{aligned} \tag{13}$$

where Δ_{ij} is the boundary between agents i and j. The first term on the right hand side of (13) is zero since the density function is not dependent on the weights. Similarly, the derivative of $\mathcal{M}_j$ with respect to w_i, is given as

$$\begin{aligned}\frac{\partial \mathcal{M}_j}{\partial w_i} &= \int_{V_j} \frac{\partial}{\partial w_i}\phi(q)dq + \int_{\partial V_j} \phi(q)\left(\frac{\partial q}{\partial w_i}\cdot \vec{n}_{ji}\right)dq,\\ &= -\int_{\Delta_{ij}} \phi(q)\left(\frac{\partial q}{\partial w_i}\cdot \vec{n}_{ij}\right)dq,\end{aligned} \tag{14}$$

where we note that $\vec{n}_{ij} = -\vec{n}_{ji}$. Therefore it is easily seen that the total gradient of $\mathcal{G}$ with respect to the vector of weights w is given by (12), with $L_a(p,w)$ defined by (4). Similarly for $p_i^{(k)}$ we get the gradient defined by (11) whose matrix $L_{b_k}(p,w)$ is given by (5). □

The following useful Proposition follows from Proposition 3 and Proposition IV.1 in [8] and is due to the monotonicity properties of the Voronoi partition.

Proposition 4 (Sign-definiteness of the partial derivatives of $\mathcal{M}$)**.** *Given $p \in \mathcal{D}$ and $w \in U$, then*

$$\frac{\partial \mathcal{M}_i(p,w)}{\partial w_i} > 0 \text{ and } \frac{\partial \mathcal{M}_i(p,w)}{\partial w_j} \le 0, \ \ j \ne i,$$

where the second inequality is strict if and only if Δ_{ij} has non-zero measure.

Next, consider the dual graph of the Voronoi partition defined by (p,w) and assign to each edge $\{i,j\}$ of this graph the (i,j) entry of $L_a(p,w)$, which is strictly-negative by Proposition 4. With this definition, the matrix $L_a(p,w)$ is the Laplacian matrix naturally associated to this weighted dual graph of the Voronoi partition defined by (p,w).

4 Area-constrained Voronoi partitions

In this section, we solve the problem of finding, given area constraints and locations of the agents, suitable weights such that the Voronoi partition generated by these locations and weights satisfies the area constraint. We begin by stating two useful results.

Proposition 5 (Existence and uniqueness of weights for area-constrained Voronoi partitions)**.** *Define constants $c \in S$. Given $p \in \mathcal{D}$, there exists a unique vector $w^* \in U$, up to translation, such that $\{V_1^f(p,w^*),\dots,V_n^f(p,w^*)\}$ satisfies $\phi(V_i^f) = c_i$ for all i.*

Proof. Existence follows from Proposition IV.4 in [8]. Before beginning the uniqueness argument, we introduce the set $V_{i\to j}(w^*,w)$ as the set of points that move from agent i to agent j due to a change in weights from w^* to w. For example, if agents i and j are neighbors, $w_j > w_j^*$, and $w_i = w_i^*$ for all $i \ne j$, then $\phi(V_{i\to j}(w^*,w)) > 0$. That is to say that some region is transferred to agent j due to its weight increasing. With this notation in mind, we begin the proof with a set of weights, $w^* \in U$, which define some arbitrary partition. Associated with this partition are a set of non-zero areas, $c_i^* > 0$ for $i \in \{1,\dots,n\}$, that correspond to each agents region. Assume there exists another set of weights, $w \ne w^*$, such that $\mathcal{M}(p,w) = \mathcal{M}(p,w^*) = c^*$. Since weights are translation invariant, we can translate w such that $w_1^* - w_1 = 0$. Without loss of generality (wlog), assume that generator 2 is the neighbor of generator 1 in $V^f(p,w^*)$, and that $w_2 - w_1 < w_2^* - w_1^*$ ($w_2 < w_2^*$, since $w \ne w^*$) or equivalently $w_2 - w_2^* < w_1 - w_1^* = 0$. Due to the monotonicity of the weights, $w_2 < w_2^*$ implies that part of the region that was once owned by agent 2 in $V^f(p,w^*)$ is either now owned by agent 1 or by some other agent in the partition $V^f(p,w)$. Thus $\phi(V_{2\to i}(w^*,w)) > 0$ for some $i \ne 2$. This means that agent 2 in $V^f(p,w)$ must own the space of some other agent (or combination of agents) in order to maintain its area-constraint. This can only happen if there exists at least one neighboring agent (wlog; agent 3) whose weight satisfies the condition $w_3 - w_2 < w_3^* - w_2^*$.Thus it must be the case that $w_3 - w_3^* < w_2 - w_2^* < w_1 - w_1^*$ and $w_3 < w_3^*$. Since every agent has at least one neighbor, we can continue in this fashion of ordering agents, until we reach agent k such that $w_j - w_j^* < w_k - w_k^*$, where $j > k$, cannot be satisfied, and so $\phi(V_k^f(p,w)) < c_k^*$. Thus, the set of weights that satisfy the area-constraint for a generalized Voronoi partition are unique, up to translation. □

Given this result, for any given set of area constraints $c \in S$, we may formally define the map $w_{\mathrm{ac}} : \mathcal{D} \to U$ as $p \mapsto$

$w_{\text{ac}}(p)$, such that $\sum_{i=1}^{n}(w_{\text{ac}}(p))_i = 0$ and $\phi(V_i^f(p, w_{\text{ac}}(p))) = c_i$ for all i.

Proposition 6 (Smoothness of mapping from positions to weights)**.** *The map $p \mapsto w_{\text{ac}}(p)$ is continuously differentiable.*

Proof. The proof makes use of the implicit function theorem in conjunction with a modified mapping of $\mathcal{M}(p,w)$, to be described later, to show continuous differentiability of $w_{\text{ac}}(p)$. Let $c \in S$ denote the vector of areas for the constraint surface. Then given the mapping $w_{\text{ac}}(p)$, we have that $\mathcal{M}(p, w_{\text{ac}}(p)) = c$. Since Voronoi partitions are translation invariant with respect to weights, w, we can define any Voronoi partition as a function of $n-1$ weights, keeping the n^{th} weight constant at zero. Define $\tilde{w} \equiv [w_1, \ldots, w_{n-1}]^T$, and define the modified mapping $(p, \tilde{w}) \mapsto \tilde{\mathcal{M}}(p, \tilde{w}) \in \mathbb{R}^{n-1}$ of $\mathcal{M}(p,w)$ by $\tilde{\mathcal{M}}(p,\tilde{w}) = \left[\int_{V_1^f(p,\tilde{w})} \phi(q)dq, \ldots, \int_{V_{n-1}^f(p,\tilde{w})} \phi(q)dq\right]^T$. Define the mapping $p \mapsto \tilde{w}_{\text{ac}}(p)$ such that $\tilde{\mathcal{M}}(p, \tilde{w}_{\text{ac}}(p)) = \tilde{c}$, where we set $\tilde{c}_i = c_i$ for $i \in \{1, \ldots, n-1\}$. Note that this is sufficient to define the constraint surface since the n^{th} constraint in $\mathcal{M}(p,w)$ and the n^{th} weight w_n are both redundant. For clarity, calculations in the rest of the proof are done with respect to one component of p_i, where $p_i \in \mathbb{R}^2$. Differentiating $\tilde{\mathcal{M}}(p,\tilde{w}) = \tilde{c}$ with respect to $\tilde{w}$ and p we obtain

$$\frac{\partial \tilde{\mathcal{M}}(p,\tilde{w})}{\partial \tilde{w}} = \tilde{L}_a(p,\tilde{w}),$$
$$\frac{\partial \tilde{\mathcal{M}}(p,\tilde{w})}{\partial p} = \tilde{L}_b(p,\tilde{w}),$$

where $\tilde{L}_b(p,\tilde{w})$ is defined as $L_b(p,w)$ with the n^{th} row removed and $\tilde{L}_a(p,\tilde{w})$ is defined as the Laplacian matrix $L_a(p,w)$ with the n^{th} column and row removed (Note $L_b(p,w)$ is dependent on which component of p_i we choose). Proposition 4 guarantees that the Laplacian of the Voronoi dual graph is always well-defined, therefore $\tilde{L}_a(p, \tilde{w}_{\text{ac}}(p)) \in \mathbb{R}^{n-1 \times n-1}$ and is full rank [11, Corollary 6.2.27]. Therefore, since $\tilde{\mathcal{M}}(p,\tilde{w})$ is continuously differentiable and $\frac{\partial \tilde{\mathcal{M}}(p,\tilde{w})}{\partial \tilde{w}}$ is invertible, then by the implicit function theorem we have that $\tilde{w}_{\text{ac}}(p)$ is continuously differentiable. Finally, note that the mapping $w_{\text{ac}}(p)$ can be written as $[\tilde{w}_{\text{ac}}(p)^T, w_{\text{ac},n}(p)]^T$, where $w_{\text{ac},n}(p)$ is a constant value of zero. Since $\tilde{w}_{\text{ac}}(p)$ and $w_{\text{ac},n}(p)$ are continuously differentiable, so is $w_{\text{ac}}(p)$. □

We now present an algorithm to compute $w_{\text{ac}}(p)$ for a specific area-constraint. Given $p \in \mathcal{D}$ and $w \in U$, the *area-constraint cost function* for the Voronoi partition generated by (p,w) is defined as

$$\begin{aligned}\mathcal{J}(p,w) &= n \log\left(\int_Q \phi(q)dq\right) - \sum_{i=1}^{n} c_i \log\left(\int_{V_i^f(p,w)} \phi(q)dq\right) \\ &= n\log(\phi(Q)) - \sum_{i=1}^{n} c_i \log\left(\phi\left(V_i^f(p,w)\right)\right) \qquad (15)\end{aligned}$$

where c_i for $i \in \{1, \ldots, n\}$ are strictly positive constants. The following result extends [7, Theorem 3.7] to (generalized) Voronoi partitions, and has the added property of being better conditioned numerically.

Theorem 7 (Gradient of the area-constraint cost function)**.** *Let Δ_{ij} denote the boundary between the i^{th} and j^{th} Voronoi region and $\vec{n}_{ij}$ the normal vector along that boundary. Define constants $c \in S$. Given $p \in \mathcal{D}$ and $w \in U$, we have*

$$\frac{\partial}{\partial w_i}\mathcal{J}(p,w) = \sum_{j \in N_i} \Xi_{ij}\left(c_j \phi(V_i^f) - c_i \phi(V_j^f)\right), \qquad (16)$$

where

$$\Xi_{ij} = \frac{1}{\phi(V_i^f)\phi(V_j^f)} \int_{\Delta_{ij}} \left(\frac{\partial q}{\partial w_i} \cdot \vec{n}_{ij}\right) \phi(q) dq, \qquad (17)$$

so that

(i) every w generating an area-constrained Voronoi partition is a critical point of the function $w \mapsto \mathcal{J}(p,w)$, and
(ii) every solution to the negative gradient flow

$$\dot{w}_i = -\frac{\partial}{\partial w_i}\mathcal{J}(p,w), \qquad (18)$$

converges asymptotically to $w_{\text{ac}}(p)$, yielding an area-constrained Voronoi partition such that $\phi(V_i^f) = c_i$.

Proof. Let $w \mapsto \mathcal{J}(p,w)$ be a candidate Lyapunov function. First, we check that $\mathcal{J}$ is continuously differentiable. Using (13) and (14) from the proof of Proposition 3 and the chain rule, we quickly have (16) with coefficients defined by (17). Therefore given $p \in \mathcal{D}$, we have that $\mathcal{J}$ is continuously differentiable with respect to the weights $w \in U$. Second, we see with $\dot{w}_i$ defined according to (18) that $\dot{\mathcal{J}} = \sum_{i=1}^{n} \frac{\partial \mathcal{J}}{\partial w_i}\dot{w}_i \leq 0$. To determine when $\dot{\mathcal{J}}$ is identically zero, let $y_i = \frac{1}{\phi\left(V_i^f(p,w)\right)}$, and rewrite (16) and (17) in vector notation to get the following:

$$\frac{\partial}{\partial w}\mathcal{J}(p,w) = y^T \operatorname{diag}([c_1, \ldots, c_n]) L_a(p,w),$$

where $L_a(p,w)$ is the Laplacian matrix defined by (4). Let $x = y^T \operatorname{diag}([c_1, \ldots, c_n])$, then $\frac{\partial}{\partial w}\mathcal{J}(p,w)$ is identically zero when $x = \alpha \mathbb{1}_n^T$ for any constant α. Given the constraints of the system this can only happen when $y_i = \frac{1}{c_i}$, or equivalently when $\phi\left(V_i^f(p,w)\right) = c_i$ for all $i \in \{1, \ldots, n\}$. Thus the invariant set for (18) is such that for $\phi\left(V_i^f(p,w)\right) = c_i$ all i and $\sum_{i=1}^{n} c_i = 1$. By these observations, we have proved claim (i). Third, we have to show that trajectories are bounded. From the gradient descent law (18) we deduce that the measures of each agent are bounded away from zero. Indeed, if the measure of an agent's region were to approach zero, then the function $\mathcal{J}$ would grow unbounded; this is impossible because we know $\mathcal{J}$ is monotonically non-increasing. Notice that the measures of the agents depend on the weights, and it is not hard to verify that the sum of weights stays constant. Hence, if a weight were to become very large, another weight would become arbitrarily small, which would cause a region to vanish. This contradicts the fact that the measures are bounded: therefore, the weights must also be bounded. After these three observations, we can invoke LaSalle Invariance Principle and deduce that the weights converge to the set of weights w such that $\phi(V_i^f(p,w)) = c_i$ for all i. Additionally, since the sum of the weights is constant and the vector of weights that satisfy the area constraint is unique by Proposition 5, we conclude that the weights converge to the vector of weights w^* such that $\sum_{i=1}^n w_i^* = \sum_{i=1}^n w_i(0)$ and $\phi(V_i^f(p,w^*)) = c_i$ for all i, proving claim (ii). □

We now specialize the gradient (18) to the case of Example 2.

Example 3 (Gradient flow for equitable power diagrams)**.** *Define the partition of the region Q according to* (2) *with $f(x) = x^2$ and let $c_i = \frac{1}{n}$ for all i in* (15)*. From [7] we have that $\frac{\partial q}{\partial w_i} \cdot \vec{n}_{ij} = \frac{1}{\|p_i - p_j\|}$ so then the gradient flow* (18) *is given by*

$$\dot{w}_i = -\sum_{j \in N_i} \left(\frac{1}{\phi(V_j^{PD})} - \frac{1}{\phi(V_i^{PD})} \right) \int_{\Delta_{ij}} \frac{\phi(q)}{\|p_i - p_j\|} dq.$$

The vector of weights w converges to the value such that $\phi\left(V_i^{PD}(p,w)\right) = \phi\left(V_j^{PD}(p,w)\right)$ for all $i \neq j$.

5 Centroidal area-constrained Voronoi partitions

Given constants $c \in S$, and given $p \in \mathcal{D}$, it is convenient to define the *area-constrained Voronoi partition* generated by p as

$$V_{\mathrm{ac}}^f(p) = V^f(p, w_{\mathrm{ac}}(p)),$$

with Voronoi regions $V_{\mathrm{ac},i}^f(p)$, $i \in \{1, \dots, n\}$ such that $\phi\left(V_{\mathrm{ac},i}^f(p)\right) = c_i$ for all i. The associated *area-constrained multicenter function* is given by

$$p \mapsto \mathcal{H}\big(p, V_{\mathrm{ac}}^f(p)\big) = \sum_{i=1}^n \int_{V_{\mathrm{ac},i}^f(p)} f(\|q - p_i\|)\phi(q) dq,$$

or equivalently by $p \mapsto \overline{\mathcal{H}}\big(p, w_{\mathrm{ac}}(p)\big)$. We are now ready for the main result of this section.

Theorem 8 (Gradient of the area-constrained multicenter function)**.** *Given $p \in \mathcal{D}$,*

$$\frac{\partial}{\partial p_i} \mathcal{H}\big(p, V_{\mathrm{ac}}^f(p)\big) = \int_{V_{\mathrm{ac},i}^f(p)} \frac{\partial}{\partial p_i} f(\|q - p_i\|)\phi(q) dq, \quad (19)$$

so that

(i) every p generating a centroidal area-constrained Voronoi partition is a critical point of the function $p \mapsto \mathcal{H}\big(p, V_{\mathrm{ac}}^f(p)\big)$, and

(ii) every solution to the negative gradient flow

$$\dot{p}_i = -\int_{V_{\mathrm{ac},i}^f(p)} \frac{\partial}{\partial p_i} f(\|q - p_i\|)\phi(q) dq \qquad (20)$$

converges asymptotically to the set of centroidal area-constrained Voronoi partitions.

Proof. Let $\overline{\mathcal{H}}(p, w_{\mathrm{ac}}(p)) = \mathcal{H}(p, V^f(p, w_{\mathrm{ac}}(p)))$, the Voronoi multicenter function restricted to the area-constraint surface, be our candidate Lyapunov function. By differentiating with respect to p we obtain

$$\begin{aligned} \frac{\partial}{\partial p} \overline{\mathcal{H}}(p, w_{\mathrm{ac}}) &= \frac{\partial \overline{\mathcal{H}}(p, w_{\mathrm{ac}})}{\partial p} + \frac{\partial \overline{\mathcal{H}}(p, w_{\mathrm{ac}})}{\partial w} \frac{d w_{\mathrm{ac}}}{dp}, \\ &= \Bigg(\left[\frac{\partial \mathcal{H}_1(p_1, V_{\mathrm{ac},1}^f(p))}{\partial p_1}, \dots, \frac{\partial \mathcal{H}_1(p_n, V_{\mathrm{ac},n}^f(p))}{\partial p_n} \right] \\ &\quad + w^T L_b(p,w) \Bigg) + w^T L_a(p,w) \frac{d w_{\mathrm{ac}}}{dp}, \end{aligned} \qquad (21)$$

where $L_a(p,w)$ and $L_b(p,w)$ are defined by (4) and (5), respectively. Differentiating $\mathcal{M}(p, w_{\mathrm{ac}}(p)) = c$ with respect to p we obtain

$$\frac{\partial \mathcal{M}(p, w_{\mathrm{ac}}(p))}{\partial p} + \frac{\partial \mathcal{M}(p, w_{\mathrm{ac}}(p))}{\partial w} \frac{d w_{\mathrm{ac}}(p)}{dp} = 0,$$

that is, $L_b(p,w) + L_a(p,w) \frac{d w_{\mathrm{ac}}(p)}{dp} = 0$, therefore $L_a(p,w) \frac{d w_{\mathrm{ac}}(p)}{dp} = -L_b(p,w)$. Substituting this equality into (21) it follows that $\frac{\partial}{\partial p} \mathcal{H}(p, V^f(p, w_{\mathrm{ac}})) = \frac{\partial}{\partial p} \overline{\mathcal{H}}(p, w_{\mathrm{ac}}) = \left[\frac{\partial \mathcal{H}_1(p_1, V_{\mathrm{ac},1}^f(p))}{\partial p_1}, \dots, \frac{\partial \mathcal{H}_1(p_n, V_{\mathrm{ac},n}^f(p))}{\partial p_n} \right]$. Therefore $\mathcal{H}(p, V^f(p, w_{\mathrm{ac}}))$ is continuously differentiable with respect to p, and its critical points are characterized, proving claim (i). For each agent the trajectories under (20) point towards the centers of their region, and since Q is convex and compact this gives that the trajectories stay in Q and are bounded. We must also show that the agents maintain distinct locations. If two agents i and j have the same weight ($w_j = w_i$), then relative to each other they generate a Standard Voronoi partition (Example 1) and the center for each agent stays in the same region as the agent, therefore the agents can not collide. Now we look at the case when the weights are different. Without loss of generality let $w_j > w_i$ and assume that agent j approaches agent i. From (2) we have that for $q \in Q$, the region of agent j satisfies $f(\|q - p_j\|) - w_j + w_i \leq f(\|q - p_i\|)$. If p_j

and p_i are close enough, all points in a neighborhood of p_i belong to region j. This implies that the measure of region i is zero; this is impossible since the flow stays along the area-constraint surface. Therefore, agents can not collide and agent locations remain distinct along the flow. Under control law (20) we have that

$$\dot{\mathcal{H}}(p,V^f(p,w_{\mathrm{ac}})) = \sum_{i=1}^{n} \frac{\partial \mathcal{H}_1(p_i, V^f_{\mathrm{ac},i}(p))}{\partial p_i}\dot{p}_i = -\sum_{i=1}^{n}\left(\frac{\partial \mathcal{H}_1(p_i, V^f_{\mathrm{ac},i}(p))}{\partial p_i}\right)^2.$$

By LaSalle's Invariance Principle the positions p converge to the invariant set of positions such that $p_i = Ce[V^f_{\mathrm{ac},i}(p)]$ for all i. Therefore, the positions converge to the set of centroidal area-constrained Voronoi partitions and claim (ii) is proved. □

There are some interesting points worth noting. First, assuming that the set of centroidal area-constrained Voronoi partitions is finite, the positions and weights converge to one of the partitions in that set. Second, the gradient descent (20) is not guaranteed to find the global minimum. Finally, the gradient restricted to the constraint surface is formally the same as the *reduced gradient* as defined in nonlinear programming [12].

We now specialize the gradient (19) to the case of Example 2.

Example 4 (Constrained gradient flow for power diagrams)**.** *Let the partition of the region Q be defined according to* (2) *with $f(x)=x^2$. Given any* powercell *A of Q the center is given by $Ce[A]=\frac{1}{\phi(A)}\int_A q\phi(q)dq$. Thus for* power diagrams*, Ce is equivalent to the well known expression for* center of mass *of a region. From [5] we have that $\frac{\partial \mathcal{H}_1\left(p, V^{PD}_{\mathrm{eq},i}(p)\right)}{\partial p_i} = \frac{1}{n}\left(Ce[V^{PD}_{\mathrm{ac},i}]-p_i\right)$, therefore the (scaled, negative) gradient flow* (20) *is given by*

$$\dot{p}_i = Ce[V^{PD}_{\mathrm{ac},i}(p)] - p_i.$$

We easily see that $\mathcal{H}(p,V^{PD}_{\mathrm{ac}})$ is minimized when $p_i = Ce[V^{PD}_{\mathrm{ac},i}(p)]$ for all $i\in\{1,\dots,n\}$.

6 Simultaneous change of agent positions and weights

The previous gradient descent laws (18) and (20), are designed to find the area-constraint surface and reach the center of a area-constrained region, respectively. In this section, we introduce a distributed algorithm which achieves both tasks simultaneously. As before, for the desired constraint surface we choose constants $c \in S$. Then, given $p \in \mathcal{D}$ and $w \in U$, the *simultaneous gradient* algorithm is given by

$$\begin{aligned}\dot{p}_i &= -\int_{V_i^f(p,w)} \frac{\partial}{\partial p_i} f(\|q-p_i\|)\phi(q)dq,\\ \dot{w}_i &= -\sum_{j\in N_i} \Xi_{ij}\big(c_j\phi(V_i^f) - c_i\phi(V_j^f)\big),\end{aligned} \tag{22}$$

where

$$\Xi_{ij} = \frac{1}{\phi(V_i^f)\phi(V_j^f)}\int_{\Delta_{ij}}\left(\frac{\partial q}{\partial w_i}\cdot \vec{n}_{ij}\right)\phi(q)dq.$$

The proposed control law is a natural combination of laws (18) and (20). If all the weights are initialized to the same value $\overline{w}$, and if $\dot{w}_i$ is set to zero, then (22) reduces to the continuous-time Lloyd algorithm presented in [5]. Simulations show that the proposed law does in fact converge to the set of centroidal area-constrained Voronoi partitions; however, a proof is not currently available.

We now specialize the gradient (22) to the case of Example 2.

Example 5 (Gradient flow for equitable power diagrams)**.** *Define the partition of the region Q according to* (2) *with $f(x)=x^2$ and let $c_i=\frac{1}{n}$ for all i in* (15)*. From [5] we have that $\frac{\partial \mathcal{H}_1\left(p, V_i^{PD}(p)\right)}{\partial p_i} = \frac{1}{n}\left(Ce[V_i^{PD}]-p_i\right)$, therefore the (scaled, negative) gradient flow* (22) *is given by*

$$\begin{aligned}\dot{p}_i &= Ce[V_i^{PD}(p)] - p_i,\\ \dot{w}_i &= -\sum_{j\in N_i}\left(\frac{1}{\phi(V_j^{PD})} - \frac{1}{\phi(V_i^{PD})}\right)\int_{\Delta_{ij}}\frac{\phi(q)}{\|p_i-p_j\|}dq.\end{aligned}$$

We easily see that the stationary set for this system is achieved when $p_i = Ce[V^{PD}_{\mathrm{ac},i}(p)]$ for all $i\in\{1,\dots,n\}$ and the vector of weights w converges to the value such that $\phi\left(V_i^{PD}(p,w)\right) = \phi\left(V_j^{PD}(p,w)\right)$ for all $i\neq j$.

7 Implementation and simulations

In this section, we discuss the practical implementation of the control law (20) and compare it against the control law (22) using representative simulation examples. Writing the area-constrained gradient flow (20), we assume that we always stay on the constraint surface and thus move along this surface continuously. In order to put this law into practice, we would need an explicit formula to instantaneously compute the weights of the current Voronoi partition, as functions of the generator locations. Since such a formula is not available, we instead have to rely on system (18) in order to determine the weights. We then design an implementation which alternates dynamics (20) and (18). Assuming that the agents start at a feasible configuration, they move their locations according to (20) for a small time duration δ, while keeping the weights fixed. After this amount of time,

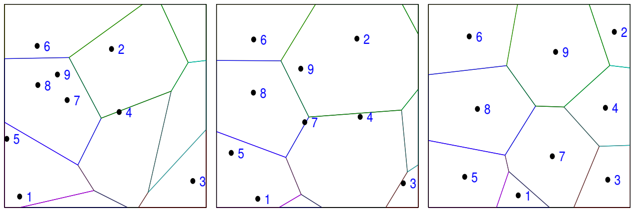

Fig. 2. Simulation of 9 agents partitioning a square environment with uniform density using the iterative gradient algorithm.

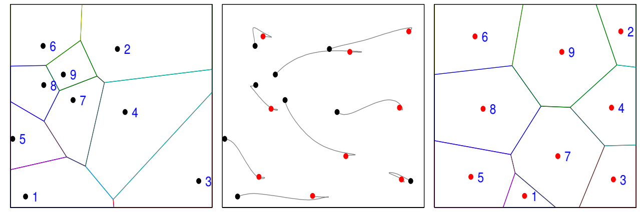

Fig. 3. Simulation of 9 agents partitioning a square environment with uniform density using the simultaneous gradient algorithm.

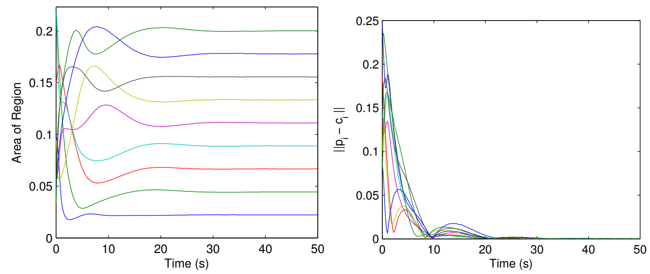

Fig. 4. Area (left) and position (right) trajectories for 9 agents partitioning a square environment with uniform density using the simultaneous gradient algorithm.

the area constraint is not satisfied: we then let the weights evolve according to (18), while the locations are fixed, until we are within the proximity of the area-constraint surface.

Some points in this implementation require attention. First, if the positions are allowed to move too much, the regions can become undefined (i.e., have measure zero), therefore care must be taken to make sure this does not happen by selecting a sufficiently small δ. Second, care must also be taken to insure that the agent location do not collide (i.e., agent locations must remain distinct). If they do, the step size δ should be reduced in order to avoid collision. Third, in order to drive the system exactly back to the constraint surface, we would need to bring the dynamics of the weights to convergence, which would take an infinitely long time: in simulations, convergence is approximated up to truncation error. In spite of these difficulties, in our simulations we have found that the algorithm is not sensitive to how far the agents deviate from the area-constraint surface (provided measures stay non-zero) during each movement step: in all our experiments, the algorithm converges to a centroidal area-constrained Voronoi diagram.

An illustrative example of the performance of the algorithm is presented in Figure 2. In the simulation, 10 agents have been randomly placed in a square region Q, where the density function $\phi(x)$ is constant. The region Q is to be partitioned according to (2) with $f(x) = x^2$, that is, as a power diagram. We define area-constraint surface such that $\phi(V_i^f) = \frac{i}{\sum_{i=1}^n i}\phi(Q)$ for all $i \in \{1,\ldots,10\}$; which means that if $i < j$, then $\phi(V_i^f) < \phi(V_j^f)$. The gradient (20) is followed during steps of duration $\delta = 0.1s$. The first panel shows the initial condition of the system at $T = 0$, where each agent has been randomly placed and the corresponding weights which generate the area-constrained partition determined. In the second panel, at $T = \delta$, the agents have moved in the gradient direction which causes each region to have a different area (the agents moved off the constraint surface of equitable area). The last panel shows the final state of the system at $T = 54.2s$. Each agent is at its region's centroid and the regions have equal measure.

Next we observe the performance of control law (22), which simultaneously optimizes the weights and the positions. We set the initial weights of the system to zero, but we keep the same initial positions and area-constraint requirements described above. The first panel of Figure 3 shows that we do in fact start with the same initial positions as the previous control algorithm, and since the weights are zero, the partition is the Standard Voronoi Diagram; the second panel shows how the position trajectories evolve over time, and the final panel shows the system's final configuration. Figure 4 shows better how the areas of each agent's region evolves over time; the second panel shows how each agent's position converges to its regions centroid. It is interesting to observe that given the same initial conditions, the iterative algorithm and control law (22) converge to the same configuration. In fact, we did not come across a case where this did not happen.

8 Conclusion

We have studied the problem of how to optimally deploy a set of agents over a convex workspace while each agent maintains a pre-specified area. We have designed a provably correct, spatially-distributed continuous-time policy that solves this optimization problem. We proposed a method for implementation of the control policy and demonstrated its effectiveness in simulation. This work leaves various extensions open for further research. First, our main approach is based on alternating phases, during which we either improve the objective function, or enforce the area constraint. In contrast to our main solution, we would like to prove convergence of the policy in which the agents converge to the constraint surface while simultaneously optimizing the coverage problem. So far, the effectiveness of this policy has been observed in simulations. Second, our policy requires synchronous and reliable communication along the edges of the dual graph associated to the Voronoi partition. It would then be worth to relax this requirement, using asynchronous, event-based, or unreliable communication: recent works in this direction include [13] and [14]. Third, our approach is based on the assumption that the environment we partition is convex, and finding policies that work over non-convex environments would be of great practical use: works in this direction include [15] and [16]. Finally, we assume that the density function ϕ is known to the agents, which may be hard to satisfy in practice; several papers have recently appeared to overcome this assumption, including [17] and [18].

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