Kron Reduction of Graphs with Applications to Electrical Networks

Florian Dörfler Francesco Bullo

***Abstract*—Consider a weighted undirected graph and its corresponding Laplacian matrix, possibly augmented with additional diagonal elements corresponding to self-loops. The Kron reduction of this graph is again a graph whose Laplacian matrix is obtained by the Schur complement of the original Laplacian matrix with respect to a specified subset of nodes. The Kron reduction process is ubiquitous in classic circuit theory and in related disciplines such as electrical impedance tomography, smart grid monitoring, transient stability assessment, and analysis of power electronics. Kron reduction is also relevant in other physical domains, in computational applications, and in the reduction of Markov chains. Related concepts have also been studied as purely theoretic problems in the literature on linear algebra. In this paper we analyze the Kron reduction process from the viewpoint of algebraic graph theory. Specifically, we provide a comprehensive and detailed graph-theoretic analysis of Kron reduction encompassing topological, algebraic, spectral, resistive, and sensitivity analyses. Throughout our theoretic elaborations we especially emphasize the practical applicability of our results to various problem setups arising in engineering, computation, and linear algebra. Our analysis of Kron reduction leads to novel insights both on the mathematical and the physical side.**

***Index Terms*—Kron reduction, equivalent circuit, algebraic graph theory, Ward equivalent, network-reduced model**

I. Introduction

Consider an undirected, connected, and weighted graph with n nodes and adjacency matrix $A \in \mathbb{R}^{n \times n}$. The corresponding loopy Laplacian matrix is the matrix $Q \in \mathbb{R}^{n \times n}$ with off-diagonal elements $Q_{ij} = -A_{ij}$ and diagonal elements $Q_{ii} = A_{ii} + \sum_{j=1}^{n} A_{ij}$. Consider now a simple algebraic operation, namely the Schur complement of the loopy Laplacian matrix Q with respect to a subset of nodes. As it turns out, the resulting lower dimensional matrix Q_{red} is again a well-defined loopy Laplacian matrix, and a graph can be naturally associated to it.

This paper investigates this Schur complementation from the viewpoint of algebraic graph theory. In particular we seek answers to the following questions. How are the spectrum and the algebraic properties of Q and Q_{red} related? How about the corresponding graph topologies and the effective resistances? What is the effect of a perturbation in the original graph on the reduced graph, its loopy Laplacian Q_{red}, its spectrum, and its effective resistance? Finally, why is this graph reduction process of practical importance and in which application areas? These are some of the questions that motivate this paper.

Electrical networks and the Kron reduction. To illustrate the physical dimension of the problem setup introduced above, we consider the circuit naturally associated to the adjacency matrix A. Consider a connected electrical network with n nodes, current injections $I \in \mathbb{R}^{n \times 1}$, nodal voltages $V \in \mathbb{R}^{n \times 1}$, branch conductances $A_{ij} \geq 0$, and shunt conductances $A_{ii} \geq 0$ connecting node i to the ground. The resulting current-balance equations are $I = QV$, where the conductance matrix $Q \in \mathbb{R}^{n \times n}$ is the loopy Laplacian. In circuit theory and related disciplines it is desirable to obtain a lower dimensional electrically-equivalent network from the viewpoint of certain boundary nodes $\alpha \subsetneq \{1, \dots, n\}$, $|\alpha| \geq 2$. If $\beta = \{1, \dots, n\} \backslash \alpha$ denotes the interior nodes, then, after appropriately labeling the nodes, the current-balance equations can be partitioned as

$$\left[\begin{array}{c} I_\alpha \\ \hline I_\beta \end{array}\right] = \left[\begin{array}{c|c} Q_{\alpha\alpha} & Q_{\alpha\beta} \\ \hline Q_{\beta\alpha} & Q_{\beta\beta} \end{array}\right] \left[\begin{array}{c} V_\alpha \\ \hline V_\beta \end{array}\right]. \tag{1}$$

Gaussian elimination of the interior voltages V_β in equations (1) gives an electrically-equivalent reduced network with the nodes α obeying the reduced current-balance equations

$$I_\alpha + Q_{\text{ac}} I_\beta = Q_{\text{red}} V_\alpha\,, \tag{2}$$

where the reduced conductance matrix $Q_{\text{red}} \in \mathbb{R}^{|\alpha| \times |\alpha|}$ is given by the Schur complement of Q with respect to the interior nodes β, that is, $Q_{\text{red}} = Q_{\alpha\alpha} - Q_{\alpha\beta} Q_{\beta\beta}^{-1} Q_{\beta\alpha}$, and the accompanying matrix $Q_{\text{ac}} = -Q_{\alpha\beta} Q_{\beta\beta}^{-1} \in \mathbb{R}^{|\alpha| \times (n - |\alpha|)}$ maps internal currents to boundary currents in the reduced network.

This reduction of an electrical network via a Schur complement of the associated conductance matrix is known as *Kron reduction* due to the seminal work of Gabriel Kron [1]. In case of a star-like network without interior current injections and shunt conductances, the Kron reduction of a network reduces to the (generalized) star-triangle transformation [2], [3].

Literature review. The Kron reduction of networks is ubiquitous in circuit theory and related applications in order to obtain lower dimensional electrically-equivalent circuits. It appears for instance in the behavior, synthesis, and analysis of resistive circuits [4]–[6], particularly in the context of large-scale integration chips [7], [8]. When applied to the impedance matrix of a circuit rather than the admittance matrix, Kron reduction is also referred to as the "shortage operator" [9], [10]. Kron reduction is a standard tool in the power systems community to obtain so-called "network-reduced" or "Ward-equivalent" models for power flow studies [11], [12], to reduce differential-algebraic power network models to purely dynamic models [13]–[16], and it is crucial for reduced order modeling, analysis, and efficient simulation of induction motors [17] and power electronics [18], [19]. A recent application of Kron reduction is monitoring in smart power grids [20] via synchronized phasor measurement units. Kron reduction

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Florian Dörfler and Francesco Bullo are with the Center for Control, Dynamical Systems and Computation, University of California at Santa Barbara, Santa Barbara, CA 93106, {dorfler, bullo}@engineering.ucsb.edu

is also known in the literature on electrical impedance tomography, where Q_{red} is referred to as the "Dirichlet-to-Neumann map" [21]–[23]. More generally, the Schur complement of a matrix and its associated graph is known in the context of Gaussian elimination of sparse matrices [24]–[26] and its application to Laplacian matrices can be found, for example, in sparse multi-grid solvers [27] and in finite-element analysis [17]. It serves as popular application example in linear algebra [28]–[31], a similar concept is employed in the cyclic reduction [32] or the stochastic complement [33] of Markov chains, and a related concept is the Perron complement [34], [35] of a matrix and its associated graph with applications in data mining [36]. Finally, Kron reduction is also crucial in model reduction of water supply networks [37] and in the context of the Yang-Baxter equation and its applications in knot theory, high-energy physics, and statistical mechanics [38].

This brief literature review shows that Kron reduction is both a practically important and theoretically fascinating problem occurring in numerous applications. Each of the aforementioned communities has different approaches and insights into Kron reduction. Engineers understand the physical dimension of Kron reduction very well, the computation community investigates the sparsity pattern of the Kron-reduced matrix, and the linear algebra community is interested in eigenvalue problems. Surprisingly, across different scientific communities little is known about the graph-theoretic properties of the Kron reduction process. Yet the graph-theoretic analysis of Kron reduction provides novel and deep insights both on the mathematical and the physical side of the considered problem.

Contributions. In this paper we provide a detailed and comprehensive graph-theoretic analysis of the Kron reduction process. Our general graph-theoretic framework and analysis of Kron reduction encompasses various theoretical problem setups as well as practical applications in a unified language.

Essentially, Kron reduction of a connected graph, possibly with self-loops, is a Schur complement of corresponding loopy Laplacian matrix with respect to a subset of nodes. We relate the topological, the algebraic, and the spectral properties of the resulting Kron-reduced Laplacian matrix to those of the non-reduced Laplacian matrix. Furthermore, we relate the effective resistances in the original graph to the elements and effective resistances induced by the Kron-reduced Laplacian. Thereby, we complement and extend various results in the literature on the effective resistance of a graph [10], [39]–[42]. In our analysis, we carefully analyze the effects of self-loops, which typically model loads and dissipation. We also present a sensitivity analysis of the algebraic, spectral, and resistive properties of the Kron-reduced matrix with respect to perturbations in the non-reduced network topology. Finally, our analysis of Kron reduction complements the literature in linear algebra [28]–[31], and we construct an explicit relationship to analogous results on the Perron complement side [33]–[36] such that our results apply also to Markov chain reductions. Throughout the paper, we will remark whenever certain basic lemmas are known or partially known to some community.

In our analysis we do not aim at deriving only mathematical elegant results but also useful tools for practical applications. Our general graph-theoretic framework encompasses the applications of Kron reduction in circuit theory [4]–[8], electrical impedance tomography [21]–[23], sensitivity in power flow studies [11], [12], monitoring in smart grids [20], transient stability assessment in power grids [13]–[16], and the stochastic reduction of Markov chains [29], [33]–[36]. Furthermore, we demonstrate how each of these applications benefits from the graph-theoretic viewpoint and analysis of the Kron reduction. We believe that our general analysis is a first step towards more detailed results in specific applications of Kron reduction.

Paper organization. The remainder of this section introduces some notation recalls some preliminaries in matrix analysis and algebraic graph theory. Section II presents the general framework of Kron reduction and reviews various application areas. Section III presents the graph-theoretic analysis of the Kron reduction process. Finally, Section IV concludes the paper and suggests some future research directions.

Preliminaries and Notation. Given a finite set $\mathcal{Q}$, let $|\mathcal{Q}|$ be its cardinality, and define for $n \in \mathbb{N}$ the set $\mathcal{I}_n = \{1, \dots, n\}$.

Vectors and matrices: Let $\mathbf{1}_{p\times q}$ and $\mathbf{0}_{p\times q}$ be the $p \times q$ dimensional matrices of unit and zero entries, and let I_n be the n-dimensional identity matrix. For vectors, we adopt the shorthands $\mathbf{1}_p = \mathbf{1}_{p\times 1}$ and $\mathbf{0}_p = \mathbf{0}_{p\times 1}$ and define e_i to be vector of zeros of appropriate dimension with entry 1 at position i. For a real-valued 1d-array $\{x_i\}_{i=1}^n$, we let $\text{diag}(\{x_i\}_{i=1}^n) \in \mathbb{R}^{n\times n}$ be the associated diagonal matrix.

Given a real-valued 2d-array $\{A_{ij}\}$ with $i, j \in \mathcal{I}_n$, let $A \in \mathbb{R}^{n\times n}$ denote the associated matrix and A^T the transposed matrix. We use the following standard notation for submatrices [43]: for two non-empty index sets $\alpha, \beta \subseteq \mathcal{I}_n$ let $A[\alpha, \beta]$ denote the submatrix of A obtained by the rows indexed by α and the columns indexed by β and define the shorthands $A[\alpha, \beta) = A[\alpha, \mathcal{I}_n \setminus \beta]$, $A(\alpha, \beta] = A[\mathcal{I}_n \setminus \alpha, \beta]$, and $A(\alpha, \beta) = A[\mathcal{I}_n \setminus \alpha, \mathcal{I}_n \setminus \beta]$. We adopt the shorthand $A[\{i\}, \{j\}] = A[i, j] = A_{ij}$ for $i, j \in \mathcal{I}_n$, and for $x \in \mathbb{R}^n$ the notation $x[\alpha, \{1\}] = x[\alpha]$ and $x(\alpha, \{1\}) = x(\alpha)$. For illustration, equation (1) can be written unambiguously as

$$\left[\begin{array}{c} I[\alpha] \\ \hline I(\alpha) \end{array}\right] = \left[\begin{array}{c|c} Q[\alpha,\alpha] & Q[\alpha,\alpha) \\ \hline Q(\alpha,\alpha] & Q(\alpha,\alpha) \end{array}\right] \left[\begin{array}{c} V[\alpha] \\ \hline V(\alpha) \end{array}\right].$$

If $A(\alpha, \alpha)$ is nonsingular, then the *Schur complement* of A with respect to the block $A(\alpha, \alpha)$ (or equivalently the indices α) is the $|\alpha| \times |\alpha|$ dimensional matrix $A/A(\alpha, \alpha)$ defined by

$$A/A(\alpha,\alpha) \triangleq A[\alpha,\alpha] - A[\alpha,\alpha)A(\alpha,\alpha)^{-1}A(\alpha,\alpha].$$

If A is Hermitian, then we implicitly assume that its eigenvalues are arranged in increasing order: $\lambda_1(A) \leq \ldots \leq \lambda_n(A)$. The reader is referred to [44] for a review of matrix analysis.

Algebraic graph theory: Consider the undirected, connected, and weighted graph $G = (\mathcal{I}_n, \mathcal{E}, A)$ with node set $\mathcal{I}_n$ and edge set $\mathcal{E} \subseteq \mathcal{I}_n \times \mathcal{I}_n$ induced by a symmetric, nonnegative, and irreducible *adjacency matrix* $A \in \mathbb{R}^{n\times n}$. A non-zero off-diagonal element $A_{ij} > 0$ corresponds to a weighted edge $\{i, j\} \in \mathcal{E}$, and a non-zero diagonal elements $A_{ii} > 0$ corresponds to a weighted self-loop $\{i, i\} \in \mathcal{E}$. We define the corresponding *degree matrix* by $D \triangleq \text{diag}\big(\{\sum_{j=1}^n A_{ij}\}_{i=1}^n\big)$.

The *Laplacian matrix* is the symmetric matrix defined by $L \triangleq D - A$. Note that self-loops, even though apparent in the adjacency matrix A, do not appear in the Laplacian matrix L.

For these reasons and motivated by the conductance matrix in circuit theory, we define the *loopy Laplacian matrix* $Q(A) = Q \triangleq L + \text{diag}(\{A_{ii}\}_{i=1}^{n}) \in \mathbb{R}^{n\times n}$. Note that adjacency matrix A can be recovered from the loopy Laplacian Q as $A = -Q + \text{diag}(\{\sum_{j=1,j\neq i}^{n} Q_{ij}\}_{i=1}^{n})$, and thus Q uniquely induces the graph G. We refer to Q as *strictly loopy* (respectively *loop-less*) Laplacian, if the graph induced by Q features at least one (respectively no) positively-weighted self-loop.

For a connected graph $\ker(L) = \text{span}(\mathbf{1}_n)$, and all $n-1$ remaining non-zero eigenvalues of L are strictly positive. Specifically, the second-smallest eigenvalue $\lambda_2(L)$ is a spectral connectivity measure called the *algebraic connectivity*. Recall that *irreducibility* of either A, L, or Q is equivalent to connectivity of G, which is again equivalent to $\lambda_2(L) > 0$. We refer to [45] for further details on algebraic graph theory.

The *effective resistance* R_{ij} between two nodes $i, j \in \mathcal{I}_n$ of an undirected connected graph with loopy Laplacian Q is

$$R_{ij} \triangleq (e_i - e_j)^T Q^\dagger (e_i - e_j) = Q^\dagger_{ii} + Q^\dagger_{jj} - 2Q^\dagger_{ij}\,, \qquad (3)$$

where $Q^\dagger$ is the Moore-Penrose pseudo inverse of Q. Since $Q^\dagger$ is symmetric (follows from the singular value decomposition), the matrix of effective resistances R is again a symmetric matrix with zero diagonal elements $R_{ii} = 0$. The effective resistance R_{ij} can be thought of as a graph-theoretic metric, and it is mostly analyzed for a loop-less and uniformly weighted graph with $Q \equiv L$. We do not restrict ourselves to this case here. We refer the reader to [10], [16], [39]–[42] for various applications and properties of the effective resistance as well as interesting results relating R, L, Q, $L^\dagger$, and Q^{-1}.

Remark I.1 (Physical interpretation) If the graph is understood as a resistive circuit with conductance matrix Q, the effective resistance R_{ij} corresponds to the potential difference between the nodes i and j when a unit current is injected in i and extracted in j. In this case, the current-balance equations are $e_i - e_j = QV$. The effective resistance R_{ij}, defined as the potential difference $R_{ij} = (e_i - e_j)^T V$, can be obtained via the impedance matrix $Q^\dagger$ as $R_{ij} = (e_i - e_j)^T Q^\dagger (e_i - e_j)$. □

II. Problem Setup, Basic Results, and Applications

A. The Kron Reduction Process

Consider an undirected, connected, and weighted graph $G = (\mathcal{I}_n, \mathcal{E}, A)$ and its associated symmetric and irreducible matrices: the adjacency matrix $A \in \mathbb{R}^{n\times n}$, Laplacian matrix $L(A)$, and loopy Laplacian matrix $Q(A)$. Furthermore, let $\alpha \subsetneq \mathcal{I}_n$ be a proper subset of nodes with $|\alpha| \geq 2$. We define the $(|\alpha| \times |\alpha|)$ dimensional *Kron-reduced matrix* Q_{red} by

$$Q_{\text{red}} \triangleq Q/Q(\alpha,\alpha)\,. \qquad (4)$$

In the following, we refer to the nodes α and $\mathcal{I}_n \setminus \alpha$ as *boundary nodes* and *interior nodes*, respectively. The following lemma establishes the existence of the Kron-reduced matrix Q_{red} as well as some structural closure properties.

Lemma II.1 (Structural Properties of Kron Reduction) *Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric irreducible loopy Laplacian and let α be a proper subset of $\mathcal{I}_n$ with $|\alpha| \geq 2$. The following statements hold for the Kron-reduced matrix $Q_{\text{red}} = Q/Q(\alpha,\alpha)$:*

1) **Existence:** *The Kron-reduced matrix Q_{red} is well defined.*
2) **Closure properties:** *If Q is a symmetric loopy, strictly loopy, or loop-less Laplacian matrix, respectively, then Q_{red} is a symmetric loopy, strictly loopy, or loop-less Laplacian matrix, respectively.*
3) **Accompanying matrix***: The accompanying matrix $Q_{\text{ac}} \triangleq -Q[\alpha,\alpha)Q(\alpha,\alpha)^{-1} \in \mathbb{R}^{|\alpha|\times(n-|\alpha|)}$ is non-negative. If the subgraph among the interior nodes is connected and each boundary node is adjacent to at least one interior node, then Q_{ac} is positive. If additionally, $Q \equiv L$ is a loop-less Laplacian, then $Q_{\text{ac}} = L_{\text{ac}} \triangleq -L[\alpha,\alpha)L(\alpha,\alpha)^{-1}$ is column stochastic.*

An interesting consequence of Lemma II.1 is that Q_{red}, as a loopy Laplacian matrix, induces again an undirected and weighted graph. Hence, Kron reduction, originally defined as an algebraic operation in equation (4), can be equivalently interpreted as a *graph-reduction process*, or as *physical reduction* of the associated circuit. This interplay between linear algebra, graph theory, and physics is illustrated in Figure 1.

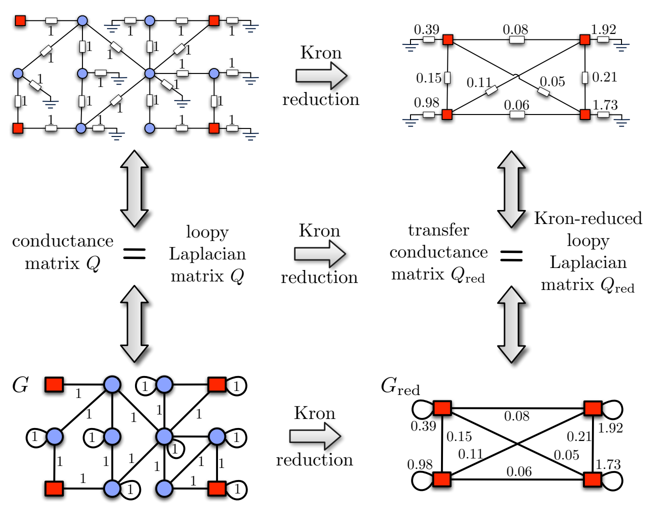

Fig. 1. Illustration of an electrical network with 4 boundary nodes, 8 interior nodes, and unit-valued branch and shunt conductances. The associated loopy Laplacian Q and the graph G are equivalent representations. Kron reduction of the interior nodes results in a reduced network among the boundary nodes with the Kron-reduced matrix Q_{red} and graph G_{red}.

In the following we denote the reduced graph induced by Q_{red} as G_{red}, and define the corresponding reduced adjacency, degree, and loop-less Laplacian matrices by $A_{\text{red}} \triangleq -Q_{\text{red}} + \text{diag}(\{\sum_{j=1,j\neq i}^{n} Q_{\text{red}}[i,j]\}_{i\in\alpha})$, $D_{\text{red}} \triangleq \text{diag}\big(\{\sum_{j=1}^{n} A_{\text{red}}[ij]\}_{i=1}^{n}\big)$, and $L_{\text{red}} \triangleq D_{\text{red}} - A_{\text{red}}$. We remark that Lemma II.1 is partially also noted in [4], [5], [13], [27], [28], [30], and we present a self-contained proof here.

Proof of Lemma II.1. First, consider the case when the graph among the interior nodes is connected, or equivalently $Q(\alpha,\alpha)$ is irreducible. By definition, Q is (weakly) diagonally dominant since $Q_{ii} = \sum_{j=1,j\neq i}^{n} |Q_{ij}| + A_{ii}$ for all $i \in \mathcal{I}_n$. Due to the irreducibility of Q the strict inequality $Q_{ii} > \sum_{j=1,j\neq i,j\notin\alpha}^{n} |Q_{ij}| + A_{ii}$ holds at least for one $i \in \mathcal{I}_n \setminus \alpha$. It follows that $Q(\alpha,\alpha)$ is also irreducible, diagonally dominant, and has at least one row with strictly positive row sum. Hence, $Q(\alpha,\alpha)$ is invertible [44, Corollary 6.2.27]. If the graph among the interior nodes consists of multiple connected components, then, after appropriately labeling the interior nodes, the matrix $Q(\alpha,\alpha)$ is block-diagonal with irreducible diagonal blocks corresponding to the connected components. The previous arguments applied to each diagonal block yield that $Q(\alpha,\alpha)$ is nonsingular, and statement 1) follows.

Statement 2) is a consequence of the closure properties of the Schur complement [43, Chapter 4], which includes the classes of symmetric, positive definite, and M-matrices. Since Q is a symmetric M-matrix, we conclude that $Q_{\text{red}} = Q/Q(\alpha,\alpha)$ is also a symmetric M-matrix. Hence, Q_{red} is a symmetric loopy Laplacian. This fact together with the closure of positive definite matrices reveals that the class of symmetric strictly loopy Laplacians is closed under Kron reduction. To prove the closure of symmetric loop-less Laplacians, assume without loss of generality that $\alpha = \mathcal{I}_{|\alpha|}$, and consider the following equality for the row sums of the loop-less Laplacian Q:

$$\left[\begin{array}{c|c} Q[\alpha,\alpha] & Q[\alpha,\alpha) \\ \hline Q(\alpha,\alpha] & Q(\alpha,\alpha) \end{array}\right]\left[\begin{array}{c} \mathbf{1}_{|\alpha|} \\ \hline \mathbf{1}_{n-|\alpha|} \end{array}\right] = \left[\begin{array}{c} \mathbf{0}_{|\alpha|} \\ \hline \mathbf{0}_{|\alpha|} \end{array}\right]. \tag{5}$$

Elimination of the second block of equations in (5) results in $\mathbf{0}_{|\alpha|} = Q_{\text{red}}\mathbf{1}_{|\alpha|}$, which shows that Q_{red} is a loop-less Laplacian and concludes the proof of statement 2).

The second block of equations in (5) can be rewritten as $\mathbf{1}_{n-|\alpha|} = Q_{\text{ac}}^T\mathbf{1}_{|\alpha|}$. Hence, Q_{ac} is a column stochastic matrix in the loop-less case. In general, $Q_{\text{ac}} = -Q[\alpha,\alpha)Q(\alpha,\alpha)^{-1}$ is nonnegative, since $-Q[\alpha,\alpha)$ and the inverse of the M-matrix $Q(\alpha,\alpha)$ are both nonnegative. If additionally each boundary node is connected to at least one interior node and the graph among the interior nodes is connected, then each row of $-Q[\alpha,\alpha)$ has at least one positive entry. Moreover, since $Q(\alpha,\alpha)$ is an irreducible non-singular M-matrix, $Q(\alpha,\alpha)^{-1}$ is positive [28, Theorem 5.12]. The latter two facts guarantee positivity of Q_{ac} and complete the proof of statement 3). ■

As mentioned in Section I, the Kron reduction has various applications, and its general purpose is to construct low dimensional "equivalent" matrices, graphs, or circuits. In the following we describe different examples arising in Markov chains, circuit theory, impedance tomography, power flow studies, transient stability assessment, and smart grid monitoring.

B. Stochastic Complements and Markov Chain Reduction

A concept related to Kron reduction is the reduction of nonnegative, irreducible, and row stochastic matrices via the Perron complement [29], [34], [35]. The latter concept finds application in Markov chain reduction [33] and in data mining [36], where it is termed *stochastic complement*. Here we relate the Schur complement of a Laplacian with the stochastic complement of the corresponding Markov chain transition matrix. Hence, our results pertaining to Kron reduction can be analogously stated for the stochastic complement. For instance, the topological properties are identical, and the spectral, algebraic, and resistive properties can be easily and naturally related via the degree matrix of the boundary nodes.

Given a loop-less graph induced by a symmetric, nonnegative, and irreducible adjacency matrix $A \in \mathbb{R}^{n\times n}$ with corresponding degree matrix D, we define the corresponding *transition matrix* by $P \triangleq D^{-1}A$. The transition matrix P induces the state transition map $x^+ = Px$ of a finite-state Markov chain, it is nonnegative, irreducible, and row stochastic, that is, $P\mathbf{1}_n = \mathbf{1}_n$. Generally, P is not symmetric. By the definitions of D, $L = D - A$, and $P = D^{-1}A$, we have that $L = D(I_n - P)$. For $\alpha \in [2, n-1]$, the Kron-reduced Laplacian is given by the Schur complement $L_{\text{red}} = L/L(\alpha,\alpha) = D_{\text{red}} - A_{\text{red}}$, and we define the *reduced transition matrix* P_{stc} by the stochastic complement [33]

$$P_{\text{stc}} \triangleq P[\alpha,\alpha] + P[\alpha,\alpha)(I_\alpha - P(\alpha,\alpha))^{-1}P(\alpha,\alpha]\,.$$

The reduced transition matrix P_{stc} has various interesting properties. For instance, analogously to Lemma II.1, P_{stc} is again nonnegative, irreducible, and row-stochastic [33, Theorem 2.3]. We refer to [29], [33]–[35] for further details. Finally, we define the *pseudo-reduced adjacency matrix* $A_{\text{stc}} \triangleq A[\alpha,\alpha] + A[\alpha,\alpha)(D(\alpha,\alpha) - A(\alpha,\alpha))^{-1}A(\alpha,\alpha]$. Then, based on the fundamental relation between Schur and Perron complements shown in [33]–[36], we can state the following lemma relating Kron reduction and the stochastic complement.

Lemma II.2 *(***Kron Reduction and Stochastic Complementation)** *Consider a loop-less graph induced by a symmetric, nonnegative, and irreducible adjacency matrix $A \in \mathbb{R}^{n\times n}$ with degree matrix D, Laplacian $L = D - A$, and transition matrix $P = D^{-1}A$. Let α be a proper subset of $\mathcal{I}_n$ with $|\alpha| \geq 2$, and consider the Kron-reduced Laplacian $L_{\text{red}} = L/L(\alpha,\alpha) = D_{\text{red}} - A_{\text{red}}$, the reduced transition matrix P_{stc}, and the pseudo-reduced adjacency matrix A_{stc}. The following identities hold:*

$$P_{\text{stc}} = D[\alpha,\alpha]^{-1}A_{\text{stc}}\,, \tag{6}$$

$$L_{\text{red}} = D_{\text{red}} - A_{\text{red}} = D[\alpha,\alpha] - A_{\text{stc}} = D[\alpha,\alpha](I_\alpha - P_{\text{stc}}). \tag{7}$$

Identity (6) gives an intuitive relation of the reduced transition matrix, the degree matrix $D[\alpha,\alpha]$, and the pseudo-reduced adjacency matrix A_{stc} among the boundary nodes. Identity (7) implies that $A_{\text{red}}[i,j] = A_{\text{stc}}[i,j] = P_{\text{stc}}[i,j]\cdot D_i$ for all distinct $i,j \in \alpha$, that is, the matrics A_{red} and A_{stc} induce the same reduced graph besides self-loops. The diagonal elements satisfy $A_{\text{red}}[i,i] = 0$ and $A_{\text{stc}}[i,i] = D_i - D_{\text{red}}[i,i] = P_{\text{stc}}[i,i]\cdot D_i$. In case that the original graph features self-loops, then the identities stated later in Theorem III.6 allow to directly relate the Kron-reduced strictly loopy Laplacian Q_{red} and identity (7).

Proof of Lemma II.2. To prove identity (6), recall that $P = D^{-1}A$, and consider the following set of equalities

$$\begin{aligned} P_{\text{stc}} =& D[\alpha,\alpha]^{-1}(A[\alpha,\alpha] + A[\alpha,\alpha) \\ &\times ((I_\alpha - D(\alpha,\alpha)^{-1}A(\alpha,\alpha))^{-1}D(\alpha,\alpha)^{-1}A(\alpha,\alpha] \\ =& D[\alpha,\alpha]^{-1}(A[\alpha,\alpha] + A[\alpha,\alpha) \\ &\times ((D(\alpha,\alpha) - A(\alpha,\alpha))^{-1}A(\alpha,\alpha] = D[\alpha,\alpha]^{-1}A_{\text{stc}}\,, \end{aligned}$$

where we used $(I_\alpha - V^{-1}U)^{-1} = (V-U)^{-1}V$ (for a nonsingular $(\alpha\times\alpha)$-matrix V), see [46, Equation (13)].

To prove identity (7) consider the following set of equalities,

$$\begin{aligned} &L_{\text{red}} = L[\alpha,\alpha] - L[\alpha,\alpha)L(\alpha,\alpha)^{-1}L(\alpha,\alpha] \\ &= (D[\alpha,\alpha] - A[\alpha,\alpha]) - A[\alpha,\alpha)(D(\alpha,\alpha) - A(\alpha,\alpha))^{-1}A(\alpha,\alpha] \\ &= D[\alpha,\alpha] - A_{\text{stc}} = D[\alpha,\alpha] - D[\alpha,\alpha]P_{\text{stc}}\,, \end{aligned}$$

where we used identity (6) in the last inequality. ■

C. Kron Reduction in Large-Scale Integration Chips

In large-scale integration chips, it is of interest to reduce the complexity of large-scale circuits by replacing them with equivalent lower dimensional circuits with the same terminals (boundary nodes) [7], [8]. The circuit reduction problem also stimulated a matrix-theoretic and behavioral analysis from the viewpoint of boundary nodes [4]–[6], [13]. For resistive networks, Kron reduction leads to such an equivalent reduced circuit. A particular reduction goal in [7] is to reduce the fill-in of the Kron-reduced matrix Q_{red} for computation of the effective resistance. The proper choice of the boundary nodes has a tremendous effect on the sparsity of the Kron-reduced matrix and saves numerical effort in subsequent computations, see Figure 2. Reduction of the fill-in is also a pervasive objective in the computational applications [24]–[27].

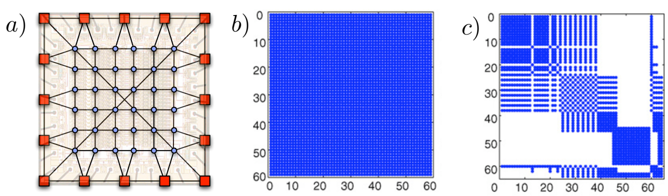

Fig. 2. a) Illustration of an integration chip with a symmetric top-level network connecting the interior nodes with the terminals. The Kron reduction of all interior nodes of a network with 59 terminals results in a Kron-reduced matrix of dimension 59^2 with $59^2 = 3481$ non-zero entries. If all but five specifically chosen interior nodes are eliminated, then the Kron-reduced matrix is of dimension 64^2 but has only 1506 non-zero entries. The corresponding sparsity patterns are shown in subfigures b) and c), which are taken from [7].

In [7] it is argued that reduction of a connected component of Q results in a dense component in Q_{red} and the effective resistance among boundary nodes is invariant under Kron reduction. We remark that these arguments are based on numerical observations and physical intuition. This paper puts the statements of [7] on solid mathematical ground. We prove invariance of the effective resistance under Kron reduction and rigorously show under which conditions a sparse topology becomes dense or even complete. Moreover, our setup encompasses shunt loads and currents drawn from the interior network, thereby generalizing results in [4]–[6], [13].

D. Electrical Impedance Tomography

In electrical impedance tomography the goal is to determine the conductivity inside a compact spatial domain $\Omega \subset \mathbb{R}^2$ from simultaneous measurements of currents and voltages at the boundary of Ω, that is, from measurement of the *Dirichlet-to-Neumann map*. Electrical impedance tomography finds applications in geophysics and medical imaging. A natural approach is a discretization of the spatial domain to a resistor network with conductance matrix Q. As seen in equations (2) with $I_\beta = \mathbf{0}_{n\times 1}$, when a unit potential is imposed at boundary node j and a zero potential at all other boundary nodes, the current measured at boundary node i gives the reduced transfer conductance $Q_{\text{red}}[i,j]$. Other methods iteratively construct the reduced impedance matrix $Q_{\text{red}}^\dagger$ from measurements of the effective resistance R [23]. The goal is then to invert the Kron reduction and recover the original network Q from the reduced

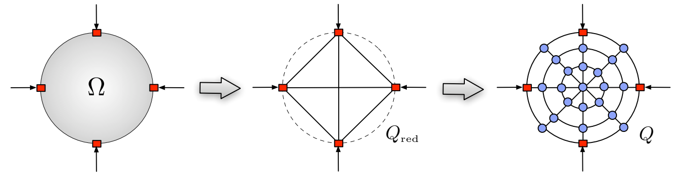

Fig. 3. In electric impedance tomography the conductivity of the spatial domain Ω is estimated by measuring the Kron-reduced matrix Q_{red} at the boundary nodes ■. From these measurements the conductance matrix Q is re-constructed and serves as a spatial discretization of Ω.

network Q_{red}, as illustrated in Figure 3. This is feasible only for highly symmetric networks [21]–[23], but generally it is not possible to infer structural properties from Q_{red} to Q.

This paper provides non-iterative identities relating the effective resistance matrix R and the Kron-reduced impedance matrix $Q_{\text{red}}^\dagger$ as well as explicit identities relating R and Q_{red} for uniform networks. Furthermore, our analysis allows to partially invert the Kron reduction by estimating the spectrum of Q or its effective resistance from the spectrum or resistance of Q_{red}. Finally, our framework allows also for dissipation of energy in the spatial domain via loads in the resistor network.

E. Sensitivity of Reduced Power Flow

Large-scale power transmission networks can be modeled as circuits, with generators and load buses as nodes, see Figure 4. Each transmission line $\{i,j\}$ is weighted by a (typically

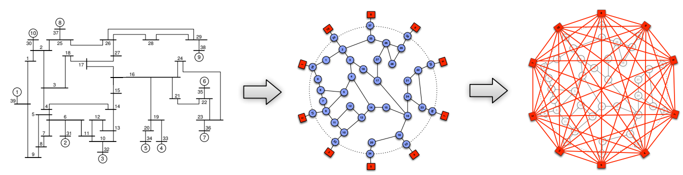

Fig. 4. Single line diagram of the New England Power Grid [14], an equivalent schematic representation with generators ■ and buses ●, and the corresponding Kron-reduced network

inductive) admittance $A_{ij} = A_{ji} \in \mathbb{C}$. Whereas generator i injects a current $I_i \in \mathbb{C}$, the load at a bus j draws a current $I_j \in \mathbb{C}$ and features a shunt admittance $A_{jj} \in \mathbb{C}$. Hence, the power network obeys the current-balance equations $I = QV$, where the nodal *admittance matrix* $Q \in \mathbb{C}^{n\times n}$ is the loopy Laplacian induced by the admittances A_{ij}. Depending on the application, the current balance equations are sometimes converted to the *power flow equations* $S = V \circ (QV)^*$, where $\circ$ is the Hadamard product, * denotes the conjugate transposed, and $S = V \circ I^*$ is the vector of power injections.

A critical task in power network operation is monitoring and control of the power flow. The determining equations $S = V \circ (QV)^*$ are too complicated to admit an analytic solution and often too onerous for a computational approach [11], [12]. If a set of nodes α is identified for sensing or control purposes, then all remaining nodes can be eliminated via Kron reduction leading to the reduced current-balance equations (2). The corresponding reduced power flow is obtained as $S_{\text{red}} = V[\alpha] \circ (Q_{\text{red}} V[\alpha])^*$, where $S_{\text{red}} = V[\alpha] \circ I[\alpha]^* + V[\alpha] \circ (Q_{\text{ac}} I(\alpha))^*$.

For the lossless case when Q is purely imaginary, this paper provides insightful and explicit results showing how perturbations in weights or topology of Q affect the reduced *transfer admittance matrix* Q_{red}. We also show the effect of shunt and current loads on the reduced network. For instance, a positive shunt load $Q_{ii} > 0$ in the non-reduced network weakens the mutual transfer admittances $Q_{\text{red}}[i,j]$ in the reduced network and increases the reduced loads $Q_{\text{red}}[i,i]$.

F. Monitoring of DC Power Flow in Smart Grid

The linearized *DC power flow equations* are $P = B\theta$, where $P = \Re(S) \in \mathbb{R}^n$ are the real power injections, $\theta \in \mathbb{R}^n$ are the voltage phase angles, and $B = -\Im(Q) \in \mathbb{R}^{n\times n}$ is the susceptance matrix. Consider the problem of monitoring an area Ω of a *smart power grid* equipped with synchronized phasor measurement units at the buses $\alpha = \{\alpha_1, \alpha_2\}$ bordering Ω [20]. Kron reduction of the DC power flow with respect to the interior nodes $\mathcal{I}_n \setminus \alpha$ yields the reduced DC power flow $P[\alpha] + B_{\text{ac}} P(\alpha) = B_{\text{red}}\theta[\alpha]$, where B_{red} and B_{ac} are defined analogously to Q_{red} and Q_{ac}. From here various scalar stress measures over the area Ω can be defined [20]. Let $\sigma \in \mathbb{R}^{|\alpha|}$ be the indicator vector for the boundary buses α_1, that is, $\sigma_i = 1$ if $i \in \alpha_1$ and zero otherwise. The cutset power flow over the area Ω is $P_{\text{cut}} = \sigma^T P[\alpha] + \sigma^T B_{\text{ac}} P(\alpha)$, the cutset susceptance is $b_{\text{cut}} = \sigma^T B_{\text{red}} \sigma$, and the corresponding cutset angle is $\theta_{\text{cut}} = P_{\text{cut}}/b_{\text{cut}}$. Hence, the area Ω is reduced to two nodes $\{1,2\}$ exchanging the power flow P_{cut} with angle θ_{cut} over the susceptance b_{cut}, see Figure 5. These scalar quantities

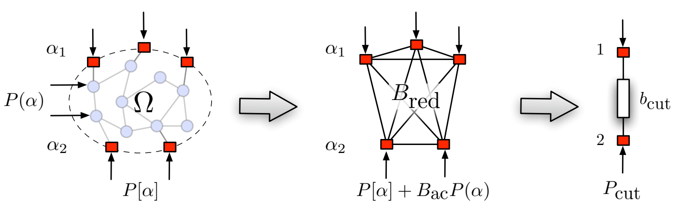

Fig. 5. Reduction of the area Ω with boundary buses ■ to a single line equivalent $\{1, 2\}$ describing the electrical characteristics between the set of boundary buses α_1 and the set of boundary buses α_2.

indicate the stress within the area Ω. For instance, a large cutset angle θ_{cut} could be a blackout risk precursor. Of special interest are how load changes, line outages, or loss of nodes within Ω or on its boundary α affect the cutset angle θ_{cut}.

This paper provides a comprehensive and detailed analysis of how changes in topology and weighting of the network affect the Kron-reduced matrix B_{red}. These results include the self-loops in the graph (modeling shunt loads) and can be easily translated to the cutset angle θ_{cut} to show its sensitivity with respect to perturbations in the network.

G. Transient Stability Assessment in Power Networks

Transient stability is the ability of a power network to remain in synchronism when subjected to large disturbances such as faults of system components or severe fluctuations in generation or load. If the power transmission is lossless with purely imaginary admittance matrix Q and the loads are modeled as constant current injections and shunt admittances, the network can be reduced to the generators nodes via Kron reduction. In this case, Q_{red} is also purely imaginary, and the dynamics of generator i are given by the *swing equations* [14]–[16]

$$M_i \ddot{\theta}_i = -D_i \dot{\theta}_i + P_i - \sum\nolimits_{j=1}^{|\alpha|} P_{ij} \sin(\theta_i - \theta_j), \quad i \in \mathcal{I}_n\,, \tag{8}$$

where $(\theta_i, \dot{\theta}_i)$ are the generator rotor angle and speed, $M_i > 0$ and $D_i > 0$ are the inertia and damping constant, the *coupling weight* $P_{ij} = |V_i||V_j|\Im(Q_{\text{red}}[i,j]) > 0$ is the maximum power transfer between generators i and j, and the *effective power input* $P_i = P_{\text{m},i} + \Re(V_i \sum_{j=1}^{n-|\alpha|} Q^*_{\text{ac}}[i,j] I^*_{|\alpha|+j}))$ results from the mechanical power input $P_{\text{m},i}$ and the current loads $I_{|\alpha|+j}$.

In [47], we derived sufficient conditions under which the reduced model (8) *synchronizes*, that is, all frequency differences $\dot{\theta}_i(t) - \dot{\theta}_j(t)$ converge to zero. For notational simplicity, we assume uniform damping here, that is, $D_i = D$ for all $i \in \alpha$. Then two sufficient conditions for synchronization are

$$|\alpha| \min_{i \neq j}\{P_{ij}\} > \max_{i,j \in \mathcal{I}_{|\alpha|}} \{P_i - P_j\}\,, \tag{9}$$

$$\lambda_2(L(P_{ij})) > \Big(\sum\nolimits_{i,j=1,\, i<j}^{|\alpha|} (P_i - P_j)^2\Big)^{1/2}. \tag{10}$$

The right-hand sides of conditions (9)-(10) measure the non-uniformity in effective power inputs P_i, and the left-hand sides reflect the connectivity in the reduced network: the term $|\alpha| \min_{i\neq j}\{P_{ij}\}$ lower-bounds $\min_i \sum_{j=1}^{|\alpha|} P_{ij}$, the worst coupling of one generator to the network, and $\lambda_2(L(P_{ij}))$ is the algebraic connectivity of the coupling. In summary, conditions (9)-(10) read as "the reduced network connectivity has to dominate the non-uniformity in effective power inputs."

For uniformly lower-bounded voltage magnitudes at all generators $|V_i| \geq V > 0$ the analysis of this paper will reveal that the spectral condition (9) in the reduced network can be converted to the spectral synchronization condition

$$\lambda_2(L) > \Big(\sum\nolimits_{i,j=1,\, i<j}^{|\alpha|} (P_i - P_j)^2\Big)^{1/2} \frac{1}{V^2} + \max_{i \in \mathcal{I}_n}\{A_{\text{red}}[i,i]\}\,, \tag{11}$$

where L is the Laplacian of the original lossless power network (weighted by $\Im(-A_{ij})$) and $A_{\text{red}}[i,i]$ is the ith shunt load in the reduced network. Similarly, if the effective resistance among all generators takes the uniform value R and the effective resistance between the generators and the ground is uniform as well, then the results of this paper render the element-wise condition (10) in the reduced network to a resistive synchronization condition in the non-reduced network:

$$\frac{1}{R} > \max_{i,j \in \mathcal{I}_{|\alpha|}} \{P_i - P_j\} \frac{1}{2V^2} + \max_{i \in \mathcal{I}_n}\{A_{\text{red}}[i,i]\}\,. \tag{12}$$

Conditions (11)-(12) state that the network connectivity has to overcome the non-uniformity in effective power inputs and the dissipation by the loads, such that the network synchronizes.

III. Kron Reduction of Graphs

This section analyzes the algebraic, topological, spectral, and sensitivity properties, as well as the effective resistance of the Kron-reduced matrix Q_{red} and its associated graph. Throughout this section we assume that $Q \in \mathbb{R}^{n\times n}$ is a symmetric and irreducible loopy Laplacian matrix (corresponding to an undirected, connected, and weighted graph with

n nodes), and we let α be a proper subset of $\mathcal{I}_n$ with $|\alpha| \geq 2$. For notational simplicity and without loss of generality, we assume that the n nodes are labeled such that $\alpha = \mathcal{I}_{|\alpha|}$.

A. The Augmented Laplacian and Iterative Kron Reduction

The concepts presented in this subsection will be central to the subsequent developments both for illustration and analysis.

The role of the self-loops induced by a *strictly* loopy Laplacian $Q \in \mathbb{R}^{n \times n}$ can be better understood by introducing the additional *grounded node* with index $n+1$. Then the strictly loopy Laplacian Q is the principal $n \times n$ block embedded in the $(n+1) \times (n+1)$ dimensional *augmented Laplacian matrix*

$$\widehat{Q} \triangleq \left[\begin{array}{c|c} Q & -\mathrm{diag}(\{A_{ii}\}_{i=1}^{n})\mathbf{1}_n \\ \hline -\mathbf{1}_n^T \mathrm{diag}(\{A_{ii}\}_{i=1}^{n}) & \sum_{i=1}^{n} A_{ii} \end{array}\right], \quad (13)$$

where $A \in \mathbb{R}^{n \times n}$ is the adjacency matrix corresponding to Q. The augmented Laplacian $\widehat{Q}$ is the Laplacian of the augmented graph $\widehat{G}$ with node set $\widehat{\mathcal{V}} = \{\mathcal{I}_n, n+1\}$ and edge set $\widehat{\mathcal{E}} = \{\mathcal{E}, \mathcal{E}_{\text{augment}}\}$. Here a node $i \in \mathcal{I}_n$ is connected to the grounded node $n+1$ via a weighted edge $\{i, n+1\} \in \mathcal{E}_{\text{augment}}$ if and only if $A_{ii} > 0$, see Figure 6 for an illustration.

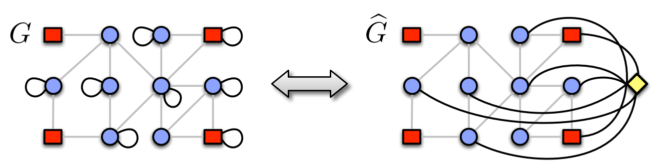

Fig. 6. Illustration of the graph G associated with the circuit from Figure 1 and the corresponding augmented graph $\widehat{G}$ with additional grounded node ◊.

Lemma III.1 **(Properties of the augmented Laplacian)** *Consider the symmetric and irreducible strictly loopy Laplacian $Q \in \mathbb{R}^{n \times n}$ and the corresponding augmented Laplacian matrix $\widehat{Q} \in \mathbb{R}^{(n+1)\times(n+1)}$. The following statements hold:*

1) **Algebraic properties:** *$\widehat{Q}$ is an irreducible and symmetric loop-less Laplacian matrix.*
2) **Spectral properties:** *The eigenvalues of Q and $\widehat{Q}$ interlace each other, that is, $0 = \lambda_1(\widehat{Q}) < \lambda_1(Q) \leq \lambda_2(\widehat{Q}) \leq \lambda_2(Q) \leq \dots \leq \lambda_n(\widehat{Q}) \leq \lambda_n(Q) \leq \lambda_{n+1}(\widehat{Q})$.*
3) **Kron reduction:** *Consider the strictly loopy Laplacian Q_{red} and the loop-less Laplacian $\widehat{Q}_{\text{red}} \triangleq \widehat{Q}/\widehat{Q}(\{\alpha, n+1\},\{\alpha, n+1\})$, both obtained by Kron reduction of the interior nodes $\mathcal{I}_n \setminus \alpha$. The following diagram commutes:*

$$\begin{tikzcd} Q \arrow[r, "\text{augment}"] \arrow[d, "\begin{array}{c}\text{Kron reduction}\\ \text{of } \mathcal{I}_n \setminus \alpha\end{array}"'] & \widehat{Q} \arrow[d, "\begin{array}{c}\text{Kron reduction}\\ \text{of } \mathcal{I}_n \setminus \alpha\end{array}"] \\ Q_{\text{red}} \arrow[r, "\text{augment}"'] & \widehat{Q}_{\text{red}} \end{tikzcd}$$

In equivalent words, $\widehat{Q}_{\text{red}}$ is the augmented Laplacian associated to Q_{red}, that is, $\widehat{Q}_{\text{red}}$ takes the form

$$\left[\begin{array}{c|c} Q_{\text{red}} & -\mathrm{diag}(\{A_{\text{red}}[i,i]\}_{i\in\alpha})\mathbf{1}_{|\alpha|} \\ \hline -\mathbf{1}_{|\alpha|}^T \mathrm{diag}(\{A_{\text{red}}[i,i]\}_{i\in\alpha}) & \sum_{i=1}^{n} A_{\text{red}}[i,i] \end{array}\right]. \quad (14)$$

Properties 2) and 3) of Lemma III.1 intuitively illustrate the effect of self-loops on the spectrum of Q and its Kron-reduced matrix. Specifically, the elegant relationship 3) implies that the Kron reduction can be equivalently applied to the strictly loopy network G or to the augmented loop-less network $\widehat{G}$.

Proof of Lemma III.1. Property 1) follows trivially from the construction of the augmented Laplacian $\widehat{Q}$. Property 2) is a direct application of the *interlacing theorem for bordered matrices* [44, Theorem 4.3.8], where $0 = \lambda_1(\widehat{Q}) < \lambda_1(Q)$ since $\widehat{Q}$ is an irreducible loop-less Laplacian and Q is non-singular. In property 3), the upper left block of the matrix on the right-hand side of identity (14) follows by writing out the Schur complement of a matrix partitioned in 3×3 blocks, as in the proof of the *Quotient Formula* [43, Theorem 1.4]. The remaining blocks follow immediately since Kron reduction of the loop-less Laplacian $\widehat{Q}_{\text{red}}$ yields again a loop-less Laplacian by Lemma II.1. This completes the proof of property 3). ■

Gaussian elimination of interior voltages from the current-balance equations $I = QV$ can either be performed via Kron reduction in a single step, as in equation (2), or in multiple steps, each interior node $\ell \in \{1, \dots, n - |\alpha|\}$ at a time. The following concept addresses exactly this point.

Definition III.2 **(Iterative Kron reduction)** *Iterative Kron reduction associates to a symmetric irreducible loopy Laplacian matrix $Q \in \mathbb{R}^{n \times n}$ and indices $\{1, \dots, |\alpha|\}$, a sequence of matrices $Q^\ell \in \mathbb{R}^{(n-\ell)\times(n-\ell)}$, $\ell \in \{1, \dots, n-|\alpha|\}$, defined by*

$$Q^\ell = Q^{\ell-1}/Q^{\ell-1}_{k_\ell k_\ell}, \quad (15)$$

where $Q^0 = Q$ and $k_\ell = n+1-\ell$, that is, $Q^{\ell-1}_{k_\ell k_\ell}$ is the lowest diagonal entry of $Q^{\ell-1}$.

If the sequence (15) is well-defined, then each Q^ℓ is a loopy Laplacian matrix inducing a graph by Lemma II.1. Before going further into the details of iterative Kron reduction, we illustrate the unweighted graph corresponding to Q^ℓ (the sparsity pattern of the corresponding adjacency matrix) in Figure 7. When no self-loops are present, then the topological

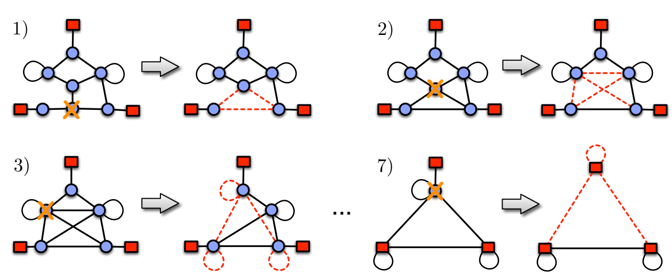

Fig. 7. Sparsity pattern (or topolgical evolution) corresponding to the iterative Kron reduction (15) of a graph with 3 boundary nodes ■ and 7 interior nodes ●. The dashed red lines indicate the newly added edges in a reduction step.

iteration illustrated in Figure 7 is also known under the name *vertex elimination* in the sparse matrix community [26].

The following observations can be made from Figure 7: (1) The connectivity is maintained. (2) At the ℓth reduction step a new edge between two nodes appears if and only if both were connected to k_ℓ before the reduction, and (3) all other edges persist. (4) Likewise, a new self-loop appears at a node $i \neq k_\ell$

if and only if i was connected to k_ℓ and k_ℓ featured a self-loop before the reduction. Theorem III.4 in the next subsection will turn these observations into rigorous theorems.

In components, Q^ℓ is defined by the celebrated Kron reduction formula illustrating the step-wise Gaussian elimination:

$$Q_{ij}^{\ell} = Q_{ij}^{\ell-1} - \frac{Q_{ik_\ell}^{\ell-1} Q_{jk_\ell}^{\ell-1}}{Q_{k_\ell k_\ell}^{\ell-1}}, \quad i,j \in \{1,\dots,n-\ell\}. \quad (16)$$

For a well-defined sequence $\{Q^\ell\}_{\ell=1}^{n-|\alpha|}$, we let A^ℓ and L^ℓ be the corresponding adjacency and loop-less Laplacian matrix of the ℓth reduction step. The following lemma states some important properties of iterative Kron reduction. In particular, the iterative Kron reduction is well-posed, it ultimately results in the Kron-reduced matrix, and the weights of the self-loops are non-decreasing due to non-decreasing diagonal dominance.

Lemma III.3 *(***Properties of iterative Kron reduction***)* *Consider the matrix sequence $\{Q^\ell\}_{\ell=1}^{n-|\alpha|}$ defined via iterative Kron reduction in equation* (15). *The following statements hold:*

1) **Well-posedness:** *Each matrix Q^ℓ, $\ell \in \{1,\dots,n-|\alpha|\}$, is well defined, and the classes of loopy, strictly loopy, and loop-less Laplacian matrices are closed throughout the iterative Kron reduction.*
2) **Quotient property:** *The Kron-reduced matrix $Q_{\text{red}} = Q/Q(\alpha,\alpha)$ can be obtained by iterative reduction of all interior nodes $k_\ell \in \mathcal{I}_n \setminus \alpha$, that is, $Q_{\text{red}} \equiv Q^{n-|\alpha|}$. Equivalently, the following diagram commutes:*

$$\begin{tikzcd}
Q = Q^0 \arrow[rrrr, bend left, "\text{Kron reduction of } \mathcal{I}_n \setminus \alpha"] \arrow[dr] & & & & Q_{\text{red}} = Q^{n-|\alpha|} \\
& Q^1 \arrow[r] & Q^2 \arrow[r] & \cdots \arrow[r] & Q^{n-|\alpha|+1} \arrow[u]
\end{tikzcd}$$

iterative Kron reduction

3) **Diagonal dominance:** *For $i \in \{1,\dots,n-\ell\}$ the ith row sum of Q^ℓ, $\sum_{j=1}^{n-\ell} Q_{ij}^\ell = A_{ii}^\ell$, is given by*

$$A_{ii}^{\ell} = \begin{cases} A_{ii}^{\ell-1}, & \text{if } A_{k_\ell k_\ell}^{\ell-1} = 0, \\ A_{ii}^{\ell-1} + A_{ik_\ell}^{\ell-1}\Big(1 - \frac{L_{k_\ell k_\ell}^{\ell-1}}{L_{k_\ell k_\ell}^{\ell-1} + A_{k_\ell k_\ell}^{\ell-1}}\Big), & \text{if } A_{k_\ell k_\ell}^{\ell-1} > 0. \end{cases}$$

Proof: Statement 2) is simply the *Quotient Formula* [43, Theorem 1.4] stating that Schur complements (or Gaussian elimination for that matter) can be taken iteratively or in a single step. Furthermore, the Quotient Formula states that all intermediate Schur complements Q^ℓ exist. This fact together with the closure properties in Lemma II.1 proves statement 1).

For notational simplicity and without loss of generality, we prove statement 3) for $\ell = 1$ and $k_1 = n$. Note that $A^0 = A$, $L^0 = L$, $Q^0 = Q$, and consider the ith row sum of Q^1 given by

$$\sum_{j=1}^{n-1} Q_{ij}^1 = \sum_{j=1}^{n-1}\left(Q_{ij} - \frac{Q_{in} Q_{jn}}{Q_{nn}}\right) = \sum_{j=1}^{n-1}\left(Q_{ij} - \frac{A_{in} A_{jn}}{L_{nn} + A_{nn}}\right)$$
$$= A_{ii} + A_{in} - \frac{A_{in}}{L_{nn} + A_{nn}} L_{nn}, \quad (17)$$

where we used equality (16), the identities $Q = L + \operatorname{diag}(\{A_{ii}\}_{i=1}^n)$, $\sum_{j=1}^{n-1} Q_{ij} = A_{ii} + A_{in}$, and $\sum_{j=1}^{n-1} A_{jn} = L_{nn}$. Since $A_{nn} \geq 0$ (due to property 1) nonnegative row sums follow also in the general case), we are left with evaluating identity (17) for the two cases presented in statement 3). ■

B. Topological, Spectral, and Algebraic Properties

In this subsection we begin our characterization of the properties of Kron reduction. We start by discussing how the graph topology of G changes under the Kron reduction process.

Theorem III.4 *(***Topological Properties of Kron Reduction***)* *Let G, G_{red}, and $\widehat{G}$ be the undirected weighted graphs associated to Q, $Q_{\text{red}} = Q/Q(\alpha,\alpha)$, and the augmented loopy Laplacian $\widehat{Q}$, respectively. The following statements hold:*

1) **Edges:** *Two nodes $i, j \in \alpha$ are connected by an edge in G_{red} if and only if there is a path from i to j in G whose nodes all belong to $\{i,j\} \cup (\mathcal{I}_n \setminus \alpha)$.*
2) **Self-loops:** *A node $i \in \alpha$ features a self-loop in G_{red} if and only if there is a path from i to the grounded node $n{+}1$ in $\widehat{G}$ whose nodes all belong to $\{i, n{+}1\} \cup (\mathcal{I}_n \setminus \alpha)$. Equivalently, a node $i \in \alpha$ features a self-loop in G_{red} if and only if i features a self-loop in G or there is a path from i to a loopy interior node $j \in \mathcal{I}_n \setminus \alpha$ whose nodes all belong to $\{i,j\} \cup (\mathcal{I}_n \setminus \alpha)$.*
3) **Reduction of connected components:** *If the interior nodes $\beta \subseteq \mathcal{I}_n \setminus \alpha$ form a connected subgraph of G, then the boundary nodes $\bar{\alpha} \subseteq \alpha$ adjacent to β in G form a clique in G_{red}. Moreover, if one node in β features a self-loop in G, then all boundary nodes adjacent to β in G feature self-loops in G_{red}.*

The topological evolution of the graph corresponding to the iterative Kron reduction (16) is illustrated in Figure 7. Statement 1) of Theorem III.4 can be observed in each reduction step, statement 2) is nicely visible in the third step, and statement 3) is visible in the final step of the reduction in Figure 7 as well as in Figures 1 and 4. We remark that Theorem III.4 is also partially stated in [13], [26], [28], [31]. Given our prior results on iterative Kron reduction and the augmented Laplacian matrix, the following proof is rather straightforward.

Proof of Theorem III.4. To prove statement 1), we initially focus on the reduction of a single interior node k via the one-step iterative Kron reduction (16). Due to the closure of loopy Laplacian matrices under iterative Kron reduction, see Lemma III.3, we restrict the discussion to the non-positive off-diagonal elements of $Q^1 \triangleq Q/Q_{kk}$ inducing the mutual edges in the graph. Any non-zero and thus strictly negative element Q_{ij} is rendered to a strictly negative element Q_{ij}^1 since the first term on the right-hand side of equation (16) is strictly negative and the second term is non-positive. Therefore, all edges in the graph induced by Q_{ij} persist in the graph induced by Q_{ij}^1. According to the iterative Kron reduction formula (16), a zero element $Q_{ij} = 0$ is converted into a strictly negative element $Q_{ij}^1 < 0$ if and only if both nodes i and j are adjacent to k. Consequently, a reduction of node k leads to a complete graph among all nodes that were adjacent to k.

Recall from Lemma III.3 that the one-step reduction of all interior nodes is equivalent to iterative reduction of each interior node. Hence, the arguments of the previous paragraph can be applied iteratively, which proves statement 1).

Statement 2) pertains to the diagonal elements. In the strictly loopy case, it follows simply by applying the previous arguments to the augmented Laplacian $\widehat{Q}$ defined in (13).

Alternatively, an element-wise analysis of A^1_{ii} together with statement 3) of Lemma III.3 lead to the same conclusion. In the loop-less case, there will be no self-loops arising in the Kron iterative reduction by statement 1) of Lemma III.3.

Finally, statement 3) of Theorem III.4 follows by applying statements 1) and 2) to the connected component β. ■

By Theorem III.4, the topological connectivity among the boundary nodes becomes only denser under Kron reduction. Hence, the algebraic connectivity $\lambda_2(L)$ – a spectral connectivity measure – should increase accordingly. Indeed, for the graph in Figure 7 (with initially unit weights) we have $\lambda_2(L) = 0.30 \leq \lambda_2(L_{\text{red}}) = 0.45$. Physical intuition suggests that loads in a circuit weaken the influence of nodes on another. Thus, self-loops should weaken the reduced algebraic connectivity $\lambda_2(L_{\text{red}})$ accordingly. We can confirm these intuitions.

Theorem III.5 *(***Spectral Properties of Kron Reduction***)* *The following statements hold for the spectrum of the Kron-reduced matrix $Q_{\text{red}} = Q/Q(\alpha,\alpha)$:*

1) **Spectral interlacing:** *For any $r \in \mathcal{I}_{|\alpha|}$ it holds that*

$$\lambda_r(Q) \leq \lambda_r(Q_{\text{red}}) \leq \lambda_r(Q[\alpha,\alpha]) \leq \lambda_{r+n-|\alpha|}(Q)\,. \quad (18)$$

In particular, in the loop-less case, it follows for the algebraic connectivity that $0 < \lambda_2(L) \leq \lambda_2(L_{\text{red}})$.

2) **Effect of self-loops:** *For any $r \in \mathcal{I}_{|\alpha|}$ it holds that*

$$\lambda_r(L_{\text{red}}) + \max_{i\in\alpha}\{A_{\text{red}}[i,i]\} \geq \lambda_r(L) + \min_{i\in\mathcal{I}_n}\{A_{ii}\}\,, \quad (19)$$

$$\lambda_r(L_{\text{red}}) + \min_{i\in\mathcal{I}_{|\alpha|}}\{A_{\text{red}}[i,i]\} \leq \lambda_{r+n-|\alpha|}(L) + \max_{i\in\mathcal{I}_n}\{A_{ii}\}. \quad (20)$$

To illustrate the effect of self-loops, consider the graph in Figure 1 with zero-valued self-loops satisfying $\lambda_2(L) = 0.39 \leq \lambda_2(L_{\text{red}}) = 0.69$. In the strictly loopy case inequalities (19)-(20) imply that self-loops weaken the algebraic connectivity tremendously.: the same graph (in Figure 1) with unit-valued self-loops satisfies $\lambda_2(L) = 0.39 \geq \lambda_2(L_{\text{red}}) = 0.29$.

Proof of Theorem III.5. To prove statement 1), recall the *spectral interlacing property* [29, Theorem 3.1] for the spectrum of a Hermitian matrix $A \in \mathbb{R}^{n\times n}$ and its Schur complement $A/A[\beta,\beta]$ (provided that $A[\beta,\beta]$ is nonsingular):

$$\lambda_r(A) \leq \lambda_r(A/A[\beta,\beta]) \leq \lambda_r(A(\beta,\beta)) \leq \lambda_{r+|\beta|}(A)\,, \quad (21)$$

where $r \in \mathcal{I}_{n-|\beta|}$. Since Q is a loopy Laplacian matrix and hence positive semidefinite, the interlacing property (21) can be applied with $\beta = \mathcal{I}_n \setminus \alpha$ and results in the bounds (18).

To prove statement 2), recall *Weyl's inequality* [44, Theorem 4.3.1] for the spectrum of the sum of two Hermitian matrices $A, B \in \mathbb{R}^{n\times n}$. Namely, for any $k \in \mathcal{I}_n$ it holds that

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_n(B)\,. \quad (22)$$

Consider now the following set of spectral (in)equalities:

$$\begin{aligned}
\lambda_r(L_{\text{red}}) &= \lambda_r(Q_{\text{red}} - \text{diag}(\{A_{\text{red}}[i,i]\}_{i\in\alpha})) \\
&\geq \lambda_r(Q_{\text{red}}) - \max\nolimits_{i\in\alpha}\{A_{\text{red}}[i,i]\} \\
&\geq \lambda_r(Q) - \max\nolimits_{i\in\alpha}\{A_{\text{red}}[i,i]\} \\
&= \lambda_r(L + \text{diag}(\{A_{ii}\}_{i=1}^n)) - \max\nolimits_{i\in\alpha}\{A_{\text{red}}[i,i]\} \\
&\geq \lambda_r(L) + \min\nolimits_{i\in\mathcal{I}_n}\{A_{ii}\} - \max\nolimits_{i\in\alpha}\{A_{\text{red}}[i,i]\}\,,
\end{aligned}$$

where we subsequently made use of the identity $L_{\text{red}} = Q_{\text{red}} - \text{diag}(\{A_{\text{red}}[i,i]\}_{i\in\alpha})$, Weyl's inequality (22), the fact $\lambda_1(-\text{diag}(\{A_{\text{red}}[i,i]\}_{i\in\alpha})) = -\max_{i\in\alpha}\{A_{\text{red}}[i,i]\}$, the spectral interlacing property (21), the identity $Q = L + \text{diag}(\{A_{ii}\}_{i=1}^n)$, and again Weyl's inequality (22) with $\lambda_1(\text{diag}(\{A_{ii}\}_{i=1}^n)) = \min_{i\in\mathcal{I}_n}\{A_{ii}\}$. This proves the spectral bound (19). The spectral bound (20) follows analogously. ■

In the following, we investigate some algebraic properties of Kron reduction. In particular, the following theorem quantifies the topological properties in Theorem III.4, it quantifies the reduced self-loops occurring in Theorem III.5, and it shows that both edge and self-loop weights among the boundary nodes are non-decreasing, as seen in Figure 1. Furthermore, the following result shows the closure of the class of undirected connected graphs under Kron reduction, and it reveals some more subtle properties concerning the effect of self-loops.

Theorem III.6 *(***Algebraic Properties of Kron Reduction***)* *Consider the Kron-reduced matrix Q_{red} and the accompanying matrices $Q_{\text{ac}} = -Q[\alpha,\alpha)Q(\alpha,\alpha)^{-1}$ and $L_{\text{ac}} = -L[\alpha,\alpha)L(\alpha,\alpha)^{-1}$. The following statements hold:*

1) **Closure of irreducibility:** *Q_{red} is irreducible if and only if Q is irreducible.*
2) **Monotonic increase of weights:** *For all $i,j \in \alpha$ it holds that $A_{\text{red}}[i,j] \geq A_{ij}$. Equivalently, it holds that $Q_{\text{red}}[i,j] \leq Q_{ij}$ for all $i,j \in \alpha$.*
3) **Effect of self-loops I:** *Define $\Delta_i \triangleq A_{ii} \geq 0$, for $i \in \mathcal{I}_n$, so that loopy and loop-less Laplacians Q and L are related by $Q = L + \text{diag}(\{\Delta_i\}_{i\in\mathcal{I}_n})$. Then the Kron-reduced matrix takes the form*

$$Q_{\text{red}} = L/L(\alpha,\alpha) + \text{diag}(\{\Delta_i\}_{i\in\alpha}) + S\,, \quad (23)$$

where $S = L_{\text{ac}}(I_{n-|\alpha|} + \text{diag}(\{\Delta_i\}_{i\in\mathcal{I}_n\setminus\alpha})L(\alpha,\alpha)^{-1})^{-1} \times \text{diag}(\{\Delta_i\}_{i\in\mathcal{I}_n\setminus\alpha})L_{\text{ac}}^T$ is a symmetric nonnegative $|\alpha| \times |\alpha|$ matrix. Furthermore, the reduced self-loops satisfy $A_{\text{red}}[i,i] = \Delta_i + \sum_{j=1}^{n-|\alpha|} Q_{\text{ac}}[i,j]\Delta_{|\alpha|+j}$ for $i \in \alpha$.
4) **Effect of self-loops II:** *If the subgraph among the interior nodes $\mathcal{I}_n \setminus \alpha$ is connected, each boundary node α is connected to at least one interior node, and at least one of the interior nodes has a positively weighted self-loop, then S and Q_{ac} are both positive matrices.*

Statements 1) and 2) are not surprising given our knowledge from Theorems III.4 and III.5. Statement 3) reveals an interesting fact that can be nicely illustrated by considering the reduction of a single interior node k with a self-loop $\Delta_k \geq 0$. In this case, the matrix S in identity (23) specializes to the symmetric and nonnegative matrix $S = c_k \cdot L(k,k]L[k,k) \in \mathbb{R}^{(n-1)\times(n-1)}$, where $c_k = \Delta_k/(L_{kk}(L_{kk}+\Delta_k)) \geq 0$. Hence, the reduction of node k decreases the mutual coupling $\{i,j\}$ in Q/Q_{kk} by the amount $c_k \cdot A_{ik}\,A_{jk} > 0$ and increases each self-loop i in Q/Q_{kk} by the corresponding amount $c_k \cdot A_{ik}\,A_{ik} > 0$. This argument can also be applied iteratively. In statement 4) the reduction of a connected set of interior nodes implies that a single positive self-loop in the interior network will affect the entire reduced network by decreasing all mutual weights and increasing all self-loops weights.

For the proof of Theorem III.6, we recall the Sherman-Morrison identities for the inverse of the sum of two matrices.

Lemma III.7 (**Sherman-Morrison Formula**, [46]). Let $A, B \in \mathbb{C}^{n\times n}$. If A and $A+B$ are nonsingular, then

$$(A+B)^{-1} = A^{-1} - A^{-1}(I + BA^{-1})^{-1}BA^{-1}\,. \tag{24}$$

If additionally $B = \Delta \cdot uv^T$ for $\Delta \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$, then

$$(A + \Delta \cdot uv^T)^{-1} = A^{-1} - \Delta \cdot \frac{A^{-1}uv^T A^{-1}}{1 + \Delta \cdot v^T A^{-1} u}\,. \tag{25}$$

Proof of Theorem III.6. First we prove the sufficiency part of statement 1). Let Q be irreducible. In the loop-less case, the spectral inequality (18) in Theorem III.5 implies non-decreasing algebraic connectivity $\lambda_2(L_{\text{red}}) \geq \lambda_2(L) > 0$ and thus irreducibility of L_{red}. In the strictly loopy case, note that the Kron-reduced graph features the same edges (excluding self-loops) as in the loop-less case, by Theorem III.4. Thus, connectivity and irreducibility follow, which proves the sufficiency part of statement 1). The necessity part of statement 1) follows directly from statement 1) of Theorem III.4.

The element-wise bound $Q_{\text{red}}[i,j] \leq Q_{ij}$ for $i,j \in \alpha$ follows directly from [48, Lemma 1], where this bound is stated for the reduction of one node. By Lemma III.3, a one-step reduction is equivalent to iterative one-dimensional reductions. Hence, [48, Lemma 1] can be applied iteratively and yields $Q_{\text{red}}[i,j] \leq Q_{ij}$. For the negative off-diagonal elements $i \neq j$, this bound is readily converted to $A_{\text{red}}[i,j] \geq A_{ij}$. The same bound follows for the diagonal elements since diagonal dominance is non-decreasing under Kron reduction, see Lemma III.3. This completes the proof of statement 2).

Identity (23) in statement 3) follows by expanding the Kron-reduced matrix Q_{red} and by applying the matrix identity (24) with $A = L(\alpha,\alpha)$ and $B = \text{diag}(\{\Delta_i\}_{i\in\mathcal{I}_n\setminus\alpha})$ as

$$\begin{aligned} Q_{\text{red}} = Q/Q(\alpha,\alpha) &= \text{diag}(\{\Delta_i\}_{i\in\alpha}) + L[\alpha,\alpha] \\ &\quad - L[\alpha,\alpha)\big(L(\alpha,\alpha) + \text{diag}(\{\Delta_i\}_{i\in\mathcal{I}_n\setminus\alpha})\big)^{-1}L(\alpha,\alpha] \\ &= L/L(\alpha,\alpha) + \text{diag}(\{\Delta_i\}_{i\in\alpha}) + S\,, \end{aligned}$$

where S is defined statement 3). This proves identity (23).

By Lemma III.3, the Schur complement $Q/Q(\alpha,\alpha)$ is equivalent to iterative one-dimensional reduction of all interior nodes $\mathcal{I}_n \setminus \alpha$, and the matrix $Q^\ell = Q^{\ell-1}/Q^{\ell-1}_{k_\ell k_\ell}$ at the ℓth reduction step is again a loopy Laplacian. If we abbreviate the self-loops at the ℓth reduction step by $\Delta_i^\ell \triangleq A_{ii}^\ell$, then Q^ℓ can be reformulated according to identity (23) as

$$Q^\ell = Q^{\ell-1}/Q^{\ell-1}_{k_\ell k_\ell} = L^{\ell-1}/L^{\ell-1}_{k_\ell k_\ell} + \text{diag}(\{\Delta_i^\ell\}_{i=1}^{n-\ell}) + S^\ell, \tag{26}$$

where S^ℓ is the symmetric and nonnegative matrix $S^\ell = c_\ell \cdot L^{\ell-1}(k_\ell,k_\ell]L^{\ell-1}[k_\ell,k_\ell) = c_\ell \cdot A^{\ell-1}(k_\ell,k_\ell]A^{\ell-1}[k_\ell,k_\ell)$ is and $c_\ell = \Delta_k^\ell/(L_{k_\ell k_\ell}(L_{k_\ell k_\ell} + \Delta_{k_\ell}^\ell)) \geq 0$. Iterative application of this argument implies that S is symmetric and nonnegative.

To obtain an explicit expression for the reduced self-loops, re-consider the identity (5) defining the self-loops of L. In the general loopy case identity (5) reads as $Q\mathbf{1}_n = \Delta$. Block-Gaussian elimination of the interior nodes yields $Q_{\text{red}}\mathbf{1}_{|\alpha|} = \Delta[\alpha] + Q_{\text{ac}}\Delta(\alpha)$. Hence, the ith row sum of Q_{red} satisfies $A_{\text{red}}[i,i] = \sum_{j=1}^{|\alpha|} Q_{\text{red}}[i,j] = \Delta_i + \sum_{j=1}^{n-|\alpha|} Q_{\text{ac}}[i,j]\Delta_{|\alpha|+j}$.

Under the assumptions of statement 4), the positivity of Q_{ac} follows from Lemma II.1. To prove positivity of S, note that iterative reduction of all but one interior node yields one remaining interior node $k_{n-|\alpha|+1} \triangleq h$. According to equality (26), reduction of this last loopy node gives the matrix $S^h = c_h \cdot A^h(h,h]A^h[h,h)$. Under the assumptions of statement 4), Theorem III.4 implies that h features a self-loop and is connected to all boundary nodes. It follows that $c_h > 0$ and $A_{ih}^h > 0$ for all $i \in \mathcal{I}_{|\alpha|+1}$. Therefore, S_h is a positive matrix, and the same can be concluded for S. ■

C. Kron Reduction and Effective Resistance

The physical intuition behind the Kron reduction and the effective resistance in Remark I.1 suggests that the transfer conductances $Q_{\text{red}}[i,j]$ are related to the corresponding effective conductances $1/R_{ij}$. The following theorem gives the exact relation between the Kron-reduced matrix Q_{red}, the effective resistance matrix R, and the augmented Laplacian $\widehat{Q}$.

Theorem III.8 (**Resistive Properties of Kron Reduction**) *Consider the Kron-reduced matrix $Q_{\text{red}} = Q/Q(\alpha,\alpha)$, the effective resistance matrix R defined in* (3)*, and the augmented Laplacian $\widehat{Q}$ defined in* (13)*. The following statements hold:*

1) **Invariance under Kron reduction:** *The effective resistance R_{ij} between any two boundary nodes is equal when computed from Q or Q_{red}, that is, for any $i,j \in \alpha$*

$$R_{ij} = (e_i - e_j)^T Q^\dagger (e_i - e_j) \equiv (e_i - e_j)^T Q_{\text{red}}^\dagger (e_i - e_j). \tag{27}$$

2) **Invariance under augmentation:** *If Q is a strictly loopy Laplacian, then the effective resistance R_{ij} between any two nodes $i,j \in \mathcal{I}_n$ is equal when computed from Q or $\widehat{Q}$, that is, for any $i,j \in \mathcal{I}_n$*

$$R_{ij} = (e_i - e_j)^T Q^{-1} (e_i - e_j) \equiv (e_i - e_j)^T \widehat{Q}^\dagger (e_i - e_j). \tag{28}$$

In other words, statements 1) and 2) imply that, if Q is a strictly loopy Laplacian, then the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[rr, "\text{augment}"] \arrow[dd, "\substack{\text{Kron reduction}\\ \text{of } \mathcal{I}_n\setminus\alpha}"'] \arrow[dr] & & \widehat{Q} \arrow[dd, "\substack{\text{Kron reduction}\\ \text{of } \mathcal{I}_n\setminus\alpha}"] \arrow[dl] \\
& \substack{R_{ij}\\ i,j\in\alpha} & \\
Q_{\text{red}} \arrow[rr, "\text{augment}"'] \arrow[ur] & & \widehat{Q}_{\text{red}} \arrow[ul]
\end{tikzcd}$$

3) **Effect of self-loops:** *If Q is a strictly loopy Laplacian and $\bar{R}_{ij} \triangleq (e_i - e_j)^T L^\dagger (e_i - e_j)$, $i,j \in \mathcal{I}_n$, is the effective resistance computed from the corresponding loop-less Laplacian L, then $R_{ij} \leq \bar{R}_{ij}$ for all $i,j \in \mathcal{I}_n$.*

Theorem III.8 is illustrated in Figure 8. Identity (27) states that the effective resistances between the boundary nodes are invariant under Kron reduction of the interior nodes. Spoken in terms of circuit theory, the effective resistance between the terminals α can be obtained from either the impedance matrix $Q^\dagger$ or the transfer impedance matrix $Q_{\text{red}}^\dagger$. Identity (28) gives a resistive interpretation of the self-loops: the effective resistance among the nodes in a strictly loopy graph G is equivalent to the effective resistance among the corresponding nodes in the augmented loop-less graph $\widehat{G}$. According to statement 3), the self-loops do not increase the effective resistance, which is in accordance with the physical interpretation in Remark I.1.

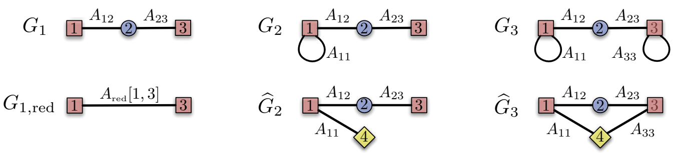

Fig. 8. Illustration of Theorem III.8: According to statement 1), the effective resistance R_{13} between the boundary nodes ■ is equal when computed in the graph G_1 or in the Kron-reduced graph $G_{1,\mathrm{red}}$. According to statement 2), the effective resistance R_{13} is equal when computed in the strictly loopy graph G_2 (respectively G_3) or in the augmented loop-less graph $\widehat{G}_2$ (respectively $\widehat{G}_3$) with grounded node ◊. According to statement 3), the effective resistance R_{13} in the strictly loopy graphs G_2 and G_3 is not larger than in the loop-less graph G_1 (with equality for $\{G_1, G_2\}$ and strict inequality for $\{G_1, G_3\}$).

For the proof of Theorem III.8 we establish some identities relating R and L via regularizations of the pseudo inverse.

Lemma III.9 *(***Laplacian and Effective Resistance Identities***)* *Let $L \in \mathbb{R}^{n\times n}$ be a symmetric irreducible loop-less Laplacian matrix. Then for any $\delta \neq 0$ it holds that*

$$(L + (\delta/n)\,\mathbf{1}_{n\times n})^{-1} = L^{\dagger} + (1/\delta n)\,\mathbf{1}_{n\times n}\,. \tag{29}$$

Consider for $i,j \in \mathcal{I}_n$ the effective resistance defined by $R_{ij} = (e_i - e_j)^T L^{\dagger}(e_i - e_j)$. For $\delta \neq 0$ it holds that

$$R_{ij} \equiv (e_i - e_j)^T (L + (\delta/n)\,\mathbf{1}_{n\times n})^{-1}(e_i - e_j),\ i,j \in \mathcal{I}_n. \tag{30}$$

If $n \geq 3$, then, by taking node n as reference, it holds that

$$R_{ij} \equiv (e_i - e_j)^T L(n,n)^{-1}(e_i - e_j)\,, \quad i,j \in \mathcal{I}_{n-1}\,. \tag{31}$$

Proof: Since $\mathbf{1}_{n\times n}\mathbf{1}_{n\times n} = n \cdot \mathbf{1}_{n\times n}$ and $LL^{\dagger} = L^{\dagger}L = I_n - (1/n)\cdot \mathbf{1}_{n\times n}$ (by definition of $L^{\dagger}$ via the singular value decomposition, see also [39, Lemma 3]), identity (29) can be verified since $(L + (\delta/n)\,\mathbf{1}_{n\times n}) \cdot \big(L^{\dagger} + (1/\delta n)\,\mathbf{1}_{n\times n}\big) = I_n$. The identity (30) follows then by multiplying equation (29) from the left by $(e_i - e_j)^T$ and from the right by $(e_i - e_j)$.

To prove identity (31), let $\bar{L} \triangleq L(n,n)$. It follows from [42, Appendix B, eq. (17)] that $\bar{L}^{-1}_{ij} = L^{\dagger}_{ij} - L^{\dagger}_{in} - L^{\dagger}_{jn} - L^{\dagger}_{nn}$. The identity (31) can then be verified by direct computation. ■

Proof of Theorem III.8. We begin by proving statement 1) in the strictly loopy case when Q is nonsingular (due to irreducible diagonal dominance [44, Corollary 6.2.27]). Note that we are interested in the effective resistances only among the nodes α, that is, the $|\alpha| \times |\alpha|$ block of Q^{-1}. The celebrated Schur complement formula [43, Theorem 1.2] gives the $|\alpha| \times |\alpha|$ block of Q^{-1} as $(Q/Q(\alpha,\alpha))^{-1} = Q^{-1}_{\mathrm{red}}$. Consequently, for $i,j \in \alpha$ the defining equation (3) for the effective resistance R_{ij} is simply rendered to $R_{ij} = (e_i - e_j)^T Q^{-1}_{\mathrm{red}}(e_i - e_j)$, which proves the claimed identity (27).

In the loop-less case when $Q \equiv L$ is singular, a similar line of arguments holds on the image of L. Let $\delta > 0$ and consider the modified and non-singular Laplacian $\tilde{L} \triangleq L + (\delta/n)\mathbf{1}_{n\times n}$. Due to identity (29) we have that $\tilde{L}^{-1} = L^{\dagger} + (1/\delta n)\mathbf{1}_{n\times n}$. We can then rewrite identity (30) in expanded form as

$$\begin{aligned} R_{ij} &= (e_i - e_j)^T (L^{\dagger} + (1/\delta n)\mathbf{1}_{n\times n})(e_i - e_j) \\ &= (e_i - e_j)^T \tilde{L}^{-1}(e_i - e_j)\,. \end{aligned} \tag{32}$$

As before, the $|\alpha| \times |\alpha|$ block of $\tilde{L}^{-1}$ is $(\tilde{L}/\tilde{L}(\alpha,\alpha))^{-1}$. Consequently, for $i,j \in \alpha$ the identity (32) is rendered to

$$R_{ij} = (e_i - e_j)^T (\tilde{L}/\tilde{L}(\alpha,\alpha))^{-1}(e_i - e_j)\,. \tag{33}$$

Since $(e_i - e_j)^T \mathbf{1}_{n\times n}(e_i - e_j) = 0$, the right-hand side of (32), or equivalently (33), is independent of δ since the matrices are evaluated on the subspace orthogonal to $\mathbf{1}_n$, the nullspace of $\tilde{L}$ as $\delta \downarrow 0$. Thus, on the image of L the limit of the right-hand side of (33) exists as $\delta \downarrow 0$. By definition, $L^{\dagger}$ acts as regular inverse on the image of L, and equation (33) is rendered to

$$R_{ij} = (e_i - e_j)^T (L/L(\alpha,\alpha))^{\dagger}(e_i - e_j) = (e_i - e_j)^T L^{\dagger}_{\mathrm{red}}(e_i - e_j),$$

which proves the claimed identity (27) in the loop-less case.

To prove statement 2), note that the strictly loopy Laplacian Q is invertible. Hence, the defining equation (3) for the resistance features a regular inverse. The matrix Q can also be seen as the principal $n \times n$ block of the augmented Laplacian $\widehat{Q}$, that is, $Q = \widehat{Q}(n+1,n+1)$. The identity (28) follows then directly from identity (31) (with n replaced by $n+1$).

To prove statement 3), we appeal to Rayleigh's celebrated *monotonicity law and short/cut principle* [41]. Since the Laplacian L induces the same graph as $\widehat{Q}$ with node $n+1$ removed, the monotonicity law states that the effective resistance $\bar{R}_{ij}$ in the graph induced by L is not smaller than the effective resistance R_{ij} in the graph induced by $\widehat{Q}$. The latter again equals the effective resistance in the graph induced by Q due to identity (28). Equivalently, for $i,j \in \mathcal{I}_n$ it holds that $\bar{R}_{ij} = (e_i - e_j)^T L^{\dagger}(e_i - e_j) \geq (e_i - e_j)^T \widehat{Q}^{\dagger}(e_i - e_j) = (e_i - e_j)^T Q^{-1}(e_i - e_j) = R_{ij}$, which proves statement 3). ■

Theorem III.8 allows to compute the effective resistance matrix R from the transfer impedance matrix $Q^{\dagger}_{\mathrm{red}}$. We are now interested in a converse result to construct $Q^{\dagger}_{\mathrm{red}}$ from R. Iterative methods constructing $Q^{\dagger}_{\mathrm{red}}$ from R can be found in [23]. However, it is also possible to recover the (pseudo) inverses of the loopy Laplacian Q, the augmented Laplacian $\widehat{Q}$, or the corresponding Kron-reduced Laplacians *directly* from R.

Lemma III.10 *(***Impedance and Effective Resistance Identities***)* *Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric irreducible loopy Laplacian matrix. Consider the following three cases:*

1) **Loop-less case:** *Let $R \in \mathbb{R}^{n\times n}$ be the effective resistance matrix. Then for $i,j \in \mathcal{I}_n$ it holds that*

$$Q^{\dagger}_{ij} = -\frac{1}{2}\Big(R_{ij} - \frac{1}{n}\sum_{k=1}^{n}(R_{ik} + R_{jk}) + \frac{1}{n^2}\sum_{k,\ell=1}^{n} R_{k\ell}\Big). \tag{34}$$

2) **Strictly loopy case:** *Consider the grounded node $n+1$, the corresponding augmented Laplacian matrix $\widehat{Q} \in \mathbb{R}^{(n+1)\times(n+1)}$ defined in* (13)*, and the corresponding matrix of effective resistances $R \in \mathbb{R}^{(n+1)\times(n+1)}$ defined in* (3)*. Then the following two identities hold:*

$$\begin{aligned} \widehat{Q}^{\dagger}_{ij} = &-\frac{1}{2}\Big(R_{ij} - \frac{1}{n+1}\sum_{k=1}^{n+1}(R_{ik} + R_{jk}) \\ &+ \frac{1}{(n+1)^2}\sum_{k,\ell=1}^{n+1} R_{k\ell}\Big)\,, \qquad i,j \in \mathcal{I}_{n+1}\,, \end{aligned} \tag{35}$$

$$Q^{-1}_{ij} = \frac{1}{2}\big(R_{i,n+1} + R_{j,n+1} - R_{ij}\big)\,, \qquad i,j \in \mathcal{I}_n\,. \tag{36}$$

3) **Kron reduced case:** *The identities* (34)*,* (35)*, and* (36) *also hold when $Q^{\dagger}$, $\widehat{Q}^{\dagger}$, and Q^{-1} on the left-hand sides are replaced by $Q^{\dagger}_{\mathrm{red}}$, $\widehat{Q}^{\dagger}_{\mathrm{red}}$, and Q^{-1}_{red}, respectively, and n on the right-hand sides is replaced by $|\alpha|$.*

Proof: Identity (34) is stated in [40, Theorem 4.8] for the weighted case and in [49, Theorem 7] for the unweighted case. According to statement 2) of Theorem III.8, the resistance is invariant under augmentation. Hence, identity (34) applied to the augmented Laplacian $\widehat{Q}$ yields identity (35). Identity (36) follows directly from [40, Theorem 4.9]. According to Theorem III.8, the effective resistance is invariant under Kron reduction. Thus, the effective resistance corresponding to Q_{red} is simply $R[\alpha, \alpha]$. Hence, the formulas (34), (35), and (36) can be applied to the Kron-reduced matrix as stated in 3). ■

By Theorem III.8 and Lemma III.10, the effective resistance matrix R in the original non-reduced network can be computed from the (pseudo) inverse of the Kron-reduced Laplacian Q_{red}, and vice versa. In some applications, it is desirable to know an *explicit* algebraic relationship between R and Q_{red} without the (pseudo) inverse. However, such an explicit relationship between can be found only if closed-form solutions of $Q_{\text{red}}^\dagger$, Q_{red}^{-1}, or $\widehat{Q}_{\text{red}}^\dagger$ are known. These are generally not available. Generally, it is also infeasible to relate bounds on R to bounds on Q_{red} since element-wise bounding of inverses of interval matrices is known to be NP-hard [50]. Fortunately, closed forms of $Q_{\text{red}}^\dagger$, Q_{red}^{-1} or $\widehat{Q}_{\text{red}}^\dagger$ can be derived in an *ideal* electric network, with uniform effective resistances among the boundary nodes as well as between the boundary nodes and the ground. In fact, this ideal case is equivalent to uniform transfer conductances (weights) in the Kron-reduced network.

Theorem III.11 (Equivalence of Uniformity in Effective Resistance and Kron Reduction) *Consider the Kron-reduced Laplacian $Q_{\text{red}} = Q/Q(\alpha, \alpha)$ and the corresponding adjacency matrix A_{red}. Consider the following two cases:*

Loop-less case: *Let $R \in \mathbb{R}^{n \times n}$ be the matrix of effective resistances. Then the following two statements are equivalent:*

1) *The effective resistances among the boundary nodes α are uniform, that is, there is $r > 0$ such that $R_{ij} = r$ for all distinct $i, j \in \alpha$; and*
2) *The weighting of the edges in the Kron-reduced network is uniform, i.e., there is $a > 0$ such that $A_{\text{red}}[i, j] = a > 0$ for all distinct $i, j \in \alpha$.*

If both statements 1) and 2) are true, then it holds that $r = \frac{2}{|\alpha| a}$.

Strictly loopy case: *Consider additionally the grounded node $n+1$ and the augmented Laplacian matrices $\widehat{Q}$ and $\widehat{Q}_{\text{red}}$ defined in* (13) *and* (14)*, respectively. Let $R \in \mathbb{R}^{(n+1) \times (n+1)}$ be the matrix of effective resistances in the augmented network. Then the following two statements are equivalent:*

3) *The effective resistances both among the boundary nodes α and between all boundary nodes α and the grounded node $n+1$ are uniform, that is, there is $r > 0$ and $g > 0$ such that $R_{ij} = r$ for all distinct $i, j \in \alpha$ and $R_{i,n+1} = g$ for all $i \in \alpha$; and*
4) *The weighting of the edges and the self-loops in the Kron-reduced network is uniform, that is, there are $a > 0$ and $b > 0$ such that $A_{\text{red}}[i, j] = a > 0$ and $A_{\text{red}}[i, i] = b > 0$ for all distinct $i, j \in \alpha$.*

If both statements 3) and 4) are true, then it holds that $r = \frac{2}{|\alpha| a + b}$ and $g = \frac{a+b}{b(a|\alpha| + b)}$.

Remark III.12 (Engineered networks and uniform graph topologies) The uniformity assumption in statements 1) and 3) corresponds to an *ideal network*, where all boundary nodes are electrically uniformly distributed with respect to each other and with respect to the shunt loads. In the applications of electrical impedance tomography and smart grid monitoring, this assumption can be met by choosing the boundary nodes corresponding to sensor locations. In the transient stability problem, the generators corresponding to boundary nodes are distributed over the power grid ideally in such a way that the loads can be effectively and uniformly sustained . Hence, the uniformity assumptions are ideally met in man-made networks.

Independently of engineered networks, uniform resistances occur for various graph topologies, even when weights as additional degrees of freedom are neglected. In the trivial case, $|\alpha| = 2$, Theorem III.11 reduces to [10, Corollary 4.41] and the resistance among the boundary nodes is clearly uniform. Second, if the boundary nodes are 1-connected leaves of a highly symmetric graph among the interior nodes, such as a star, a complete graph, or a combination of these two, then the resistance among the boundary nodes is uniform. Third, the effective resistance in large random geometric graphs, small word networks, and lattices and their fuzzes becomes uniform among sufficiently distant nodes, see [16] for further details.□

To prove Theorem III.11, we need the following identities.

Lemma III.13 (Inverses of Uniform Laplacian Matrices) *Let $a > 0$ and $b \geq 0$ and consider the loopy Laplacian matrix $Q \triangleq a\big(nI_n - \mathbf{1}_{n \times n}\big) + bI_n$ corresponding to a complete graph with n nodes, uniform positive edge weights $a > 0$ between any two distinct nodes, and nonnegative and uniform self-loops $b \geq 0$ attached to every node. The following statements hold:*

1) *For zero self-loops $b = 0$, $Q^\dagger$ is the loop-less Laplacian*
$$Q^\dagger = \frac{1}{n^2 a^2} \cdot Q = \frac{1}{n^2 a} \cdot \big(nI_n - \mathbf{1}_{n \times n}\big).$$
2) *For positive self-loops $b > 0$, Q^{-1} is the positive matrix*
$$Q^{-1} = -\frac{a}{b(an+b)} \big(nI_n - \mathbf{1}_{n \times n}\big) + \frac{1}{b} I_n.$$
3) *Consider the augmented Laplacian $\widehat{Q}$ given by*
$$\widehat{Q} = \left[\begin{array}{c|c} a\big(nI_n - \mathbf{1}_n\big) + bI_n & -b\mathbf{1}_n \\ \hline -b\mathbf{1}_n^T & n \cdot b \end{array}\right].$$
Then $\widehat{Q}^\dagger$ is given by the (augmented) loop-less Laplacian
$$\widehat{Q}^\dagger = \left[\begin{array}{c|c} c\big(nI_n - \mathbf{1}_n\big) + dI_n & -d\mathbf{1}_n \\ \hline -d\mathbf{1}_n^T & n \cdot d \end{array}\right],$$
where $d = \frac{1}{b(n+1)^2}$ and $c = d \cdot \frac{(n+2)b-a}{an+b}$.

Proof: The identities can be verified by direct computation. Since Q and $Q^\dagger$ (respectively $\widehat{Q}$ and $\widehat{Q}^\dagger$) satisfy the Penrose equations [44], the loop-less Laplacian $Q^\dagger$ (respectively $\widehat{Q}^\dagger$) is the unique pseudo inverse, which proves statements 1) and 3). Statement 2) follows since $QQ^{-1} = Q^{-1}Q = I_n$. ■

We now have all three ingredients to prove Theorem III.11: the invariance formulas (27)-(28) for the effective resistance stated in Theorem III.8, the relations between effective resistance and the Kron-reduced impedance matrix in statement 3) of Lemma III.10, and the Laplacian identities in Lemma III.13. Given these formulas, the proof of Theorem III.14 reduces to mere computation. For the sake of brevity, it will be omitted.

D. Sensitivity of Kron Reduction to Perturbations

In the final subsection of our analysis of Kron reduction we discuss the sensitivity of the Kron-reduced matrix Q_{red} to perturbations in the original matrix Q. A number of interesting perturbations can be modeled by adding symmetric matrix $W \in \mathbb{R}^{n \times n}$ and considering the perturbed loopy Laplacian $\tilde{Q} = Q + W$, where Q is the nominal loopy Laplacian matrix.

The case when W is diagonal is fully discussed in Theorem III.6. A perturbation of the form when $W[\alpha, \alpha]$ is a non-zero matrix and all other entries of W are zero can model the emergence, loss, or change of a self-loop or an edge among boundary nodes. Such a perturbation acts additively on Q_{red} as

$$\tilde{Q}_{\text{red}} \triangleq \tilde{Q}/\tilde{Q}(\alpha, \alpha) = Q_{\text{red}} + W[\alpha, \alpha]\,. \tag{37}$$

If the perturbation affects the interior nodes, then $W(\alpha, \alpha)$ is a non-zero matrix. Inspired by [12], [13], we put more structure on the perturbation matrix W and consider symmetric rank one perturbations of the form $W = \Delta \cdot (e_i - e_j)(e_i - e_j)^T$, where $\Delta \in \mathbb{R}$. Such a perturbation changes the weight of the edge $\{i, j\}$ from A_{ij} to $A_{ij} + \Delta$ and also can model the loss or emergence of the edge $\{i, j\}$. Since a perturbation among the boundary nodes is fully captured by (37), we consider now perturbations of the edge between the ith and jth interior node.

Theorem III.14 (Perturbation of the Interior Network) *Consider the Kron-reduced matrix $Q_{\text{red}} = Q/Q(\alpha, \alpha)$, the accompanying matrix $Q_{\text{ac}} = -Q[\alpha, \alpha)Q(\alpha, \alpha)^{-1}$, and a symmetric rank one perturbation. $W \triangleq \Delta \cdot (e_{i+|\alpha|} - e_{j+|\alpha|})(e_{i+|\alpha|} - e_{j+|\alpha|})^T$ for distinct $i, j \in \mathcal{I}_{n-|\alpha|}$ and such that the perturbed matrix $\tilde{Q} \triangleq Q + W$ remains an irreducible loopy Laplacian. The following statements hold:*

1) **Algebraic perturbation:** Q_{red} *undergoes the rank one perturbation $\tilde{Q}/\tilde{Q}(\alpha, \alpha) = \tilde{Q}_{\text{red}}$ given by*

$$\tilde{Q}_{\text{red}} \triangleq Q_{\text{red}} + \frac{Q_{\text{ac}}(e_i - e_j)\Delta(e_i - e_j)^T Q_{\text{ac}}^T}{1 + \Delta \cdot R_{\text{int}}[i, j]}\,, \tag{38}$$

where $R_{\text{int}}[i, j] \triangleq (e_i - e_j)^T Q(\alpha, \alpha)^{-1}(e_i - e_j) \geq 0$.

2) **Resistive perturbation:** *Let R and $\tilde{R}$ be the matrices of effective resistances corresponding to Q and $\tilde{Q}$, respectively. For any $k, \ell \in \mathcal{I}_n$ it holds that*

$$\tilde{R}_{k\ell} = R_{k\ell} - \frac{\Delta \cdot \|(e_k - e_\ell)^T Q^\dagger (e_{i+|\alpha|} - e_{j+|\alpha|})\|_2^2}{1 + \Delta \cdot R_{i+|\alpha|, |j+\alpha|}}. \tag{39}$$

If $\Delta > 0$ (respectively $\Delta < 0$) then it holds that $\tilde{R}_{k\ell} \leq R_{k\ell}$ (respectively $\tilde{R}_{k\ell} \geq R_{k\ell}$).

The term $R_{\text{int}}[i, j]$ in (38) is the effective resistance between the perturbed nodes in the interior network. Likewise, the physical interpretation of the term $Q_{\text{ac}}(e_i - e_j)\Delta(e_i - e_j)^T = Q_{\text{ac}} W(\alpha, \alpha)$ is well-known in network theory. The perturbation W has the same effect on the equations $I = (Q + W)V$ as the current injection $\tilde{I} = -WV$, that is, the perturbation of the interior edge $\{i, j\}$ by a value Δ is equivalent to injecting the current $\Delta \cdot (V_{i+|\alpha|} - V_{j+|\alpha|})$ into the jth interior node and extracting it from the ith interior node. In the reduced network equations (2) the current injection $\tilde{I}$ translates to the current injection $Q_{\text{ac}}\tilde{I}(\alpha) = -Q_{\text{ac}}(e_i - e_j)\Delta(e_i - e_j)^T V(\alpha)$ into the boundary nodes. Finally, the additive term in identity (39) resembles the *sensitivity factor* in network theory [12], [20]. From Remark I.1, notice that $(e_k - e_\ell)^T Q^\dagger (e_{i+|\alpha|} - e_{j+|\alpha|})$ is the potential drop between nodes k and ℓ if a unit current is injected in the ith interior node and extracted at the jth interior node. As before, the current flowing along the perturbed edge is redistributed in the network according to identity (39).

Various spectral bounds can be derived from identity (38). For instance, for $\Delta < 0$, Weyl's inequalities (22) give

$$\lambda_r(Q_{\text{red}}) \geq \lambda_r(\tilde{Q}_{\text{red}}) \geq \lambda_r(Q_{\text{red}}) + \frac{\Delta \cdot \|Q_{\text{ac}}(e_i - e_j)\|_2^2}{1 + \Delta \cdot R_{\text{int}}[i, j]}\,,$$

where $r \in \mathcal{I}_{|\alpha|}$. These bounds can be further related to Q and $\tilde{Q}$ via the interlacing inequalities (18) or [44, Theorem 4.3.4].

Proof of Theorem III.14. Since the perturbed matrix $\tilde{Q} = Q + W$ is a symmetric and irreducible loopy Laplacian, the reduced matrix $\tilde{Q}_{\text{red}} = \tilde{Q}/\tilde{Q}(\alpha, \alpha)$ exists by Lemma II.1. By the matrix identity (25), the Schur complement $\tilde{Q}_{\text{red}}$ given by

$$\tilde{Q}_{\text{red}} = \big(Q + W\big) / \big(Q(\alpha, \alpha) + \Delta(e_i - e_j)^T (e_i - e_j)\big)$$

further simplifies to identity (38) in statement 1). For the proof of statement 2), we initially consider the strictly loopy case. Here, $\tilde{Q}^{-1} = (Q + W)^{-1}$ can be obtained from identity (25) as

$$\tilde{Q}^{-1} = Q^{-1} - \frac{\Delta \cdot Q^{-1}(e_{i+|\alpha|} - e_{j+|\alpha|})^T (e_{i+|\alpha|} - e_{j+|\alpha|}) Q^{-1}}{1 + \Delta(e_{i+|\alpha|} - e_{j+|\alpha|})^T Q^{-1} (e_{i+|\alpha|} - e_{j+|\alpha|})}.$$

A multiplication of $\tilde{Q}^{-1}$ from the left by $(e_k - e_\ell)^T$ and from the right by $(e_k - e_\ell)$ yields then identity (39). In the loop-less case when Q is singular, the same arguments can be applied on the image of Q by considering the non-singular matrix $Q + (\delta/n)\, \mathbf{1}_{n \times n}$ for $\delta \neq 0$ and identity (30). This results in the more general identity (39). The second part of statement 2) follows again from Rayleigh's monotonicity law [41]. ■

IV. Conclusions

We studied the Kron reduction process from the viewpoint of algebraic graph theory. Our analysis is motivated by various applications spanning from classic circuit theory over electrical impedance tomography to power network applications and Markov chains. Prompted by these applications, we presented a detailed and comprehensive graph-theoretic analysis of Kron reduction. In particular, we carried out a thorough topological, algebraic, spectral, resistive, and sensitivity analysis of the Kron-reduced matrix. This analysis led to novel results in algebraic graph theory and new physical insights in the application domains of Kron reduction. We believe our results can be directly employed in the application areas of Kron reduction.

Of course, the results contained in this paper can and need to be further refined to meet the specific problems in each particular application area. Our analysis also demands answers to further general questions, such as the extension of this work to directed graphs or complex-valued weights occurring in all disciplines of circuit theory [8], [11], [12], [14]. Finally, it would be of interest to analyze the effects of Kron reduction on centrality measures, clustering coefficients, and more general graph-theoretic metrics than the effective resistance.

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