Distributed Sequential Algorithms for Regional Source Localization ☆ ,☆☆

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Abstract

We study the problem of source localization as a multiple hypothesis testing, where each hypothesis corresponds to the event that the source belongs to a particular region. We use sequential hypothesis tests based on posterior computations to solve for the correct hypothesis. Measurements corrupted with noise are used to calculate conditional posteriors. We prove that the regional localization problem has geometric properties that allow correct detection almost surely in the limit of infinite measurements. We present the *Sense, Transmit & Test* algorithm that allows sequential sensing, communication and testing and we analyze the accuracy of this distributed algorithm and show that the test ends in a finite time. We also present numerical results illustrating properties of the suggested algorithm.

Key words: Localization, Hyphothesis Testing, Distributed Algorithms, Sequential Decision Making

1. Introduction

1.1. Problem description and motivation

Applications where source localization is of great concern, vary between finding the source of oil spills in the ocean, determining cellular locations, detecting an earthquake's epicenter, locating an acoustic source, or simply finding an intruder in a protected environment. For most of these applications, it is sufficient to find a region that contains the source rather than pinpointing the exact source position, which relies most of the time on approximations.

In this work we consider the following problem: A source at an unknown location in a bounded region Q transmits a power signal. N sensors receive noisy and decayed versions of the signal, they can communicate and exchange measurements. The environment Q is divided into M regions W_α, where $\alpha \in \{1, \dots, M\}$. The objective of the sensors is to find which region contains the source.

We pose the problem as a multiple hypothesis testing problem, where hypothesis H_α is true if the source lies in the region W_α. We assume no prior knowledge about the location of the source and therefore model the source location as a uniformly distributed random variable over the environment Q, any prior information about the source location can be incorporated in the location density function. We adopt the log-normal fading model for the propagation of the received signal power. The noise added to the log of the power is Gaussian with zero mean and a known variance σ^2.

1.2. Literature review

In the classical source localization problem, a number of sensors collaborate to locate the exact position of a source. The relation between the position of a source and the received signal strength (RSS) is described in (Rappoport, 1996; Proakis and Salehi, 2001; Sayed et al., 2005; Chen et al., 2002). Several authors treat localization as a nonconvex optimization problem (Hero III and Blatt, 2005; Rabbat and Nowak, 2004a). Gradient descent algorithms and weighted least squares approximations can be used to solve the maximum likelihood estimation problems but such algorithms tend to get stuck at local optimas (Rabbat and Nowak, 2004b; Mao et al., 2007). Meng et al. (2008) approximate the nonlinear nonconvex optimization problem by a linear and convex problem. Hero III and Blatt (2005) use a method of projection onto convex sets. A necessary and sufficient condition for the convergence of this algorithm is that the source lies inside the convex hull of the sensors. Properly placing the sensors assumes knowledge of the position of the source.

Designing distributed algorithms is in general a problem specific task, and many researchers from various communities have looked at this problem. We refer the reader to (Nedic and Ozdaglar, 2009; Lynch, 1997; Boyd et al., 2006) and references therein for more details about this topic.

The multiple hypothesis problems are considerably more difficult than the binary problem and optimality of the proposed algorithms is usually hard to prove. Some tests that have some asymptotic optimality properties were developed in the literature, but these tests tend to be very complex (Savin, 1984; Baum and Veeravalli, 1994; Armitage, 1950). Alternatively ad hoc tests based on repeated pairwise applications of optimal sequential hypothesis tests (Wald, 1945) were developed but these tests have little optimality results, e.g., see Eisenberg (1991).

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1.3. Contributions

The contributions of this paper are three-folds.

First, we formulate the source localization problem in a novel multi-hypothesis testing setting. We analyze properties of the Maximum A Posteriori (MAP) algorithm that requires the computation of a finite number of integrals which is to be compared to the need to solve a non-linear, non-convex problem in the classical source localization problem. We provide a proof of almost sure convergence of the MAP solution asymptotically in the limit of a large number of measurements, a step that tends to be missing in all of the work presented earlier in the source localization literature.

Second, inspired by the proof of convergence of the MAP solution, we propose and implement a distributed sequential regional localization algorithm: *Sense, Transmit & Test*. This algorithm allows for sequential sensing, transmission and testing at each processor. We allow each processor to have one or multiple regions of responsibility and relate the probability of error for each processor in the case of multiple regions to the probability of error in the case of a single region. We also show that the test ends in a finite time under mild conditions on the sensor locations.

Third, we illustrate the results of the *Sense, Transmit & Test* and show how the expected decision time for a network increases with the required accuracy and noise. We also provide numerical results illustrating how it is possible to increase the level of localization accuracy at the expense of the expected decision time for the network for a fixed decision accuracy.

1.4. Paper organization

The paper proceeds as follows: we formulate the problem as a multi-hypotheses testing problem in Section 2. We present a distributed algorithm to solve the problem in Section 3. We present in Section 4 numerical results showing the performance of the algorithm as various parameters are changed. We conclude in Section 5.

2. Source localization as multi-hypothesis testing

We start this section by introducing the model and the problem definition.

2.1. Model and problem definition

Consider a compact connected environment $Q \subset \mathbb{R}^2$. Suppose that there are N sensors placed at positions $q_i \in Q$ with $i \in \{1, \ldots, N\}$, and that the source located at an unknown location $s \in Q$ transmits a signal whose power undergoes lognormal shadowing summarized as follows. The average power loss for an arbitrary Transmitter-Receiver separation is expressed as a function of distance by using a path loss exponent $\rho > 2$. For reasons to be explained shortly, we work with a slight modification of the traditionally used model. The adopted model for the received power at a sensor i is $P_i = \frac{P d_0}{d_0 + \|q_i - s\|^\rho}$, where ρ indicates the rate at which the power loss increases with distance. The nominal distance d_0 is chosen so that the received power in the vicinity of the source is almost equal to the transmitted power P at the source. Note that while this model gets rid of the singularity at the source, it converges to the same behavior as the classical model used in communication literature $P_i = \frac{P}{\|q_i - s\|^\rho}$, when the distance $\|q_i - s\|$ is large. Here P is the power received at a unit distance from the source. The received power becomes

$$\ln P_i = \ln(P d_0) - \ln(d_0 + \|q_i - s\|^\rho) + n_i, \qquad (1)$$

where n_i is the noise associated with sensor i, and all n_i are independent and identically distributed (i.i.d) Gaussian random variables with zero mean and known variance σ^2. The joint probability density function of the received power $P_r = [P_1, \ldots, P_N]^T$, conditioned on the source location $y \in Q$ is

$$\mathbb{p}(P_1, \ldots, P_N | y) = \frac{1}{(2\pi\sigma^2)^{N/2}} \times \exp\Big(-\frac{\sum_{i=1}^N \Big(\ln P_i - \ln(\frac{P d_0}{d_0 + \|q_i - y\|^\rho})\Big)^2}{2\sigma^2}\Big). \qquad (2)$$

Problem 2.1 (MAP point localization problem) *Compute the position that maximizes the conditional density of the joint observations, that is compute*

$$y^* = \operatorname*{argmax}_{y \in Q} \mathbb{p}(P_1, \ldots, P_N | y)\mathbb{P}(y).$$

Problem 2.1 is a nonlinear nonconvex optimization problem. Attempts to solve this problem, usually revert to relaxing the problem or approximating its solution without providing a convergence analysis. In this paper we look for a regional localization, so the conditioning on the exact position y in (2) is replaced by a conditioning on the source being in a region W_i. The environment Q with area A is divided into M regions $\{W_1, \ldots, W_M\}$ with positive areas $\{A_1, \ldots, A_M\}$. The hypothesis H_α is true if and only if $s \in W_\alpha$.

Problem 2.2 (MAP regional localization problem) *Compute the hypothesis H_α that maximizes the posterior of the joint observations, that is, compute*

$$\alpha^* = \operatorname*{argmax}_{\alpha \in \{1, \ldots, M\}} \mathbb{p}(P_1, \ldots, P_N | H_\alpha)\mathbb{P}(H_\alpha). \qquad (3)$$

2.2. Regional posterior density

Assuming no prior knowledge about the location of the source, the density describing $s \in Q$ is

$$\mathbb{p}(s) = \begin{cases} 1/A, & \text{if } s \in Q, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3 (Repeated measurements) *The ith sensor takes k repeated i.i.d. noisy measurements and computes the average of the logarithms of the measurements*

$$\ln \mathbf{P}_i(k) = \sum_{t=1}^{k} \frac{\ln P_i(l)}{k}. \qquad (4)$$

In the infinite measurement case, we write

$$\ln \mathbf{P}_i = \lim_{k\to\infty} \sum_{t=1}^{k} \frac{\ln P_i(t)}{k}.$$

and the variance $\lim_{k\to\infty} \sigma^2(k) = 0$.

Proposition 2.4 (Expressions for posteriors) *In the case of k repeated measurements, the regional posterior for sensor i about region* W_α *is*

$$\mathbb{p}(\mathbf{P}_i(k)|H_\alpha)\mathbb{P}(H_\alpha) = \frac{1}{A}\int_{W_\alpha} \frac{1}{(2\pi\sigma^2(k))^{1/2}} \times \exp\Big(- \frac{\Big(\ln \mathbf{P}_i(k) - \ln(\frac{Pd_0}{d_0+||q_i-y||^\rho})\Big)^2}{2\sigma^2(k)}\Big)dy,$$

and the joint regional posterior for sensors $\{1,\dots,N\}$ *about region* W_α *is*

$$\mathbb{p}(\mathbf{P}_1(k),\dots,\mathbf{P}_N(k)|H_\alpha)\mathbb{P}(H_\alpha) = \frac{1}{A}\int_{W_\alpha} dy \prod_{l=1}^{N} \frac{1}{(2\pi\sigma^2(k))^{1/2}} \exp\Big(-\frac{\Big(\ln \mathbf{P}_l(k) - \ln(\frac{Pd_0}{d_0+||q_l-y||^\rho})\Big)^2}{2\sigma^2(k)}\Big).$$

Proof: Call $z = \ln \mathbf{P}_i(k)$. We compute

$$\mathbb{p}(z|H_\alpha) = \frac{d}{dz}\frac{\mathbb{P}(Z\le z, H_\alpha)}{\mathbb{P}(H_\alpha)} = A\frac{d}{dz}\frac{\int_{-\infty}^{z}\int_{W_\alpha}\mathbb{p}(z|y)p(y)dydz}{A_\alpha}$$
$$= A\frac{d}{dz}\frac{\int_{-\infty}^{z}\int_{W_\alpha}\big(\mathbb{p}(z|y)/A\big)dydz}{A_\alpha} = \frac{\int_{W_\alpha}\mathbb{p}(z|y)dy}{A_\alpha}.$$

Since $z = \ln \mathbf{P}_i(k) = \sum_{t=1}^{k}\frac{\ln P_i(t)}{k}$, the conditional probability is

$$\mathbb{p}(z|y) = \frac{1}{(2\pi\sigma^2(k))^{1/2}} \times \exp\Big(- \frac{\Big(\ln \mathbf{P}_i(k) - \ln(\frac{Pd_0}{d_0+||q_i-y||^\rho})\Big)^2}{2\sigma^2(k)}\Big)dy.$$

The regional posterior is

$$\mathbb{p}(\mathbf{P}_i(k)|H_\alpha)\mathbb{P}(H_\alpha) = \frac{\int_{W_\alpha}\frac{1}{(2\pi\sigma^2(k))^{1/2}}}{A_\alpha} \times \exp\Big(- \frac{\Big(\ln \mathbf{P}_i(k) - \ln(\frac{Pd_0}{d_0+||q_i-y||^\rho})\Big)^2}{2\sigma^2(k)}\Big)dy \times \frac{A_\alpha}{A}$$
$$= \frac{\int_{W_\alpha}\frac{1}{(2\pi\sigma^2(k))^{1/2}}\cdot\exp\Big(- \frac{\big(\ln \mathbf{P}_i(k)-\ln(\frac{Pd_0}{d_0+||q_i-y||^\rho})\big)^2}{2\sigma^2(k)}\Big)dy}{A}.$$

Equations for the joint regional posterior follow by independence of measurements. ∎

2.3. Asymptotic properties of regional source localization

We show here some properties of the MAP algorithm when applied to regional source localization for a general number of sensors and regions. We start by presenting a property of non-collinear sensors when applied to source localization using measurements undergoing log-normal shadowing.

Lemma 2.5 (Three non-collinear sensors) *For* $d_0 > 0$ *and* $\rho > 0$*, given a source* $s \in \mathbb{R}^2$ *and three non-collinear sensors* q_1*,* q_2 *and* $q_3 \in \mathbb{R}^2$*, the only solution for the equation* $\sum_{i=1}^{3}\Big(\ln\frac{d_0+\|z-q_i\|^\rho}{d_0+\|s-q_i\|^\rho}\Big)^2 = 0$ *is* $z = s$.

Proof: In fact, it is easy to check that the sum is zero at $z = s$. Uniqueness of this solution is verified by noting that the sum of the square terms is zero only if all the summands are zero. Let $q = (x, y)$ and $q_i = (q_{i1}, q_{i2})$. The solution $z = s$ is unique if and only if the following system has a unique solution:

$$\begin{bmatrix} -2(q_{11}-q_{21}) & -2(q_{12}-q_{22}) \\ -2(q_{11}-q_{31}) & -2(q_{12}-q_{32}) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \tag{5}$$

where k_1 and k_2 are known values determined by the measurements and the positions of the sensors. The system presented in Equation (5) has a unique solution if and only if the system is consistent and the determinant of the matrix is non zero, i.e., the three points are non-collinear. ∎

As usual, assume that N sensors are at positions q_i, $i \in \{1,\dots,N\}$ and that the environment is partitioned into closed regions. For a region W_α, define the two scalar quantities

$$U_\alpha = \max_{\substack{y\in W_\alpha \\ i\in\{1,\dots,N\}}} \left|\ln\frac{d_0+\|y-q_i\|^\rho}{d_0+\|s-q_i\|^\rho}\right|, \tag{6}$$

$$L_\alpha = \min_{y\in W_\alpha}\sum_{i=1}^{N}\left(\ln\frac{d_0+\|y-q_i\|^\rho}{d_0+\|s-q_i\|^\rho}\right)^2. \tag{7}$$

Both quantities are well posed because they are the maximum and minimum value of a continuous function over a compact domain. Additionally, U_α is strictly positive for all source locations $s \in Q$ and L_α is strictly positive for all source locations $s \in Q \setminus W_\alpha$. The latter statement follows from Lemma 2.5 and from the fact that the distance from s to W_α is strictly positive for all $s \notin W_\alpha$. Define

$$\eta_\alpha = \sqrt{U_\alpha^2 + \frac{L_\alpha}{2N}} - U_\alpha > 0, \tag{8}$$

for all $s \notin W_\alpha$. We state the following result on the magnitude of sums of powers.

Lemma 2.6 (On the posterior of a wrong hypothesis) *Consider* L_α*,* U_α *and* η_α *as defined in* (6), (7) *and* (8). *Assume the source s is outside* W_α *and the noise* n_i *satisfies* $|n_i| \le \eta_\alpha$ *for all* $i \in \{1,\dots,N\}$ *and* $\alpha \in \{1,\dots,M\}$. *The following statements hold:*

1. *the joint measurement is lower bounded as*

$$\min_{y\in W_\alpha}\sum_{i=1}^{N}\Big(\ln\frac{d_0+\|y-q_i\|^\rho}{d_0+\|s-q_i\|^\rho}+n_i\Big)^2\geq\frac{1}{2}L_\alpha,\ \textit{and}$$

2. *the posterior probability for the wrong hypothesis α is upper bounded as*

$$\mathbb{p}(P_1,\dots,P_N|H_\alpha)\mathbb{P}(H_\alpha)\leq\frac{A_\alpha\exp\left(-L_\alpha/4\sigma^2\right)}{A(2\pi\sigma^2)^{N/2}}.$$

Proof: To prove the first statement, consider the expansion

$$\Big(\ln\frac{d_0+\|y-q_i\|^\rho}{d_0+\|s-q_i\|^\rho}+n_i\Big)^2=\Big(\ln\frac{d_0+\|y-q_i\|^\rho}{d_0+\|s-q_i\|^\rho}\Big)^2+2\left(\ln\frac{d_0+\|y-q_i\|^\rho}{d_0+\|s-q_i\|^\rho}\right)n_i+\sum_{i=1}^{N}n_i^2.$$

By computing lower bounds for each term and substituting the definition of η_α, obtain

$$\begin{aligned}\min_{y\in W_\alpha}\sum_{i=1}^{N}\Big(\ln\frac{d_0+\|y-q_i\|^\rho}{d_0+\|s-q_i\|^\rho}+n_i\Big)^2&\geq L_\alpha-2NU_\alpha\eta_\alpha-N\eta_\alpha^2\\&=L_\alpha+2NU_\alpha^2-2NU_\alpha\sqrt{U_\alpha^2+\frac{L_\alpha}{2N}}-N\Big(U_\alpha^2+\frac{L_\alpha}{2N}\Big)\\&\quad-NU_\alpha^2+2NU_\alpha\sqrt{U_\alpha^2+\frac{L_\alpha}{2N}}=\frac{1}{2}L_\alpha.\end{aligned}$$

The second statement follows directly from the first statement because of the equality

$$\ln P_i-\ln\frac{Pd_0}{d_0+\|y-q_i\|^\rho}=\ln\frac{d_0+\|y-q_i\|^\rho}{d_0+\|s-q_i\|^\rho}+n_i,$$

and because of the fact that the surface integral of a function f is upper bounded by the surface integral of the maximum value of f. ■

We are now ready for the convergence theorem. We introduce the standard function $Q:\mathbb{R}\to\mathbb{R}_{>0}$ by

$$Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^{+\infty}\exp(-y^2/2)dy.$$

Theorem 2.7 (Elimination of wrong hypothesis) *Consider sensors at positions $q_1,\dots,q_N$. Let σ be the noise variance. If the source $s\notin W_\alpha$, then*

$$\mathbb{P}\Big[\mathbb{p}(P_1,\dots,P_N|H_\alpha)\mathbb{P}(H_\alpha)\leq\epsilon_\alpha(\sigma)\Big]\geq\mu_\alpha(\sigma),$$

where

$$\epsilon_\alpha(\sigma)=\frac{A_\alpha\exp(-L_\alpha/4\sigma^2)}{A(2\pi\sigma^2)^{N/2}},\quad\mu_\alpha(\sigma)=(1-2Q(\eta_\alpha/\sigma))^N.$$

Furthermore, in the k repeated measurement case, if at least 3 sensors are non-collinear, then $\lim_{k\to\infty}\epsilon_\alpha(\sigma_k)=0^+$ *and* $\lim_{k\to\infty}\mu_\alpha(\sigma_k)=1^-$.

Proof: From Lemma 2.6, we compute

$$\begin{aligned}&\mathbb{P}\big[\mathbb{p}(P_1,\dots,P_N|H_\alpha)\mathbb{P}(H_\alpha)\leq\epsilon_\alpha(\sigma)\big]\\&\geq\mathbb{P}\left[[n_1,\dots,n_N]^T\in[-\eta_\alpha,\eta_\alpha]^N\right]\\&=\prod_{i=1}^{N}\Big(\frac{1}{2}-\mathbb{P}[n_i>\eta_\alpha]+\frac{1}{2}-\mathbb{P}[n_i<-\eta_\alpha]\Big)\\&=\big(1-2Q(\eta_\alpha/\sigma)\big)^N.\end{aligned}$$

The first inequality follows from the fact that Lemma 2.6 holds whenever all $|n_i|\leq\eta_\alpha$. The proofs of the two limits of $\lim_{k\to\infty}\epsilon_\alpha(\sigma_k)$ and $\lim_{k\to\infty}\mu_\alpha(\sigma_k)$ are immediate when there are at least 3 non-collinear sensors. Indeed, if there are at least 3 non-collinear sensors and if $s\notin W_\alpha$, then Lemma 2.5 applies and one can show $L_\alpha>0$ and $\eta_\alpha>0$. ■

This theorem states that, as $\sigma\to0^+$, the joint regional posterior $\mathbb{p}(P_1,\dots,P_N|H_\alpha)\mathbb{P}(H_\alpha)$ takes an arbitrarily small value with a probability that goes arbitrarily close to 1 when H_α is not the correct hypothesis. This is so as $Q(x)\to0$ as $x\to\infty$. To complement the Theorem 2.7, we prove below that for the correct hypothesis, the probability density is lower bounded by a positive term w.p.1.

Theorem 2.8 (Strict positivity of correct hypothesis) *Consider sensors at positions $q_1,\dots,q_N$. Let σ be the noise variance. If the source $s\in W_{\overline{\alpha}}$, then*

$$\mathbb{P}\left[\mathbb{p}(P_1,\dots,P_N|H_{\overline{\alpha}})\mathbb{P}(H_{\overline{\alpha}})\geq\Psi(\sigma)\right]\geq\Omega(\sigma),$$

where

$$\Psi(\sigma)=\mathbb{p}(P_1,\dots,P_N)-\sum_{\substack{\alpha=1,\dots,M\\\alpha\neq\overline{\alpha}}}\frac{A_\alpha\exp(-L_\alpha/4\sigma^2)}{A(2\pi\sigma^2)^{N/2}},$$

$$\Omega(\sigma)=\prod_{\substack{\alpha=1,\dots,M\\\alpha\neq\overline{\alpha}}}\mu_\alpha(\sigma)=\prod_{\substack{\alpha=1,\dots,M\\\alpha\neq\overline{\alpha}}}(1-2Q(\eta_\alpha/\sigma))^N.$$

Furthermore, in the k repeated measurement case, if at least 3 sensors are non-collinear, then $\lim_{k\to\infty}\Psi(\sigma_k)=\mathbb{p}(P_1,\dots,P_N)>0$ *and* $\lim_{k\to\infty}\Omega(\sigma_k)=1^-$.

Proof: The proof of this theorem follows directly from Theorem 2.7 and from the total probability theorem. Call $z=[P_1,\dots,P_N]^T$. We know from the total probability theorem that

$$\mathbb{p}(z)=\sum_{\alpha=1}^{M}\mathbb{p}(z|H_\alpha)\mathbb{P}(H_\alpha)=\mathbb{p}(z|H_{\overline{\alpha}})\mathbb{P}(H_{\overline{\alpha}})+\sum_{\substack{\alpha=1,\dots,M\\\alpha\neq\overline{\alpha}}}\mathbb{p}(z|H_\alpha)\mathbb{P}(H_\alpha)$$

and, in turn, that

$$\mathbb{p}(z|H_{\overline{\alpha}})\mathbb{P}(H_{\overline{\alpha}})=\mathbb{p}(z)-\sum_{\substack{\alpha=1,\dots,M\\\alpha\neq\overline{\alpha}}}\mathbb{p}(z|H_\alpha)\mathbb{P}(H_\alpha).$$

From Theorem 2.7

$$\mathbb{P}\left[\mathbb{p}(z|H_{\overline{\alpha}})\mathbb{P}(H_{\overline{\alpha}}) \geq \mathbb{p}(z) - \sum_{\substack{\alpha=1,\ldots,M\\ \alpha\neq\overline{\alpha}}} \epsilon_\alpha(\sigma)\right]$$

$$\geq \prod_{\substack{\alpha=1,\ldots,M\\ \alpha\neq\overline{\alpha}}} \mathbb{P}\Big[\mathbb{p}(z|H_\alpha)\mathbb{P}(H_\alpha) \leq \epsilon_\alpha(\sigma)\Big] \geq \prod_{\substack{\alpha=1,\ldots,M\\ \alpha\neq\overline{\alpha}}} \mu_\alpha(\sigma)$$

As $\lim_{k\to\infty}\sigma_k = 0^+$, $\lim_{k\to\infty}\Psi(\sigma_k) = \mathbb{p}(z)$ and $\lim_{k\to\infty}\Omega(\sigma_k) = 1^-$. ■

Theorem 2.8 complements Theorem 2.7 in that is shows that as $\lim_{k\to\infty}\sigma_k = 0^+$, the largest regional posterior is the one associated with the correct hypothesis.

Remark 2.9 (Almost sure convergence of MAP) Using a MAP algorithm to solve the problem of regional localization, is assured to provide a correct answer, almost surely, in the limit of infinite measurements. This follows directly from Theorem 2.8 and Theorem 2.7.

3. Distributed sequential regional localization

In this section we assume that each sensor is a processor that can perform computational tasks as well as communicate to other processors according to a specified communication graph. Each processor takes measurements and computes a conditional posterior that it communicates to all its neighbors and then makes a decision if a desired accuracy is reached. A group of regions is associated with each processor. The processor will need to provide a decision about which of these regions if any contains the source. We call such a group, the regions of responsibility of the processor. We do not assume any constraints on the assignment of regions of responsibilities. We present the algorithm in Subsection 3.1 and describe its properties in Subsection 3.2.

3.1. Distributed algorithm based on sequential sensing, communication and hypothesis testing

We present below a distributed algorithm where each processor decides whether or not its region of responsibility contains the source. The algorithm as presented, dictates until when a processor needs to continue to take measurements, as well as the information that needs to be communicated.

For each processor $i \in \{1,\ldots,N\}$, the set of neighbors $\mathcal{N}_i$ consists of the processor itself along with the processors that can communicate with it. The ith processor is responsible for a set $\mathcal{R}_i$ of M_i regions. We denote these M_i regions by W_α for $\alpha \in \mathcal{R}_i$. The processor collects the measurements from its neighboring processors, and calculates two posteriors for all regions W_α, $\alpha \in \mathcal{R}_i$. The first posterior corresponds to the hypothesis that the source is in W_α, the second posterior corresponds to the hypothesis that the source is outside W_α. Once the processor reaches a pre-defined level of confidence, it provides a decision about whether or not W_α contains the source. The ith processor stops running its test when it reaches a decision about all W_α, $\alpha \in \mathcal{R}_i$. The processor then sets its decision to either `yes`, the source is in W_α, or `no`, no source is in $\cup_{\alpha\in\mathcal{R}_i} W_\alpha$. Each processor continues to sense and transmit its measurements until all its neighbors $j \in \mathcal{N}_i$ have reached a decision. We give here a formal description of the algorithm.

Algorithm : *Sense, Transmit & Test*

algorithm tolerance: $0 < \epsilon \ll \frac{1}{2}$
network processors: $i \in \{1,\ldots,N\}$
regions: W_α, $\alpha \in \{1,\ldots,M\}$
state of processor i contains:
  $\texttt{a-dcsn}_i \in \{\texttt{yes}\ \textit{source} \in W_\alpha, \texttt{no}\ \textit{source} \in \cup_\alpha W_\alpha, \texttt{unknown}\}$,
  for all $j \in \mathcal{N}_i : q_j, \texttt{a-stop}_j \in \{\texttt{false},\texttt{true}\}$,
  for all $\alpha \in \mathcal{R}_i : W_\alpha, \texttt{r-stop}_\alpha \in \{\texttt{false},\texttt{true}\}$,
    $\texttt{r-dcsn}_\alpha \in \{\texttt{yes},\texttt{no},\texttt{unknown}\}$

Processor i with set of neighbors $\mathcal{N}_i$ executes:

transmit q_i to $j \in \mathcal{N}_i$
set $k := 0$, $\texttt{a-stop}_i := \texttt{false}$ and $\texttt{a-dcsn}_i := \texttt{unknown}$
set $\texttt{r-stop}_\alpha := \texttt{false}$ and $\texttt{r-dcsn}_\alpha := \texttt{unknown}$ for $\alpha \in \mathcal{R}_i$
While $\exists\ j \in \mathcal{N}_i$ with $\texttt{a-stop}_j == \texttt{false}$ **do**
  update $k := k+1$ and take measurement $P_i(k)$
  compute $\ln \mathbf{P}_i(k) = \sum_{t=1}^{k} \frac{\ln \mathbf{P}_i(t)}{k}$
  transmit $\ln \mathbf{P}_i(k)$ to $j \in \mathcal{N}_i$
  store $\ln \mathbf{P}_{\mathcal{N}_i}(k) = \{\ln \mathbf{P}_i(k)\} \cup \{\ln \mathbf{P}_j(k)$ for all $j \in \mathcal{N}_i\}$
  For all $\alpha \in \mathcal{R}_i$ with $\texttt{r-stop}_\alpha == \texttt{false}$ **do**
    If $\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|s \in W_\alpha)\mathbb{P}(s \in W_\alpha) > (1-\epsilon)\ \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k))$
      $\texttt{dcsn}_\alpha := \texttt{yes}, \texttt{r-stop}_\alpha := \texttt{true},\ \texttt{a-dcsn}_i := \texttt{true}$
    If $\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|s \notin W_\alpha)\mathbb{P}(\notin W_\alpha) > (1-\epsilon)\ \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k))$
      $\texttt{r-dcsn}_\alpha := \texttt{no}$ and $\texttt{r-stop}_\alpha := \texttt{true}$
  End For
If $\texttt{r-stop}_\alpha == \texttt{true}$ **for all** $\alpha \in \mathcal{R}_i$
  $\texttt{a-stop}_i := \texttt{true}$
  If $\texttt{a-dcsn}_i == \texttt{unknown}$
    $\texttt{a-dcsn}_i := \texttt{no}$
transmit $\texttt{a-stop}_i$ to all $j \in \mathcal{N}_i$
return $\texttt{a-dcsn}_i$
End While

3.2. *Properties of* Sense, Transmit & Test

We present below properties involving the accuracy and decision time of the *Sense, Transmit & Test* algorithm.

Theorem 3.1 (Accuracy and decision time for *Sense, Transmit & Test* algorithm) *Assume that only one source exists in the environment Q, that each processor has at least 2 neighboring processors with which it forms a non-collinear triplet, and that each processor is assigned M_i regions. Given an accuracy $\epsilon \in (0, \frac{1}{2})$, the* Sense, Transmit & Test *algorithm enjoys the following two properties:*

1. *the algorithm ends in a finite time, and*
2. *each processor i has a probability of error no larger than $2M_i\epsilon$ if $2 \leq M_i \leq 1 + \frac{1}{\epsilon}$, and no larger than ϵ if $M_i = 1$.*

Proof: It is well known (Wald, 1945; Varshney, 1996) that given two hypothesis H_1 and H_0 with known posteriors, $P(H_1)$

and $P(H_0)$, a hypothesis test that ensures that the decision under hypotheses H_0 and H_1 is correct with a probability greater than τ_0 and τ_1 respectively is the following:

1. Calculate $\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1), \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0)$
2. if $\frac{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1)}{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0)} \geq \frac{\tau_1}{1-\tau_1}$ decide in favor of H_1,
3. if $\frac{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1)}{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0)} \leq \frac{1-\tau_0}{\tau_0}$, decide in favor of H_1,
4. othewise repeat measurements and go to 1.

We show below that the *Sense, Transmit & Test* algorithm satisfies the description above. Applying the total probability theorem we get

$$\begin{aligned}
&\mathbb{p}(P_{\mathcal{N}_i}|s \notin W_\alpha)\mathbb{P}(s \notin W_\alpha) \\
&= \mathbb{p}(P_{\mathcal{N}_i}|y \in Q)\mathbb{P}(s \in Q) - \mathbb{p}(P_{\mathcal{N}_i}|s \in W_\alpha)\mathbb{P}(s \in W_\alpha) \\
&= \mathbb{p}(P_{\mathcal{N}_i}) - \mathbb{p}(P_{\mathcal{N}_i}|s \in W_\alpha)\mathbb{P}(s \in W_\alpha). \qquad (9)
\end{aligned}$$

Call $(H_1 := s \in W_\alpha)$ and $(H_0 := s \notin W_\alpha)$. If we set $\tau_0 = \tau_1 = (1-\epsilon)$, and the thresholds to accept a hypothesis H_1, to be

$$\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1) \geq \tau_1 \mathbb{p}(P_{\mathcal{N}_i}),$$

and the thresholds to reject a hypothesis to be

$$\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0) \geq \tau_0 \mathbb{p}(P_{\mathcal{N}_i}),$$

then using Eq. (9), one can show that

$$\begin{aligned}
&\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1) \geq \tau_1\ \mathbb{p}(P_{\mathcal{N}_i}) \\
&\Rightarrow \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0) \leq (1-\tau_1)\mathbb{p}(P_{\mathcal{N}_i}) \\
&\Rightarrow \frac{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1)}{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0)} \geq \frac{\tau_1}{1-\tau_1}. \qquad (10)
\end{aligned}$$

Similarly, assuming H_0 is correct, one can show that

$$\begin{aligned}
&\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0) \geq \tau_0\ \mathbb{p}(P_{\mathcal{N}_i}) \\
&\quad \Rightarrow \frac{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1)}{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0)} \leq \frac{1-\tau_0}{\tau_0}. \qquad (11)
\end{aligned}$$

Assuming H_1 is correct, the probability of correct decision for the ith processor is no smaller than τ_1, for each of the regions $W_\alpha, \alpha \in \mathcal{N}_i$. Similar result hold assuming H_0 is correct.

The maximum numbers of errors that a processor can make in a decision is two: a mis-detection and a false-alarm for W_α where $\alpha \in \mathcal{R}_i$. Alternatively all the other combinations of choices result in at most one error, since the ith processor can declare at most one hypothesis H_α to be correct for all $\alpha \in \mathcal{R}_i$.

The scenarios where the decision of the processor is erroneous are presented below:

1. If one of the regions W_α satisfies $\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1) \geq \tau_1\mathbb{p}(P_{\mathcal{N}_i})$, then for all $\beta \in \mathcal{R}_i \setminus \alpha$, the following holds (from the complete probability theorem)

   $$\begin{aligned}
   &\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|s \in W_\beta)\mathbb{P}(s \in W_\beta) < (1-\tau_1)\mathbb{p}(P_{\mathcal{N}_i}) \\
   &\Rightarrow \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|s \notin W_\beta)\mathbb{P}(s \notin W_\beta) \geq \tau_1\ \mathbb{p}(P_{\mathcal{N}_i}) = \tau_0\mathbb{p}(P_{\mathcal{N}_i}). \qquad (12)
   \end{aligned}$$

   From Eqs. (11) and (12) it follows that the source can be detected in at most one region W_α. It follows that at most one false alarm can happen, which might or might not be accompanied with one mis-detection.
2. If none of the regions of responsibilities of the processor contain the source, then the processor can make at most one mistake by having at most one false alarm.

To write a formal proof, we introduce p_f and p_m to be the probability of false alarm and mis-detection. Where p_f corresponds to choosing `yes` while the correct decision is `no` and mis-detection corresponds to choose `no` when the correct decision is `yes` for any region W_α. Here that $p_f = p_m = \epsilon$. A processor makes an error if it wrongly decides `yes`/`no` on W_α for any $\alpha \in \mathcal{R}_i$. Following the analysis above, the probability of error for the ith processor is:

$$\begin{aligned}
P_e < \binom{M_i}{1}\Big(&p_m P(s \notin \cup_{\alpha\in\mathcal{R}_i} W_\alpha) \\
&+ (p_f + \binom{M_i - 1}{1} p_f p_m) P(s \in \cup_{\alpha\in\mathcal{R}_i} W_\alpha)\Big) \leq 2M_i\epsilon,
\end{aligned}$$

if $\epsilon(M_i - 1) \leq 1$. If the processor has only one region of responsibility, it is straightforward to see that the processor has a probability of error no larger than ϵ.

We show now that the test ends after a finite number of measurements. For a region W_α, processor needs to decide whether the source is in W_α (H_1) or outside it (H_0).

Without loss of generality, assume that H_1 is correct for a region W_α. We know from Theorem 2.7 that

$$\lim_{k\to\infty} \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0) = 0^+,$$

almost surely. We also know from Theorem 2.8 that

$$\lim_{k\to\infty} \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1) = \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)) > 0,$$

almost surely. This has the following implication

$$\lim_{k\to\infty} \frac{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1)}{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k))} = 1,$$

which implies that for all $\bar{\epsilon} > 0$, there exists $0 < K < \infty$, s.t.

$$\begin{aligned}
&|\frac{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(K)|H_1)\mathbb{P}(H_1) - \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(K))}{\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(K))}| < \bar{\epsilon} \\
&\iff -\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(K)|H_1)\mathbb{P}(H_1) + \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(K)) < \bar{\epsilon}\ \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(K)) \\
&\iff \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(K)|H_1)\mathbb{P}(H_1) > (1-\bar{\epsilon})\ \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(K)).
\end{aligned}$$

So for any $\frac{1}{2} < \tau < 1$, there exists, almost surely, $K < \infty$, s.t.

$$\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_1)\mathbb{P}(H_1) > \tau\ \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)),$$

here $\tau = 1 - \bar{\epsilon}$, where $0 < \bar{\epsilon} < \frac{1}{2}$.

Similarly one can prove that if H_0 is correct, then there exists, almost surely, $K < \infty$, such that

$$\mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)|H_0)\mathbb{P}(H_0) > \tau\ \mathbb{p}(\mathbf{P}_{\mathcal{N}_i}(k)).$$

To complete the proof, we cover the cases where the algorithm makes a wrong decision. This is possible if the thresholds corresponding to a wrong decision are crossed at a time $K_1 < K < \infty$.

This completes the proof that the *Sense, Transmit & Test* algorithm has a finite decision time. ∎

4. Numerical results

We present in this section three sets of simulations. The first two sets illustrate some properties of the *Sense, Transmit & Test* algorithm, while the third presents a modification of the algorithm that introduces an interesting extension of the work. In the first simulations, there are as many regions as there are sensors, i.e., $N = M = 10$. We start by presenting in Figure 1 a sample of the results obtained by the *Sense, Transmit & Test* algorithm. The figure shows the positions of the processors (equipped with sensors) as well as the partition of Q. As partition we adopt the Voronoi partition generated by the processors positions; each processor is responsible for its corresponding Voronoi region. As stated in the caption, after 113 observations all decisions have been made and the source has been correctly localized.

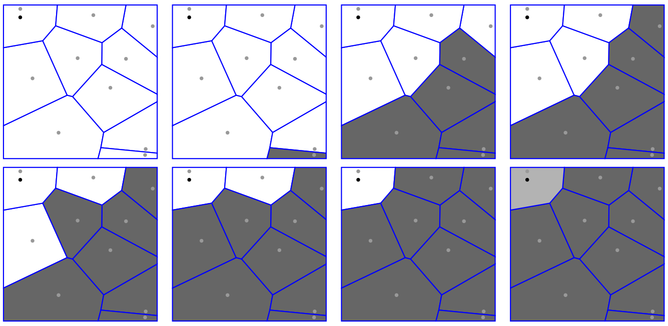

Figure 1: This picture illustrates an evolution of the output of the *Sense, Transmit & Test* algorithm. At each instant a region is colored in white, light gray or dark gray, indicating `unknown`, `yes` or `no` respectively. The output of the distributed algorithm is shown at times $0, 1, 3, 4, 6, 8, 11, 113$ respectively. In this run we $\epsilon = 0.01$ and $\sigma = 0.5$ with $N = M = 10$.

We then present in Figure 2 a plot that shows how the expected number of observations needed to reach a decision varies with the accuracy ϵ in the algorithm. Clearly, the probability of correct detection increases for decreasing values of ϵ.

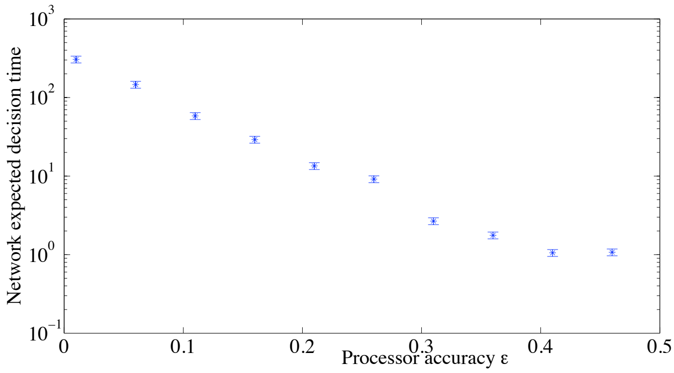

Figure 2: This plot shows the expected time it takes a network of 10 processors implementing the *Sense, Transmit & Test* algorithm to reach a decision for a noise standard of deviation $\sigma = 0.1$ when the probability of error ϵ varies. We show the logarithm of the decision time. Note that the network decision time seems to grow exponentially with the desired accuracy as is standard in sequential hypothesis testing. The network is assumed to have reached a decision when all processors have decided. The expected decision time is calculated over 1000 runs.

In Figure 3 we show how the expected number of observations needed to reach a decision increases with the standard deviation of the noise.

Next, we report the second sets of simulations, where we have differing numbers of regions and sensors. Specifically, we have $N = 4$ sensors and $M = 16$ regions. Figure 4 illustrates

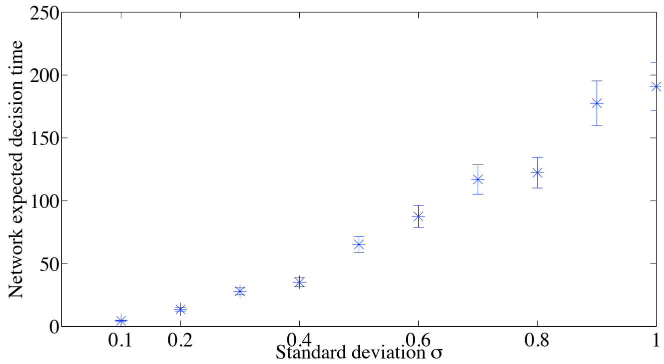

Figure 3: This plot shows the expected time it takes a network of 10 processors implementing the *Sense, Transmit & Test* algorithm to reach a decision with a probability of error no larger than $\epsilon = 0.01$ as the noise standard of deviation σ varies. The network is assumed to have reached a decision when all processors have decided. The expected decision time is calculated over 100 runs.

the evolution of the *Sense, Transmit & Test* algorithm in this case. The overall accuracy for each processor is 0.9. This is achieved by setting $\epsilon = 0.1/8$.

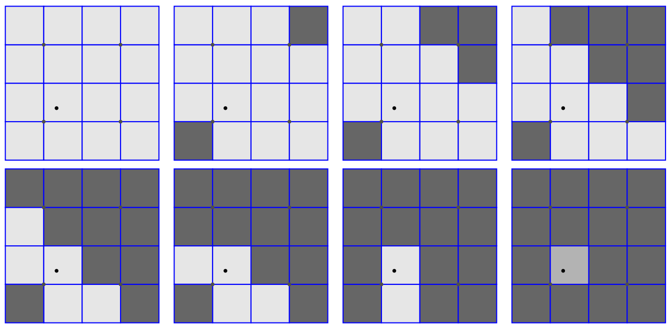

Figure 4: This picture illustrates an evolution of the *Sense, Transmit & Test* algorithm. At each instant a region is colored in white, light gray or dark gray, indicating `unknown`, `yes` or `no` respectively. The output of the distributed algorithm is shown at times $1, 4, 6, 7, 13, 131, 142, 202$ respectively. In this run we set $\epsilon = 0.1/8$ and $\sigma = 0.5$ and $N = 4$ and $M = 16$.

In this third set of simulations, we show the output of a modified, multi-resolution version of the *Sense, Transmit & Test* algorithm. This multi-resolution version runs over multiple stages, at each stage the environment under consideration is divided in two regions. Observations are taken at each stage until one of the two regions is rejected with an accuracy of $1-\epsilon$. The rejected region is removed from the environment, and the remaining region is again divided in two regions. Observations are transferred from one stage to another and re-used to reach a decision about the more fine environment division. A sample output of the modified algorithm is shown in Figure 5. In order to reach the same precision in localization as that shown in Figure 4, we divide the regions 4 times. Note that the original *Sense, Transmit & Test* algorithm reached its decision after an average of 290 observations whereas the multi-resolution algorithm did so after an average of 100 observations. We calculated these values from 1000 Monte-Carlo runs, that is with an error of $\pm 3\%$ to show a similar probability of error with the same level of fineness. We leave a rigorous analysis of the multi-resolution algorithm to future work.

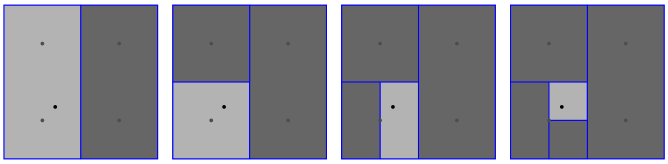

Figure 5: This picture illustrates an evolution of the modified version of the *Sense, Transmit & Test* algorithm. The output is shown at times $7, 11, 71, 72$ respectively. The rejected regions are shown in dark grey and the ones accepted at each stage are shown in light grey. In this run we set $\epsilon = 0.1$ and $\sigma = 0.5$ and $N = 4$ and $M = 2$ at each set of tests.

We conclude with a general remark. The *Sense, Transmit & Test* algorithm presented in this work might at first glance look similar to sequential multiple hypothesis testing algorithms by elimination such as the one presented in (Bauer, 1989). A closer comparison of the two algorithms shows that while in this work at most $2M$ tests are run at each sample, the hypothesis test by sequential elimination requires a number of tests of the order 2^M as it proceeds by a pairwise comparison over all hypothesis. Nonetheless, it is worth mentioning that while the sequential elimination algorithm leads to a decision as soon as all but one hypothesis is eliminated, we wait here until the last hypothesis reaches the required certainty level. This can be seen in Figure 1 where all but one hypothesis were canceled at the 11th observation, yet the algorithm did not end until the 113th observation when the last processor reached its required accuracy. The geometric aspects and the properties associated with the regional localization problem, made it possible to propose the simpler, yet less general, *Sense, Transmit & Test* algorithm.

5. Conclusion

In this work, we looked at the problem of source localization in a multiple hypothesis testing setting. We based our formulation on the geometric properties of the MAP algorithm when applied to regional localization. We proved that when measurements are available from three or more non-collinear sensors, MAP based algorithms choose the correct region almost surely in the limit of infinite measurements. We then presented a sequential distributed algorithm where each processor senses, transmits and tests to provide a decision. We analyzed the algorithm and provided a measure of its accuracy and showed that it ends in a finite time. We concluded the paper by numerically illustrating the algorithm's performance.

There are two direct extensions for this work that we are considering. The first is using an adaptive hierarchical methods based on quadtrees (de Berg et al., 2000) to increase the level of details in the choice of regions. The regions could be finely divided as fewer candidate regions are left, an example of such adaptation is shown at the end of the manuscript. It would be interesting to study the trade off between the accuracy and the decision time as a function of the fine-gridding of the regions.

The second is allowing the algorithm to stop as soon as a processor decides that its region contains the source. As presented in this manuscript, the algorithm has a proven accuracy performance based on the assumption that all processors reach their decisions independently of each other, and although we assume only one source, a processor will continue applying the *Sense, Transmit & Test* algorithm until it decides that its region does not contain the source even if the source was detected by one of the other processors. It will be interesting to see what happens to the accuracy if an individual can broadcast a `yes` to everyone in the group, allowing them to stop. Alternatively, as we showed in Figure 1, it is possible that only one hypothesis is left by elimination. It will also be interesting to analyze, if possible, the accuracy of an algorithm that makes use of such scenarios when they occur.

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