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Opinion Dynamics and the Evolution of Social Power in Influence Networks*

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Abstract. This paper studies the evolution of self-appraisal, social power, and interpersonal influences for a group of individuals who discuss and form opinions about a sequence of issues. Our empirical model combines the averaging rule of DeGroot to describe opinion formation processes and the reflected appraisal mechanism of Friedkin to describe the dynamics of individuals' self-appraisal and social power. Given a set of relative interpersonal weights, the DeGroot–Friedkin model predicts the evolution of the influence network governing the opinion formation process. We provide a rigorous mathematical formulation of the influence network dynamics, characterize its equilibria, and establish its convergence properties for all possible structures of the relative interpersonal weights and corresponding eigenvector centrality scores. The model predicts that the social power ranking among individuals is asymptotically equal to their centrality ranking, that social power tends to accumulate at the top of the hierarchy, and that an autocratic (resp., democratic) power structure arises when the centrality scores are maximally nonuniform (resp., uniform).

Key words. opinion dynamics, reflected appraisal, influence networks, mathematical sociology, network centrality, dynamical systems, coevolutionary networks

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1. Introduction.

The investigation of social networks has regularly attracted contributions from applied mathematicians and social scientists over the last several decades. Graph theory and matrix algebra have natural applications to such investigations; e.g., see the early monograph by Harary, Norman, and Cartwright [39]. Classic problems of interest include comparative static analyses of social network structures [75, 24], functional implications of network structures [60], and numerical taxonomies of nodes [69, 10]. Much ongoing interest is focusing on dynamic models

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of structural change [1, 25, 43, 54, 70] and on a broad range of dynamic processes unfolding over static networks; examples include the study of social learning [35, 2], opinion formation [23, 32], and information propagation [60, 57, 25]. The study of dynamic models directly addresses one of the key problems of the field, which is to understand the implications of social structures for relevant dynamical states of the network. As Newman [60, p. 224] notes, "...the ultimate goal of the study of the structure of networks is to understand and explain the workings of systems built upon those networks."

Many of the research problems of the field, which may be addressed with dynamic models, are old ones that remain unsettled. A core set of these problems is defined on social networks of individuals and their interpersonal relations. For such networks, which may or may not be static, the literature features an accelerating number of proposals of dynamic models for (a) mechanisms of network formation and transformation (e.g., see [68, 42]) and (b) mechanisms by which individuals' attitudes, opinions, and behaviors toward particular objects (specific issues, events, institutions, leaders) are modified by the displayed attitudes, opinions, and behaviors of other individuals toward the same object (e.g., see [22, 40, 11, 2]). Research on these mechanisms is now being rapidly advanced by an influx of investigators into the sociology field from the natural and engineering sciences. The online social networks enabled by internet and cell phone technologies provide accessible data for the investigation of social network dynamics, and investigations of these data are being encouraged by large scale corporate and government investments. This work is now appearing regularly in the premier journals of science.

Opinion Dynamics in Networks. Inquiries into opinion dynamics draw on a large preexisting empirical literature in experimental social psychology, i.e., the discipline of science devoted to the study of how individuals' thoughts, feelings, and behaviors are influenced by the actual, imagined, or implied presence of others [4]. It should not be surprising that the accumulated findings of this discipline have a useful bearing on formulations of opinion formation mechanisms. These findings point to the social cognition foundations of interpersonal influence systems and the importance of individuals' automatic-heuristic responses to objects. Individuals' attitudes toward objects, i.e., signed evaluative orientations of particular strengths, are often automatically generated without conscious effort [7], and these attitudes are important antecedents of displayed cognitive and behavioral orientations toward objects [3]. Automatically activated heuristic mechanisms of the mind appear to be more generally important bases of displayed opinions than rational calculations [46]. See [29, 32] for reviews of these and other relevant lines of work in experimental social psychology. The available empirical evidence is also consistent with the assumption that individuals update their opinions as convex combinations of their own and others' displayed opinions, based on weights that are automatically generated by individuals in their responses to the displayed opinions of other individuals. This specification appeared in the literature on opinion dynamics in the early works by French [27], Harary [38], and DeGroot [23]. Anderson's information integration theory [5] is seminal in its effort to secure the convex combination mechanism as a fundamental "cognitive algebra" of the mind's synthesis of heterogeneous information. Thus, interpersonal influence networks are social cognition structures assembled by individuals who are dealing with a common issue.

In summary, independent work by investigators from different disciplines has formulated a social influence network as a weight matrix $W = [w_{ij}]$ satisfying $w_{ij} \in$

$[0,1]$ for all i and j and $\sum_j w_{ij} = 1$ for all i (that is, W is *row-stochastic*). Each edge of this network $i \xrightarrow{w_{ij}} j$, including loops $i \xrightarrow{w_{ii}} i$, represents the influence and weight accorded by agent i to agent j. Representing the individuals' opinions with a real-valued vector y, the classic *DeGroot model* [23] is

$$y(t+1) = Wy(t), \quad t = 0, 1, 2, \ldots. \tag{1.1}$$

According to the recent empirical data and comparative analysis in [18], the DeGroot model outperforms Bayesian methods in describing social learning processes. An attractive generalization of this model, proposed by Friedkin and Johnsen [31, 32], is based on the introduction of a positive diagonal matrix Λ quantifying the extent to which each individual is open to the influence of others rather than anchored to her initial opinion:

$$y(t+1) = \Lambda W y(t) + (I_n - \Lambda)y(0), \quad t = 0, 1, 2, \ldots. \tag{1.2}$$

Starting from these classic models, there is a developing line of work on convex combinations of real-valued opinions as a model for opinion dynamics in social networks. In bounded-confidence models, each individual interacts with only those individuals whose opinions are close enough to its own: synchronous [40, 52, 11, 58] as well as pairwise asynchronous [22, 74] updates have been studied. Significant attention [35, 2, 8, 76, 59] has focused on opinion dynamic processes with exogenous inputs as models for Bayesian and non-Bayesian learning in networks. Large societies have been modeled as probability distributions over the opinion space in [9, 15]. The convex averaging model is related to the study of Markov chains [65] and is relevant in several other fields, including politics and economics [70], multiagent dynamical systems [44], flocking models in biophysics [71], and behavioral ecology [64].

A separate line of work on opinion formation (e.g., see [63, 36, 72, 17]) focuses on the development of binary-response threshold models and on social influence in collective decision making. Central problems in these models are how individuals in a society make (binary) decisions under the influence of others, and how individual decisions aggregate under such influence. Many of these models are also constructed on the assumption of a row-stochastic influence network W, which, with binary responses, presents the proportions of individuals' neighbors who have adopted a particular position on an issue.

Evolving Influence Networks over Issue Sequences. This article studies the evolution of an influence network in a group of individuals who form opinions on a sequence of issues. Small groups within firms, deliberative bodies of government, and other associations of individuals may be assembled ad hoc to deal with one issue, or they may be constituted to deal with sequences of issues within particular issue domains. For the latter enduring groups, their repetitive engagement with issues opens the possibility of an evolution of the group's influence network over an issue sequence. Any of the existing models for opinion dynamics (over a single issue) may be modified and extended to deal with the evolution of interpersonal influence structures over a sequence of issues. Here we elaborate on the seminal DeGroot model

$$y(s, t+1) = W(s)y(s,t), \quad s = 0, 1, 2, \ldots, \quad t = 0, 1, 2, \ldots,$$

where $y(s,t) \in \mathbb{R}^n$ and $W(s)$ is static during $t = 0, 1, 2, \ldots.$ Our inquiry deals with the evolution of the influence network over repetitions of the opinion formation process, that is, the evolution of $W(s)$ over a sequence of issues $s = 0, 1, 2, \ldots$. The literature on opinion dynamics includes models for influence networks altering during the discussion of a single issue, e.g., see the bounded-confidence models cited above. Apart from Friedkin's work [30], we have found no prior investigation on the evolution of social power and influence networks across issues, even though groups that deal with issue sequences are a prevalent feature of social organizations. Considering issue sequences leads to new forms of network evolution that are of potential importance in the fields of social organization and social psychology.

Our analysis of issue sequences and our proposed formalization of this evolution is motivated by the sociological hypothesis of reflected appraisals; see the seminal work by Cooley [21]. The general hypothesis is that individuals' self-appraisals on some dimension (e.g., self-confidence, self-esteem, self-efficacy) are influenced by the appraisals by other individuals of them. This classic hypothesis is widely accepted and empirically validated; e.g., see [33, 66, 77]. In the context of social influence networks, the hypothesis is empirically supported by our empirical findings [30] that individuals' self-reported self-weights, i.e., the values on the main diagonal of the weight matrix $W(s)$, $s = 1, 2, \ldots$, are elevated or dampened in correspondence with individuals' relative net control over the outcome of the previous issues discussed by the group. In the context of the DeGroot model, self-weights correspond to individuals' levels of closure-openness to influence, and the relative control of an individual over an issue outcome is naturally defined to be the average effect of that individual's opinion on the final opinions of all other individuals. In the language of Cartwright [16], individual power is the ability to control outcomes of interest in social systems. Accordingly, we adopt the term *social power* as a synonym for relative control over issue outcomes in this paper.

Based on these empirical observations, this article combines DeGroot's model (1.1) of opinion dynamics, in which the influence network for a particular issue is fixed, with Friedkin's [30] formalization of the evolution of interpersonal influences in an issue sequence. We refer to the resulting dynamical process as the *DeGroot–Friedkin model*. This dynamical model explains via a reflected appraisal mechanism the evolution of individuals' self-weights, that is, the evolution of the diagonal elements of the weight matrix. Following the original study [30], we adopt the simplest possible assumption on the off-diagonal values corresponding to the interpersonal weights. Consistent with the unit row-sum constraint, we assume that each interpersonal weight $w_{ij}(s)$, $i \neq j$, satisfies $w_{ij}(s) = (1 - w_{ii}(s))c_{ij}$, where the *relative interpersonal weights* c_{ij} are static and issue independent. A row-stochastic matrix $[c_{ij}]$ results by assuming zero diagonal elements. The assumption of static relative interpersonal weights may be relaxed with an additional specification of a mechanism that also alters them across issues, but we do not do so here.

As our treatment reveals, this static constraint structure plays an important role in the evolution of self-weights solely through its eigenvector centrality. Eigenvector centrality was first proposed by Bonacich [12] and has since been widely adopted to determine the relative importance of an individual in a social influence network. Other applications of eigenvector centrality and its variations include the ranking of college football teams [48], the measure of producer status in a market [61], the prediction of social mobility in a biological network [62], and the spread of behaviors (e.g., obesity, smoking cessation, and happiness) in social networks [19, 20, 26]. Recent work [53] details how localization and accumulation of centrality in power-law networks may be undesirable; these features, however, appear as natural phenomena in our proposed model. Theoretical approaches to centrality have recently become exceptionally useful in Google's PageRank algorithm [13]. We refer the reader to [34] for a recent extensive survey of eigenvector centrality, related notions, and applications. In its seminal application to social networks, eigenvector centrality posits that individuals' centralities are a function of the centralities of other individuals adjacent to them. This paper contributes a new perspective on eigenvector centrality as the ultimate unique driver of an individual's self-appraisal and social power in sequences of opinion formation processes.

Contributions. We propose and analyze the DeGroot–Friedkin model for the evolution of social influence networks subject to reflected appraisal. As a first step, we provide an explicit and concise mathematical formulation of the reflected appraisal mechanism for network evolution as a discrete-time nonlinear system defined over a simplex. The state of this dynamical system is the measure of self-weight and social power of the individuals. We show that the only parameter in the network dynamics is the dominant left eigenvector of the row-stochastic matrix $[c_{ij}]$ describing the relative interpersonal weights. As a second and extensive set of contributions, we characterize the equilibria and the asymptotic convergence properties of this nonlinear dynamical system. We provide a complete mathematical analysis under various assumptions on the structural properties of the relative interpersonal weights; we allow the matrix representation of these weights to be doubly stochastic, to have star topology, and to be irreducible or reducible with globally reachable nodes (we review these notions below). Finally, we examine our results numerically by applying the DeGroot–Friedkin model to networks with different numbers of nodes, varying from a few dozen to a few thousand, and to networks of different types, varying from highly clustered networks to Erdős–Rényi networks. In particular, we illustrate the results on four social networks observed in field settings.

The DeGroot–Friedkin model predicts the final asymptotic values for self-weight and social power for each individual along the sequence of opinion formation processes. These final values of self-weight and social power are independent of the corresponding initial values and depend uniquely upon the relative interpersonal weights c_{ij} accorded among individuals or, more precisely, upon the eigenvector centrality scores defined by these weights. The final values have the following interpretations: (i) the social power ranking among individuals is asymptotically equal to their eigenvector centrality ranking; (ii) social power tends to accumulate in the hands of the top tier of individuals at the expense of the individuals with lower eigenvector centrality scores; and (iii) an autocratic (resp., democratic) power structure arises when the eigenvector centrality scores are maximally nonuniform (resp., uniform). An autocratic power structure features an autocratic individual, who is maximally closed to interpersonal influence, and $n-1$ accommodative individuals, who are maximally open to interpersonal influence. A democratic power structure features n individuals equally contributing to the final opinion outcome.

These findings are of sociological interest in their advancement of the dynamical foundations of power concentration in social groups. Our rigorous results for the DeGroot–Friedkin model and the more pronounced simulation-based results for the Friedkin–Johnsen model [30] suggest that influence networks evolve toward a concentration of social power over issue outcomes, consistent with Michels' [56] important postulate of the existence of an "iron law of oligarchy" in social organizations. These findings also more generally contribute to the rapidly growing literature on coevolutionary networks, that is, networks in which feedback loops link structure and dynamics. We refer the interested reader to the survey by Gross and Blasius [37]. The work is also related to the literature on social network formation and coordination games [42, 68] and, more broadly, to the study of complex networks and evolutionary rules [6, 73].

Organization. The rest of the paper is organized as follows. Section 2 features the DeGroot–Friedkin model and the notion of eigenvector centrality. Section 3 presents the analysis results for the two meaningful scenarios in which the relative interpersonal influences either are doubly stochastic or have star topology. These scenarios correspond to the uniform centrality and the maximally nonuniform centrality situations, respectively. The DeGroot–Friedkin model with general irreducible interpersonal influences is characterized in section 4. Section 5 completes our analysis by considering reducible relative interactions. Section 6 contains our conclusions and all proofs are presented in the appendices.

Notation. For a vector $x \in \mathbb{R}^n$, we let $x \geq 0$ and $x > 0$ denote component-wise inequalities. We adopt the shorthands $\mathbb{1}_n = [1, \ldots, 1]^T$ and $\mathbb{0}_n = [0, \ldots, 0]^T$. For $i \in \{1, \ldots, n\}$, we let $\mathbb{e}_i$ be the ith basis vector with all entries equal to 0 except for the ith entry equal to 1. Given $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, we let $\mathrm{diag}(x)$ denote the diagonal $n \times n$ matrix whose diagonal entries are $x_1, \ldots, x_n$. The n-simplex Δ_n is the set $\{x \in \mathbb{R}^n \mid x \geq 0,\ \mathbb{1}_n^T x = 1\}$; recall that the vertices of the simplex are the vectors $\{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$. A nonnegative matrix is *row-stochastic* (resp., *doubly stochastic*) if all its row sums are equal to 1 (resp., all its row and column sums are equal to 1). For a nonnegative matrix $M = \{m_{ij}\}_{i,j \in \{1,\ldots,n\}}$, the *associated digraph* $G(M)$ of M is the directed graph with node set $\{1, \ldots, n\}$ and with edge set defined as follows: (i, j) is a directed edge if and only if $m_{ij} > 0$. A nonnegative matrix M is *irreducible* if its associated digraph is strongly connected; a nonnegative matrix is *reducible* if it is not irreducible. An irreducible matrix M is *aperiodic* if it has only one eigenvalue of maximum modulus. A vertex of a digraph is *globally reachable* if it can be reached from any other vertex by traversing a directed path.

2. The DeGroot–Friedkin Model.

In this section we incrementally introduce and motivate the dynamical model for the evolution of the social influence network and, in particular, of the self-weights. This model combines the DeGroot model for the dynamics of opinions over a single issue and the Friedkin model for the dynamics of self-weight and social power over a sequence of issues.

2.1. Origins and Model Derivation.

We consider a group of $n \geq 2$ individuals who discuss a sequence of issues $s \in \mathbb{Z}_{\geq 0}$ according to a DeGroot opinion formation model with issue-dependent influence matrices. Specifically, we assume the individuals' opinions about each issue s are described by a trajectory $t \mapsto y(s,t) \in \mathbb{R}^n$ that is determined by the DeGroot averaging model

$$y(s, t+1) = W(s) y(s, t), \tag{2.1}$$

with given initial conditions $y_i(s, 0)$ for each individual i. Here, each influence matrix in the sequence $\{W(s)\}_{s \in \mathbb{Z}_{\geq 0}}$ is row-stochastic, i.e., for each issue s, each entry of $W(s)$ is nonnegative and each row sum of $W(s)$ equals 1. On the discussion of issue s, each individual i updates her opinion according to the convex combination

$$y_i(s, t+1) = w_{ii}(s) y_i(s, t) + \sum\nolimits_{j=1, j \neq i}^{n} w_{ij}(s) y_j(s, t), \tag{2.2}$$

in which, from a psychological viewpoint, the diagonal and the off-diagonal entries of an influence matrix play conceptually distinct roles. Specifically, the diagonal *self-weight* $w_{ii}(s)$ is the individual's self-appraisal (e.g., self-confidence, self-esteem, self-worth) and corresponds to the extent of closure-openness to the interpersonal influence of the ith individual. The off-diagonal entries $w_{ij}(s)$, $j \neq i$, are *interpersonal weights accorded* by each individual i to particular individual j based on j's displayed opinions.

The central object studied in this paper is the set of self-weights of the individuals. For simplicity of notation, we adopt the shorthand $x_i(s) \in [0,1]$ to denote the self-weight $w_{ii}(s)$ of the ith individual. Because $1-x_i$ is the aggregated allocation of weight to other individuals, we decompose the off-diagonal entries as $w_{ij}(s) = (1-x_i(s))c_{ij}$, where the coefficients c_{ij} are the *relative interpersonal weights* that the ith individual accords to other individuals. With $c_{ii} = 0$, the matrix C, which we refer to as the *relative interaction matrix*, is row-stochastic with zero diagonal. This construction assumes that, while the self-weights $s \mapsto x(s)$ are issue-dependent, the matrix C is issue-independent, that is, constant. With these notations and assumptions, each influence matrix in the sequence is written as

$$W(x(s)) = \mathrm{diag}(x(s)) + (I_n - \mathrm{diag}(x(s)))C, \tag{2.3}$$

and the opinion dynamic process (2.1) is equivalently rewritten as

$$y(s, t+1) = W(x(s))y(s,t). \tag{2.4}$$

Now, for simplicity of exposition, we assume that the relative interaction matrix C is irreducible or, equivalently, that its associated digraph is strongly connected. (A more general treatment is possible and partly discussed in section 5.) Based on this assumption and on some simple calculations reported in Appendix A, the Perron–Frobenius Theorem for nonnegative matrices implies that the influence matrix $W(x)$ admits a unique left eigenvector $w(x)^T \geq 0$ associated with the eigenvalue 1, with nonnegative entries and normalized to have unit sum so that $\mathbb{1}_n^T w(x) = 1$. In other words, $w(x) \in \Delta_n$. We refer to the row vector $w(x)^T$ as the *dominant left eigenvector* of $W(x)$ and we know it satisfies

$$\lim_{t\to\infty} W(x)^t = \mathbb{1}_n w(x)^T$$

for a broad range of self-weight vectors x specified below. According to this limit, the DeGroot process (2.4) results in the well-understood opinion consensus

$$\lim_{t\to\infty} y(s,t) = \Big(\lim_{t\to\infty} W(x)^t\Big) y(s) = \big(w(x)^T y(s)\big)\mathbb{1}_n, \tag{2.5}$$

that is, the individuals' opinions converge to a consensus value $w(x)^T y(s)$ equal to a convex combination of their initial opinions $y(s)$. The convex combination coefficients $w(x)$ mathematically describe the relative control of each individual. Note that relative control in influence networks, i.e., the ability to control issue outcomes, is precisely a manifestation of individual power, as defined in the seminal work by Cartwright [16]. Alternative mechanisms and other forms of power exist (e.g., see the concept of situational power in [67]), but in this paper we focus on power that is based on interpersonal influence networks and sequences of issues. For this scenario, we equivalently refer to $w_i(x)$ as both the relative control over discussion outcomes as well as the social power of the ith individual.

Finally, our model is completed by prescribing how the self-weights $s \mapsto x(s)$ evolve from issue to issue. We adopt the psychological mechanism of *reflected appraisal*, as reviewed in the introduction and mathematized by [30]. In this straightforward model, the self-weight of an individual is updated after each issue discussion

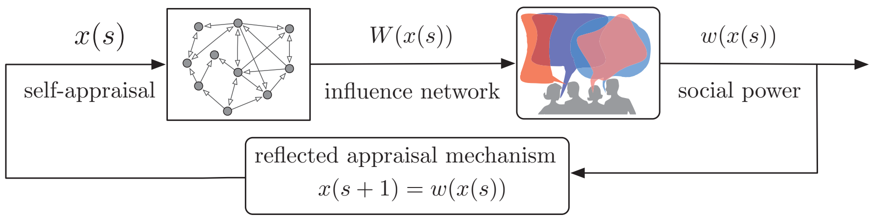

Fig. 1 *The dynamic feedback nature of the DeGroot–Friedkin model.*

and is set equal to the relative control that the individual exerted over the prior issue outcome. In short, the reflected appraisal mechanism "self-weight := relative control over prior issue," as illustrated in Figure 1, is written as

$$x(s+1) = w(x(s)), \tag{2.6}$$

where $w(x(s))^T$ is the dominant left eigenvector for the influence matrix $W(x(s))$. Notice that, for issue $s \geq 1$, the self-weight vector $x(s)$ necessarily takes values inside Δ_n. It is therefore natural to assume that the self-weight vector takes values in Δ_n for all issues.

We conclude this modeling discussion with a summary definition.

Definition 2.1 (DeGroot–Friedkin model for the evolution of social influence networks). *Consider a group of $n \geq 2$ individuals discussing a sequence of issues $s \in \mathbb{Z}_{\geq 0}$. Let the row-stochastic zero-diagonal irreducible matrix C be the relative interaction matrix encoding the relative interpersonal weights among the individuals. The DeGroot–Friedkin model for the evolution of the self-weights $s \mapsto x(s) \in \Delta_n$ is*

$$x(s+1) = w(x(s)),$$

where $w(x(s)) \in \Delta_n$ and $w(x(s))^T$ is the dominant left eigenvector of the influence matrix

$$W(x(s)) = \mathrm{diag}(x(s)) + (I_n - \mathrm{diag}(x(s)))C.$$

From a sociological viewpoint, it is interesting to note that the evolution and limiting values of the self-weights depend *only* upon the network structure and network parameters as embodied in the relative interaction matrix C: the DeGroot–Friedkin model is therefore a mechanistic explanation of how the social structure affects the evolution of individuals' extents of closure-openness to interpersonal influences and relative control over group outcomes.

Generally speaking, the DeGroot–Friedkin model belongs to a class of coevolutionary networks where the DeGroot dynamics describe the evolution of opinions over a possibly constant influence network and the reflected appraisal mechanism describes how the influence network evolves. In this paper, for simplicity, we assume that the timescales for the two processes are separate: the opinion dynamics are faster than the reflected appraisal dynamics in the influence network. In other words, opinion consensus is achieved before individual self-weights are updated. We leave to future work the study of scenarios in which the two processes take place over comparable timescales.

2.2. The Scope of the Model. A fundamental implicit assumption present in the DeGroot–Friedkin model is that each individual perceives her relative control over discussion outcomes. This assumption is well justified in two distinct settings. First, for small and moderate size social groups, we argue that individuals are typically able to directly perceive who shaped the discussion and whose opinion had an impact in the final decisions. Such groups, composed of a few to a hundred individuals, include deliberative assemblies, boards of directors, judiciary bodies (e.g., the U.S. Supreme Court), policy making groups (e.g., the U.S. Senate), and faculty committees, to name a few. These small and moderate size deliberative assemblies play an extraordinarily important role in modern society.

Second, we believe our model is relevant even in large social groups, provided that the individuals in those groups deals with a common issue sequence. While it is less plausible for individuals to directly perceive their relative control over the outcomes of these common issues, we propose here a natural dynamical process that allows each individual to accurately estimate her perceived power. The dynamical process is distributed in the sense that each individual only needs to interact with her influenced neighbors (i.e., those who accord positive interpersonal weights to the individual). By assuming that she is aware of the direct interpersonal weights accorded to her and the perceived powers of her influenced neighbors, each individual updates her perceived power as a convex combination of her own and her influenced neighbors' perceived powers. That is, in the discussion of each issue s, each individual i estimates her perceived power $p_i(s,t)$ according to

$$p_i(s,t+1) = w_{ii}(s)p_i(s,t) + \sum\nolimits_{j=1, j\neq i}^{n} w_{ji}(s)p_j(s,t), \tag{2.7}$$

or, equivalently, $p(s,t+1) = W(s)^T p(s,t)$, where $W(s)$ represents the influence matrix associated to issue s. Following the same analysis leading to (2.5), we know that $\lim_{t\to\infty} p(s,t) = w(s)$ for all initial states $p(s,0)$ such that $\mathbb{1}_n^T p(s,0) = 1$, where $w(s)^T$ is the dominant left eigenvector of $W(s)$. That is, for each issue s, the equilibrium individual perceived power $p_i^*(s) := \lim_{t\to\infty} p_i(s,t)$ obtained via the dynamical system (2.7) is equal to the individual relative control $w_i(s)$ manifested in the DeGroot process (2.5). In summary, we argue that even in large networks the relative control over discussion outcomes can be perceived by individuals via the natural dynamics (2.7), so long as the individuals are dealing with a common sequence of issues.

2.3. Problem Statement. We are now able to ask several interesting questions. For example, we seek an explicit formulation of the DeGroot–Friedkin dynamics in Definition 2.1. More importantly, we are interested in characterizing the existence, stability, and region of attraction of the equilibria for the DeGroot–Friedkin model. We begin by defining two specific vectors of self-weights that correspond to power configurations of sociological interest.

First, suppose that at some issue s the vector of self-weights $x(s)$ is equal to $\mathbb{e}_i$, for some individual i. Recall that $\mathbb{e}_i$ is a vertex in the simplex, where the self-weight of agent i is maximal. Then one can show that, independent of the relative interaction matrix C, the social power of the individual at issue s is also maximal: $w(\mathbb{e}_i) = \mathbb{e}_i$. In other words, each $\mathbb{e}_i$ is an equilibrium of the DeGroot–Friedkin model. We refer to this configuration of self-weights and corresponding social power as an *autocratic configuration* with the ith individual being the *autocrat*. Autocracy is an equilibrium point of the DeGroot–Friedkin model.

Second, we call $x = \frac{1}{n}\mathbb{1}_n$ the *democratic configuration*, whereby each individual has an identical self-weight. One can show that $\frac{1}{n}\mathbb{1}_n$ is an equilibrium of the DeGroot–Friedkin model if and only if the relative interaction matrix C is doubly stochastic. In other words, precisely when C is doubly stochastic, the model admits the democracy configuration as an equilibrium, whereby the self-weight and social power of each individual is equal to $1/n$. In this case, the influence matrix is doubly stochastic and the final opinion in (2.5) is the exact average of the initial opinions. If such structures are unusual, then so is a democratic equilibrium.

In summary, this paper will address the following relevant questions: (i) Given an arbitrary relative interaction matrix C, when is autocracy attractive? (ii) Is democracy attractive when C is doubly stochastic, and what is its region of attraction? (iii) Do equilibrium configurations exist that are similar to democracy for general matrices C, and are they attractive?

2.4. Explicit Mathematical Modeling and Eigenvector Centrality. In this subsection we provide an explicit expression for the evolution of the DeGroot–Friedkin model and establish some preliminary properties. Given a relative interaction matrix C (row-stochastic with zero diagonal) that is irreducible, let $c^T = [c_1, \dots, c_n]$ be its dominant left eigenvector, i.e., the left eigenvector associated with the eigenvalue 1, with positive entries and normalized so that $\mathbb{1}_n^T c = 1$. The existence and uniqueness of this vector follow from the Perron–Frobenius theorem for nonnegative irreducible matrices. (For simplicity we call c dominant, even if this wording is a slight abuse of notation for irreducible matrices C that are periodic.)

LEMMA 2.2 (explicit formulation of the DeGroot–Friedkin model). *For $n \geq 2$, let c^T be the dominant left eigenvector of the relative interaction matrix $C \in \mathbb{R}^{n\times n}$ that is row-stochastic, zero-diagonal, and irreducible. The DeGroot–Friedkin model is equivalent to $x(s+1) = F(x(s))$, where $F : \Delta_n \to \Delta_n$ is a continuous map defined by*

(2.8)
$$F(x) = \begin{cases} \mathbb{e}_i & \text{if } x = \mathbb{e}_i \text{ for all } i \in \{1,\dots,n\}, \\ \Big(\dfrac{c_1}{1-x_1}, \dots, \dfrac{c_n}{1-x_n}\Big)^T \Big/ \displaystyle\sum_{i=1}^{n} \dfrac{c_i}{1-x_i} & \text{otherwise.} \end{cases}$$

The result of Lemma 2.2 has several consequences. First, the map F defined in Lemma 2.2 is continuous. This property is very convenient as it will allow us to establish the existence and stability of certain critical points via a fixed-point theorem and Lyapunov analysis, respectively. (The theory of Lyapunov functions for discrete-time systems is discussed in [47], [49, Exercises 4.62–68], and [14, section 1.3].) Second, Lemma 2.2 implies that the dominant left eigenvector c^T of the relative interaction matrix C plays a key role in the definition and analysis of the DeGroot–Friedkin model. Specifically, the relative interaction matrix C plays no direct role and the only parameter appearing in the DeGroot–Friedkin dynamic model is $c \in \Delta_n$.

In the language of [12], the entries of c are the *eigenvector centrality scores* for the weighted digraph with adjacency matrix C^T. In our setup, if one regards the row-stochastic matrix C as an adjacency matrix, then its dominant right eigenvector $\mathbb{1}_n$ is not informative, whereas it is precisely the left dominant eigenvector c that measures the influence of a node on all others. In what follows, we refer to c_i as the eigenvector centrality score of the ith individual and we refer to the individual with the largest entry of c, if it is unique, as the *eigenvector center*.

Motivated by the importance of the dominant left eigenvector, we briefly characterize the eigenvector centrality scores associated to a row-stochastic, zero-diagonal, and irreducible matrix. If C is doubly stochastic, then we know c is maximally uniform in the sense that all its entries are identical to $1/n$. It is useful to study the case when the entries of c are maximally nonuniform in some sense. Let $G(C)$ be the digraph associated to C. The digraph $G(C)$ has *star topology* if there exists a node i, called the *center node*, such that all directed edges of $G(C)$ are either from or to node i.

LEMMA 2.3 (eigenvector centrality for a digraph with star topology). *For $n \geq 3$, let C be row-stochastic, irreducible, and zero-diagonal. Let c^T be its dominant left eigenvector and $G(C)$ be its associated digraph. Then*

(i) *the centrality score of each node is at most $1/2$, and*

(ii) *the centrality score of the ith node is $1/2$ and is strictly larger than that for every other node if and only if the digraph $G(C)$ has star topology with center node i.*

For example, consider the star digraph in Figure 2 with row-stochastic adjacency matrix

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

As predicted, the dominant left eigenvector of C is $c^T = [1/4, \;\; 1/2, \;\; 1/4]$ and the center node in the star topology is also the eigenvector center.

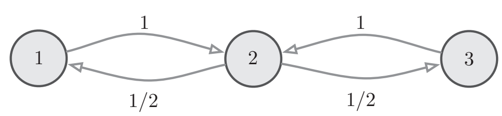

Fig. 2 *Star topology with center node 2.*

3. Influence Dynamics in Two Special Scenarios.

In this section we begin the mathematical analysis of the asymptotic behavior of DeGroot–Friedkin model. We consider the two meaningful and extreme situations where the relative interaction matrix C is doubly stochastic and where the digraph associated to C has star topology. In the first situation, convergence to a uniform self-weight configuration is observed from almost all initial conditions and a democratic power structure is achieved across issues. The second situation instead leads to the emergence of an autocratic power structure with a single leader from all initial conditions.

3.1. Doubly Stochastic Interactions and Democratic Influence Networks.

Consider the first case where the relative interaction matrix C is doubly stochastic, i.e., each of its rows and columns sums to 1. Then its dominant left eigenvector is $\mathbb{1}_n^T/n$ and so the DeGroot–Friedkin map simplifies to

$$F(x) = \begin{cases} \mathbb{e}_i & \text{if } x = \mathbb{e}_i \text{ for all } i \in \{1, \dots, n\}, \\ \left(\dfrac{1}{1-x_1}, \dots, \dfrac{1}{1-x_n}\right)^T \Big/ \displaystyle\sum_{i=1}^{n} \dfrac{1}{1-x_i} & \text{otherwise.} \end{cases}$$

Note that if $n = 2$, then C is always doubly stochastic and that, for any $(x_1, x_2) \in \Delta_2$ with strictly positive components, F satisfies $F(x_1, x_2) = (x_1, x_2)$. We therefore discard the trivial case $n = 2$. If $n \geq 3$ and C is doubly stochastic, then the digraph associated to C cannot have star topology.

Lemma 3.1 (DeGroot–Friedkin model with doubly stochastic interactions). *For $n \geq 3$, consider the DeGroot–Friedkin dynamical system $x(s+1) = F(x(s))$ defined by a relative interaction matrix $C \in \mathbb{R}^{n\times n}$ that is row-stochastic, irreducible, and has zero diagonal. If C is doubly stochastic, then*

(i) (Equilibria:) *the equilibrium points of F are the autocratic vertices $\{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ and the democratic configuration $\frac{1}{n}\mathbb{1}_n$.*

(ii) (Convergence property:) *for all nonautocratic initial conditions $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, the self-weights $x(s)$ and the social power $w(x(s))$ converge to the democratic configuration $\frac{1}{n}\mathbb{1}_n$ as $s \to \infty$.*

Some remarks are in order. First, property (ii) implies that the DeGroot processes (2.4) result in opinion consensus on each issue along the sequence of issues, where consensus opinions are equal to the average of the initial opinions. In other words, a doubly stochastic relative interaction matrix C leads to a doubly stochastic influence matrix $W(x(s))$ as $s \to \infty$. A doubly stochastic influence matrix indicates a democratic system where the social power of each individual is uniform. Second, let us mention that the lemma follows from a Lyapunov function analysis: one can show that, whenever $x(0) \neq \mathbb{1}_n/n$, the function $s \mapsto \max\{x_1(s), \dots, x_n(s)\}/\min\{x_1(s), \dots, x_n(s)\}$ is strictly decreasing and converges to 1 as $s \to \infty$. In other words, the self-weight difference between the two individuals with maximum self-weight and minimum self-weight is monotonic and vanishes asymptotically (see Figure 3). Along the same lines, one can show a monotonicity property: if the self-weight of the ith individual is greater than that of the jth individual at the initial issue, then it will remain so for all issues. In short, $x_i(0) > x_j(0)$ implies $x_i(s) > x_j(s)$ for all $s \in \mathbb{Z}_{\geq 0}$.

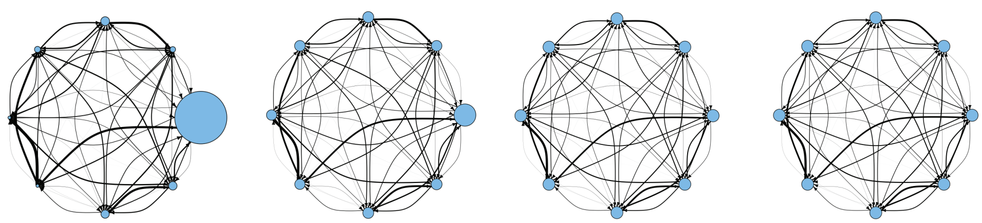

Fig. 3 *Emergence of democratic configurations: a trajectory for the DeGroot–Friedkin system with eight nodes and a doubly stochastic C. The size of the nodes is proportional to the individual self-weights $x(s)$ and the width of the edges is proportional to the off-diagonal entries of the influence matrix $W(x(s))$.*

We present some example simulations in dimension $n = 3$. Trajectories of the DeGroot–Friedkin dynamics with a doubly stochastic C are depicted in Figure 4. As predicted, all trajectories converge to the democratic configuration $\mathbb{1}_3/3$.

3.2. Interactions with Star Topology and Autocratic Influence Networks. Having characterized doubly stochastic and democratic structures, we now consider a diametrically opposite scenario in which the digraph associated to the relative interaction matrix has star topology. We assume $n \geq 3$ because the case $n = 2$ is trivial (where C is necessarily symmetric and doubly stochastic).

Lemma 3.2 (DeGroot–Friedkin model with star topology). *For $n \geq 3$, consider the DeGroot–Friedkin dynamical system $x(s+1) = F(x(s))$ defined by a relative interaction matrix $C \in \mathbb{R}^{n\times n}$ that is row-stochastic, irreducible, and has zero diagonal. If C has star topology with center node 1, then*

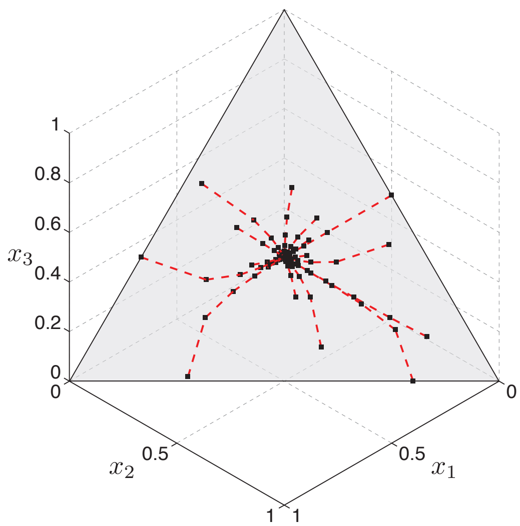

Fig. 4 *DeGroot–Friedkin dynamics with a doubly stochastic C: every self-weight trajectory starting from arbitrary initial states in $\Delta_3 \setminus \{\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3\}$ converges to the democratic configuration $\mathbb{1}_3/3$.*

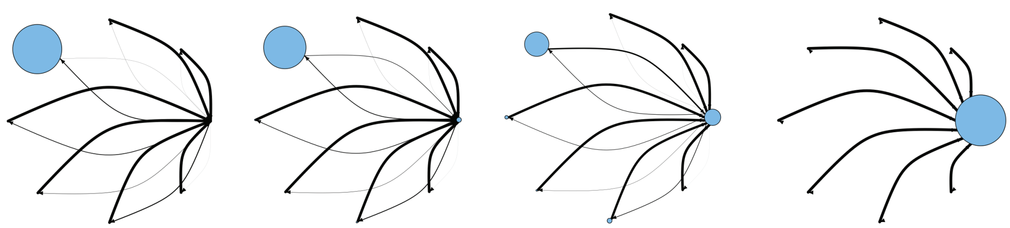

Fig. 5 *Emergence of autocratic configurations: a trajectory for the DeGroot–Friedkin system with eight nodes and star topology. The size of the nodes is proportional to the individual self-weights $x(s)$ and the width of the edges is proportional to the off-diagonal entries of the influence matrix $W(x(s))$.*

(i) (Equilibria:) *the equilibrium points of F are the autocratic vertices $\{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$.*

(ii) (Convergence property:) *for all nonautocratic initial conditions $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$, the self-weights $x(s)$ and the social power $w(x(s))$ converge to the autocratic configuration $\mathbb{e}_1$ as $s \to \infty$.*

The result is interpreted as follows: for a DeGroot–Friedkin model with star topology, the autocrat is predicted to appear on the center node along the sequence of opinion formation processes, independent of the initial values in most scenarios (except those autocratic states corresponding to the equilibrium points of F); see Figure 5. The proof of the lemma is based on a Lyapunov function argument: the social power of the center individual is strictly increasing across issues and, asymptotically, the opinion consensus resulting from the DeGroot process is equal to the initial opinion of the autocrat individual.

For the relative interaction matrix of the digraph in Figure 2, Lemma 3.2 establishes that the vertices $\{\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3\}$ are the only equilibria and that all trajectories

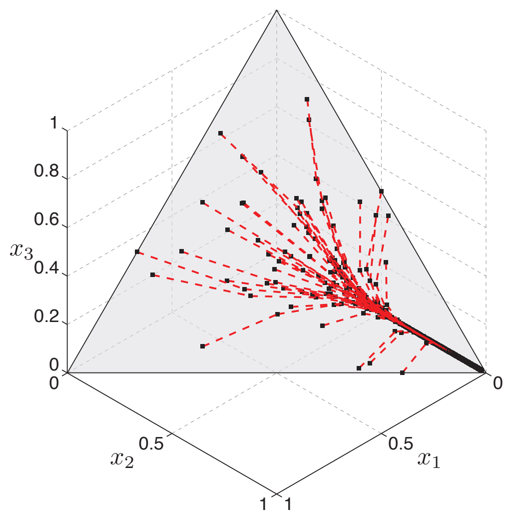

Fig. 6 *DeGroot–Friedkin dynamics with star topology as shown in Figure* 2*: every state trajectory starting from several sample initial states in* $\Delta_3 \setminus \{\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3\}$ *converges to the vertex* $\mathbb{e}_2$*.*

starting away from the equilibria converge to $\mathbb{e}_2$; these statements are illustrated by Figure 6.

4. Influence Dynamics with Irreducible Relative Interactions. We now consider the fairly general situation of a DeGroot–Friedkin dynamical system associated with an irreducible relative interaction matrix C.

Theorem 4.1 (DeGroot–Friedkin model with row-stochastic interactions). *For* $n \geq 3$*, consider the DeGroot–Friedkin dynamical system* $x(s+1) = F(x(s))$ *defined by a relative interaction matrix* $C \in \mathbb{R}^{n\times n}$ *that is row-stochastic, irreducible, and has zero diagonal. Assume the digraph associated to* C *does not have star topology and let* c^T *be the dominant left eigenvector of* C*. Then*

- (i) (Equilibria:) *the set of equilibrium points of* F *is* $\{\mathbb{e}_1, \dots, \mathbb{e}_n, x^*\}$*, where* x^* *lies in the interior of the simplex* Δ_n *and the ordering of the entries of* x^* *is equal to the ordering of the eigenvector centrality scores* c*.*
- (ii) (Convergence property:) *for all nonautocratic initial conditions* $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$*, the self-weights* $x(s)$ *and the social power* $w(x(s))$ *converge to the equilibrium configuration* x^* *as* $s \to \infty$*.*

According to this result, for a general C (i.e., an irreducible row-stochastic matrix that is not necessarily column-stochastic nor has star topology), the vector of self-weights $x(s)$ converges to a unique equilibrium value x^* from all initial conditions, except the autocratic states. This equilibrium value x^* is uniquely determined by the eigenvector centrality score c. The entries of x^* are strictly positive and have the same ordering as that of c, that is, if the centrality scores satisfy $c_i > c_j$, then the equilibrium social power x^* satisfies $x_i^* > x_j^*$, and if $c_i = c_j$, then $x_i^* = x_j^*$. Our model exhibits an additional interesting phenomenon, formally stated as follows.

Proposition 4.2 (social power accumulation). *Under the same assumptions as in Theorem* 4.1*, there exists a unique threshold* $c_{\text{thrshld}} := 1 - (\sum_{i=1}^{n} \frac{c_i}{1-x_i^*})^{-1} \in [0,1]$ *such that*

(i) *if* $c_{\text{thrshld}} < 0.5$*, then every individual with centrality above the threshold* ($c_i > c_{\text{thrshld}}$) *has social power larger than centrality* ($x_i^* > c_i$) *and, conversely, every individual with centrality below the threshold* ($c_i < c_{\text{thrshld}}$) *has social power smaller than centrality* ($x_i^* < c_i$)*; moreover, individuals with* $c_i = c_{\text{thrshld}}$ *satisfy* $x_i^* = c_i$*;*

(ii) *if* $c_{\text{thrshld}} \geq 0.5$*, then there exists only one individual with social power larger than centrality* $x_i^* > c_i$ *and all other individuals have* $x_i^* < c_i$*.*

In other words, individuals with the large centrality scores have an equilibrium social power that is larger than their respective centrality scores; in turn, the individual with the lowest centrality score has a lower equilibrium social power. There is an accumulation of social power in the central nodes of the network. The accumulation phenomenon is most evident for the star topology case (studied in Lemma 3.2): the center individual with $c_i = 0.5$ has a self-weight of 1, and all other individuals have zero social powers even though they may have strictly positive centrality scores.

An Example Application to the Reduced Krackhardt's Advice Network. In [50] Krackhardt presents data about an advice network (partly illustrated in Figure 7) in a manufacturing organization on the West Coast of the United States. The organization has 21 managers and the directed advice network C characterizes who sought advice from whom. If individual i asks for advice from n_i different individuals, then we assume, as done, for example, in [43], that $c_{ij} = 1/n_i$ for j in these n_i individuals, and $c_{ik} = 0$ for all other individuals k. Moreover, self-weighting is not considered in C, that is, $c_{ii} = 0$ for all $i \in \{1, \dots, 21\}$.

The complete Krackhardt's network includes four managers (i.e., individuals 6, 13, 16, and 17) from whom no other individual requests advice. We will analyze the case of reducible relative interaction matrices in the next section; for now, in this section, we simulate a reduced Krackhardt's advice network (as shown in Figure 7) without these four *source* nodes (i.e., the nodes with zero in-degree and positive out-degree) in the digraph associated to C. The complete Krackhardt's advice network will be analyzed in section 5 after the DeGroot–Friedkin influence dynamics with reducible relative interactions are considered.

The reduced matrix C has a unique dominant left eigenvector

$$c^T = [0.0609,\ 0.1302,\ 0.0383,\ 0.0547,\ 0.0022,\ 0.1378,\ 0.0078,\ 0.0141,\ 0.0239, \dots,\\ 0.0521,\ 0.0498,\ 0.0699,\ 0.0141,\ 0.0997,\ 0.0066,\ 0.0360,\ 0.2018].$$

We simulate the DeGroot–Friedkin model on this reduced Krackhardt's advice network with various initial states $x(0) \in \Delta_{17}$. The simulations show that all dynamical trajectories converge to a unique equilibrium self-weight vector x^* given by

$$x^* = [0.0441,\ 0.1339,\ 0.0355,\ 0.048,\ 0.0018,\ 0.1473,\ 0.0062,\ 0.0134,\ 0.0215, \dots,\\ 0.049,\ 0.047,\ 0.0668,\ 0.0134,\ 0.1039,\ 0.0018,\ 0.0374,\ 0.229]^T.$$

Comparing these two vectors c and x^*, it is clear that the ordering of the vector components of x^* is consistent with that of c, that is, $x_i^* > x_j^*$ if and only if $c_i > c_j$ for $i, j \in \{1, \dots, 17\}$. This observation verifies statement (i) of Theorem 4.1. Meanwhile, we can calculate $c_{\text{thrshld}} = 0.1183$. In Figure 8, the social power accumulation is observed such that $c_i > c_{\text{thrshld}}$ implies $x_i^* > c_i$, and $c_i < c_{\text{thrshld}}$ implies $x_i^* < c_i$, for all $i \in \{1, \dots, 17\}$. This is consistent with Proposition 4.2.

The dynamical trajectories of the self-weights in the reduced Krackhardt's advice network generated by the DeGroot–Friedkin model are illustrated in Figure 9. Individual 5 has the minimal eigenvector centrality score and her equilibrium self-weight

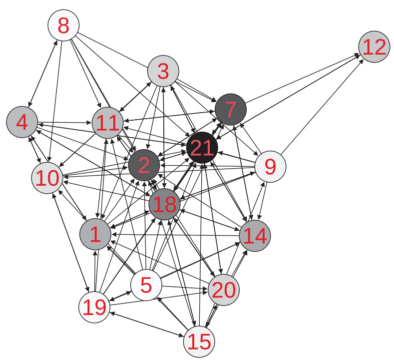

Fig. 7 *Reduced Krackhardt's advice network with 17 nodes: source nodes 6, 13, 16, and 17 are excluded. This and all following network layouts are obtained via the graph drawing algorithm described in* [41]*. The grayscale of the nodes represents* c_i.

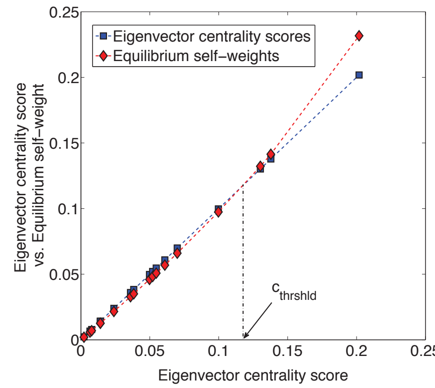

Fig. 8 *Comparison between eigenvector centrality scores and DeGroot–Friedkin equilibrium self-weights for the reduced Krackhardt's advice network: we compute* $c_{\text{thrshld}} = 0.1183$ *and verify* $c_i > c_{\text{thrshld}}$ *implies* $x_i^* > c_i$ *and* $c_i < c_{\text{thrshld}}$ *implies* $x_i^* < c_i$.

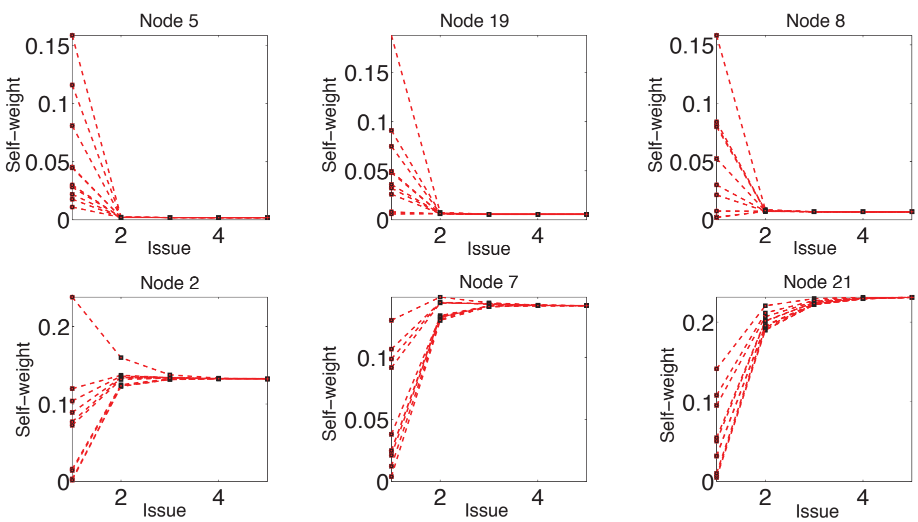

Fig. 9 *DeGroot–Friedkin self-weight dynamics for the reduced Krackhardt's advice network: we simulate the dynamics for ten distinct initial conditions and we display the trajectories of six nodes under these ten initial conditions. The ten distinct initial states converge to a unique self-weight configuration* x^* *with the properties that* x^* *is strictly positive and* $c_i > c_j$ *implies* $x_i^* > x_j^*$.

(social power) is the minimum; individual 19 has the second smallest score and her equilibrium social power is the second smallest; individual 8 advises only one neighbor but her score is not the smallest, hence her social power is not the smallest; individual 2 advises the most neighbors but her score is not the largest, nor is her equilibrium social power; individual 7, the head of the organization, has the second largest score, and her social power in the equilibrium is also the second largest; individual 21 has the maximal score and indeed has the maximum equilibrium social power.

Further Discussion on the Convergence Behaviors. In the following numerical examples we illustrate the dynamical behaviors of the DeGroot–Friedkin model on three social influence networks, including a discussion and advice network of a commercial organization (for which the unpublished data were collected by Friedkin), a research relation network of Biological Sciences faculty at the University of Chicago (circa 1978) [28], and a Facebook circle network [55]. The first two moderate size networks contain 101 and 141 individuals, respectively, and the third relatively large network has 4031 individuals. Just like the reduced Krackhardt's advice network, the relative errors of self-weights (with respect to the equilibria) are reduced to $O(10^{-8})$ in 4 or 5 iterations on all three networks. As shown in Figure 10, the trajectory convergence rates of the DeGroot–Friedkin dynamical systems are essentially independent of network size and initial self-weights, and there are no obvious fluctuations along the trajectories. Additionally, we simulated our model extensively over various network sizes composed of three to thousands of nodes and over different types of networks, e.g., highly clustered networks or Erdős–Rényi networks. In all our numerical experiments the DeGroot–Friedkin dynamical trajectories starting from $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ converge to sufficiently small neighborhoods of the equilibria in only a few iterations. Moreover, in most numerical experiments we have observed monotonic trajectories for all self-weights.

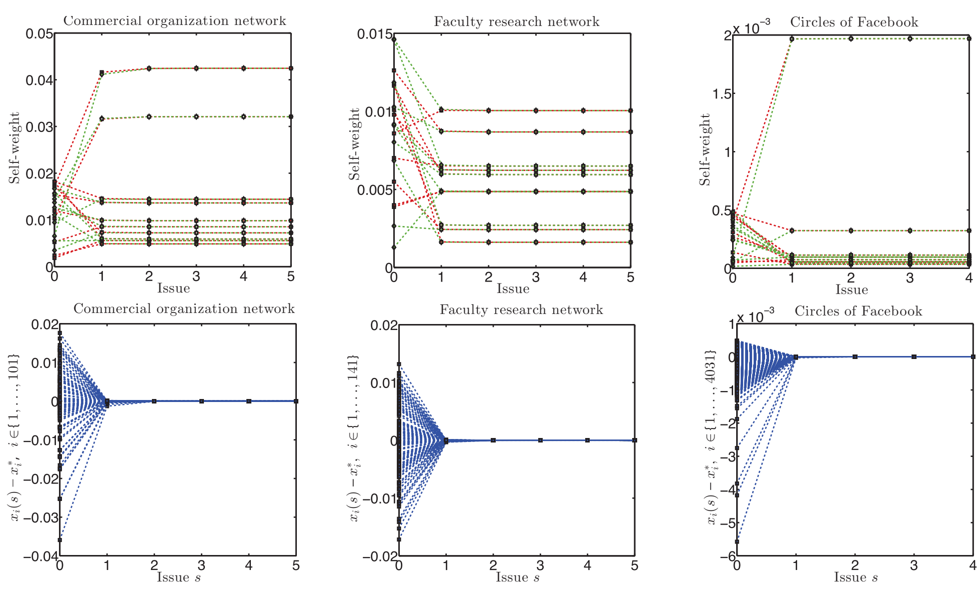

Fig. 10 *Three social networks: The images in the left column, middle column, and right column represent the influence evolution on a commercial organization network with* 101 *nodes, a faculty network with* 141 *nodes, and a Facebook circle network with* 4031 *nodes, respectively. The top images illustrate two distinct trajectories (drawn with different colors) converging to the unique equilibrium. In each network, ten nodes are randomly picked for easy reading. The bottom images illustrate the relative errors between the self-weight trajectories and the self-weight equilibria of all nodes.*

5. Influence Dynamics with Reducible Relative Interactions. The analysis in the previous sections relies on the assumption that the relative interaction matrix C is irreducible, i.e., the associated digraph is strongly connected. This assumption does not always hold (e.g., in Krackhardt's advice network): when C is reducible, the social influence network is not strongly connected. In this section we assume that the matrix C is reducible and its associated digraph has one or multiple globally reachable nodes. In this case, the matrix C admits a dominant left eigenvector, the DeGroot opinion dynamics (2.1) is always convergent, and the analysis of the DeGroot–Friedkin model is essentially similar to that for an irreducible C. We leave for future work the complete study of reducible cases.

We generalize Theorem 4.1 to the setting of reducible relative interaction matrices C with globally reachable nodes. Without loss of generality, assume the globally reachable nodes are $\{1,\dots,m\}$ for $m \leq n$ and let $G(C_m)$ be the subgraph induced by the globally reachable nodes. One can show that there exists no row-stochastic matrix C with zero diagonal and a unique globally reachable node; we therefore assume $m \geq 2$. For simplicity of analysis, we assume that the subgraph $G(C_m)$ is aperiodic (otherwise, the dynamics of opinions about a single issue may exhibit oscillations and not converge). Under these assumptions the DeGroot opinion dynamics is always convergent. Indeed, the matrix C admits a unique dominant left eigenvector c with the property that $c_1,\dots,c_m$ are strictly positive and $c_{m+1},\dots,c_n$ are zero. Moreover, if $x \in \Delta_n \setminus \{\mathbb{e}_1,\dots,\mathbb{e}_n\}$, then we can show that there exists a unique $w(x) \in \Delta_n$ such that $w(x)^T W(x) = w(x)^T$, $w_{m+1}(x) = \cdots = w_n(x) = 0$, and $\lim_{t\to\infty} W(x)^t = \mathbb{1}_n w(x)^T$. In other words, opinion consensus is always achieved and the individuals who are not globally reachable in $G(C)$ have no influence on the final opinion. Consequently, there exists a unique nonautocratic equilibrium point x^* such that $x_1^*,\dots,x_m^*$ are strictly positive and $x_{m+1}^*,\dots,x_n^*$ are zero, and $x(s)$ converges to x^* asymptotically as $s \to \infty$ for all nonautocratic initial states $x(0)$. This implies that the individuals who are not globally reachable in $G(C)$ have zero social power in the equilibrium influence networks. Moreover, as for irreducible matrices C, social power accumulation is observed in this case: there exists a threshold $c_{\mathrm{thrshld}} := 1 - (\sum_{i=1}^n \frac{c_i}{1-x_i^*})^{-1}$ such that any individual i with c_i greater than c_{thrshld} has more social power than that predicted by the eigenvector centrality. In summary, the properties specified in Theorem 4.1 and Proposition 4.2 for irreducible matrices can be naturally adapted and remain mostly unchanged for reducible matrices C with globally reachable nodes.

An Example Application to the Krackhardt's Advice Network. The original Krackhardt's advice network (as illustrated in Figure 11 and introduced in [50]) is connected and reducible with 17 globally reachable nodes. The associated matrix C has a unique dominant left eigenvector

$$\begin{aligned} c^T = [&0.0470,\ 0.1320,\ 0.0388,\ 0.0516,\ 0.0022,\ 0,\ 0.1434,\ 0.0074,\ 0.0143,\ 0.0237,\dots,\\ &0.0528,\ 0.0512,\ 0,\ 0.0716,\ 0.0143,\ 0,\ 0,\ 0.1012,\ 0.0065,\ 0.0366,\ 0.2053]. \end{aligned} \tag{5.1}$$

As predicted, the four source nodes $\{6, 13, 16, 17\}$ in the digraph associated to C have zero eigenvector centrality scores. We simulate the DeGroot–Friedkin model on this Krackhardt's advice network with 27000 randomly selected initial states $x(0) \in \Delta_{21}$. The simulations show that all dynamical trajectories converge to a unique equilibrium self-weight vector x^* given by

$$\begin{aligned} x^* = [&0.0432,\ 0.1339,\ 0.0354,\ 0.0476,\ 0.0020,\ 0,\ 0.1478,\ 0.0065,\ 0.0128,\ 0.0212,\dots,\\ &0.0487,\ 0.0472,\ 0,\ 0.0674,\ 0.0127,\ 0,\ 0,\ 0.0987,\ 0.0058,\ 0.0332,\ 0.2360]^T. \end{aligned} \tag{5.2}$$

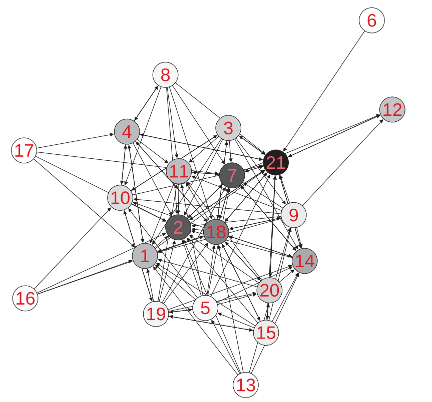

Fig. 11 *Krackhardt's advice network with* 21 *nodes: the four source nodes* $\{6, 13, 16, 17\}$ *are included and they have zero eigenvector centrality scores.*

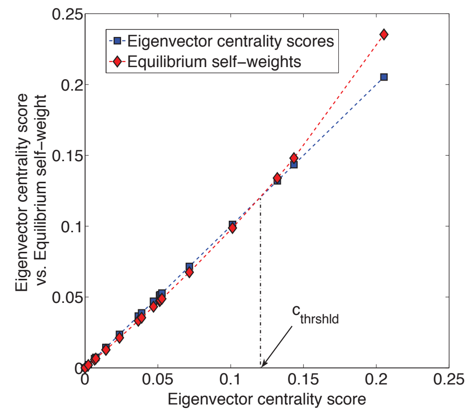

Fig. 12 *Comparison between eigenvector centrality scores and DeGroot–Friedkin equilibrium self-weights for the Krackhardt's advice network: we compute* $c_{\text{thrshld}} = 0.1216$ *and verify* $c_i > c_{\text{thrshld}}$ *implies* $x_i^* > c_i$ *and* $0 < c_i < c_{\text{thrshld}}$ *implies* $0 < x_i^* < c_i$.

Comparing the two vectors c and x^* given in (5.1) and (5.2), the ordering of the entries of x^* is consistent with that of c, that is, $x_i^* > x_j^*$ if and only if $c_i > c_j$ for $i, j \in \{1, \ldots, 21\}$. Moreover, we can calculate $c_{\text{thrshld}} = 0.1216$ and observe social power accumulation in Figure 12, whereby $c_i > c_{\text{thrshld}}$ implies $x_i^* > c_i$ and $0 < c_i < c_{\text{thrshld}}$ implies $0 < x_i^* < c_i$ for all $i \in \{1, \ldots, 21\}$. This numerical example validates our claim that Theorem 4.1 and Proposition 4.2 essentially hold true for the DeGroot–Friedkin dynamical system associated with a reducible matrix C and with globally reachable nodes.

6. Conclusion. This article studies the dynamics of opinions and influence relationships over a sequence of issues and, in doing so, extends and generalizes existing models that focus on opinion dynamics over a single issue. Issue sequences are natural phenomena for enduring groups, and their occurrence raises the possibility of the evolution of influence network typology across issues. Such evolution is poorly understood. Our investigation is focused on one such evolutionary process, that is, the adjustments of individuals' levels of closure-openness to influence and their effect on the content of the consensus that is generated by the DeGroot process [23] in a sequence of issues. Our influence network evolution model is based on a natural "reflected appraisal" mechanism [30] that is well accepted in sociology. A fundamental implicit assumption in this novel model is that individuals perceive their social power, "know their place" in a social group, and adjust their levels of closure-openness and accommodation accordingly. As we have discussed, we believe that this assumption may hold not only for small and moderate size social groups, but also for large groups that are dealing with a common issue sequence.

We have presented several novel results on the modeling and analysis of the dynamic evolution of influence networks via reflected appraisal. Based upon the classic DeGroot averaging model for opinion dynamics and the recently proposed model of reflected appraisal, we have derived a concise explicit dynamical model for DeGroot–Friedkin evolution and characterized completely its asymptotic properties. Our analysis leads to several important properties of the asymptotic influence network as a function of an appropriately defined eigenvector centrality score: (i) there exists a unique, invariant self-weight configuration associated with the eigenvector centrality score vector; (ii) for all nontrivial (i.e., nonvertex) initial conditions, the individuals' self-weights converge asymptotically to this unique equilibrium; (iii) the equilibrium self-weights have the same ordering as that of the eigenvector centrality scores; and (iv) the equilibrium self-weights (i.e., the social power) of the individuals with the largest centrality scores are larger than their centrality scores. In other words, our model predicts a tendency to accumulation of social power in the individual(s) with the largest centrality score(s), except for the implausible special case of doubly stochastic relative interaction matrices. Moreover, the proposed mechanism encourages autocracy, because the mechanism dampens protest movements assembled by individuals with low levels of relative control.

This paper presents only an introduction to social power and interpersonal influence evolution models and much work remains to be done in order to understand the robustness of our formulation and its results. First, it would be valuable to extend our analysis to opinion formation processes such as the Friedkin–Johnsen model (1.2), where individuals have a tendency to anchor their evolving opinion in their initial values. For this case, the simulation-based results in [30] indicated a more pronounced tendency to autocracy than what is predicted by our DeGroot–Friedkin model. On the other hand, there may be conditions for which this tendency is less pronounced. At the present time, we cannot assert robustness.

Second, interesting unexplored variations on our analysis include ones in which the process of opinion dynamics and the process of reflected appraisal take place over comparable timescales (that is, the individual self-weight x_i is set equal to the individual perceived power p_i in (2.7) right after each opinion discussion iteration), and under specifications that allow heterogeneous individual closure responses to relative control. A large literature exists in social psychology on conditions that may affect individuals' closure-openness to influence. We believe there are opportunities for an investigative debate on useful alternative mechanisms that adjust the extents of closure-openness to influence across issue sequences. Nevertheless, these extensions may not be critical to our present results as some preliminary analysis on these modified models indicates that their asymptotic convergence behaviors and equilibria are identical to those in the DeGroot–Friedkin model.

Third and finally, an assessment of the validity of the model has just begun. The experiment reported by Friedkin [30] supports the postulated linkage for small groups and short issue sequences. The strength of tendency for larger groups and longer issue sequences is presently unknown. Future research will be directed at validating the assumptions upon which our present work depends.

Appendix A. Properties of the Influence Matrix.

LEMMA A.1. *Given a self-weight vector $x \in \Delta_n$ and a relative interaction (row-stochastic and zero-diagonal) matrix $C \in \mathbb{R}^{n\times n}$ that is irreducible, the following statements hold:*

(i) *the influence matrix $W(x)$, as defined in the decomposition* (2.3)*, is nonnegative and row-stochastic;*
(ii) *for all $x \in \Delta_n$, there exists a unique vector $w(x) \in \Delta_n$ such that $w(x)^T W(x) = w(x)^T$ and $\lim_{t\to\infty} W(x)^t = \mathbb{1}_n w(x)^T$;*

(iii) *if* $x \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$*, then* $W(x)$ *is irreducible, the digraph associated to* $W(x)$ *is strongly connected, and* $w(x) > 0$*;*

(iv) *if* $x = \mathbb{e}_i$ *for some* $i \in \{1, \dots, n\}$*, then* $W(\mathbb{e}_i)$ *is reducible, the node* i *is the only globally reachable node in the digraph associated to* $W(\mathbb{e}_i)$*, and* $w(x) = \mathbb{e}_i$*; and*

(v) $w(\mathbb{1}_n/n) = \mathbb{1}_n/n$ *if and only if* C *is doubly stochastic.*

Proof. Statement (i) is an immediate consequence of C being row-stochastic. Moreover, if $x \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, then the diagonal matrix $I_n - \mathrm{diag}(x) = \mathrm{diag}(\mathbb{1}_n - x)$ has all diagonal elements strictly positive. Hence, the matrix $(I_n - \mathrm{diag}(x))C$ has the same pattern of zero and positive entries as C. Because C is irreducible, $W(x)$ is irreducible. Hence, the existence, uniqueness, and other properties of the left eigenvector $w(x)$ are a restatement of the Perron–Frobenius theorem for irreducible matrices, whereby the eigenvector $w(x)$ is referred to as the Perron vector for $W(x)^T$. This completes the proof of statement (ii) for $x \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ and of statement (iii).

Next, regarding statement (iv), assume $x = \mathbb{e}_i$ for some $i \in \{1, \dots, n\}$. Without loss of generality, let $i = 1$. Let $C_{\{2,\dots,n\}}$ be the $(n-1) \times n$ matrix obtained by removing the first row from C. Simple calculations show

$$W(\mathbb{e}_1) = \mathrm{diag}(1, 0, \dots, 0) + \mathrm{diag}(0, 1, \dots, 1)C = \begin{bmatrix} \mathbb{e}_1^T \\ C_{\{2,\dots,n\}} \end{bmatrix},$$

which is a reducible matrix. In the strongly connected digraph associated to the irreducible C, there exists a directed path from any node to node 1. Identical directed paths exist in the digraph associated to $W(\mathbb{e}_1)$. Therefore, the node 1 is globally reachable and, since it has no out-edge, it is the only globally reachable node in the digraph associated to $W(\mathbb{e}_1)$. These assumptions are known to imply $w(\mathbb{e}_1) = \mathbb{e}_1$ and $\lim_{t\to\infty} W(\mathbb{e}_1)^t = \mathbb{1}_n \mathbb{e}_1^T$.

Finally, regarding statement (v), let us compute

$$W(\mathbb{1}_n/n) = \mathrm{diag}(\mathbb{1}_n/n) + (I_n - \mathrm{diag}(\mathbb{1}_n/n))C = I_n/n + (n-1)C/n.$$

If C is doubly stochastic, then $\mathbb{1}_n^T W(\mathbb{1}_n/n) = \mathbb{1}_n^T/n + (n-1)\mathbb{1}_n^T/n = \mathbb{1}_n^T$, which implies $W(\mathbb{1}_n/n)$ is doubly stochastic and $w(\mathbb{1}_n/n) = \mathbb{1}_n/n$. On the other hand, if $w(\mathbb{1}_n/n) = \mathbb{1}_n/n$, then $\mathbb{1}_n^T W(\mathbb{1}_n/n) = \mathbb{1}_n^T = \mathbb{1}_n^T/n + (n-1)\mathbb{1}_n^T C/n$, which is equivalent to $\mathbb{1}_n^T C = \mathbb{1}_n^T$; that is, C is doubly stochastic, as claimed. ∎

Appendix B. Proof of Lemma 2.2. Given the self-weight $x(s) \in \Delta_n$ at issue s, the subsequent self-weight vector is defined by $W(x(s))^T x(s+1) = x(s+1)$ and $x(s+1) \in \Delta_n$. We are therefore interested in the equality

$$\big(\mathrm{diag}(x(s)) + (I_n - \mathrm{diag}(x(s)))C\big)^T x(s+1) = x(s+1).$$

Straightforward manipulation leads to $\big(I_n - C^T\big)\,\mathrm{diag}\big(\mathbb{1}_n - x(s)\big)x(s+1) = \mathbb{0}_n$. This equality implies

$$\mathrm{diag}\big(\mathbb{1}_n - x(s)\big)x(s+1) = C^T\,\mathrm{diag}\big(\mathbb{1}_n - x(s)\big)x(s+1), \tag{B.1}$$

which implies that the vector $x^T(s+1)\,\mathrm{diag}\big(\mathbb{1}_n - x(s)\big)$ is a left eigenvector of C associated with eigenvalue 1. Therefore, $x(s+1) \in \Delta_n$ satisfies $\mathrm{diag}\big(\mathbb{1}_n - x(s)\big)x(s+1) = \alpha(s)c$, where the scaling coefficient $\alpha(x) = 1/\sum_{i=1}^n \frac{c_i}{1-x_i}$ is computed so that $\mathbb{1}_n^T x(s+1) = 1$. In other words, we have $(1 - x_j(s))x_j(s+1) = \alpha(s)c_j$ for all $j \in \{1, \dots, n\}$. If $x(s) = \mathbb{e}_i$ for some $i \in \{1, \dots, n\}$, we have proved that $x(s+1) = \mathbb{e}_i$

in Lemma A.1. If instead $x(s)$ is not a vertex of the simplex, then $x_i(s) < 1$ for all i so that $\alpha(s) \neq 0$ and, therefore, $x_i(s+1) = \frac{\alpha(s)c_i}{1-x_i(s)}$, or equivalently

$$x(s+1) = \alpha(s)\Big(\frac{c_1}{1-x_1(s)}, \dots, \frac{c_n}{1-x_n(s)}\Big)^T.$$

In other words, $x(s+1) = F(x(s))$ as claimed in (2.8). Note that $c > 0$ as C is irreducible. It remains to prove that the map F is continuous. By definition, F is an analytic function on the domain $\Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ and, therefore, it is continuous in $\Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$. Next, we show that the function F is locally Lipschitz at the vertex $\mathbb{e}_i$ for each $i \in \{1, \dots, n\}$. For all $x \in \Delta_n \setminus \{\mathbb{e}_i\}$, we write $x = (1-\delta)\mathbb{e}_i + \delta z$, where $\delta = (1 - x_i) > 0$ and $z = (x - x_i\mathbb{e}_i)/\delta$ is a point on the simplex and is perpendicular to $\mathbb{e}_i$. Note that $x - \mathbb{e}_i = ((1-\delta)\mathbb{e}_i + \delta z) - \mathbb{e}_i = -\delta\mathbb{e}_i + \delta z$, hence $\|x - \mathbb{e}_i\| = \delta\|z - \mathbb{e}_i\| > \delta$. We restrict our attention to a neighborhood of $\mathbb{e}_i$ where $\delta < 1/2$ so that we have $\min_{j\in\{1,\dots,n\}}(1 - \delta z_j) > 1/2$. As a result,

$$\begin{aligned}
\|F(x) - F(\mathbb{e}_i)\| &= \left\| \frac{\Big(\frac{c_1}{1-\delta z_1}, \dots, \sum_{j=1,j\neq i}^n \frac{-c_j}{1-\delta z_j}, \dots, \frac{c_n}{1-\delta z_n}\Big)^T}{\sum_{j=1,j\neq i}^n \frac{c_j}{1-\delta z_j} + \frac{c_i}{\delta}} \right\| \\
&< \left\| \left(\frac{c_1}{1-\delta z_1}, \dots, \sum_{j=1,j\neq i}^n \frac{-c_j}{1-\delta z_j}, \dots, \frac{c_n}{1-\delta z_n}\right)^T \Big/ \Big(\frac{c_i}{\delta}\Big) \right\| \\
&< \frac{\delta}{c_i \min_{j\in\{1,\dots,n\}}(1-\delta z_j)} \|(c_1, \dots, (c_i - 1), \dots, c_n)^T\| \\
&< \frac{2\delta}{c_i}\|c - \mathbb{e}_i\| < \frac{2\sqrt{2}}{c_i}\delta < \frac{2\sqrt{2}}{c_i}\|x - \mathbb{e}_i\|.
\end{aligned}$$

This inequality shows that F is locally Lipschitz continuous at $\mathbb{e}_i$ with Lipschitz constant $2\sqrt{2}/c_i$ for all $i \in \{1, \dots, n\}$. Therefore, F is continuous on Δ_n. ∎

Appendix C. Proof of Lemma 2.3. From the definition of the dominant left eigenvector, $c_i = \sum_{j\in\{1,\dots,n\}\setminus\{i\}} c_j C_{ji}$ for all $i \in \{1, \dots, n\}$. Since C is row-stochastic, $C_{ij} \in [0, 1]$ for all $i, j \in \{1, \dots, n\}$, so that

$$c_i = \sum_{j\in\{1,\dots,n\}\setminus\{i\}} c_j C_{ji} \leq \sum_{j\in\{1,\dots,n\}\setminus\{i\}} c_j = 1 - c_i.$$

That implies $\max\{c_1, \dots, c_n\} \leq 0.5$. As C is irreducible, $c > 0$ implies that there exists $i \in \{1, \dots, n\}$ such that $\sum_{j\in\{1,\dots,n\}\setminus\{i\}} c_j C_{ji} = \sum_{j\in\{1,\dots,n\}\setminus\{i\}} c_j$ if and only if $C_{ji} = 1$ for all $j \in \{1, \dots, n\} \setminus \{i\}$. That is to say, the corresponding $G(C)$ has star topology. ∎

Appendix D. Proof of Lemma 3.1. Suppose C is doubly stochastic. Since $c_i = c_j$ for all $i, j \in \{1, \dots, n\}$, we have $x_i^* = x_j^*$ directly from Theorem 4.1 (i) and, therefore, $x^* = c = \mathbb{1}_n/n$. The remaining statements are simply the special case of Theorem 4.1. ∎

Appendix E. Proof of Lemma 3.2. Regarding statement (i), the equilibria of the DeGroot–Friedkin dynamical system are the fixed points of the map F defined by (2.8). It is easy to see that the vertices of Δ_n are always fixed points. It remains to show that there does not exist an equilibrium in $\Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ for $G(C)$ with star topology. By contradiction, assume there exists a vector $x \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ such that $x = F(x)$. The fixed-point equation $x = F(x)$ implies $x_i(1 - x_i) = c_i\alpha(x) > 0$ for all $i \in \{1, \dots, n\}$, where the quantity $\alpha(x) = \big(\sum_{i=1}^n c_i/(1 - x_i)\big)^{-1}$ is well-posed because $x_i < 1$ for all $i \in \{1, \dots, n\}$ and c_i is positive. Hence, $x_i > 0$ for all $i \in \{1, \dots, n\}$. Because $1 = c_1 + \sum_{j=2}^n c_j$ and $c_1 = 0.5$,

$$\sum_{j=2}^{n} x_j(1 - x_j) = (1 - c_1)\alpha(x) = c_1\alpha(x) = x_1(1 - x_1). \tag{E.1}$$

Note that $n \geq 3$ and $x_j > 0$ for $j \in \{2, \dots, n\}$ together imply $\dfrac{x_j}{1 - x_1} < 1$. Also note that $f(z) = z(1 - z)$ is a concave function for $z \in [0, 1]$ so that $f(az) > af(z)$ for all $0 < a < 1$. Therefore, for $a = \frac{x_j}{1-x_1} < 1$ and $z = 1 - x_1$, we have, for $j \in \{2, \dots, n\}$,

$$f(az) = x_j(1 - x_j) > \frac{x_j}{1 - x_1}(1 - x_1)x_1 = af(z).$$

In other words, $\sum_{j=2}^n x_j(1 - x_j) > \sum_{j=2}^n \frac{x_j}{1-x_1}(1 - x_1)x_1 = x_1(1 - x_1)$, which contradicts (E.1).

Regarding statement (ii), consider an initial state $x \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$. As $x_1 \neq 1$, $x_1 - F_1(x) = x_1 - \alpha(x)c_1/(1 - x_1)$, where $F_1(x)$ is the first component of $F(x)$ and $\alpha(x) = (c_1/(1 - x_1) + \sum_{j=2}^n c_j/(1 - x_j))^{-1}$. If $x_1 = 0$, then $F_1(x) = \alpha(x)c_1 > 0$, and hence $x_1 - F_1(x) < 0$. If $x_1 > 0$, we claim that $c_1 = 0.5$ implies

$$\sum_{j=2}^{n} \frac{c_j}{1 - x_j} < \frac{c_1}{x_1}. \tag{E.2}$$

To show this claim we consider two possibilities: (1) if there exists $k \in \{2, \dots, n\}$ such that $1 - x_k = x_1$, then

$$\sum_{j=2}^{n} \frac{c_j}{1 - x_j} = \frac{c_k}{1 - x_k} + \sum_{j=2, j\neq k}^{n} c_j = \frac{c_k}{x_1} + (c_1 - c_k) = \frac{(1 - x_1)c_k + x_1c_1}{x_1} < \frac{c_1}{x_1},$$

where the last inequality follows from $c_1 > c_k$; (2) otherwise, if two or more entries x_j are strictly positive, then $1 - x_j > x_1$ and $\sum_{j=2}^n \frac{c_j}{1-x_j} < \frac{1}{x_1}\sum_{j=2}^n c_j = \frac{c_1}{x_1}$. This establishes our claim.

Now, (E.2) implies

$$\sum_{j=2}^{n} \frac{c_j}{1 - x_j} + \frac{c_1}{1 - x_1} < \frac{c_1}{x_1} + \frac{c_1}{1 - x_1} = \frac{c_1}{(1 - x_1)x_1},$$

which is equivalent to $\alpha(x) > (1 - x_1)x_1/c_1$. Hence, $x_1 - F_1(x) = \frac{c_1}{1-x_1}(\frac{(1-x_1)x_1}{c_1} - \alpha(x)) < 0$ for all $x_1 \neq 1$ and $x_1 - F_1(x) = 0$ for $x_1 = 1$ by definition (2.8).

Define a Lyapunov function candidate $V(x) = 1 - x_1$ for $x \in \Delta_n$. A sublevel set of V is defined as $\{x \mid V(x) \leq \beta\}$ for a given constant β. It is clear that (1) any sublevel set of V is compact and invariant, (2) V is strictly decreasing anywhere in $\Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, and (3) V and F are continuous. Consider now a sublevel set W_ϵ of V with $\beta = 1 - \epsilon$ for small ϵ. Then, by the LaSalle Invariance Principle as stated in [14, Theorem 1.19], every trajectory starting in W_ϵ converges asymptotically to the equilibrium point $\mathbb{e}_1$. Moreover, by property (2), any initial condition $x(0) \notin \{\mathbb{e}_2, \dots, \mathbb{e}_n\}$ will satisfy $V(F(x(0))) < 1$. Therefore, by selecting $1 - \epsilon > V(F(x(0)))$, we prove that every trajectory starting in $\Delta_n \setminus \{\mathbb{e}_2, \dots, \mathbb{e}_n\}$ converges asymptotically to the equilibrium point $\mathbb{e}_1$.

Finally, given $\lim_{s\to\infty} x(s) = \mathbb{e}_1$ and given the definition of $W(x)$ in (2.3),

$$\lim_{s\to\infty} W(x(s)) = W(\mathbb{e}_1) = \begin{bmatrix} \mathbb{e}_1^T \\ C_{\{2,\dots,n\}} \end{bmatrix}.$$

It is clear that $\mathbb{e}_1^T W(\mathbb{e}_1) = \mathbb{e}_1^T$. Moreover, since the dominant left eigenvector of $W(x)$ is an analytic function of x near $\mathbb{e}_1$ (see [51]), we conclude $\lim_{s\to\infty} w(x(s)) = \mathbb{e}_1$. ∎

Appendix F. Proof of Theorem 4.1. We start by proving statement (i) on *the existence and uniqueness of the equilibria*. The vertices of Δ_n are always fixed points of the map F by (2.8). If there exists $x \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ with the property that $x = F(x)$ for nonstar $G(C)$, then $\alpha(x) > 0$ by definition. This implies that $x_i > 0$ for all $i \in \{1, \dots, n\}$. Therefore, no other point on the boundary of Δ_n is a fixed point.

Regarding the *existence* of a nonvertex fixed point x^*, we introduce a positive number $r \ll 1$ and define the set $A = \{x \in \Delta_n \mid 1 - r \ge x_i \ge 0 \text{ for all } i \in \{1, \dots, n\}\}$. We claim that $F(A) \subset A$. For any $x \in A$ and $j \in \{1, \dots, n\}$, we compute

$$F_j(x) = \frac{\alpha(x) c_j}{1 - x_j} = \frac{1}{1 + \frac{\sum_{k\neq j} c_k/(1-x_k)}{c_j/(1-x_j)}} \le \frac{1}{1 + \sum_{k\neq j} \frac{c_k r}{c_j(1-x_k)}}.$$

Because $G(C)$ does not have star topology, Lemma 2.3 implies $c_j < 0.5$ and, in turn,

$$\sum_{k\neq j} \frac{c_k}{c_j(1-x_k)} > \frac{\sum_{k\neq j} c_k}{c_j} > 1,$$

which implies that there exists a sufficiently small $r \ll 1$ such that

$$\left(\frac{\sum_{k\neq j} c_k}{c_j(1-x_k)} - 1\right) r - \frac{\sum_{k\neq j} c_k}{c_j(1-x_k)} r^2 > 0$$
$$\iff \qquad \frac{1}{1 + \frac{\sum_{k\neq j} c_k r}{c_j(1-x_k)}} < 1 - r \qquad \iff \qquad F_j(x) < 1 - r.$$

This fact establishes our claim that $F(A) \subset A$. Next, since F is a continuous map on the compact set A, the Brouwer fixed-point theorem implies the existence of at least one fixed point $x^* \in A$. Moreover, since $A \subset \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ and there is no other fixed point on the boundary of Δ_n besides all vertices, then we know $x^* \in \text{interior}(\Delta_n)$.

In the following we first prove that *the entry ordering of x^* is consistent with that of c* on which the uniqueness of x^* is built. If $c_i > c_j$, it is clear that $x_i^*(1 - x_i^*) > x_j^*(1 - x_j^*)$. Since $0 < x_i^* + x_j^* \le 1$, we obtain $x_i^* - x_j^* > 0$ and $x_i^* + x_j^* < 1$, that is, $c_i > c_j$ implies that $x_i^* > x_j^*$ for all $i, j \in \{1, \dots, n\}$. Moreover, if $c_i = c_j$, we know

$$x_i^*(1 - x_i^*) = x_j^*(1 - x_j^*), \tag{F.1}$$

which means $x_i^* = x_j^*$ or $x_i^* = 1 - x_j^*$. Since all components of x^* are nonzero and $n \ge 3$, it is clear that $x_i^* < 1 - x_j^*$. Hence, the only solution of (F.1) is $x_i^* = x_j^*$.

Regarding the *uniqueness* of x^*, if there exist two vectors $x, z \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ with the property that $x = F(x)$ and $z = F(z)$, then, without loss of generality, we can write $x_i(1-x_i) = \gamma z_i(1-z_i)$ for all $i \in \{1, \dots, n\}$ and for some $0 < \gamma \leq 1$.

If $\gamma = 1$, then $x_i(1-x_i) = z_i(1-z_i)$ for all $i \in \{1, \dots, n\}$. This implies that $x_i = z_i$ or $x_i = 1 - z_i$. If there exists at least one $x_j = 1 - z_j \neq z_j$ for some $j \in \{1, \dots, n\}$, then

$$1 = \sum_{i=1}^{n} x_i = \sum_{i=1, i\neq j}^{n} x_i + x_j = \sum_{i=1, i\neq j}^{n} x_j + \sum_{i=1, i\neq j}^{n} z_j. \tag{F.2}$$

For the remaining individuals, two cases may arise: First, if there exists another individual $k \neq j$ such that $x_k = 1 - z_k$, then $\sum_{i=1, i\neq j}^{n} x_j + \sum_{i=1, i\neq j}^{n} z_j > 1$, which is a contradiction. Second, if all other $i \in \{1, \dots, n\}, i \neq j$, satisfy $x_i = z_i$, then (F.2) implies

$$1 = 2(1 - z_j) \quad \iff \quad z_j = 1 - z_j = 0.5,$$

which is another contradiction. Therefore, if $\gamma = 1$, then $x = z$.

If $\gamma < 1$, by assuming that $c_1 = \max\{c_1, \dots, c_n\}$ we have $x_1 = \max\{x_1, \dots, x_n\}$ and $z_1 = \max\{z_1, \dots, z_n\}$ from the consistent ordering statement above, which imply $x_j < 0.5$ and $z_j < 0.5$, or equivalently $x_j + z_j < 1$ for all $j \in \{2, \dots, n\}$.

Since $x_j(1-x_j) = \gamma z_j(1-z_j) < z_j(1-z_j)$ for $\gamma < 1$ and $x_j + z_j < 1$, we have $x_j < z_j$ for all $j \in \{2, \dots, n\}$, and, hence, $x_1 > z_1$. Moreover, for any $j \in \{2, \dots, n\}$,

$$\begin{aligned} \frac{x_j}{x_1} < \frac{z_j}{z_1} &\implies \frac{\sum_{i=2, i\neq j}^{n} x_i}{x_1} < \frac{\sum_{i=2, i\neq j}^{n} z_i}{z_1} \\ &\iff \frac{x_1 + \sum_{i=2, i\neq j}^{n} x_i}{x_1} < \frac{z_1 + \sum_{i=2, i\neq j}^{n} z_i}{z_1} \\ &\iff \frac{1 - x_j}{x_1} < \frac{1 - z_j}{z_1} \iff \frac{1 - x_j}{1 - z_j} < \frac{x_1}{z_1}. \end{aligned} \tag{F.3}$$

From $x_j(1-x_j) = \gamma z_j(1-z_j)$ and the inequality (F.3), we obtain $x_j x_1 > \gamma z_j z_1$ for all $j \in \{2, \dots, n\}$, and $\sum_{j=2}^{n} x_j x_1 > \gamma \sum_{j=2}^{n} z_j z_1$, i.e., $x_1(1-x_1) > \gamma z_1(1-z_1)$, which is a contradiction to $x_1(1-x_1) = \gamma z_1(1-z_1)$. Therefore, there exists a unique x such that $x = F(x)$ and $x \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$.

Next, we prove statement (ii) on the *convergence analysis of the influence dynamics*. We claim that for all initial conditions $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, the solution $\{x(s)\}_{s \in \mathbb{Z}_{\geq 0}}$ has the following properties:

(ii.1) if $i_{\max} = \operatorname{argmax}_{k \in \{1,\dots,n\}} x_k(0)/x_k^*$ and $i_{\min} = \operatorname{argmin}_{k \in \{1,\dots,n\}} x_k(0)/x_k^*$, then $i_{\max} = \operatorname{argmax}_{k \in \{1,\dots,n\}} x_k(s)/x_k^*$ and $i_{\min} = \operatorname{argmin}_{k \in \{1,\dots,n\}} x_k(s)/x_k^*$ for all future issues $s \in \mathbb{Z}_{\geq 0}$;

(ii.2) if $x(0) \neq x^*$, the function $s \mapsto \frac{\max_{j \in \{1,\dots,n\}} x_j(s)/x_j^*}{\min_{j \in \{1,\dots,n\}} x_j(s)/x_j^*}$, for $s \geq 1$ is bounded and strictly decreasing;

(ii.3) $\lim_{s\to\infty} x(s) = x^*$; and

(ii.4) $W(x(s))$ converges to $W(x^*)$ and $w(x(s))$ converges to x^*.

Given the equilibrium $x^* \in \text{interior}(\Delta_n)$, we define the shorthands $\bar{x}_i(s) = x_i(s)/x_i^*$, $\bar{x}_{\max}(s) = \max_{j \in \{1,\dots,n\}}\{\bar{x}_j(s)\}$, and $\bar{x}_{\min}(s) = \min_{j \in \{1,\dots,n\}}\{\bar{x}_j(s)\}$. The properties of the trajectories $\bar{x}_i(s)$ are given in Lemmas F.1 and F.2. Based upon Lemma F.2, the proof of property (ii.1) follows if we can show that the inequalities

$\frac{1-x_k(s)}{1-x_{i_{\max}}(s)} \geq \frac{1-x_k^*}{1-x_{i_{\max}}^*}$ and $\frac{1-x_{i_{\min}(s)}}{1-x_k(s)} \geq \frac{1-x_{i_{\min}}^*}{1-x_k^*}$ hold for all $k \in \{1,\dots,n\}$. Indeed, these inequalities are a direct result of Lemma F.1 (iii). Therefore, if $\bar{x}_i(0) = \bar{x}_{\max}(0)$, then $\bar{x}_i(s) = \bar{x}_{\max}(s)$ for all $s \in \mathbb{Z}_{\geq 0}$. Similarly, we can show $\bar{x}_j(s) = \bar{x}_{\min}(s)$ for all $s \in \mathbb{Z}_{\geq 0}$ if $\bar{x}_j(0) = \bar{x}_{\min}(0)$.

Regarding property (ii.2), without loss of generality, we assume $\bar{x}_i(s) = \bar{x}_{\max}(s)$ and $\bar{x}_j(s) = \bar{x}_{\min}(s)$ for some $i, j \in \{1,\dots,n\}$ and for $s \geq 1$. One may check $x(s) \in \text{interior}(\Delta_n)$ for all $s \geq 1$ and for all initial states $x(0) \in \Delta_n \setminus \{\mathbb{e}_1,\dots,\mathbb{e}_n\}$, which implies $\bar{x}_{\min}(s) > 0$ when $s \geq 1$. If $\bar{x}_{\max}(s) \neq \bar{x}_{\min}(s)$, then $i \neq j$ and

$$\bar{x}_i(s) = \frac{x_i(s)}{x_i^*} \geq \frac{1-x_j(s)}{1-x_j^*} \quad \text{and} \quad \bar{x}_j(s) = \frac{x_j(s)}{x_j^*} \leq \frac{1-x_i(s)}{1-x_i^*} \tag{F.4}$$

from Lemma F.1 (ii). Moreover, for $n \geq 3$, we can show that the two inequalities in (F.4) cannot hold as equalities at the same time: If both $\bar{x}_i(s) = \frac{1-x_j(s)}{1-x_j^*}$ and $\bar{x}_j(s) = \frac{1-x_i(s)}{1-x_i^*}$, we compute

$$\bar{x}_i(s) = \frac{1-x_j(s)}{1-x_j^*} = \frac{1-x_j(s)-x_i(s)}{1-x_j^*-x_i^*} \quad \text{and} \quad \bar{x}_j(s) = \frac{1-x_i(s)}{1-x_i^*} = \frac{1-x_j(s)-x_i(s)}{1-x_j^*-x_i^*},$$

which means $\bar{x}_i(s) = \bar{x}_j(s)$ and $\bar{x}_{\max}(s) = \bar{x}_{\min}(s)$, which is a contradiction. Therefore,

$$\begin{aligned}
\frac{x_i(s)}{x_i^*}\frac{1-x_i(s)}{1-x_i^*} > \frac{x_j(s)}{x_j^*}\frac{1-x_j(s)}{1-x_j^*} &\iff \frac{\frac{x_i(s)(1-x_i(s))}{c_i}}{\frac{x_i^*(1-x_i^*)}{c_i}} > \frac{\frac{x_j(s)(1-x_j(s))}{c_j}}{\frac{x_j^*(1-x_j^*)}{c_j}} \\
&\iff \frac{x_i(s)(1-x_i(s))}{c_i} > \frac{x_j(s)(1-x_j(s))}{c_j} \\
&\iff \frac{x_i(s)}{x_j(s)} > \frac{c_i(1-x_j(s))}{c_j(1-x_i(s))} = \frac{x_i(s+1)}{x_j(s+1)},
\end{aligned} \tag{F.5}$$

since $\frac{x_i^*(1-x_i^*)}{c_i} = \frac{x_j^*(1-x_j^*)}{c_j}$. As a result of inequality (F.5), for $\bar{x}_{\max}(s) > \bar{x}_{\min}(s)$ we have

$$\frac{x_i(s)}{x_j(s)} > \frac{x_i(s+1)}{x_j(s+1)} \iff \frac{x_i(s)/x_i^*}{x_j(s)/x_j^*} > \frac{x_i(s+1)/x_i^*}{x_j(s+1)/x_j^*} \iff \frac{\bar{x}_{\max}(s)}{\bar{x}_{\min}(s)} > \frac{\bar{x}_{\max}(s+1)}{\bar{x}_{\min}(s+1)}.$$

It is also clear that $\frac{\bar{x}_{\max}(s)}{\bar{x}_{\min}(s)} = \frac{\bar{x}_{\max}(s+1)}{\bar{x}_{\min}(s+1)}$ if and only if $\bar{x}_{\max}(s) = \bar{x}_{\min}(s)$, or equivalently $x(s) = x^*$, since x^* is unique. Given an initial state $x(0) \in \Delta_n \setminus \{\mathbb{e}_1,\dots,\mathbb{e}_n\}$, the boundedness of $\bar{x}_{\max}(s)/\bar{x}_{\min}(s)$ is obvious from nonzero unique x^* and positive $x(s)$ for $s \geq 1$.

Regarding property (ii.3), define the Lyapunov function candidate $V(x(s)) = \frac{\bar{x}_{\max}(s)}{\bar{x}_{\min}(s)}$ and note that (1) any sublevel set of V is compact and invariant, (2) V is strictly decreasing anywhere in interior$(\Delta_n) \setminus \{x^*\}$, and (3) the function V and the map F are continuous. Consider now a sublevel set $W_\beta = \{x \mid V(x) \leq \beta\}$ of V for $\beta \geq 1$. Then, by the LaSalle Invariance Principle as stated in [14, Theorem 1.19], every trajectory starting in W_β converges asymptotically to the equilibrium point x^*. Moreover, any initial condition $x(0) \in \Delta_n \setminus \{\mathbb{e}_1,\dots,\mathbb{e}_n\}$ will satisfy $V(F(x(0))) \leq \epsilon$ for some $\epsilon \geq 1$ depending upon $x(0)$. Therefore, by selecting $\beta = \epsilon$, we prove that every trajectory starting in $\Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ converges asymptotically to the equilibrium point x^*.

Finally, we prove statement (ii.4) on the convergence of influence matrices across issues. Given $\lim_{s\to\infty} x(s) = x^*$ and given the definition of $W(x)$ in (2.3), we have

$$\lim_{s\to\infty} W(x(s)) = W(x^*) = \operatorname{diag}(x^*) + \operatorname{diag}(\mathbb{1}_n - x^*)C.$$

Since ${x^*}^T \operatorname{diag}(\mathbb{1}_n - x^*)$ is the dominant left eigenvector of C (see (B.1)), we compute

$$\begin{aligned} {x^*}^T W(x^*) &= {x^*}^T \operatorname{diag}(x^*) + {x^*}^T \operatorname{diag}(\mathbb{1}_n - x^*)C \\ &= {x^*}^T \operatorname{diag}(x^*) + {x^*}^T \operatorname{diag}(\mathbb{1}_n - x^*) = {x^*}^T. \end{aligned}$$

Moreover, since the dominant left eigenvector of $W(x)$ is an analytic function of x near x^* (see [51]), we conclude $\lim_{s\to\infty} w(x(s)) = x^*$. □

Properties of $x_i(s)/x_i^*$. Consider a DeGroot–Friedkin dynamical system $x(s+1) = F(x(s))$. If there exists a unique equilibrium $x^* \in \operatorname{interior}(\Delta_n)$ such that $x^* = F(x^*)$, then we denote $\bar{x}_i(s) = x_i(s)/x_i^*$, $\bar{x}_{\max}(s) = \max_{i\in\{1,\dots,n\}}\{\bar{x}_i(s)\}$, and $\bar{x}_{\min}(s) = \min_{i\in\{1,\dots,n\}}\{\bar{x}_i(s)\}$.

LEMMA F.1. *For $x(s) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, if $\bar{x}_i(s) = \bar{x}_{\max}(s)$, then the following statements hold true:*

(i) *$\bar{x}_i(s) \geq (1 - x_i(s))/(1 - x_i^*)$ and $\bar{x}_i(s) \geq 1$, and, moreover, $\bar{x}_i(s) > (1 - x_i(s))/(1 - x_i^*)$ and $\bar{x}_i(s) > 1$ for $\bar{x}_{\max}(s) \neq \bar{x}_{\min}(s)$;*

(ii) *$\bar{x}_i(s) \geq (1 - x_k(s))/(1 - x_k^*)$ for all $k \in \{1, \dots, n\}$;*

(iii) *$(1 - x_k(s))/(1 - x_k^*) \geq (1 - x_i(s))/(1 - x_i^*)$ for all $k \in \{1, \dots, n\}$, and, moreover, $(1 - x_k(s))/(1 - x_k^*) > (1 - x_i(s))/(1 - x_i^*)$ for $\bar{x}_{\max}(s) \neq \bar{x}_{\min}(s)$ and $i \neq k$.*

Similarly, if $\bar{x}_j(s) = \bar{x}_{\min}(s)$, then the reverse properties of (i)–(iii) hold for $\bar{x}_j(s)$ and $x_j(s)$.

Proof. Regarding statement (i), since $\bar{x}_i(s) = \bar{x}_{\max}(s)$, for any $k \in \{1, \dots, n\}$,

$$\begin{aligned} x_i(s)/x_i^* \geq x_k(s)/x_k^* &\iff x_i(s)x_k^*/x_i^* \geq x_k(s) \\ &\iff \sum_{k\in\{1,\dots,n\}} x_i(s)x_k^*/x_i^* \geq \sum_{k\in\{1,\dots,n\}} x_k(s) \iff x_i(s)/x_i^* \geq 1 \\ &\iff \bar{x}_i \geq 1 \iff x_i(s)/x_i^* \geq (1 - x_i(s))/(1 - x_i^*). \end{aligned}$$

Moreover, if $\bar{x}_{\max}(s) \neq \bar{x}_{\min}(s)$, there exists at least one individual j, $\bar{x}_j(s) = \bar{x}_{\min}(s)$, such that $x_i(s)x_j^*/x_i^* > x_j(s)$. Therefore,

$$\begin{aligned} \sum_{k\in\{1,\dots,n\}} x_i(s)x_k^*/x_i^* > \sum_{k\in\{1,\dots,n\}} x_k(s) &\iff \bar{x}_i > 1 \\ &\iff x_i(s)/x_i^* > (1 - x_i(s))/(1 - x_i^*). \end{aligned}$$

Regarding statement (ii),

$$\begin{aligned} x_i(s)/x_i^* \geq x_k(s)/x_k^* \ \ \forall k \in \{1, \dots, n\} &\iff \sum_{l\neq k} x_i(s)x_\ell^*/x_i^* \geq \sum_{l\neq k} x_\ell(s) \\ &\iff x_i(s)/x_i^* \geq (1 - x_k(s))/(1 - x_k^*). \end{aligned}$$

Regarding statement (iii),

$$\frac{x_i(s)-x_k(s)}{x_i^*-x_k^*} \geq \frac{x_i(s)}{x_i^*} \geq \frac{1-x_i(s)}{1-x_i^*} \iff \frac{1-x_i(s)+x_i(s)-x_k(s)}{1-x_i^*+x_i^*-x_k^*} \geq \frac{1-x_i(s)}{1-x_i^*}$$
$$\iff \frac{1-x_k(s)}{1-x_k^*} \geq \frac{1-x_i(s)}{1-x_i^*}.$$

Moreover, if $\bar{x}_{\max}(s) \neq \bar{x}_{\min}(s)$, based upon the results in statement (i), we have $\frac{x_i(s)}{x_i^*} > \frac{1-x_i(s)}{1-x_i^*}$, which implies that $\frac{1-x_j(s)}{1-x_j^*} > \frac{1-x_i(s)}{1-x_i^*}$.

The discussion of $\bar{x}_{\min}(s)$ is similar. ∎

LEMMA F.2. *For any $i, j \in \{1,\ldots,n\}$ and $x(s) \in \Delta_n \setminus \{\mathbb{e}_1,\ldots,\mathbb{e}_n\}$, either* (1) *if $\frac{1-x_j(s)}{1-x_i(s)} \geq \frac{1-x_j^*}{1-x_i^*}$, then $\bar{x}_i(s+1) \geq \bar{x}_j(s+1)$; or* (2) *if $\frac{1-x_j(s)}{1-x_i(s)} < \frac{1-x_j^*}{1-x_i^*}$, then $\bar{x}_i(s+1) < \bar{x}_j(s+1)$.*

Proof. Since $x_i(s+1) = \alpha(s)c_i/(1-x_i(s))$ and $x_i^* = \alpha^* c_i/(1-x_i^*)$, we have

$$\frac{\bar{x}_i(s+1)}{\bar{x}_j(s+1)} = \frac{x_j^* x_i(s+1)}{x_i^* x_j(s+1)} = \frac{(1-x_i^*)(1-x_j(s))}{(1-x_j^*)(1-x_i(s))},$$

which implies the lemma statement immediately. ∎

Appendix G. Proof of Proposition 4.2. Denote $\alpha^* = 1/(\sum_{i=1}^n \frac{c_i}{1-x_i^*})$. $c_{\text{thrshld}} = 1-\alpha^*$, or equivalently $\frac{1}{1-c_{\text{thrshld}}} = \sum_{i=1}^n \frac{c_i}{1-x_i^*}$, which implies that $\min\{x_1^*,\ldots,x_n^*\} < c_{\text{thrshld}} < \max\{x_1^*,\ldots,x_n^*\}$ for a non-doubly-stochastic C. Moreover, since $F(x^*) = x^*$, for all $i \in \{1,\ldots,n\}$,

$$x_i^*(1-x_i^*)/c_i = \alpha^* = c_{\text{thrshld}}(1-c_{\text{thrshld}})/c_{\text{thrshld}}. \tag{G.1}$$

For $c_{\text{thrshld}} < 0.5$: First, if $c_i > c_{\text{thrshld}}$, then $x_i^*(1-x_i^*) > c_i(1-c_i)$. Since $c_i < 0.5$, it is clear that $x_i^* > c_i$. Second, if $c_i < c_{\text{thrshld}}$, then $x_i^*(1-x_i^*) < c_i(1-c_i)$, which implies $x_i^* < c_i$ or $x_i^* > 1-c_i > 0.5$. Furthermore, since $c_{\text{thrshld}} < 0.5$, we can show $c_{\text{thrshld}} < \max\{c_1,\ldots,c_n\}$ (otherwise, if $0.5 > c_{\text{thrshld}} \geq \max\{c_1,\ldots,c_n\}$, then by statement (i) of Theorem 4.1 and (G.1) we can show $c_{\text{thrshld}} \geq \max\{x_1^*,\ldots,x_n^*\}$, which is a contradiction). Thus, there exists another individual j such that $c_j > c_i$, which by statement (i) of Theorem 4.1 implies $x_j^* > x_i^*$. Therefore, $x_i^* < c_i$ for $c_i < c_{\text{thrshld}}$, otherwise, $x_j^* > x_i^* > 0.5$ contradicts the fact that $x_j^* + x_i^* < 1$. Third, if $c_i = c_{\text{thrshld}}$, then $x_i^*(1-x_i^*) = c_i(1-c_i)$ from (G.1). Similarly, we can show $x_i^* < 0.5$ and hence $x_i^* = c_i$.

For $c_{\text{thrshld}} \geq 0.5$: Denote $x_{\max}^* = \max\{x_1^*,\ldots,x_n^*\}$ and $c_{\max} = \max\{c_1,\ldots,c_n\}$. By statement (i) of Theorem 4.1 and the fact that $0.5 \leq c_{\text{thrshld}} < x_{\max}^*$, there exists only one individual denoted by $i_{\max}$ associated with $c_{\max}$ and her equilibrium self-weight is $x_{\max}^*$. Since $c_{\text{thrshld}} < x_{i_{\max}}^*$, (G.1) implies $c_{i_{\max}} < x_{i_{\max}}^*$. For any other individual $j \neq i_{\max}$, we have $c_j < 0.5 \leq c_{\text{thrshld}}$, which implies $x_j^*(1-x_j^*) < c_{\text{thrshld}}(1-c_{\text{thrshld}})$ from (G.1). As $c_{\text{thrshld}} + x_j^* < x_{i_{\max}}^* + x_j^* < 1$, we obtain $x_j^* < 0.5 \leq c_{\text{thrshld}}$ and hence $x_j^* < c_j$ from (G.1). ∎

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