CENTROIDAL AREA-CONSTRAINED PARTITIONING FOR ROBOTIC NETWORKS

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ABSTRACT

We consider the problem of optimal coverage with area-constraints in a mobile multi-agent system. For a planar environment with an associated density function, this problem is equivalent to dividing the environment into optimal subregions such that each agent is responsible for the coverage of its own region. In this paper, we design a continuous-time distributed policy which allows a team of agents to achieve a convex area-constrained partition of a convex workspace. Our work is related to the classic Lloyd algorithm, and makes use of generalized Voronoi diagrams. We also discuss practical implementation for real mobile networks. Simulation methods are presented and discussed.

1 Introduction

Problem description and motivation The applications of multi-agent systems to accomplish complex tasks in a complex environment are vast. They include but are not limited to tasks such as search and rescue operations, ad-hod mobile wireless networks, warehouse management and environmental monitoring [1–4]. When working with large robotic networks, it is often desired to partition the environment amongst all agents in an optimal way so that the workload can be equalized across all agents. For example, in ocean surveillance the cost of travel is large so it is desirable to survey one contiguous region as opposed to traveling around multiple ones. A partitioning policy is one in which an environment $Q \subset \mathbb{R}^d$ is partitioned into n disjoint subregions $W_i \subset Q$, $i \in \{1,\dots,n\}$ whose union is Q. Given some measure ϕ, the partitioning policy is equitable if $\phi(W_i) = \phi(W_j)$ for all $i \neq j$. Placement of agents within such regions is also of importance. In the case of surveillance or warehouse management, it is desirable to be at the center of your region so tasks are more easily serviced. Placing agents optimally and defining optimal regions simultaneously can be complicated as the region of an agent and its position are related to one another. We provide methods to accomplish this simultaneous operation using generalized Voronoi partitions.

Literature review Partitioning and coverage control is discussed in detail in [5] along with the application of multicenter functions to robotic networks. Results on specific manifestations of generalized Voronoi partitions and partitioning can be found in [6]. Results on the existence of area-constrained power diagrams along with a method to determine them are presented in [7]. More detailed results on existence of generalized Voronoi partitions for arbitrary area constraints are presented in [8]. Linear programming is used to handle generalized Voronoi partitions in [9] for fixed agents, showing that generalized Voronoi partitions are optimal for a certain class of multicenter functions. Similar results are obtained in [8], together with a discrete-time algorithm to solve the problem of optimal deployment of agents, while satisfying constraints on the areas.

Contributions The contributions of this work are several. First, we design a provably correct, spatially distributed continuous time algorithm to compute area-constrained generalized Voronoi partitions of a convex environment. The approach builds on work in [7]. Second, we build on work in [8], by introducing a continuous time spatially distributed algorithm to compute centroidal area-constrained generalized Voronoi partitions of a convex environment. More precisely, the continuous-time algorithm presented in this paper and the discrete-time algorithm in [8] both converge to the set of centroidal area-constrained Voronoi partitions, which shall be formally defined below. Finally, we introduce a practical method for implementation of our algorithms, and show their performance in simulation. We introduce methods from nonlinear optimization to achieve our results. Due to

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space constraints, all proofs are omitted and will be made available in a forthcoming full-length paper.

Paper organization The paper is organized as follows. In section 2 we setup preliminary notation, introduce the concept of generalized Voronoi partitions and present our problem in technical detail. In section 3 we state existence properties of area-constrained generalized Voronoi partitions and present algorithms to reach the set of area-constrained Voronoi partitions. In section 4 we state the main result of our paper on centroidal equitable generalized Voronoi partitions. In section 5 we discuss the application of our algorithm to real systems and provide numerical simulations. In the final section we present our conclusions and future directions for research.

2 Preliminaries and problem statement

Let us have a convex compact set $Q \subset \mathbb{R}^2$, endowed with a density function $\phi : Q \to \mathbb{R}_{\geq 0}$, so that the measure (or area) of a region $A \subset Q$ is defined as

$$\phi(A) = \int_A \phi(q)dq, \tag{1}$$

provided the set A is measurable in the sense of Lebesgue. Without loss of generality, we assume that Q has unit measure, that is, $\phi(Q) = \int_Q \phi(q)dq = 1$. Let $p_1, \dots, p_n$ denote the positions of n robotic agents in Q. We assume that each agent is associated with a (measurable) sub-region $W_i \subset Q$, where $\{W_i\}_{i=1}^n$ partitions Q into sets whose interiors are pairwise disjoint. A vector can be defined to collect the measures of the regions of a partition, as $\phi(W) = [\phi(W_1), \dots, \phi(W_n)]^T$. By our assumptions on Q, we have $\sum_{i=1}^n \phi(W_i) = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex, increasing, and differentiable function. Then, given n locations $p = (p_1, \dots, p_n)$ and a partition $W = (W_1, \dots, W_n)$, the *multicenter function* is defined by

$$\mathcal{H}(p,W) = \sum_{i=1}^{n} \int_{W_i} f(\|q - p_i\|)\phi(q)dq. \tag{2}$$

Our goal in this work is minimizing the function $\mathcal{H}$ under certain constraints, namely, that the areas of each region are fixed.

Problem 1 (Multicenter optimization with area constraints)**.** *Given $c_i > 0$ for $i \in \{1, \dots, n\}$ such that $\sum_i c_i = 1$, determine the locations of the agents $p = (p_1, \dots, p_n)$ and the partition $W = (W_1, \dots, W_n)$ solving:*

$$\begin{aligned} \min_{p,W} \quad & \mathcal{H}(p,W) \\ \text{subject to} \quad & \phi(W_i) = c_i, \;\; i \in \{1, \dots, n-1\}. \end{aligned} \tag{3}$$

Note that the nth constraint $\phi(W_n) = c_n$ is omitted because redundant.

In order to solve this problem, we introduce a useful partitioning scheme. Given the function f as above, n distinct locations $p = (p_1, \dots, p_n)$, and n scalar weights $w = (w_1, \dots, w_n)$, the *generalized Voronoi partition* of Q is the collection of subsets $V^f(p,w) = (V_1^f(p,w), \dots, V_n^f(p,w))$ of Q, defined by

$$\begin{aligned} V_i^f(p,w) = \{q \in Q \;\mid\; & f(\|q - p_i\|) - w_i \\ & \leq f(\|q - p_j\|) - w_j, \;\; \forall j \neq i\}. \end{aligned} \tag{4}$$

Generalized Voronoi partitions enjoy several important properties. First, any generalized Voronoi partition of Q is in fact a partition of Q. Second, the generalized partition generated by (p,w) is equal to the generalized partition generated by $(p, w + \alpha \mathbb{1}_n)$, for any $\alpha \in \mathbb{R}$ (Here, $\mathbb{1}_n$ is the vector in $\mathbb{R}^n$ whose entries are all equal to 1). From here onwards we will refer to generalized Voronoi partitions simply as *Voronoi partitions*. Two important special cases are described below.

Example 1 (Standard Voronoi Diagram)**.** *Given n distinct locations $p = (p_1, \dots, p_n)$ and n scalar weights $w = (w_1, \dots, w_n)$, the* Standard Voronoi Diagram *of Q is given by (4) with $w = 0$. The partition is given by*

$$V_i^{SD}(p,w) = \{q \in Q \;\mid\; f(\|q - p_i\|) \leq f(\|q - p_j\|)\},$$

regardless of the choice of $f(x)$. We call each V_i^{SD} a standard Voronoi region*. These regions are convex and have boundaries that are given by straight line segments; moreover, every generator p_i is contained in its respective region V_i^{SD}.*

Example 2 (Power Diagrams)**.** *Given n distinct locations $p = (p_1, \dots, p_n)$ and n scalar weights $w = (w_1, \dots, w_n)$, the* power diagram *of Q is given by (4) with $f(x) = x^2$. The partition is given by*

$$V_i^{PD}(p,w) = \{q \in Q \;\mid\; \|q - p_i\|^2 - w_i \leq \|q - p_j\|^2 - w_j\},$$

and we call each Voronoi region V_i^{PD} a power cell. Note that Standard Voronoi Diagrams are a special case of Power Diagrams, since $V_i^{PD}(p,0) = V^{SD}(p)$. These regions are convex and their boundaries that are line segments; however, it is possible that the generators p_i are not contained by their respective power cells V_i^{PD}.

We are now ready to define a second problem of interest, which is a simplified version of Problem 1.

Problem 2 (Multicenter Voronoi partition optimization with area constraints)**.** *Given $c_i > 0$ for $i \in \{1, \dots, n\}$ such that $\sum_i c_i = 1$, determine the locations of the agents $p = (p_1, \dots, p_n)$ and a set of weights $w = (w_1, \dots, w_n)$ solving:*

$$\begin{aligned} \min_{p,w} \quad & \mathcal{H}(p, V^f(p,w)) \\ \text{subject to} \quad & \phi(V_i^f(p,w)) = c_i, \;\; i \in \{1, \dots, n-1\}. \end{aligned} \tag{5}$$

Preliminarily, we should make sure that this problem has feasible solutions: this fact is shown in Section 3, which also provides a method to find a feasible set of weights for every set of locations. Problems 1 and 2 are known to be equivalent, in the following sense.

Proposition 1 (Proposition V.1 in [8])**.** *Let $p \in Q^n$ be the agent locations and $w \in \mathbb{R}^n$ a weight assignment which satisfies the area constraint. Then, the Voronoi partition $V^f(p,w)$ optimizes $\mathcal{H}(p,W)$ among all partitions satisfying the area constraint.*

In order to derive a useful consequence of this fact, we consider the simpler case when there is only one agent in Q: then, the multicenter function becomes

$$p \mapsto \mathcal{H}_1(p,Q) := \int_Q f(\|q-p\|)\phi(q)dq. \qquad (6)$$

Since f is strictly convex, $\mathcal{H}_1$ is too, and the following holds: If Q is convex, then there is a unique minimizer of (6), which we denote by $\mathrm{Ce}(Q)$. The Voronoi partition $V^f(p,w)$ generated by (p,w) is said to be *centroidal* if

$$\mathrm{Ce}[V_i^f(p,w)] = p_i,$$

for all $i \in \{1,\dots,n\}$. This notation allows us to state the following fact [8]: for every solution (p^*,W^*) of Problem 1, there exists a weight assignment $w^* \in \mathbb{R}^n$ such that $W^* = V^f(p^*,w^*)$ and $p_i^* = \mathrm{Ce}(W_i^*)$ for all $i \in \{1,\dots,n\}$. Equivalently, the solutions to Problem 1 are centroidal Voronoi partitions whose regions have the prescribed areas.

In the rest of this paper, we will go beyond this abstract characterization of the optimal solutions and give an optimization algorithm which is amenable to practical implementation.

3 Area-constrained Voronoi partitions

In this section, we solve the problem of finding, given area constraints and locations of the agents, suitable weights such that the Voronoi partition which is generated by these locations and weights satisfies the area constraint. We begin by stating a useful result which follows from Proposition IV.1 and Proposition IV.4 in [8].

Proposition 2 (Existence and uniqueness of area-constrained Voronoi partitions)**.** *Define a set of constants $c_i > 0$ for $i \in \{1,\dots,n\}$ such that $\sum_{i=1}^n c_i = 1$. Given n distinct locations $p = (p_1,\dots,p_n)$ in Q, there exists a locally unique vector $w_{\mathrm{ac}}(p) \in \mathbb{R}^n$ (i.e, there exists a neighborhood of weights such that no set of weights w other than those equivalent by translation to $w_{\mathrm{ac}}(p)$) such that $\{V_1^f(p,w_{\mathrm{ac}}),\dots,V_n^f(p,w_{\mathrm{ac}})\}$ satisfies $\phi(V_i^f) = c_i$ for all $i \in \{1,\dots,n\}$.*

Based on this result, for any given set of area constraints $\{c_i\}_{i=1}^n$, we define the map $p \mapsto w_{\mathrm{ac}}(p)$ as the map from p to the set of points w such that the $\phi(V_i^f(p,w_{\mathrm{ac}}(p))) = c_i$.

We now present an algorithm to compute $w_{\mathrm{ac}}(p)$. Given n distinct locations $p = (p_1,\dots,p_n)$ and n scalar weights $w = (w_1,\dots,w_n)$, the Voronoi partition generated by the corresponding weights and locations is defined by $V^f = (V_1^f(p,w),\dots,V_n^f(p,w))$. The *energy function* for the Voronoi partition is defined as

$$\mathcal{U}(p,w) = \sum_{i=1}^n \frac{c_i^2}{\int_{V_i^f(p,w)} \phi(q)dq} = \sum_{i=1}^n \frac{c_i^2}{\phi(V_i^f(p,w))}, \qquad (7)$$

where c_i for $i \in \{1,\dots,n\}$ are strictly positive constants. The following result extends [7, Theorem 3.7] to (generalized) Voronoi partitions.

Theorem 3 (Gradient of the weight energy function)**.** *Let Δ_{ij} denote the boundary between the i^{th} and j^{th} Voronoi region and $\vec{n}_{ij}$ the normal vector along that boundary. Define constants c_i for $i \in \{1,\dots,n\}$ such that $c_i > 0$ for all i and $\sum_{i=1}^n c_i = 1$. Given n distinct locations $p = (p_1,\dots,p_n)$ and n scalar weights $w = (w_1,\dots,w_n)$, we have*

$$\frac{\partial}{\partial w_i}\mathcal{U}(p,w) = \sum_{j \in N_i}\left(\frac{c_j^2}{\phi(V_j^f)^2} - \frac{c_i^2}{\phi(V_i^f)^2}\right)\int_{\Delta_{ij}}\left(\frac{\partial q}{\partial w_i}\cdot\vec{n}_{ij}\right)\phi(q)dq, \qquad (8)$$

so that

(i) every w generating an area-constrained Voronoi partition is a critical point of the function $w \mapsto \mathcal{U}(p,w)$, and
(ii) every solution to the negative gradient flow

$$\dot{w}_i = -\frac{\partial}{\partial w_i}\mathcal{U}(p,w), \qquad (9)$$

converges asymptotically to the vector of weights yielding an area-constrained Voronoi partition such that $\phi(V_i^f) = c_i$.

We now specialize the gradient (8) to the case of Example 2.

Example 3 (Gradient flow for equitable power diagrams)**.** *Define the partition of the region Q according to (4) with $f(x) = x^2$ and let $c_i = \frac{1}{n}$ for all i in (7). From [7] we have that $\frac{\partial q}{\partial w_i}\cdot\vec{n}_{ij} = \frac{1}{\|p_i - p_j\|}$ so then the gradient flow (9) is given by*

$$\dot{w}_i = -\sum_{j \in N_i}\left(\frac{1}{\phi(V_j^{PD})^2} - \frac{1}{\phi(V_i^{PD})^2}\right)\int_{\Delta_{ij}}\frac{\phi(q)}{\|p_i - p_j\|}dq.$$

The vector of weights w converges to the value such that $\phi\left(V_i^{PD}(p,w)\right) = \phi\left(V_j^{PD}(p,w)\right)$ for all $i \neq j$.

4 Centroidal area-constrained Voronoi partitions

Given constants c_i for $i \in \{1,\dots,n\}$ such that $c_i > 0$ for all i and $\sum_{i=1}^n c_i = 1$, and given n distinct locations $p = (p_1,\dots,p_n)$, it is convenient to define the *area-constrained Voronoi partition* generated by p as

$$V^f_{\text{ac}}(p) = V^f(p, w_{\text{ac}}(p)), \tag{10}$$

with Voronoi regions $V^f_{\text{ac},i}(p)$, $i \in \{1,\dots,n\}$ such that $\phi\left(V^f_{\text{ac},i}(p)\right) = c_i$ for all i. The associated *area-constrained multicenter function* is given by

$$p \mapsto \mathcal{H}\big(p, V^f_{\text{ac}}(p)\big) = \sum_{i=1}^n \int_{V^f_{\text{ac},i}(p)} f(\|q-p_i\|)\phi(q)dq. \tag{11}$$

We are now ready for the main result of this section.

Theorem 4 (Gradient of the area-constrained multicenter function)**.** *Given n distinct locations $p = (p_1,\dots,p_n)$,*

$$\frac{\partial}{\partial p_i}\mathcal{H}\big(p, V^f_{\text{ac}}(p)\big) = \frac{\partial \mathcal{H}_1(p_i, V^f_{\text{ac},i}(p))}{\partial p_i}, \tag{12}$$

so that

(i) every p generating a centroidal area-constrained Voronoi partition is a critical point of the function $p \mapsto \mathcal{H}\big(p, V^f_{\text{ac}}(p)\big)$, and

(ii) every solution to the negative gradient flow

$$\dot{p}_i = -\frac{\partial \mathcal{H}_1(p_i, V^f_{\text{ac},i}(p))}{\partial p_i} \tag{13}$$

converges asymptotically to the set of centroidal area-constrained Voronoi partitions, provided the initial locations are distinct.

We now specialize the gradient (12) to the case of Example 2.

Example 4 (Constrained gradient flow for power diagrams)**.** *Let the partition of the region Q be defined according to (4) with $f(x) = x^2$. Given any* powercell *A of Q the center is given by $Ce[A] = \frac{1}{\phi(A)}\int_A q\phi(q)dq$. Thus for* power diagrams*, Ce is equivalent to the well known expression for* center of mass *of a region. From [5] we have that $\frac{\partial \mathcal{H}_1\left(p, V^{PD}_{\text{eq},i}(p)\right)}{\partial p_i} = \frac{1}{n}\left(Ce[V^{PD}_{\text{ac},i}] - p_i\right)$, therefore the (scaled, negative) gradient flow (13) is given by*

$$\dot{p}_i = Ce[V^{PD}_{\text{ac},i}(p)] - p_i. \tag{14}$$

We easily see that $\mathcal{H}(p, V^{PD}_{\text{ac}})$ is minimized when $p_i = Ce[V^{PD}_{\text{ac},i}(p)]$ for all $i \in \{1,\dots,n\}$.

5 Implementation and simulations

In this section, we discuss the practical implementation of the control law (13) and provide a simulation example. Writing the area-constrained gradient flow (13), we assume that we always stay on the constraint surface and thus move along this surface continuously. In order to put this law into practice, we would need an explicit formula to instantaneously compute the weights of the current Voronoi partition, as functions of the generator locations. Since such a formula is not available, we instead have to rely on system (9) in order to determine the weights. We then design an implementation which alternates dynamics (13) and (9). Assuming that the agents start at a feasible configuration, they move their locations according to (13) for a small time duration δ, while keeping the weights fixed. After this amount of time, the area constraint is not satisfied: we then let the weights evolve according to (9), while the locations are fixed, until we are within the proximity of the area-constraint surface.

Some points in this implementation require attention. First, if the positions are allowed to move too much, the regions can become undefined (i.e., have measure zero), therefore care must be taken to make sure this does not happen by selecting a sufficiently small δ. Second, care must also be taken to insure that the agent location do not collide (i.e., agent locations must remain distinct). If they do, the step size δ should be reduced in order to avoid collision. Third, in order to drive the system exactly back to the constraint surface, we would need to bring the dynamics of the weights to convergence, which would take an infinitely long time: in simulations, convergence is approximated up to truncation error. In spite of these difficulties, in our simulations we have found that the algorithm is not sensitive to how far the agents deviate from the area-constraint surface (given that measures stay non-zero) during each movement step: in all our experiments, the algorithm converges to a centroidal area-constrained Voronoi diagram.

An illustrative example of the performance of the algorithm is presented in Figure 1. In the simulation, 10 agents have been randomly placed in a convex polygon Q, where the density function $\phi(x)$ is non-uniform: more precisely, two exponential density spikes are placed in the polygon. The region Q is to be partitioned according to (4) with $f(x) = x^2$, that is, as a power diagram. We define an equitable area-constraint, which is to say that $\phi(V^f_i) = \frac{\phi(Q)}{10}$ for all $i \in \{1,\dots,10\}$. The gradient (13) is followed during steps of duration $\delta = 0.1s$. The first panel shows the initial condition of the system at $T = 0$, where each agent has been randomly placed and the corresponding weights which generate a equitable partition determined. In the second panel, at $T = \delta$, the agents have moved in the gradient direction which causes each region to have a different area (the agents moved off the constraint surface of equitable area). The last panel shows the final state of the system at $T = 54.2s$. Each agent is at its region's centroid and the regions have equal measure.

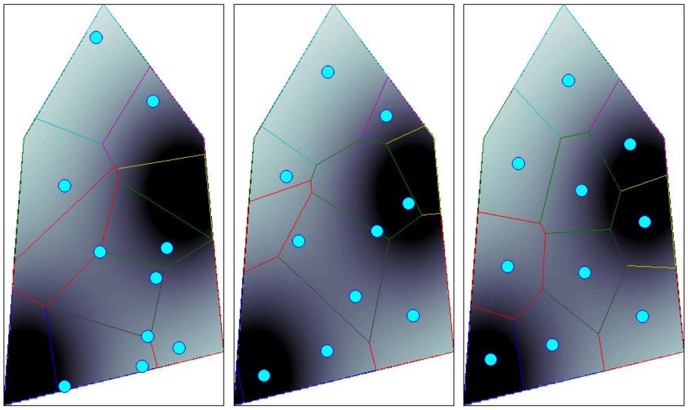

Figure 1. SIMULATION OF 10 AGENTS PARTITIONING A POLYGON ENVIRONMENT. DOTS ARE AGENTS' LOCATIONS, AND THE DENSITY FUNCTION IS SHOWN BY THE BACKGROUND COLOR

6 Conclusion

We have studied the problem of how to optimally deploy a set of agents over a convex workspace while each agent maintains a pre-specified area. We have designed a provably correct, spatially distributed continuous-time policy that solves this optimization problem. We also proposed a method for implementation of the control policy and demonstrated its effectiveness in simulation. This work leaves various extensions open for further research. First, our approach is based on alternating phases, during which we either improve the objective function, or enforce the area constraint. In contrast to our current solution, we would like to find a policy in which the agents converge to the constraint surface while simultaneously optimizing the coverage problem. Second, our policy requires synchronous and reliable communication along the edges of the dual graph associated to the Voronoi partition. It would then be worth to relax this requirement, using asynchronous, event-based, or unreliable communication: recent works in this direction include [10] and [11]. Third, our approach is based on the assumption that the environment we partition is convex: finding policies that work over non-convex environments would be of great practical use: works in this direction include [12] and [13].

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