Spectral Analysis of Synchronization in a Lossless Structure-Preserving Power Network Model

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***Abstract*—This paper considers the synchronization and transient stability analysis in a simple model of a structure-preserving power system. We derive sufficient conditions relating synchronization in a power network directly to the underlying network state, parameters, and topology. In particular, we provide a spectral condition based on the algebraic connectivity of the network and a second condition based on the effective resistance among generators. These conditions build upon the authors' earlier results on synchronization in network-reduced power system models. Central to our analysis is the reduced admittance matrix of the network, which is obtained by a Schur complement of the network's topological admittance matrix with respect to its bus nodes. This network-reduction process, termed Kron reduction, relates the structure-preserving and the network-reduced power system model. We provide a detailed graph-theoretic, algebraic, and spectral analysis of the Kron reduction process leading directly to the novel synchronization conditions.**

I. Introduction

In face of the rising complexity of the envisioned future power grid and the stochastic disturbances caused by renewable energy sources such as wind and solar power, an important form of power network stability is the so-called *transient stability*. Transient stability considers the stability of a synchronous operating point arising after large transient disturbances such as faults of system components or significant changes in load or generation. The more general concept of synchronization encompasses transient stability, is defined independent of specific equilibria, and is loosely speaking the ability of a power system to remain in synchronism when subjected to transient disturbances in parameters or topology.

The problem of synchronization and transient stability is well-studied and surveyed in [1]–[3]. The mathematical model considered in transient stability analysis consists of a set of differential-algebraic equations representing the rotor dynamics of each generator as well as the power flow at each bus in the network. In a classic setting, the loads in the network are modeled as constant impedances, which allows the reduction of the power system model to the well-known *swing equations* featuring an all-to-all coupling among the generators [4], [5]. This so-called *network-reduced model* is mathematically tractable but the original network topology representing the system components is lost. Analytic approaches to *structure-preserving* (or network-preserving) models have been considered in [6]–[9]. The cited approaches for both network-reduced and structure-preserving models [1]–[9] rely on Hamiltonian arguments, which also lead to computational procedures providing precise estimates of the region of attraction of synchronous equilibria. An open problem, recognized by Hill and Chen [10] and not resolved by classical analysis methods, is the quest for explicit and concise conditions for synchronization as a function of state, parameters, and graph-theoretical properties of the power network.

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Recently, different scientific communities showed an burgeoning interest in synchronization, including the networked control community interested in the *consensus protocol* [11], the dynamical systems community analyzing the *Kuramoto model of coupled oscillators* [12], and the physics community studying synchronization in *complex networks* and its relation to the algebraic connectivity [13] and the effective resistance [14], which are spectral and graph-metric connectivity measures. In the earlier work [15] the authors combined classic transient stability analysis and synchronization theory to approach the outstanding problem of relating synchronization in a network-reduced power system to the underlying network structure. In particular, the synchronization conditions read as "the network connectivity has to dominate the network's non-uniformity (in effective power inputs) and the network's losses (due to transfer conductances)." Since network-reduced power system models feature all-to-all coupling, the conditions in [15] did not capture the original power network topology.

As a first contribution of this paper, we provide a rigorous algebraic analysis and graph-theoretic interpretation of the Kron reduction process relating the structure-preserving and the network-reduced power system model. In essence, Kron reduction of a network is a Schur complement of the Laplacian matrix with respect to a set of nodes. We relate the spectrum of the resulting Kron-reduced Laplacian matrix to the spectrum of the non-reduced Laplacian matrix, relate its elements to the effective resistances in the non-reduced network, and give various interpretations in the spirit of algebraic graph theory. The analysis of the effective resistance is presented in detail in [16] whereas this article focuses on the spectral analysis.

The detailed analysis of the Kron reduction process leads to the second contribution of this paper, the extension of the synchronization conditions derived by [15] to structure-preserving (topological) power network models. As a result, two sufficient conditions for synchronization among generators are presented depending on the algebraic connectivity and the effective resistance among the generators, both in the non-reduced network. These conditions bridge the gap from transient stability in power networks to the synchronization analysis carried out by the physics community [13], [14] and are derived for a lossless network under the assumptions of uniform voltage levels at all generator nodes and zero shunt admittances. The second condition relies upon the additional assumption of uniform effective resistances; this assumption can be justified for various examples. We are aware that the considered network-preserving power system model is idealistic and no estimates for the region of attraction of synchronous equilibria are provided. Rather, our analysis aims at the open problem proposed by [10]: we provide explicit and concise conditions that relate synchronization to the state, parameters, and graph-theoretical properties of the power network.

Paper organization: The remainder of this section introduces some notation. Section II recalls the structure-preserving and network-reduced power system models, the network-reduction process, and a short summary of the authors' earlier results. Section III presents the analysis of the Kron reduction process resulting in novel synchronization conditions stated in Section IV. Finally, Section V concludes the paper.

Notation: Given a finite set $\mathcal{Q}$, we let $|\mathcal{Q}|$ be its cardinality and define for $n \in \mathbb{N}$ the index set $\mathcal{I}_n := \{1, \dots, n\}$. Let $\mathbf{1}_{p\times q}$ and $\mathbf{0}_{p\times q}$ be the $p\times q$ dimensional matrices of unit and zero entries. Given a complex-valued 2d-array $\{A_{ij}\}$ with $i, j \in \mathcal{I}_n$, let $A \in \mathbb{C}^{n\times n}$ denote the associated matrix and A^* the conjugate transposed matrix, and define $A_{\max} = \max_{ij}\{|A_{ij}|\}$ and $A_{\min} = \min_{ij}\{|A_{ij}|\}$. We use the following standard notation for submatrices [17]: for two non-empty index sets $\alpha, \beta \subseteq \mathcal{I}_n$ let $A[\alpha, \beta]$ denote the submatrix of A obtained by the rows indexed by α and the columns indexed by β and define the shorthands $A[\alpha, \beta) = A[\alpha, \mathcal{I}_n \setminus \beta]$, $A(\alpha, \beta] = A[\mathcal{I}_n \setminus \alpha, \beta]$, and $A(\alpha, \beta) = A[\mathcal{I}_n \setminus \alpha, \mathcal{I}_n \setminus \beta]$. We adopt the shorthand $A[\{i\}, \{j\}] = A[i, j] = A_{ij}$ for $i, j \in \mathcal{I}_n$. If the matrices $A[\alpha, \alpha)$ and $A(\alpha, \alpha]$ have zero entries, then A is a block-diagonal matrix denoted by $A = \mathrm{blkdiag}(A[\alpha, \alpha], A(\alpha, \alpha))$.

For a nonsingular $A(\alpha, \alpha)$, the *Schur complement* of A w.r.t. the block $A(\alpha, \alpha)$ (or equivalently the indices α) is the $|\alpha|\times|\alpha|$ dimensional matrix denoted by $A/A(\alpha, \alpha)$ and defined by

$$A/A(\alpha, \alpha) = A[\alpha, \alpha] - A[\alpha, \alpha)A(\alpha, \alpha)^{-1}A(\alpha, \alpha] .$$

If A is Hermitian, then we implicitly assume that its eigenvalues are arranged in increasing order: $\lambda_1(A) \leq \ldots \leq \lambda_n(A)$.

For a weighted undirected graph induced by a symmetric and nonnegative adjacency matrix $A = A^T \in \mathbb{R}^{n\times n}$, the *Laplacian matrix* is defined as $L(A) = \mathrm{diag}(\sum_{j=1}^{n} A_{ij}) - A = L(A)^T$. Recall that *irreducibility* of the Laplacian matrix is equivalent to connectivity of the corresponding graph.

II. Review of the Power Network Model

This section recalls the structure-preserving and network-reduced power system model to be found in [1], [18], [19] and relates network-reduction process to algebraic graph theory.

A. The Structure-Preserving Power System Model

Consider the single-line diagram of a power network G_{network}, such as the *New England Power Grid* which can be found in [1] and is schematically illustrated in Figure 1. The nodes of the network can be classified as n generator nodes $\mathcal{V}_G$, n generator terminal buses $\mathcal{V}_{GB}$, and m load buses $\mathcal{V}_{LB}$. The network has the following topology:

- **(i)** each generator node $i_{\text{G}} \in \mathcal{V}_G$ is connected to exactly one generator terminal bus $i_{\text{GB}} \in \mathcal{V}_{GB}$,
- **(ii)** each generator terminal bus $i_{\text{GB}} \in \mathcal{V}_{GB}$ is connected to at least one load bus $i_{\text{LB}} \in \mathcal{V}_{LB}$, and
- **(iii)** the buses $\mathcal{V}_{GB} \cup \mathcal{V}_{LB}$ form a connected network.

In essence, this topology corresponds to a connected network among the bus nodes $\mathcal{V}_{GB} \cup \mathcal{V}_{LB}$, and the generator nodes $\mathcal{V}_G$ are coupled to the interior network via $\mathcal{V}_{GB}$. Adopting nomenclature of circuit theory, the generators and the buses are also denoted as boundary nodes and interior nodes.

Each edge connecting two nodes i and j is weighted by a non-zero *line admittance* $Y_{ij} = Y_{ji} \in \mathbb{C}$ which is typically of inductive nature, i.e., a negative imaginary part dominates a small positive real part. This weighting of the network G_{network} gives rise to the complex-valued adjacency matrix

$$A(G_{\text{network}}) := \left[\begin{array}{c|c|c} \mathbf{0}_{n\times n} & Y_{\text{G-GB}} & \mathbf{0}_{n\times m} \\ \hline Y_{\text{G-GB}}^T & \mathbf{0}_{n\times n} & Y_{\text{GB-LB}} \\ \hline \mathbf{0}_{m\times n} & Y_{\text{GB-LB}}^T & Y_{\text{LB-LB}} \end{array}\right] \in \mathbb{C}^{(2n+m)\times(2n+m)},$$

where the matrices $Y_{\text{G-GB}}$, $Y_{\text{GB-LB}}$, and $Y_{\text{LB-LB}} = Y_{\text{LB-LB}}^T$ induce the topology **(i)-(iii)**. Finally, the loads are modeled as passive *shunt admittances* connecting the buses to the ground:

- **(iv)** each bus $i \in \mathcal{V}_{GB} \cup \mathcal{V}_{LB}$ is connected to the ground via a shunt admittance $Y_{i\text{-ground}}$.

In case the shunt admittance at a bus is zero, the bus is said to be *floating*. From a viewpoint of circuit theory, the topology **(i)-(iv)** and Kirchhoff's and Ohm's laws give rise to the network equations relating voltages and currents as

$$I = Y_{\text{network}} V \, , \tag{1}$$

where $V = [V_{\text{G}} \,|\, V_{\text{GB}} \,|\, V_{\text{LB}}]^T \in \mathbb{C}^{2n+m}$ is the vector of nodal voltages, $I = [I_{\text{G}} \,|\, \mathbf{0} \,|\, \mathbf{0}]^T \in \mathbb{C}^{2n+m}$ is the vector of currents injected into the nodes, and $Y_{\text{network}} \in \mathbb{C}^{(2n+m)\times(2n+m)}$ is the nodal *admittance matrix*. The matrix Y_{network} is the

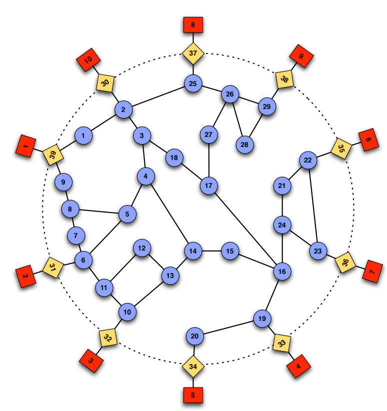

Fig. 1. Schematic representation of the power network topology **(i)-(iii)** for the New England Power Grid. The symbols ■, ◇, and ● correspond to the generators $\mathcal{V}_G = \{1, \dots, 10\}$, generator terminal buses $\mathcal{V}_{GB} = \{30 - 39\}$, and the load buses $\mathcal{V}_{LB} = \{11, 29\}$.

sum of the complex-valued Laplacian (or Kirchhoff) matrix $L(A(G_{\text{network}}))=\text{diag}(\sum_{j=1}^{n} A(G_{\text{network}})_{ij})-A(G_{\text{network}})$ and a diagonal matrix containing the shunt admittances:

$$\begin{aligned} Y_{\text{network}} &= L(A(G_{\text{network}})) \\ &+ \text{blkdiag}(\mathbf{0}_{n\times n}, \text{diag}(Y_{i_{\text{GB}}\text{-ground}}), \text{diag}(Y_{i_{\text{LB}}\text{-ground}}))\,. \end{aligned} \tag{2}$$

In the completely floating case, where all shunt admittances are zero, Y_{network} is simply a complex-valued Laplacian matrix.

We assume that the rotor dynamics of generator i are given by the constant-voltage behind reactance model [18], [19]

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + P_{\text{m},i} - P_{\text{e},i}, \quad i \in \{1,\dots,n\}\,, \tag{3}$$

where the rotor angle θ_i is measured with respect to a rotating frame with frequency f_0, $P_{\text{m},i} > 0$ is the mechanical power input, and $M_i > 0$ and $D_i > 0$ are the inertia and damping constant. The active output power injected by generator i into the adjacent generator terminal bus (with index j_{GB}) is $P_{\text{e},i} = \Re(V_i I_i^*) = \Re(V_i Y_{ij_{\text{GB}}}^* (V_i^* - V_{j_{\text{GB}}}^*))$. Typically, the loads are modeled as passive shunt admittances. Thus, the interior network is governed by the network equations (1) and for each bus $i \in \mathcal{V}_{GB} \cup \mathcal{V}_{LB}$ the power flow equations are obtained as

$$0 = V_i \sum\nolimits_{j\in\mathcal{V}_G\cup\mathcal{V}_{GB}\cup\mathcal{V}_{LB}} Y_{\text{network}}[i,j]^* V_j^*\,. \tag{4}$$

The constant-voltage behind reactance generator dynamics (3) and the algebraic power flow equations (4) define the classic differential-algebraic structure-preserving power system model.

B. The Network-Reduced Power System Model

Since the loads are assumed to be constant, all passive nodes $\mathcal{V}_{GB}\cup\mathcal{V}_{LB}$ can be eliminated, i.e., the algebraic equations (4) are removed and the network is reduced to its n active nodes $\mathcal{V}_{\text{G}}$, where the current I_{G} is injected. Spoken in terms of circuit theory, we look for the *reduced admittance matrix* Y_{red} relating boundary voltages and currents according to $I_{\text{G}} = Y_{\text{red}} V_{\text{G}}$.

Remark II.1 (Physical interpretation of Y_{red}) The (i,j) element of Y_{red} corresponds to the current at generator i due to a potential of 1 at generator j and 0 at all other generators.□

For the subsequent network reduction it is assumed that $Y_{\text{network}}(\mathcal{I}_n,\mathcal{I}_n)$ is non-singular, which can be easily verified due to diagonal dominance (as seen in (2)) and irreducibility of Y_{red} (due to connectivity of the corresponding graph). Standard Gaussian elimination of the lower $n+m$ nodes in equation (1) leads to $I_{\text{G}} = Y_{\text{red}} V_{\text{G}}$, where the reduced admittance matrix Y_{red} is obtained as the Schur complement

$$Y_{\text{red}} = Y_{\text{network}}/Y(\mathcal{I}_n,\mathcal{I}_n)\,.$$

This Schur-complementation is referred to as *reduction by structural partitioning* in the power systems literature.

Instead of obtaining Y_{red} as the Schur complement of the suitably partitioned matrix Y_{network}, the interior (passive) nodes $\mathcal{V}_{GB}\cup\mathcal{V}_{LB}$ can be eliminated equivalently by *iterative Kron reduction*. Given an admittance matrix $Y\in\mathbb{R}^{k\times k}$, the removal of the kth node in $\mathcal{V}_{GB}\cup\mathcal{V}_{LB}$, leads to the reduced admittance matrix $Y^+\in\mathbb{R}^{(k-1)\times(k-1)}$ defined component-wise as

$$Y_{ij}^+ = Y_{ij} - Y_{ik}\,Y_{jk}/Y_{kk}\,, \quad i,j\in\{1,\dots,k-1\}\,. \tag{5}$$

Kron's reduction formula (5) corresponds the Schur complement $Y^+ = Y/Y_{kk}$, or equivalently, Gaussian elimination of the kth line in equation (1). The following lemma follows directly from the *Quotient Formula* [17, Theorem 1.4].

Lemma II.2 *Iterative Kron reduction of Y_{network} with respect to all nodes $\mathcal{V}_{\text{GB}}\cup\mathcal{V}_{\text{LB}}$ is equivalent to the Schur complement $Y_{\text{red}} = Y_{\text{network}}/Y(\mathcal{I}_n,\mathcal{I}_n)$.* □

Iterative Kron reduction of all interior nodes $\mathcal{V}_{\text{GB}}\cup\mathcal{V}_{\text{LB}}$ leads to a complete graph among the boundary nodes $\mathcal{V}_{\text{G}}$, which will be formally shown in the next section. Note also that purely real (respectively imaginary) matrices Y_{network} have purely real (respectively imaginary) reduced matrices Y_{red}.

The off-diagonal elements of Y_{red} are referred to as *transfer admittances* and the diagonal elements as *self-admittances*. Typically, the admittances are inductive on the transmission level, the line admittances dominate the shunt admittances, and Y_{red} is a fully populated Laplacian-like matrix, which is verified later in Theorem III.4. For these reasons Y_{red} typically satisfies $\Re(Y_{\text{red}}[i,j])\geq 0$ and $\Im(Y_{\text{red}}[i,j]) > 0$ for all $i\neq j$, and $\Re(Y_{\text{red}}[i,i]) \geq 0$ and $\Im(Y_{\text{red}}[i,i]) < 0$ for the self-admittances, which we assume from now on.

In the reduced network the electrical output power is $P_{\text{e},i} = \Re(V_i\sum_{j=1}^{n} Y_{\text{red}}[i,j])^* V_j^*)$ given by the *power-angle relation*

$$\begin{aligned} P_{\text{e},i} = \sum\nolimits_{j=1}^{n} |V_i||V_j| \big(&\Re(Y_{\text{red}}[i,j])\cos(\theta_i-\theta_j) \\ &+ \Im(Y_{\text{red}}[i,j])\sin(\theta_i-\theta_j)\big). \end{aligned} \tag{6}$$

Equations (3) and (6) give the classic constant-voltage behind reactance model of interconnected *swing equations*.

C. Review of Sufficient Synchronization Conditions for a Lossless Network-Reduced Power System Model

Exponential synchronization in the reduced power network model given by (3) and (6) means that all angular (geodesic) distances $|\theta_i(t)-\theta_j(t)|$ become bounded below $\pi/2$ and all frequency differences $\dot{\theta}_i(t)-\dot{\theta}_j(t)$ converge to zero with exponential decay rate. When writing the system given by (3) and (6) in relative angular coordinates, the classic definition of *transient stability* is stability of an equilibrium $(\theta^*,\dot{\theta}^*)$ arising after a fault-clearance. Note that transient stability is a special case of synchronization, and the latter is defined independently of an explicit equilibrium point $(\theta^*,\dot{\theta}^*)$ and even independently of the existence of an equilibrium for (3) and (6).

In the earlier work [15], the authors derived sufficient conditions under which the network-reduced model given by (3) and (6) synchronizes exponentially. In the following, we review some of the results of [15] in the case of a lossless power network model with $\Re(Y_{\text{red}}[i,j])=0$ for all $i,j\in\mathcal{I}_n$. Define the *coupling weights* $P_{ij} := |V_i||V_j|\Im(Y_{\text{red}}[i,j]) > 0$ (maximum power transferred between generators i and j) with $P_{ii} := 0$ for $i\in\mathcal{I}_n$, then the dynamics (3) and (6) read as

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + P_{\text{m,i}} - \sum\nolimits_{j=1}^{n} P_{ij}\sin(\theta_i-\theta_j)\,. \tag{7}$$

To quantify the non-uniformity of the damping constants, define $\mu = \sum_{k=1}^{n}(D_k/n)\sqrt{\min_{i\neq j}\{D_iD_j\}/\max_{i\neq j}\{D_iD_j\}^3}$ (if all $D_i\equiv D$, then $\mu=1/D$). Also let $\epsilon := M_{\max}/(\pi f_0 D_{\min})$ be the maximal inertia over damping ratio. Two sufficient conditions for synchronization of the model (7) are as follows [15]:

Theorem II.3 *Consider the power network model* (7)*. Assume that either one of the following two conditions hold*

$$n\frac{\min_{i\neq j}\{P_{ij}\}}{D_{\max}} > \max_{\{i,j\}}\left\{\frac{P_{\text{m,i}}}{D_i}-\frac{P_{\text{m,j}}}{D_j}\right\}, \tag{8}$$

$$\lambda_2(L(P_{ij})) > \left\|(P_{\text{m},2}/D_2 - P_{\text{m},1}/D_1,\dots)\right\|_2/\mu\,. \tag{9}$$

If initially all angles $\theta_i(0)$ are contained in an arc of length strictly less than $\pi/2$, then for any bounded initial frequencies $\dot\theta_i(0)$ there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$ the power network model synchronizes exponentially.

Remark II.4 (Physical interpretation of Theorem II.3) For uniform voltages, $|V_i|\equiv V$, we have that $P_{ij} = V^2\Im(Y_{\text{red}}[i,j])$. Thus, the left-hand sides of (8)-(9) reflect the connectivity of the weighted graph induced by Y_{red}: the term $n\min_{i\neq j}\{P_{ij}\}/D_{\max}$ is a lower bound for $\min_i\sum_j P_{ij}/D_j$, the worst coupling of one generator to the network, and $\lambda_2(L(P_{ij}))$ is the algebraic connectivity of the coupling. The right-hand side describes the non-uniformity in power inputs $P_{\text{m,i}}$ scaled by the damping D_i. In summary, conditions (8) and (9) can be interpreted as "the network connectivity has to dominate the network non-uniformity in power inputs." □

Remark II.5 (Technical comments) It is also possible to extend Theorem II.3 to non-zero transfer conductances, give explicit conditions on the initial and asymptotic phase differences, the synchronization frequency and rates, and conditions for phase synchronization. The interested reader is referred to [15]. The smallness assumption on ϵ is a singular perturbation assumption such that the dimension-reduced model

$$D_i\dot\delta_i = P_{\text{m},i} - \sum\nolimits_{j=1}^{n} P_{ij}\sin(\delta_i-\delta_j),\quad i\in\{1,\dots,n\}\,. \tag{10}$$

can be analyzed. This assumption is well justified in [15]: among other justifications, (i) ϵ is indeed a small physical quantity when considering overdamped generators subject to local excitation controllers, (ii) the reduced model (10) is (ii) topologically equivalent to the full model (7) independent of the magnitude of ϵ, and (iii) the model (10) equivalent to the model analyzed by the classic and industrially applied PEBS and BCU algorithms [3]. The authors' analysis in [15] approaches the reduced model (10) as a generalization of the Kuramoto model [12] and the consensus protocol [11]. The synchronization conditions (8)-(9) are derived for the reduced model (10) and it can be shown that the approximation error $\theta_i(t)-\delta_i(t)$ is of order ϵ and ultimately vanishes, c.f. [15]. □

Remark II.6 (Necessary synchronization conditions) We emphasize that the conditions (8) and (9) are only sufficient. To derive a necessary condition for synchronization in the reduced model (10), consider the equation $\dot\delta_i - \dot\delta_j \equiv 0$, which has no solution if $|P_{\text{m},i}/D_i - P_{\text{m},j}/D_j| > \sum_{k=1}^n (P_{ik}/D_i + P_{jk}/D_j)$. Thus, the frequencies cannot be synchronized if the non-uniformity dominates the coupling. Our ongoing research reveals that condition (8) is necessary and sufficient when reducing (10) to the classic Kuramoto model with arbitrary and unknown distribution of the oscillators' natural frequencies. □

III. Analysis of the Kron Reduction Process

Consider a connected, weighted, and undirected graph with n nodes with Laplacian $L = L^T \in \mathbb{R}^{n\times n}$. Given a subset of nodes α with $2 \le |\alpha| \le n-1$, the *Kron-reduced Laplacian matrix* is defined as $L_{\text{red}} := L/L(\alpha,\alpha) \in \mathbb{R}^{|\alpha|\times|\alpha|}$. For notational simplicity, assume that $\alpha = \{1,\dots,|\alpha|\}$, that is, $L(\alpha,\alpha)$ is the lower diagonal block of L.

In a lossless network with purely-inductive line admittances and zero shunt admittances, the nodes α and $\mathcal{I}_n\setminus\alpha$ obviously correspond to the generators and buses, and the matrices L and L_{red} correspond to $\Im(-Y_{\text{network}})$ and $\Im(-Y_{\text{red}})$.

Remark III.1 (Related literature) In electrical impedance tomography L_{red} is also referred to as the *Dirichlet-to-Neumann map* [20]. The Schur complement of a matrix and its graph is also referred to as *Schur contraction* [21], it is known in the context of Gaussian elimination [22], and also as popular application example in linear algebra [23], [24]. □

A. Algebraic, Spectral, and Topological Properties

Recall that certain classes of matrices we are interested in are closed under Schur complementation [17, Chapter 4].

Lemma III.2 *The following classes of matrices are closed under Schur complementation: Hermitian matrices, (strictly) diagonally dominant matrices, and M-matrices.*

Another property of the Schur complement that we will make use of is the *interlacing property* [23, Theorem 3.1]:

Lemma III.3 *Let A be a Hermitian positive semidefinite matrix of order n and let β be a non-empty proper subset of $\mathcal{I}_n$ such that $A[\beta,\beta]$ is nonsingular. Then, for any $r\in\{1,\dots,n-|\beta|\}$,*

$$\lambda_r(A) \le \lambda_r(A/A[\beta,\beta]) \le \lambda_r(A(\beta,\beta)) \le \lambda_{r+|\beta|}(A)\,.$$

The following theorem shows that also the class of symmetric and irreducible Laplacian matrices is closed under Schur complementation with non-decreasing algebraic connectivity.

Theorem III.4 (Algebraic and Spectral Properties of Kron Reduction) *The following statements hold for the reduced Laplacian matrix $L_{\text{red}} = L/L(\alpha,\alpha)$:*

1) *the Schur complement $L_{\text{red}} = L/L(\alpha,\alpha)$ always exists;*
2) *L_{red} is a symmetric and irreducible Laplacian matrix;*
3) *for any $r\in\{1,\dots,|\alpha|\}$,*

$$\lambda_r(L) \le \lambda_r(L_{\text{red}}) \le \lambda_r(L[\alpha,\alpha]) \le \lambda_{r+n-|\alpha|}(L), \tag{11}$$

   and, in particular, $\lambda_2(L_{\text{red}}) \ge \lambda_2(L)$; and

4) *$L_{\text{red}}[i,j] \le L[\alpha,\alpha][i,j]$ for all $i,j\in\mathcal{I}_n\setminus\alpha$.*

Proof: By definition, L is weakly diagonally dominant since $L_{ij} = \sum_{j=1}^n |L_{ij}|$ for all $i\in\mathcal{I}_n$. Due to connectivity L is irreducible and thus $L(\alpha,\alpha)$ is strictly diagonally dominant, i.e., $L_{ii} = \sum_{j=1}^n |L_{ij}| > \sum_{j=|\alpha|+1}^n |L_{ij}|$ for all $i\in\alpha$. Hence, $L(\alpha,\alpha)$ is invertible and statement 1) follows.

Due to the Quotient Formula for Schur complements [17, Theorem 1.4], $L_{\text{red}} = L/L(\alpha,\alpha)$ can be obtained by iterative application of the Kron reduction formula (5) with respect to the lowest diagonal element. In the first step, a matrix $L^+ := L/L_{nn} \in \mathbb{R}^{(n-1)\times(n-1)}$ is obtained with

$$L^+_{ij} = L_{ij} - L_{in}L_{jn}/L_{nn}\,.$$

Since L is a symmetric and diagonally dominant M-matrix, so is L^+ by Lemma III.2. Moreover, L^+ has zero row sum since

$$\sum_{j=1}^{n-1} L^+_{ij} = \sum_{j=1}^{n-1}\left(L_{ij} - \frac{L_{in}\,L_{jn}}{L_{nn}}\right) = -L_{in} - L_{in}\sum_{j=1}^{n-1}\frac{L_{jn}}{L_{nn}} = 0,$$

where the last step follows since $\sum_{j=1}^{n-1} L_{jn} = -L_{nn}$ due to symmetry of L. Hence, L^+ is a symmetric Laplacian matrix, which can also be concluded for L_{red} by repetitive arguments.

Since L is a Laplacian matrix and hence positive semidefinite, Lemma III.3 can be applied with $\beta = \mathcal{I}_n \setminus \alpha$ and results in statement 3). In particular, (11) gives that $\lambda_2(L_{\text{red}}) \geq \lambda_2(L) > 0$, which implies non-decreasing algebraic connectivity and thus also irreducibility of L_{red}. This completes the proof of statement 2). Statement 4) follows from [25, Lemma 1]. ■

The following theorem gives an intuitive understanding of the Kron reduction process, the resulting reduced Laplacian matrix, and the corresponding graph.

Theorem III.5 (Topological Properties of Kron Reduction) *The following statements hold for the graph induced by the Kron-reduced matrix* $L_{\text{red}} = L/L(\alpha,\alpha)$*:*

1) *All existing edges in the graph induced by* L_{ij}, $i,j \in \alpha$, *persist in the graph induced by* L_{red}*;*
2) *Kron reduction of the nodes* $\mathcal{I}_n \setminus \alpha$ *leads to a complete graph among all nodes* α *that were adjacent to the nodes* $\mathcal{I}_n \setminus \alpha$ *prior to the reduction; and*
3) *If the nodes* $\mathcal{I}_n \setminus \alpha$ *are connected and each node* α *is adjacent to at least one node in* $\mathcal{I}_n \setminus \alpha$*, then Kron reduction of all nodes* $\mathcal{I}_n \setminus \alpha$ *leads to a complete graph among the* α *nodes. Equivalently, the Kron-reduced Laplacian matrix* L_{red} *induces a complete graph.*

Proof: Since the class of Laplacian matrices is closed under the Schur complement, by Theorem III.4, we restrict the discussion to off-diagonal elements L_{ij} with $i \neq j$.

First, we prove statements 1) and 2) for a single-step Kron reduction (5). Any nonzero and thus strictly negative element L_{ij} is rendered to a strictly negative element L^+_{ij} since the first term on the right-hand side of equation (5) is strictly negative and the second term is non-positive. Therefore, all existing edges in the graph induced by L_{ij} persist in the graph induced by L^+_{ij}. A zero element L_{ij} is converted into a strictly negative element L^+_{ij} iff both node i and node j are adjacent to node k. Since iterative Kron reduction is equivalent to the Schur complement $L_{\text{red}} = L/L(\alpha,\alpha)$, statements 1) and 2) follow.

Under the assumption of statement 3), iterative Kron reduction of $n - |\alpha| + 1$ (all but one) "interior nodes" $\mathcal{I}_n \setminus \alpha$ renders a connected graph among the interior nodes to a single interior node connected to all nodes α. Reduction of this last interior node results in a complete graph among the nodes α. ■

We remark that Theorem III.5 is stated in Theorems 4.20 and 4.23 in [21] using entirely different proof techniques.

B. Kron Reduction and Effective Resistance

This subsection briefly summarizes some properties established in [16]. The *effective resistance* or *resistance distance* R_{ij} between two nodes $i,j \in \mathcal{I}_n$ of an undirected, connected, and uniformly weighted graph with Laplacian L is defined as

$$R_{ij} := L^\dagger_{ii} + L^\dagger_{jj} - 2L^\dagger_{ij}\,, \tag{12}$$

where $L^\dagger$ is the Moore-Penrose pseudo inverse of L. Since $L^\dagger$ is symmetric and $R_{ii} := 0$ by definition, the resistance matrix R is again a symmetric matrix. For $i \neq j$ the reciprocal $1/R_{ij}$ is referred to as the *effective conductance* between $i,j \in \mathcal{I}_n$.

Remark III.6 (Physical interpretation) If the graph is understood as an electrical network, R_{ij} corresponds to the potential difference between the nodes i and j when a unit current is injected in i and extracted in j. Definition (12) can be readily extended to weighted graphs if the weights are understood as conductances in the electrical circuit. □

The effective resistance captures various global properties of the graph topology such as distance and connectivity measures. Many interesting results relating R, L, and $L^\dagger$ can be found in [26], and applications of effective resistance range from the connectivity of biochemical molecules to performance of distributed estimation algorithms and random walks in graphs.

The physical intuition in the Remarks II.1 and III.6 suggests that the elements $L_{\text{red}}[i,j]$ are related to the corresponding effective conductances $1/R_{ij}$. The following theorem gives the exact relation between the reduced Laplacian and the effective resistance. In essence, the effective resistance among a set of nodes α is invariant under Kron reduction of the nodes $\mathcal{I}_n \setminus \alpha$.

Theorem III.7 (Invariance of the Effective Resistance) *Consider the reduced Laplacian* $L_{\text{red}} = L/L(\alpha,\alpha)$*, and the matrix* R *of effective resistances defined in* (12)*. Then for* $i,j \in \alpha$*,* $i \neq j$*, it holds that* $R_{ij} = L^\dagger_{\text{red}}[i,i] + L^\dagger_{\text{red}}[j,j] - 2L^\dagger_{\text{red}}[i,j]\,.$

Theorem III.7 establishes a simple but implicit relationship among R and $L^\dagger_{\text{red}}$. If the effective resistances among the nodes α are uniform, then an explicit relationship can be found:

Corollary III.8 *The following statements are equivalent:*

1) *the off-diagonal elements of* L_{red} *are uniform: there is* $\lambda > 0$ *such that* $L_{\text{red}}[i,j] = -\lambda$ *for all* $i,j \in \mathcal{I}_{|\alpha|}$, $i \neq j$*;*
2) *the effective resistances* R_{ij} *among* α *nodes are uniform: there is* $r > 0$ *such that* $R_{ij} = r$ *for all* $i,j \in \alpha$, $i \neq j$.

Moreover, if both cases are true, then $\lambda = (2/|\alpha|)/r$.

Uniform effective resistances among a set of nodes α occur in a variety of graphs, as the following examples demonstrate.

Example III.9 (Uniform Effective Resistances) In the trivial case, $|\alpha| = 2$, the effective resistance among the α nodes is clearly uniform. Second, if the α nodes are 1-connected leaves of a highly symmetric graph among the nodes $\mathcal{I}_n \setminus \alpha$, such as a star-shaped tree, a complete graph, or a combination of these two, then the effective resistance among the nodes α is uniform. Third, the effective resistance in large-scale random small-world networks is known to become uniform among sufficiently distant nodes. Fourth, the effective resistance in random geometric graphs converges to a uniform limit as the number of nodes increases. Fifth and finally, geometric graphs such as lattices and their fuzzes are special random geometric graphs with vertices sampled on a grid. According to the previous arguments, the resistance among sufficiently distant lattice nodes becomes uniform in the large limit [16].□

IV. Sufficient Spectral and Resistance-Based Conditions for Synchronization

In the following, the results of Section III are applied to a lossless power network, where Y_{network} is purely inductive. Assume that the voltages are uniform, and that the shunt admittances can be modeled equivalently as admittances with respect to an auxiliary reference bus, and thus all buses are floating. In this case, $\Im(-Y_{\text{network}})$ is a real-valued Laplacian and it follows that $\Im(-Y_{\text{red}}) = L(P_{ij})/V^2$. Under this assumption we can state the following corollary to Theorem II.3.

Corollary IV.1 (Spectral and Resistance-based Synchronization Condition) *Consider the network-reduced model* (7) *derived from the structure-preserving model* (3)-(4)*, and assume floating buses and uniform generator voltages* $|V_i| \equiv V$*. Assume that either one of the two following conditions hold:*

(i) the effective conductance $1/R$ *among all generator nodes in* $\Im(-Y_{\text{network}})$*) is uniform and larger than a critical value, i.e.,*

$$\frac{1}{R} > \max_{\{i,j\}} \left\{ \frac{P_{\text{m,i}}}{D_i} - \frac{P_{\text{m,j}}}{D_j} \right\} \frac{D_{\max}}{2V^2}\,, \tag{13}$$

or (ii) the algebraic connectivity of the power network G_{network} *is larger than a critical value, i.e.,*

$$\lambda_2(\Im(-Y_{\text{network}}))) > \left\| (P_{\text{m},2}/D_2 - P_{\text{m},1}/D_1, \dots) \right\|_2 / (V^2 \mu). \tag{14}$$

If initially all angles $\theta_i(0)$ *are contained in an arc of length strictly less than* $\pi/2$*, then for any bounded initial frequencies* $\dot{\theta}_i(0)$ *there exists* $\epsilon^* > 0$ *such that for all* $\epsilon < \epsilon^*$ *the power network model synchronizes exponentially.*

Proof: Under the assumptions in case *(i)*, it follows from Corollary III.8 that $|Y_{\text{red}}[i,j]| = 2/(nR)$ and consequently also $\min_{i \neq j}\{P_{ij}\} = 2V^2/(nR)$. Thus condition (8) in Theorem II.3 is rendered to (13). In case *(ii)*, condition (14) guarantees condition (9) in Theorem II.3 due to statement 3) in Theorem III.4. Synchronization follows directly from Theorem II.3. ■

Condition (13) requires uniform effective resistances among the generators, which can be verified for the Examples III.9 and is also reasonable from a physical viewpoint: the generators are spread over the network such that they can effectively balance the loads. Thus, the potential difference (the effective resistance) should ideally be equal for all generator pairs.

V. Conclusions

This paper studied synchronization in a simple structure-preserving power network model. The network-reduction to the classic swing equations was related to the reduced Laplacian matrix for which various algebraic and graph-theoretic properties were established. These results allowed the extension of the authors' earlier synchronization conditions for network-reduced models to network-preserving models.

The following assumptions should be removed to render the power network model more realistic: purely inductive line admittances, zero shunt admittances, and uniform voltages during transients. Ongoing work addresses sharper synchronization conditions, the effects of loads modeled as shunt admittances, and further properties of the Kron reduction process.

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