Contraction Theory on Riemannian Manifolds

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Abstract

Contraction theory is a methodology for assessing the stability of trajectories of a dynamical system with respect to one another. In this work, we present the fundamental results of contraction theory in an intrinsic, coordinate-free setting. The presentation highlights both the underlying geometric foundations of contraction theory, and the coordinate invariance of the resulting stability properties. We provide coordinate-free proofs of the main results for autonomous vector fields, and clarify the assumptions under which the results hold. In addition, we state and prove several interesting corollaries, study cascade and feedback interconnections, and highlight how contraction theory has arisen independently in other scientific disciplines. We conclude by illustrating the developed theory for the case of gradient dynamics.

Keywords: contraction theory, incremental stability, differential geometric methods, nonlinear control

1. Introduction

In the stability analysis of dynamical systems, one is sometimes interested not in the stability of trajectories with respect to specific attractor, but rather with the stability of trajectories with respect to *one another*. For linear systems, these two concepts are identical, but for nonlinear systems the latter is indeed a much stronger property.

Literature Review: Both differential and integral approaches have been developed for studying this relative stability of trajectories. The former, popularized in the control community by J.-J.E. Slotine *et al.* under the title of contraction theory, involves examining the dynamics of a "virtual displacement" between two infinitesimally separated trajectories. The root piece of literature for this methodology is [1], with a plethora of extensions including graph-theoretic characterizations [2], backstepping design [3], extensions to distributed systems [4], and algorithmic searches for contraction metrics [5]. Rigorous proofs of the main result — a sufficient condition for a vector field to be "contracting" — are varied in style, with some revolving around the use of the *matrix measure* [6, 7] while others utilize the perspective of a *contraction metric* [8, 9, 10]. As noted a recent historical review [11], the main ideas of the theory trace to the works [12, 13] in the early 1950's, with some similar concepts presented slightly later [14, 15]. A notion closely related to contraction is that of a *convergent system*, first introduced in [16]. Demidovich gave a sufficient condition for this type of convergence, with contraction theory as described above representing a generalization of this body of work. A simple proof of how Demidovich's condition leads to the convergent system property can be found in the tribute paper [17], with a thorough account of convergent systems available in [18].

The integral approach to the relative stability of trajectories — referred to as *incremental stability* — revolves around the use of an appropriate *incremental Lyapunov function*, and was first defined and put on firm ground in [19]. Extensions and generalizations of this line of work are presented in [20, 21, 22, 3, 23]. Of particular interest are [3, 23], which provide a coordinate-invariant formulation of incremental stability. As noted in [3], related ideas can be traced to [24]. Recently in [25], Finsler geometry has been used to formulate a characterization of contracting systems using Lyapunov functions defined on the tangent bundle.

Application areas of contraction theory and incremental stability have grown as the topics have proliferated, and now include symbolic models and control [26, 9], output regulation [18], synchronization and consensus [27, 21, 28], bio-molecular systems [7], intrinsic observer design [8, 29, 30], mechanical system controller design [31], and power system dynamics [3]. Among these works, we also note the useful tutorial paper [32].

Limitations of the Literature: Despite a flurry of activity in the area, the basic theory of contracting vector fields has never been presented rigorously in a coordinate-free setting. Moreover, the literature is often unclear on exactly what the key assumptions are for the main sta-

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bility result to hold, with corollaries presented as remarks without detailed proofs.

Contributions: The contributions of this work are fivefold. First, we use the tools of affine differential geometry to provide a purely geometric and intrinsic description of contracting vector fields on Riemannian manifolds (Definition 2.1). Second, we state and rigorously prove the main contraction stability theorem in an intrinsic setting, highlighting all assumptions (Theorem 2.3). In particular, the presentation unequivocally shows that contraction is a coordinate-invariant stability concept. Third, we present with proofs some ancillary facts about contracting systems, none of which are proven rigorously in the contraction literature. In particular, we show that contraction implies the existence of an exponentially stable fixed point for the vector field, and we show how the Riemannian contraction metric can be used to construct Lyapunov functions for this fixed point (Proposition 2.5). Fourth, we rigorously examine the modularity properties of contracting systems by studying cascade and feedback interconnections (Lemmas 3.1 and 3.2). Our analysis results in cascades always being contracting in an appropriate metric, while feedback systems require a suitable small gain condition. Finally, we connect the concept of contraction to ideas studied in the fields of differential geometry and general relativity (Section 4), and illustrate the theory with the important example of gradient dynamics on general Riemannian manifolds (Corollary 5.1).

Notation and Preliminaries: The presentation that follows relies heavily on concepts from affine differential geometry. Our notation follows [33], and the reader is referred to standard references such as [34, 35, 36] for details. For simplicity, all objects are assumed to be smooth, and we employ the Einstein summation convention in which repeated indices are implicitly summed over. Throughout, M denotes a manifold of dimension n, and $\mathsf{T}_x\mathsf{M}$ ($\mathsf{T}^*_x\mathsf{M}$) denotes the tangent (resp. cotangent) vector space at $x \in \mathsf{M}$. We denote by $v_x \in \mathsf{T}_x\mathsf{M}$ a vector in the tangent space at x. The tangent bundle TM (resp. cotangent bundle $\mathsf{T}^*\mathsf{M}$) is the disjoint union of these tangent (resp. cotangent) vector spaces over all $x \in \mathsf{M}$. A ***vector field*** (resp. ***covector field***) X on $\mathcal{U} \subset \mathsf{M}$ is a map assigning to each point $x \in \mathcal{U}$ a vector $X(x) \in \mathsf{T}_x\mathsf{M}$ (resp. $X(x) \in \mathsf{T}^*_x\mathsf{M}$). We let $\Gamma^\infty(\mathsf{TM})$ (resp. $\Gamma^\infty(\mathsf{T}^*\mathsf{M})$) denote the set of vector (resp. covector) fields on M. In a set of local coordinates $(x^1, \dots, x^n)$ in a neighborhood $\mathcal{V}$ of $x \in \mathsf{M}$, we denote by $\partial/\partial x^i$ (resp. $\mathrm{d}x^i$) the i^{th} basis vector of $\mathsf{T}_x\mathsf{M}$ (resp. $\mathsf{T}^*_x\mathsf{M}$). We denote the canonical pairing between a tangent vector $v_x \in \mathsf{T}_x\mathsf{M}$ and a covector $\alpha_x \in \mathsf{T}^*_x\mathsf{M}$ by $\langle \alpha_x; v_x\rangle \in \mathbb{R}$. A point $\bar{x} \in \mathsf{M}$ is a ***fixed point*** of X if $X(\bar{x}) = 0_{\bar{x}}$. An ***integral curve*** of $X \in \Gamma^\infty(\mathsf{TM})$ at $x \in \mathsf{M}$ is a curve $\gamma : J_x \subset \mathbb{R} \to \mathsf{M}$ satisfying $\gamma(0) = x$ and $\gamma'(t) = X(\gamma(t))$ for each $t \in J_x$. If J_x is the largest interval containing the origin over which γ is well defined, the integral curve γ at x is ***maximal***. The ***flow*** of X is the unique map $\Phi : J \times \mathcal{V} \to \mathsf{M}$, $(t, x) \mapsto \Phi_t(x)$, with $\mathcal{V}$ an open neighborhood of $x \in \mathsf{M}$, such that $t \to \Phi_t(x)$ is the maximal integral curve of X through x. If $\sup J_x = +\infty$ for each $x \in \mathcal{U}$, X is said to be forward complete on $\mathcal{U}$. If for each $t \geq 0$, $x \in \mathcal{U} \Rightarrow \Phi_t(x) \in \mathcal{U}$, the set $\mathcal{U}$ is said to be ***forward X-invariant***. We denote the set of functions from M to $\mathbb{R}$ by $C^\infty(\mathsf{M})$. The ***differential*** of a function $f \in C^\infty(\mathsf{M})$ is the covector field $\mathrm{d}f \in \Gamma^\infty(\mathsf{T}^*\mathsf{M})$. In local coordinates, $\mathrm{d}f = (\partial f/\partial x^i)\, \mathrm{d}x^i$. The ***Lie derivative*** $\mathcal{L}_X f$ of a function $f \in C^\infty(\mathsf{M})$ with respect to a vector field $X \in \Gamma^\infty(\mathsf{TM})$ is the function in $C^\infty(\mathsf{M})$ defined by $(\mathscr{L}_X f)(x) = \langle \mathrm{d}f(x); X(x)\rangle$. The ***Lie Bracket*** of two vector fields $X, Y \in \Gamma^\infty(\mathsf{TM})$ is the vector field $[X, Y] \in \Gamma^\infty(\mathsf{TM})$ defined by $[X, Y]f = \mathscr{L}_X\mathscr{L}_Y f - \mathscr{L}_Y\mathscr{L}_X f$ for any $f \in C^\infty(\mathsf{M})$. A ***Riemannian metric*** $\mathbb{G}$ is a $(0, 2)$-tensor field on M having the property that $\mathbb{G}(x) : \mathsf{T}_x\mathsf{M} \times \mathsf{T}_x\mathsf{M} \to \mathbb{R}$ is an inner product on $\mathsf{T}_x\mathsf{M}$. We denote this inner product at the point $x \in \mathsf{M}$ by $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{\mathbb{G}(x)}$, with the induced norm denoted by $\| \cdot \|_{\mathbb{G}(x)}$. The pair $(\mathsf{M}, \mathbb{G})$ is a ***Riemannian manifold***. In local coordinates, $\mathbb{G} = \mathbb{G}_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$. We will write $[\mathbb{G}_{ij}]$ for the associated $n \times n$ matrix. The metric can be used to uniquely relate elements of TM and elements of $\mathsf{T}^*\mathsf{M}$. Specifically, for each $x \in \mathsf{M}$ we define the ***flat map*** $\mathbb{G}(x)^\flat : \mathsf{T}_x\mathsf{M} \to \mathsf{T}^*_x\mathsf{M}$, by $\langle \mathbb{G}^\flat(v_x); u_x\rangle = \langle\!\langle v_x, u_x\rangle\!\rangle_{\mathbb{G}(x)}$, and the ***sharp map*** $\mathbb{G}^\sharp : \mathsf{T}^*_x\mathsf{M} \to \mathsf{T}_x\mathsf{M}$ as the inverse of $\mathbb{G}^\flat$, for each $v_x, u_x \in \mathsf{T}_x\mathsf{M}$. In local coordinates, $\mathbb{G}^\sharp = \mathbb{G}^{ij}\mathrm{d}x^i \otimes \partial/\partial x^j$, where $\mathbb{G}^{ik}\mathbb{G}_{kj} = \delta^i_j$. A Riemannian metric $\mathbb{G}$ induces a unique affine connection $\overset{\mathbb{G}}{\nabla}$ on M, called the ***Levi-Civita connection***, satisfying (i) $\overset{\mathbb{G}}{\nabla}_X Y - \overset{\mathbb{G}}{\nabla}_Y X = [X, Y]$, and (ii) $\mathscr{L}_Z \langle\!\langle X, Y\rangle\!\rangle_{\mathbb{G}} = \langle\!\langle \overset{\mathbb{G}}{\nabla}_Z X, Y\rangle\!\rangle_{\mathbb{G}} + \langle\!\langle X, \overset{\mathbb{G}}{\nabla}_Z Y\rangle\!\rangle_{\mathbb{G}}$ for vector fields $X, Y, Z \in \Gamma^\infty(\mathsf{TM})$. The vector field $\overset{\mathbb{G}}{\nabla}_Y X$ is the ***covariant derivative*** of X with respect to Y. In local coordinates the ***Christoffel symbols*** for $\overset{\mathbb{G}}{\nabla}$ are $(i, j, k \in \{1, \dots, n\})$

$$\Gamma^k_{ij} = \Gamma^k_{ji} = \frac{1}{2}\mathbb{G}^{k\ell}\left(\frac{\partial \mathbb{G}_{i\ell}}{\partial x^j} + \frac{\partial \mathbb{G}_{j\ell}}{\partial x^i} - \frac{\partial \mathbb{G}_{ij}}{\partial x^\ell}\right). \tag{1}$$

Given a curve $\gamma : J \to \mathsf{M}$, a ***vector field along*** γ is a map that assigns to each $t \in J$ an element of $\mathsf{T}_{\gamma(t)}\mathsf{M}$. Let $\xi(t)$ and $\eta(t)$ be vector fields along γ. The ***covariant derivative of ξ along γ*** — denoted by $\nabla_{\gamma'(t)}\xi(t)$ — is linear in ξ, and satisfies (i) $\nabla_{\gamma'(t)}(f(t)\xi(t)) = \dot{f}(t)\xi(t) + f(t)\nabla_{\gamma'(t)}\xi(t)$, and (ii) $\nabla_{\gamma'(t)}\xi(t) = \nabla_{\gamma'(t)}Y(\gamma(t))$, for any vector field Y satisfying $Y \circ \gamma = \xi$ and any $f : \mathbb{R} \to \mathbb{R}$. In particular for a Levi-Civita connection it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle \xi(t), \eta(t)\rangle\!\rangle_{\mathbb{G}(\gamma(t))} = {}& \langle\!\langle \overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\xi(t), \eta(t)\rangle\!\rangle_{\mathbb{G}(\gamma(t))} \\ &+ \langle\!\langle \xi(t), \overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\eta(t)\rangle\!\rangle_{\mathbb{G}(\gamma(t))}.\end{aligned} \tag{2}$$

In components, the covariant derivative of a vector field $Y \in \Gamma^\infty(\mathsf{TM})$ along the curve γ is

$$(\nabla_{\gamma'(t)}Y(t))^k = \dot{Y}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{x}^i(t)Y^j(t), \tag{3}$$

where $t \mapsto (x^1(t), \dots, x^n(t))$ is the local representation of γ. The ***Lie derivative*** of a metric $\mathbb{G}$ along a vector field X is defined by $\mathscr{L}_X\mathbb{G} = \mathbb{G}^\flat(\overset{\mathbb{G}}{\nabla}X)$, where $\overset{\mathbb{G}}{\nabla}X$ is the unique $(1,1)$-tensor field defined by $\overset{\mathbb{G}}{\nabla}X(Y) = \overset{\mathbb{G}}{\nabla}_Y X$ for each $Y \in \Gamma^\infty(\mathsf{TM})$. A vector field X is said to be a ***Killing*** vector field if $\langle\langle v_x, v_x\rangle\rangle_{\mathscr{L}_X\mathbb{G}} = 0$ for each $v_x \in \mathsf{TM}$. The ***arclength*** of a piecewise smooth curve $\gamma : J \to \mathsf{M}$ is defined to be

$$\ell_{\mathbb{G}}(\gamma) \triangleq \int_J \sqrt{\langle\langle \gamma'(t), \gamma'(t)\rangle\rangle_{\mathbb{G}(\gamma(t))}}\,\mathrm{d}t\,.$$

For each pair of points $x_1, x_2 \in \mathsf{M}$, we define the path space $\Omega(x_1,x_2) \triangleq \{\gamma : [0,1] \to \mathsf{M}\,|\,\gamma$ is piecewise smooth and $\gamma(0) = x_1, \gamma(1) = x_2\}$. The ***Riemannian distance*** between x_1 and x_2 is then defined as $\mathrm{d}_{\mathbb{G}}(x_1,x_2) \triangleq \inf\{\ell_{\mathbb{G}}(\gamma) : \gamma \in \Omega(x_1,x_2)\}$. It follows that $(\mathsf{M}, \mathrm{d}_{\mathbb{G}})$ is a metric space [37]. For a fixed $\bar{x} \in \mathsf{M}$ and $r > 0$, we define the r-ball $\mathcal{B}_r(\bar{x})$ by $\mathcal{B}_r(\bar{x}) \triangleq \{x \in \mathsf{M}\,|\,\mathrm{d}_{\mathbb{G}}(x,\bar{x}) \le r\}$. We denote by $\omega : \mathsf{T}^*\mathsf{M} \times \cdots \times \mathsf{T}^*\mathsf{M} \to \mathbb{R}$ the canonical volume form associated with $(\mathsf{M},\mathbb{G})$. The ***volume*** of a set $N \subset \mathsf{M}$ is given by $\mathrm{Vol}(N) \triangleq \int_N \omega$. Moreover, for $X \in \Gamma^\infty(\mathsf{TM})$ it holds that $\mathcal{L}_X\omega = \mathrm{div}(X)\omega$. In components, $\mathrm{div}X = \partial X^i/\partial x^i + \Gamma^i_{ij}X^j$.

2. Theory of Contracting Vector Fields

We now present the theory of contracting vector fields in a coordinate-free setting. We begin our presentation with the key definition.

Definition 2.1. (Contracting System). *Let* M *be a manifold. A* contracting system *is a quadruple* $(\mathcal{U}, X, \mathbb{G}, \lambda)$ *where*

- $\mathcal{U} \subset \mathsf{M}$ *is a connected set, called the contraction region,*
- $X \in \Gamma^\infty(\mathsf{TM})$ *is a vector field,*
- $\mathbb{G}$ *is a Riemannian metric, called the contraction metric, and*
- $\lambda > 0$ *is the contraction rate.*

Moreover, for each $x \in \mathcal{U}$ *and each* $v_x \in \mathsf{T}_x\mathsf{M}$*,* X *satisfies*

$$\langle\langle \overset{\mathbb{G}}{\nabla}_{v_x} X, v_x\rangle\rangle_{\mathbb{G}(x)} \le -\lambda \|v_x\|^2_{\mathbb{G}(x)}\,. \tag{4}$$

The vector field X *is then said to be infinitesimally contracting on* $\mathcal{U}$ *with respect to the metric* $\mathbb{G}$.

Condition (4) can be equivalently rewritten as

$$\langle\langle v_x, v_x\rangle\rangle_{\mathscr{L}_X\mathbb{G}(x)} \le -2\lambda\|v_x\|^2_{\mathbb{G}(x)}\,.$$

This formulation gives the intuition that lengths measured using $\mathbb{G}$ "shrink" as one travels along the integral curves of X. As we will see this intuition is essentially correct, but we will find the form presented in (4) more relevant. We now require one small definition regarding the contraction region $\mathcal{U}$.

Definition 2.2. (K-Reachable Set). *Let* $(\mathsf{M},\mathbb{G})$ *be a Riemannian manifold. For* $K \ge 1$*, a set* $\mathcal{U} \subseteq \mathsf{M}$ *is* K*-reachable if, for any two points* $x_0, x_1 \in \mathcal{U}$*, there exists a continuously differentiable curve* $\gamma : [0,1] \to \mathcal{U}$ *such that* $\gamma(0) = x_0$*,* $\gamma(1) = x_1$*, and* $\ell_{\mathbb{G}}(\gamma) \le K\mathrm{d}_{\mathbb{G}}(x_0,x_1)$*.*

This particular geometry serves to weaken the condition that the contraction region must be geodesically convex (see [38, Theorem 4.1]), in exchange for a potentially weaker estimate on the convergence. In particular, for the case of Euclidean space, Definition 2.2 relaxes the assumption that the contraction region must be convex, c.f. [7]. We now present the main stability result for contracting systems.

Theorem 2.3. (Contraction Theorem). *Let* M *be a manifold, and let* $X \in \Gamma^\infty(\mathsf{TM})$ *be a vector field on* M*. Denote by* $\Phi_t(x)$ *the flow of* X*. Suppose there exists a Riemannian metric* $\mathbb{G}$*, constants* $\lambda, K > 0$*, and a set* $\mathcal{U} \subseteq \mathsf{M}$ *such that*

(i) $(\mathcal{U}, X, \mathbb{G}, \lambda)$ *is a contracting system,*

(ii) $\mathcal{U}$ *is a* K*-reachable, forward* X*-invariant set, and*

(iii) X *is forward complete on* $\mathcal{U}$*.*

Then for each pair of points $x_0, x_1 \in \mathcal{U}$*, it holds for each* $t \ge 0$ *that*

$$\mathrm{d}_{\mathbb{G}}(\Phi_t(x_0), \Phi_t(x_1)) \le Ke^{-\lambda t}\mathrm{d}_{\mathbb{G}}(x_0,x_1)\,. \tag{5}$$

Proof. Let $\gamma : [0,1] \to \mathcal{U}$ be a continuously differentiable curve joining the points x_0, $x_1 \in \mathcal{U}$ as in Definition 2.2, with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. For each $t \ge 0$, let $L(t)$ be the length of the curve $s \mapsto \Phi_t(\gamma(s))$, for $s \in [0,1]$, given by

$$L(t) = \int_0^1 \sqrt{\left\|\frac{\mathrm{d}}{\mathrm{d}s}\Phi_t(\gamma(s))\right\|^2_{\mathbb{G}(\Phi_t(\gamma(s)))}}\,\mathrm{d}s\,. \tag{6}$$

From now on we suppress the point at the inner product is being taken. Since X is forward complete, this is well defined for each $t > 0$. Differentiating (6) with respect to time gives the length dynamics

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} = \int_0^1 \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\frac{\mathrm{d}}{\mathrm{d}s}\Phi_t(\gamma(s))\|^2_{\mathbb{G}}\right)}{2\sqrt{\|\frac{\mathrm{d}}{\mathrm{d}s}\Phi_t(\gamma(s))\|^2_{\mathbb{G}}}}\,\mathrm{d}s\,. \tag{7}$$

Define $S(s,t) \triangleq \frac{\mathrm{d}}{\mathrm{d}s}\Phi_t(\gamma(s))$, and note that for $s \in [0,1]$, the map $t \mapsto S(s,t) \in \mathsf{TM}$ is a vector field along the curve $t \mapsto \Phi_t(\gamma(s))$. Similarly, define $T(s,t) \triangleq \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(\gamma(s))$, the velocity field of the curve $t \mapsto \Phi_t(\gamma(s))$. These constructions are depicted in Figure 1. Since $\overset{\mathbb{G}}{\nabla}$ is a Levi-Civita connection, the numerator of the integrand in (7) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S(s,t)\|^2_{\mathbb{G}} = 2\langle\langle \overset{\mathbb{G}}{\nabla}_{T(s,t)}S(s,t), S(s,t)\rangle\rangle_{\mathbb{G}}\,.$$

Again by the properties of the Levi-Civita connection,

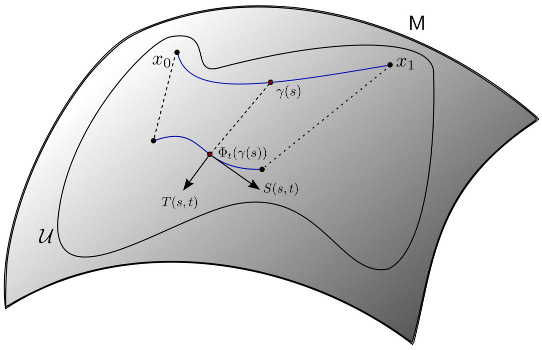

Figure 1: Construction for proof of Theorem 2.3. The Riemannian length of the curve $\gamma(s)$ connecting two arbitrary initial conditions is shown to shrink under the flow.

Lemma 1 implies that $\overset{\mathbb{G}}{\nabla}_T S = \overset{\mathbb{G}}{\nabla}_S T$. We therefore have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|S(s,t)\|^2_{\mathbb{G}} &= 2\langle\!\langle \overset{\mathbb{G}}{\nabla}_{T(s,t)} S(s,t), S(s,t)\rangle\!\rangle_{\mathbb{G}} \\ &= 2\langle\!\langle \overset{\mathbb{G}}{\nabla}_{S(s,t)} T(s,t), S(s,t)\rangle\!\rangle_{\mathbb{G}} \\ &= 2\langle\!\langle \overset{\mathbb{G}}{\nabla}_{S(s,t)} X(\Phi_t(\gamma(s))), S(s,t)\rangle\!\rangle_{\mathbb{G}} \\ &\leq -2\lambda \langle\!\langle S(s,t), S(s,t)\rangle\!\rangle_{\mathbb{G}}\,.\end{aligned} \tag{8}$$

The first equality above follows from (2), the second from Lemma 1 of Appendix A, and the third from the fact that $T(s,t)$ is the maximal integral curve of X through the point $\gamma(s)$. The final inequality uses the fact that the system is contracting and that $\mathcal{U}$ is forward X-invariant. Substituting the result of (8) back into (7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) \leq -\lambda L(t)\,, \quad \forall t \geq 0\,,$$

and we therefore conclude via the Bellman-Gronwall lemma that $L(t) \leq L(0)e^{-\lambda t}$, for each $t \geq 0$. Since by definition $\mathrm{d}_{\mathbb{G}}(\Phi_t(x_0), \Phi_t(x_1)) \leq L(t)$, and $L(0) \leq K\mathrm{d}_{\mathbb{G}}(x_0, x_1)$, we arrive at (5) and the proof is complete. □

We note that the *tensor* nature of the contraction condition (4) along with Theorem 2.3 imply that the stability properties of contracting systems are *coordinate-invariant.*

Proposition 2.4. (Coordinate Description of Contraction). *The following two statements are equivalent:*

(i) The contraction condition (4) holds;

(ii) For every $x \in \mathcal{U}$ and in every set of admissible coordinates $(x^1, \ldots, x^n)$ in a neighborhood $\mathcal{V}$ of x, it holds that[1]

$$\left[\mathbb{G}_{ki}\frac{\partial X^k}{\partial x^\ell} + \frac{\partial X^k}{\partial x^i}\mathbb{G}_{k\ell} + \frac{\partial \mathbb{G}_{i\ell}}{\partial x^j}X^j\right] \preceq -2\lambda[\mathbb{G}_{i\ell}]. \tag{9}$$

Proof. Choosing a set of coordinates $(x^1, \ldots, x^n)$ and using the component expression (3) for the covariant derivative, the contraction condition of Definition 2.1 becomes

$$\left(\mathbb{G}_{k\ell}\frac{\partial X^k}{\partial x^i} + \mathbb{G}_{k\ell}\Gamma^k_{ij}X^j\right)v^i v^\ell \leq -\lambda \mathbb{G}_{i\ell}v^i v^\ell, \tag{10}$$

where Γ^k_{ij}, are the Christoffel symbols for the Levi-Civita connection. Due to the interchange symmetry between i and ℓ on the left side of (10), only the symmetric part of the $(0,2)$-tensor in parentheses contributes to the overall result. Symmetrizing the tensor in parenthesis in (10) and using the Christoffel symbols (1) gives the coordinate formula (9). The converse statement holds since admissible local coordinates satisfy a smooth overlap condition [33, Chapter 3]. □

Inequality (9) is the *generalized Demidovich condition* for contraction [1, 3]. In the special case of $\mathsf{M} = \mathbb{R}^n$ with Euclidean metric $\mathbb{G} = \mathbb{G}_{\mathbb{R}^2}$, inequality (9) reduces further to the classical stability condition of Krasovskiĭ that the symmetric part of the Jacobian matrix be negative definite [39]. The following proposition collects some fascinating ancillary facts about contracting systems. Some of these facts have been noted in the literature, but have lacked formal proofs.

Proposition 2.5. (Properties of Contracting Systems). *Let $(\mathcal{U}, X, \mathbb{G}, \lambda)$ be a contracting system. The following statements hold:*

(i) **Existence of Stable Fixed Point:** *if $(\mathcal{U}, \mathrm{d}_{\mathbb{G}})$ is a complete metric space, X is forward complete, and $\mathcal{U}$ is a forward X-invariant K-reachable set, then X has a unique fixed point $\bar{x} \in \mathcal{U}$, and for each $x \in \mathcal{U}$ it holds that $\Phi_t(x) \to \bar{x}$ exponentially fast as $t \to +\infty$;*

(ii) **Krasovskiĭ's Method:** *if (i) holds, then $V(x) \triangleq \|X(x)\|^2_{\mathbb{G}(x)}$ serves as a strict local Lyapunov function for the unique fixed point $\bar{x} \in \mathcal{U}$;*

(iii) **Incremental Lyapunov Function:** *if (i) holds, then the Riemannian distance $x \mapsto \mathrm{d}_{\mathbb{G}}(x, \bar{x})$ serves locally as a strict Lyapunov function for $\bar{x}$. Moreover, for any $r > 0$ such that $\mathcal{B}_r(\bar{x}) \subset \mathcal{U}$, the system $(\mathcal{B}_r(\bar{x}), X, \mathbb{G}, \lambda)$ is contracting and the ball $\mathcal{B}_r(\bar{x})$ is forward X-invariant and 1-reachable;*

(iv) **Contraction of Volume:** *for any $r > 0$ and $x \in \mathcal{U}$ such that $\mathcal{B}_r(x) \subset \mathcal{U}$, the volume $\mathrm{Vol}(\Phi_t(\mathcal{B}_r(x))) \to 0$ exponentially fast as $t \to +\infty$.*

Note that by Proposition 2.5 (i), one can infer the existence of a unique, exponentially stable fixed point for contracting systems without knowing *a priori* where the fixed point actually is. This fixed point is unique in the contraction region, but is not necessarily the only fixed point on all of M. Indeed, the contraction region is *contractible*, in the sense that after a reparameterization of

[1] For positive definite symmetric (0,2) tensors A and B, we write $A \preceq B$ if $B - A$ is a positive definite symmetric (0,2) tensor.

time, the identity map $x \mapsto x$ is *homotopic* to the constant map $x \mapsto \bar{x}$. It follows that there are no globally contracting vector fields on compact manifolds [40].

Proof. **(i):** Let $\tau > 0$ be such that $Ke^{-\lambda\tau} < 1$, and let $x \in \mathcal{U}$. Examining (5), we see that $x \mapsto \Phi_\tau(x)$ is a contraction mapping on $\mathcal{U}$. Applying the Banach Fixed Point Theorem in the complete space $(\mathcal{U}, \mathrm{d}_{\mathbb{G}})$, we conclude the existence of a unique fixed point $\bar{x} \in \mathcal{U}$ for the map $x \mapsto \Phi_\tau(x)$, and hence for the vector field X itself. Exponential stability of $\bar{x}$ follows from (5). **(ii):** Since $\bar{x}$ is a fixed point, $V(\bar{x}) = 0$, and due to the uniqueness of $\bar{x}$, $V(x)$ is strictly positive for each $x \in \mathcal{U}$ with $x \neq \bar{x}$. Taking the Lie derivative of $V(x)$ along X we obtain that $\mathcal{L}_X V(x) = \mathcal{L}_X \|X(x)\|^2_{\mathbb{G}(x)} \leq -2\lambda \|X(x)\|^2_{\mathbb{G}(x)} = -2\lambda V(x)$, where we have applied the contraction condition (2.1) with $v_x = X(x)$. Hence $V(x)$ is a strict local Lyapunov function for $\bar{x}$. **(iii):** The first claim follows, as the Riemannian distance is positive definite about $\bar{x}$, and from (5), decreases exponentially along trajectories of X. The second claim follows from the forward X-invariance of the Lyapunov sublevel sets. The "geodesic sphere" $\mathcal{B}_r(\bar{x})$ is strongly convex, and therefore 1-reachable [10]. **(iv):** For $x \in \mathcal{U}$ and $\tau > 0$, we compute that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(\Phi_t(\mathcal{B}_r(x)))\bigg|_{t=\tau} &= \int_{\Phi_\tau(\mathcal{B}_r(x))} \mathcal{L}_X \omega \\
&= \int_{\Phi_\tau(\mathcal{B}_r(x))} \mathrm{div}X\, \omega .
\end{aligned}
$$

A coordinate calculation using (10) and [41, Lemma 8.4.12] quickly shows that $\mathrm{div}X \leq -n\lambda$. The claim then follows by the Bellman-Gronwall lemma. □

Example 1. (Damped Oscillator). *For constants k,m,b > 0 consider the vector field on $\mathsf{M} = \mathbb{R}^2$ given by $X = y\frac{\partial}{\partial x} - (\frac{k}{m}x + \frac{b}{m}y)\frac{\partial}{\partial y}$. Defining $\zeta \triangleq b/(2\sqrt{km})$, for $\varepsilon \in]0, 1/(1+\zeta^2)[$ one can verify that this vector field is infinitesimally contracting on all of M with respect to the positive definite metric $\mathbb{G} = \frac{1}{2}k\,\mathrm{d}x \otimes \mathrm{d}x + b\varepsilon\,\mathrm{d}x \otimes \mathrm{d}y + \frac{1}{2}m\,\mathrm{d}y \otimes \mathrm{d}y$. The origin is a unique and globally exponentially stable fixed point, and Proposition 2.5 (iii) implies that $V(x, y) = \frac{1}{2}kx^2 + \frac{1}{2}my^2 + \varepsilon bxy$ serves as a strict Lyapunov function (the alternative Lyapunov function arising from 2.5 (ii) is somewhat messy).*

Remark 1. (Killing Vector Fields). *It is illuminating to view the results of both Theorem 2.3 and Proposition 2.5 in the limiting case of $\lambda \to 0^+$ — the case of a* Killing vector field. *For clarity, consider again Example 1 for the case of $b = 0$. One can then verify that this field is Killing with respect to the metric $\mathbb{G} = \frac{1}{2}k\,\mathrm{d}x \otimes \mathrm{d}x + \frac{1}{2}m\,\mathrm{d}y \otimes \mathrm{d}y$. The integral curves of X are ellipses around the origin, and thus trajectories of X are Lyapunov stable. Proposition 2.5 (ii) and (iii) then both imply that $V(x, y) = \frac{1}{2}kx^2 + \frac{1}{2}my^2$ serves as a Lyapunov function for the unique fixed point at the origin. Forward X-invariance of the Lyapunov sublevel sets is replaced by invariance of the corresponding level sets, and Proposition 2.5 (iv) becomes the volume preservation statement of Louville's Theorem for solenoidal vector fields [42]. Finally, note that if $(\mathcal{U}, X, \mathbb{G}, \lambda)$ is a contracting system, then so is $(\mathcal{U}, X+Y, \mathbb{G}, \lambda)$ for any Killing field Y of $(\mathsf{M}, \mathbb{G})$.*

3. Modularity Properties of Contracting Systems

In this section we consider the modularity properties of infinitesimally contracting vector fields. In particular we study cascade and feedback interconnections of contracting systems, and give conditions under which the system resulting from the interconnection is also contracting. Let M be a manifold, and consider for $k \geq 1$ the vector field with inputs $X : \mathsf{M} \times \mathbb{R}^k \to \mathsf{TM}$. That is, for each fixed $u \in \mathbb{R}^k$, $x \mapsto X(x, u)$ is a vector field on M. For a vector field with inputs X, $(\mathcal{U}, X, \mathbb{G}, \lambda, \mathbb{R}^k)$ is said to be a *contracting system with inputs* if $(\mathcal{U}, X(\cdot, u), \mathbb{G}, \lambda)$ is a contracting system for every $u \in \mathbb{R}^k$; that is, when the input is taken as a constant parameter.

Lemma 3.1. (Cascade Interconnection of Contracting Systems). *Consider the cascade interconnection of a contracting system $(\mathcal{U}_1 \subseteq \mathsf{M}_1, X_1, \mathbb{G}_1, \lambda_1)$ with smooth output map $h_1 : \mathsf{M}_1 \to \mathbb{R}^k$ and a contracting system with inputs $(\mathcal{U}_2 \subseteq \mathsf{M}_2, X_2, \mathbb{G}_2, \lambda_2, \mathbb{R}^k)$. Assume $\mathcal{U}_1$ and $\mathcal{U}_2$ are compact. Then there exists a metric $\mathbb{G}$ on $\mathsf{M}_1 \times \mathsf{M}_2$ such that the closed loop system is contracting on $\mathcal{U}_1 \times \mathcal{U}_2$ with respect to $\mathbb{G}$.*

We omit the proof of Lemma 3.1, as it is similar to the proof of Lemma 3.2 which follows. We note however that a proof of Lemma 3.1 on $\mathbb{R}^n$ using the alternative notion of a matrix measure can be found in [7]. While the previous result shows that cascades of contracting systems are always contracting in an appropriate metric, the following lemma gives sufficient conditions for the existence of a contraction metric. To our knowledge, the following small-gain type result does not appear in the contraction literature.

Lemma 3.2. (Feedback Interconnection of Contracting Systems). *Consider the feedback interconnection of two contracting systems with inputs $(\mathcal{U}_i \subseteq \mathsf{M}_i, X_i, \mathbb{G}_i, \lambda_i, \mathbb{R}^{k_i})$ and associated output maps $h_i : \mathsf{M}_i \to \mathbb{R}^{k_{(3-i)}}$, $i \in \{1, 2\}$. Assume $\mathcal{U}_1$ and $\mathcal{U}_2$ are compact, and define the induced input-output gains $\gamma_1, \gamma_2 > 0$ by*[2]

$$
\begin{aligned}
\gamma_1 &\triangleq \max_{x_1 \in \mathcal{U}_1} \max_{w_{x_2} \in \mathsf{T}\mathcal{U}_2} \frac{1}{2} \frac{\left\| \frac{\partial X_1}{\partial u_1} \circ (Th_2) w_{x_2} \right\|_{\mathbb{G}_1}}{\|w_{x_2}\|_{\mathbb{G}_2}} < \infty \,, \\
\gamma_2 &\triangleq \max_{x_2 \in \mathcal{U}_2} \max_{v_{x_1} \in \mathsf{T}\mathcal{U}_1} \frac{1}{2} \frac{\left\| \frac{\partial X_2}{\partial u_2} \circ (Th_1) v_{x_1} \right\|_{\mathbb{G}_2}}{\|v_{x_1}\|_{\mathbb{G}_1}} < \infty \,.
\end{aligned}
\tag{11}
$$

[2] Here, the mappings $\frac{\partial X_i}{\partial u_i} \circ (Th_j) : \mathsf{TM}_j \to \mathsf{TM}_i$ can be thought of as the input/output gain operators which determine how system j couples to system i.

If

$$\gamma_1\gamma_2 < \lambda_1\lambda_2\,, \tag{12}$$

then there exists a metric $\mathbb{G}$ on $\mathsf{M}_1 \times \mathsf{M}_2$ such that the closed loop system is contracting on $\mathcal{U}_1 \times \mathcal{U}_2$ with respect to $\mathbb{G}$.

Proof. The system evolves on the smooth product manifold $\mathsf{M} = \mathsf{M}_1 \times \mathsf{M}_2$ with tangent bundle $\mathsf{TM} = \mathsf{TM}_1 \times \mathsf{TM}_2$. For points $v_x = (v_{x_1}, w_{x_2}), v'_x = (v'_{x_1}, w'_{x_2}) \in \mathsf{TM}$ define the metric $\mathbb{G} : \mathsf{TM} \times \mathsf{TM} \to \mathbb{R}$ on M by

$$\langle\!\langle v_x, v'_x\rangle\!\rangle_{\mathbb{G}} = \alpha_1 \langle\!\langle v_{x_1}, v'_{x_1}\rangle\!\rangle_{\mathbb{G}_1} + \alpha_2 \langle\!\langle w_{x_2}, w'_{x_2}\rangle\!\rangle_{\mathbb{G}_2}\,,$$

where $\alpha_1, \alpha_2 > 0$. That is, the combined metric is "block diagonal". Denoting by X the total vector field on M, for $x \in \mathcal{U}_1 \times \mathcal{U}_2$ and $v_x = (v_{x_1}, w_{x_2}) \in \mathsf{TM}$ we compute that[3]

$$\begin{aligned}\langle\!\langle \overset{\mathbb{G}}{\nabla}_{v_x} X, v_x\rangle\!\rangle_{\mathbb{G}} &= \alpha_1 \langle\!\langle \overset{\mathbb{G}_1}{\nabla}_{v_{x_1}} X_1, v_{x_1}\rangle\!\rangle_{\mathbb{G}_1} + \alpha_2 \langle\!\langle \overset{\mathbb{G}_2}{\nabla}_{w_{x_2}} X_2, w_{x_2}\rangle\!\rangle_{\mathbb{G}_2} \\ &\quad + 2\alpha_1 \langle\!\langle v_{x_1}, \frac{\partial X_1}{\partial u_1} \circ (Th_2) w_{x_2}\rangle\!\rangle_{\mathbb{G}_1} \\ &\quad + 2\alpha_2 \langle\!\langle \frac{\partial X_2}{\partial u_2} \circ (Th_1) v_{x_1}, w_{x_2}\rangle\!\rangle_{\mathbb{G}_2} \\ &\le -\alpha_1\lambda_1 \|v_{x_1}\|^2_{\mathbb{G}_1} - \alpha_2\lambda_2 \|w_{x_2}\|^2_{\mathbb{G}_2} \\ &\quad + (\alpha_1\gamma_1 + \alpha_2\gamma_2) \|v_{x_1}\|_{\mathbb{G}_1} \|w_{x_2}\|_{\mathbb{G}_2}\,.\end{aligned}$$

This quantity is strictly negative for all non-zero $v_x \in \mathsf{TM}$ if and only if the matrix

$$M \triangleq \begin{bmatrix} \alpha_1\lambda_1 & -\frac{1}{2}(\alpha_1\gamma_1 + \alpha_2\gamma_2) \\ -\frac{1}{2}(\alpha_1\gamma_1 + \alpha_2\gamma_2) & \alpha_2\lambda_2 \end{bmatrix}$$

is positive definite. Note that $\alpha_1\lambda_1 > 0$, and define $z \triangleq \alpha_1/\alpha_2 > 0$. Then M is positive definite if and only if

$$\lambda_1\lambda_2 - \gamma_1\gamma_2/2 > (z\gamma_1^2 + \gamma_2^2/z)/4\,. \tag{13}$$

The right hand side of (13) is a convex function of $z > 0$ with a unique minimum value of $\gamma_1\gamma_2/2$ at $z^* = \gamma_2/\gamma_1$. Selecting $\alpha_1/\alpha_2 = z^*$ reduces (13) to (12), and hence M is positive definite. Therefore, there exists a $\lambda > 0$ such that $M > \lambda\,\mathrm{diag}(\alpha_1, \alpha_2)$, and it therefore holds that

$$\begin{aligned}\langle\!\langle \overset{\mathbb{G}}{\nabla}_{v_x} X, v_x\rangle\!\rangle_{\mathbb{G}} &\le -\lambda \begin{bmatrix} \|v_{x_1}\|_{\mathbb{G}_1} \\ \|w_{x_2}\|_{\mathbb{G}_2} \end{bmatrix}^T \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \|v_{x_1}\|_{\mathbb{G}_1} \\ \|w_{x_2}\|_{\mathbb{G}_2} \end{bmatrix} \\ &= -\lambda \|v_x\|^2_{\mathbb{G}}\,,\end{aligned}$$

which completes the proof. □

4. Connections to Contraction Theory in Mathematics and Physics

Within the differential geometry community, contracting-type vector fields have been studied under the title of ***monotone*** vector fields [43]. In [10], a vector field $X \in \Gamma^\infty(\mathsf{TM})$ is said to be strongly monotone on $\mathcal{U} \subset \mathsf{M}$ if there exists $\lambda > 0$ such that, for any geodesic γ joining two points in $\mathcal{U}$, the function $\varphi(t) \triangleq \langle\!\langle \gamma'(t), X(\gamma(t))\rangle\!\rangle_{\mathbb{G}} - \lambda t \langle\!\langle \gamma'(0), \gamma'(0)\rangle\!\rangle_{\mathbb{G}}$ is monotone increasing. That is, the function of time $\langle\!\langle \gamma'(t), X(\gamma(t))\rangle\!\rangle_{\mathbb{G}}$ must be increasing faster than linearly. We refer the reader to [43, 44, 10, 45] for further details. Since we are more interested in stable systems, we will in fact reverse this definition and say that X is strongly monotone if there exists a $\lambda > 0$ such that $\varphi(t) \triangleq \langle\!\langle \gamma'(t), X(\gamma(t))\rangle\!\rangle_{\mathbb{G}} + \lambda t \langle\!\langle \gamma'(0), \gamma'(0)\rangle\!\rangle_{\mathbb{G}}$ is monotone decreasing. The following result relates monotone and infinitesimally contracting vector fields.

Proposition 4.1. (Monotone and Contracting Vector Fields). *Let $X \in \Gamma^\infty(\mathsf{TM})$ be a smooth vector field on $(\mathsf{M}, \mathbb{G})$. The following two statements are equivalent:*

(i) X is strongly monotone on $\mathcal{U} \subset \mathsf{M}$ with parameter $\lambda > 0$;

(ii) $(\mathcal{U}, X, \mathbb{G}, \lambda)$ is a contracting system with $\mathcal{U}$ geodesically convex.

Proof. By definition, the function φ is monotone decreasing if

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) &= \langle\!\langle \gamma''(t), X(\gamma(t))\rangle\!\rangle_{\mathbb{G}} \\ &\quad + \langle\!\langle \gamma'(t), \overset{\mathbb{G}}{\nabla}_{\gamma'(t)} X(\gamma(t))\rangle\!\rangle_{\mathbb{G}} + \lambda\|\gamma'(0)\| \le 0\,.\end{aligned}$$

From the definition of a geodesic we have that $\gamma''(t) = 0$, and the remaining inequality therefore holds if and only if it holds that $\langle\!\langle v_x, \overset{\mathbb{G}}{\nabla}_{v_x} X\rangle\!\rangle_{\mathbb{G}(x)} + \lambda\|v_x\|^2_{\mathbb{G}(x)} \le 0$ for each $x \in \mathcal{U}$ and each $v_x \in \mathsf{T}_x\mathsf{M}$, which is exactly the contraction condition (4). □

Consider now the case where the contraction condition of Definition 2.1 is satisfied with equality rather than inequality. Such vector fields have been studied in the general relativity community, and are referred to as ***proper homothetic*** vector fields. The perspective of study is somewhat different, in that in this context one studies a pseudo-Riemannian manifold and asks what classes of continuous vector fields can exist. The vector space $H(\mathsf{M})$ of homothetic vector fields on a Riemannian manifold $(\mathsf{M}, \mathbb{G})$ is of dimension at most $\frac{1}{2}n(n+1)+1$, and is in fact a Lie algebra when endowed with the standard Lie bracket. One can show that the dimension of $H(\mathsf{M})$ is one greater than that of the vector space of Killing fields $K(\mathsf{M})$, and therefore that there is *exactly one* linearly independent proper homothetic field on $(\mathsf{M}, \mathbb{G})$, with all others being constructed as a linear combination of this field and an appropriate Killing field (c.f. Remark 1). Homothetic vector fields enjoy the property that geodesics of the manifold are mapped to geodesics by the flow, and that every fixed point is necessarily isolated. In addition, a quick calculation shows that $\mathscr{L}_X S \ge 2\lambda S$, where S is the scalar

[3]Here, the mappings $\frac{\partial X_i}{\partial u_i} \circ (Th_j) : \mathsf{TM}_j \to \mathsf{TM}_i$ can be thought of as the input/output gain operators which determine how system j couples to system i.

curvature of the Riemannian manifold. That is, the scalar curvature grows exponentially as one approaches a fixed point, and hence the nomenclature "singularities" in reference to fixed points of vector fields. We refer the reader to [46] for additional results.

5. Gradient Systems

We now examine how the preceding results play out for a class of example systems. A *gradient system* on $(\mathsf{M}, \mathbb{G})$ is a vector field of the form

$$X(x) = -\mathrm{grad}\psi(x) \triangleq -\mathbb{G}^{\sharp}(\mathrm{d}\psi(x)),$$

where $\psi \in C^\infty(\mathsf{M})$. Note that a gradient system is defined *only* in the presence of a Riemannian metric. In local coordinates, the components of the gradient vector field are $X^i = -\mathbb{G}^{ij}(\partial\psi/\partial x^j)$ for $i \in \{1, \dots, n\}$, which reduces as expected in Euclidean space ($\mathbb{G} = \mathbb{G}_{\mathbb{R}^n}$) to the standard "$\dot{z} = -\nabla\psi(z)$". The Hessian of ψ is the symmetric $(0,2)$-tensor field on M defined for for $v_x, w_x \in \mathsf{T}_x\mathsf{M}$ by $\mathrm{Hess}\,\psi(x) \cdot (v_x, w_x) \triangleq \langle\langle v_x, \nabla_{w_x}\mathrm{grad}\psi\rangle\rangle_{\mathbb{G}(x)}$. In coordinates, the Hessian has components

$$(\mathrm{Hess}\,\psi)_{ij} = \frac{\partial^2\psi}{\partial x^i \partial x^j} - \frac{\partial\psi}{\partial x^k}\Gamma^k_{ij}, \quad i, j \in \{1, \dots, n\}. \tag{14}$$

A ***critical zero*** for a function $\psi \in C^\infty(\mathsf{M})$ is a point $\bar{x} \in \mathsf{M}$ such that $\psi(\bar{x}) = 0$ and $\mathrm{d}\psi(\bar{x}) = 0_{\bar{x}}$. This is a point where both the function and its "slope" are zero. From (14), we see that at a critical zero, the Hessian is simply the matrix of mixed partial derivatives. A critical zero is said to be ***nondegenerate*** if, in any coordinate chart, this matrix has full rank. Given $L > 0$, we denote by $\psi^{-1}(\leq L, \bar{x}) \triangleq \{x \in \mathsf{M} \,|\, \psi(x) \leq L\}$ the (connected component of the) ***L-sublevel set of ψ containing $\bar{x}$***. We set

$$\begin{aligned} L_{\mathrm{cpt}}(\psi, x_0) &\triangleq \sup\{L \in \mathbb{R} \,|\, \psi^{-1}(\leq L, \bar{x}) \text{ is compact}\}, \\ L_{\mathrm{reg}}(\psi, x_0) &\triangleq \sup\{L \in \mathbb{R} \,|\, x \in \psi^{-1}(\leq L, x_0) \setminus \{\bar{x}\} \\ &\qquad \implies \mathrm{d}\psi(x) \neq 0\}, \end{aligned}$$

and $L_{\mathrm{cpt,reg}}(\psi, \bar{x}) \triangleq \min\{L_{\mathrm{cpt}}(\psi, \bar{x}), L_{\mathrm{reg}}(\psi, \bar{x})\}$. In other words, $L_{\mathrm{cpt,reg}}(\psi, \bar{x})$ bounds from above the largest value L such that the connected component of the L-sublevel set of ψ is compact and contains no critical point other than $\bar{x}$. The preceding constructions are depicted in Figure 2. In the presence of a nondegenerate critical zero, the application of Theorem 2.3 gives us the following result.

Corollary 5.1. (Contraction for Gradient Systems). *Let $\psi \in C^\infty(\mathsf{M})$, suppose that $\bar{x} \in \mathsf{M}$ is a critical, nondegenerate zero for ψ, and let $L \in [0, L_{\mathrm{cpt,reg}}(\psi, \bar{x})[$. If there exists $\lambda > 0$ and a Riemannian metric $\mathbb{G}$ such that for each $x \in \psi^{-1}(\leq L, \bar{x})$ the Hessian of ψ satisfies*

$$\mathrm{Hess}\,\psi(x) \succeq \lambda\mathbb{G}(x), \tag{15}$$

then Theorem 2.3 holds with $\mathcal{U} = \psi^{-1}(\leq L, \bar{x})$, $K = 1$ and $X = -\mathrm{grad}\psi$.

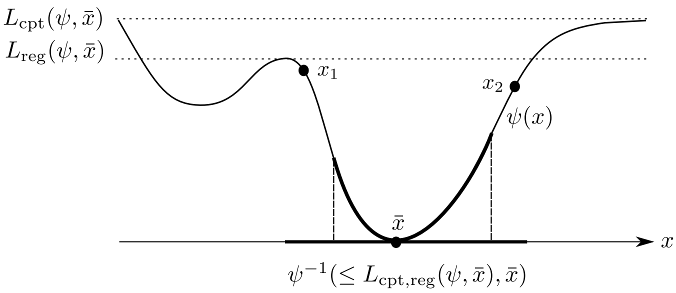

Figure 2: Illustration of sublevel set constructions for Corollary 5.1. The points x_1 and x_2 represent initial conditions which clearly converge to the stable fixed point $\bar{x}$, but do not lie within the contraction region defined the the hatched vertical lines.

Proof. We set $\mathcal{U} \triangleq \psi^{-1}(\leq L, \bar{x})$. With this, $(-\mathrm{grad}\psi, \mathcal{U}, \mathbb{G}, \lambda)$ is by assumption a contracting system, since the inequality (15) is nothing other than the contraction condition of Definition 2.1. The inequality (15) implies ψ is strongly convex on $\mathcal{U}$, and $\mathcal{U}$ is therefore 1-reachable [10, Proposition 3.4]. Forward invariance of $\mathcal{U}$ follows from the orthogonality of the gradient field of ψ and its level sets, and the vector field is therefore forward complete on $\mathcal{U}$. Thus, the criteria of Theorem 2.3 are satisfied, completing the proof. □

We note that Corollary 5.1 demonstrates that contraction is a stronger property than exponential stability of a fixed point. Indeed, as evident from Figure 2, the initial conditions $x_1, x_2 \in \psi^{-1}(\leq L_{\mathrm{cpt,reg}}(\psi, \bar{x}), \bar{x})$ both converge asymptotically to the fixed point, whereas the exponential convergence guaranteed by the contraction condition applies only in the region between the hatched vertical lines, where the curvature of the function ψ is strictly positive.

6. Conclusions

In this work we have examined contracting systems in a coordinate free setting on Riemannian manifolds. In particular, we have rigorously examined the main theory of contraction using intrinsically defined quantities. An interesting question not examined here is the relationship between contraction theory and the dual Lyapunov approach proposed in [47]; see also the Koopman theory presented in [48]. It seems plausible that there should be such a density function approach for incremental stability. A challenging problem, which to our knowledge remains open, is that of finding succinct and meaningful conditions for a simple mechanical system to exhibit contracting behavior.

Appendix A.

The following lemma records a useful property of the vector fields $S(s,t)$ and $T(s,t)$.

Lemma 1. *Let $\widetilde{S}, \widetilde{T} \in \Gamma^\infty(\mathsf{TM})$ be vector fields having the property that $\widetilde{S} \circ \Phi_t(\gamma(s)) = S(s,t)$ and $\widetilde{T} \circ \Phi_t(\gamma(s)) =$*

$T(s,t)$. *Then* $[\widetilde{S}, \widetilde{T}](\Phi_t(\gamma(s))) = 0$ *for* $(s,t) \in [0,1] \times [0, \infty[$.

Proof. The existence of the vector fields $\widetilde{S}$ and $\widetilde{T}$ follows from the smoothness of the vector field X, so the construction is well defined. Let $f \in C^\infty(\mathsf{M})$ so that $f \circ \Phi : [0,1] \times [0, \infty[\to \mathbb{R}$. The definition of the Lie bracket and a coordinate calculation yields

$$\begin{aligned} &\mathscr{L}_{[\widetilde{S},\widetilde{T}]} f \circ \Phi_t(\gamma(s)) \\ &\quad = \mathscr{L}_S \mathscr{L}_T f \circ \Phi_t(\gamma(s)) - \mathscr{L}_T \mathscr{L}_S f \circ \Phi_t(\gamma(s)) \\ &\quad = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t} f \circ \Phi_t(\gamma(s)) - \frac{\mathrm{d}^2}{\mathrm{d}t\mathrm{d}s} f \circ \Phi_t(\gamma(s))) = 0 \end{aligned}$$

by the equality of mixed derivatives. □

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