

On Frame and Orientation Localization for Relative Sensing Networks^{☆,☆☆}

Giulia Piovan^a, Iman Shames^b, Barış Fidan^c, Francesco Bullo^a, Brian D. O. Anderson^d

^aCenter for Control, Dynamical Systems and Computation, University of California, Santa Barbara, CA 93106, USA

^bACCESS Linnaeus Centre, Electrical Engineering, Royal Institute of Technology, SE-100 44 Stockholm, Sweden

^cDepartment of Mechanical and Mechatronics Engineering, University of Waterloo, Waterloo, ON N2L 3G1, Canada

^dAustralian National University and National ICT Australia, Canberra, ACT 0200, Australia

Abstract

We develop a novel localization theory for networks of nodes that measure each other's bearing, i.e., we assume that nodes do not have the ability to perform measurements expressed in a common reference frame. We begin with some basic definitions of frame localizability and orientation localizability. Based on some key kinematic relationships, we characterize orientation localizability for planar networks with angle-of-arrival sensing. We then address the orientation localization problem in the presence of noisy measurements. Our first algorithm computes a least-squares estimate of the unknown node orientations in a ring network given angle-of-arrival sensing. For arbitrary connected graphs, our second algorithm exploits kinematic relationships among the orientations of nodes in loops in order to reduce the effect of noise. We establish the convergence of the algorithm, and through some simulations we show that the algorithm reduces the mean-square error due to the noisy measurements in a way that is comparable to the amount of noise reduction obtained by the classic least-square estimator. We then consider networks in 3-dimensional space and we explore necessary and sufficient conditions for orientation localizability in the noiseless case.

1. Introduction

One of the key problems in sensor networks is localization, i.e., determining the location of each sensor in the network. Sensor networks are used in a large number of applications which cover a wide range of fields, such as, surveillance, controls, communications, monitoring areas, intrusion detection, vehicle tracking, mapping and reconstruction of environments (camera sensor networks). In particular, angle-of-arrival sensors are used for operations such as maintaining formations of robotic agents, or geographically locating cell phones and other wireless devices based on the information from multiple antennas.

We address the aforementioned problem in a distributed manner, by assuming that any node in the network has its own reference frame, and does not have any knowledge about its physical position in the environment or the position of the other nodes. Each node, through a sensor,

can detect the relative position of any other node inside a given sensor footprint. The measurements are affected by noise, so we extend our analysis to the noisy case. We call *frame localization* the problem of computing the relative location and orientation of each node of the network with respect to each other. In the 2-d case, we aim to solve the problem through a distributed algorithm, which computes the estimate of the angle associated to every edge of the graph by distributing the error of every cycle on its edges.

Network localization has been the center of extensive research work, and the various approaches are due to different assumptions on the deployment of the nodes, the dimension of the space, and the type of sensors employed. In some cases, there is the use of special nodes, whose positions are known, called beacons or anchors (e.g., see Aspnes et al. (2006), Moore et al. (2004), and Khan et al. (2009).) Aspnes et al. (2006) present a theoretical foundation of the network localization problem; the authors provide conditions for uniqueness in localization of networks with beacons and distance measurements, and study the computational complexity in uniquely localizable networks and in typical network deployment scenarios. Together with the aforementioned, other works that exploit distance informations or relative positions between node in studying networks and robot formations are available in the literature; e.g, see Zhou and Roumeliotis (2007) and Trawny et al. (2010). In the present work, we focus on angle-of-arrival measurements. Therefore, particular interest arises from the work by Rong and Sichertiu (2006), who propose a localization and orientation approach based on angle-of-arrival information between neighboring nodes. However, again, prior knowledge of the orientation of a few

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^{☆☆}A preliminary version of this work appeared as (Piovan et al., 2008); differences with this article include complete proofs of all statements, improved discussions about motivation and performance, and the treatment of 3-dimensional problems.

Email addresses: giulia@engineering.ucsb.edu (Giulia Piovan), imansh@kth.se (Iman Shames), fidan@uwaterloo.ca (Barış Fidan), bullo@engineering.ucsb.edu (Francesco Bullo), Brian.Anderson@anu.edu.au (Brian D. O. Anderson)

nodes is required. The authors of Eren et al. (2006) study the uniqueness of network localization solutions through the theory of rigid graphs, but they do not provide an algorithm to compute such solution, even for the noiseless case. Zhu et al. (2008) provide two methods for network localization using angle-of-arrival measurements. However, the problem statement differs from ours in that it utilizes the relationship between triplets of nodes, whereas we admit relationship between a greater number of nodes. Our work is more closely related to the work of Tron and Vidal (2009), where a distributed algorithm for 3-d sensor network orientation and translation localization is proposed.

This paper contains several contributions. First, we present a novel formulation of the frame localizability and frame computational localization problem for networks in 2-d or 3-d ambient space with relative sensing. Second, we define a characterization of frame localizability for planar networks, focusing on consistency for the orientation localization problem. Third, we consider arbitrary connected graphs and provide a distributed algorithm for planar orientation localization which exploits kinematic relationships among the orientation of nodes in loops in order to reduce the effect of noise. Fourth, we provide simulations in order to validate our algorithm results. Finally, we consider networks in three-dimensional space and we explore necessary or sufficient conditions for orientation localizability in the noiseless case.

The paper is organized as follows. Section II reviews some basic notions from kinematics and graph theory. Section III contains the model and the problem statement. Section IV contains the localizability results and the localization algorithm. Section V explores the orientation localizability problem in three-dimensional space.

2. Preliminaries

2.1. Elements of kinematics

Let \mathbb{R} and \mathbb{C} denote real and complex numbers, respectively. Let $\|\mathbf{v}\|$ denote the Euclidean norm of the vector $\mathbf{v} \in \mathbb{R}^d$. Define the versor operator $\text{vers}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $\text{vers}(\mathbf{0}) = \mathbf{0}$ and $\text{vers}(\mathbf{v}) = \mathbf{v}/\|\mathbf{v}\|$ for $\mathbf{v} \neq \mathbf{0}$. Define the map $\text{proj}: \mathbb{R} \rightarrow [-\pi, \pi[$ by

$$\text{proj}(x) = (x + \pi) \bmod 2\pi - \pi, \quad (1)$$

and similarly $\text{proj}: \mathbb{R}^n \rightarrow [-\pi, \pi[$ by $\text{proj}([x_1, \dots, x_n]^T) = [\text{proj}(x_1), \dots, \text{proj}(x_n)]^T$. Let $\angle z$ denote the phase of $z \in \mathbb{C}$. We are interested in measurements expressed in different reference frames. Accordingly, we review some basic kinematic conventions. We let $\Sigma_1 = \{p_1, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1\}$ be a fixed reference frame in \mathbb{R}^3 . A point q and a vector \mathbf{v} expressed with respect to frame Σ_1 are denoted by q^1 and \mathbf{v}^1 , respectively. Next, let $\Sigma_2 = \{p_2, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2\}$ be a reference frame fixed with a moving body. The origin of Σ_2 is the point p_2 , denoted by p_2^1 when expressed with respect to Σ_1 . The orientation of Σ_2 is characterized by the 3-dimensional rotation matrix \mathbf{R}_2^1 , whose columns are the frame vectors $\{\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2\}$ of Σ_2 expressed with respect to Σ_1 . We recall here the definition of the set of rotation

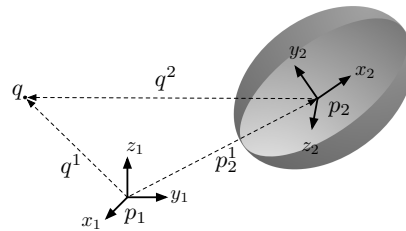


Figure 1: Two reference frames in \mathbb{R}^3 .

matrices in d -dimensions: $SO(d) = \{\mathbf{R} \in \mathbb{R}^{d \times d} \mid \mathbf{R}\mathbf{R}^T = I_d, \det(\mathbf{R}) = +1\}$. With these notations, reference frame transformations in 3-dimensions are described by

$$q^1 = \mathbf{R}_2^1 q^2 + p_2^1, \quad \text{and} \quad \mathbf{v}^1 = \mathbf{R}_2^1 \mathbf{v}^2. \quad (2)$$

Recall also $\mathbf{R}_2^1 = (\mathbf{R}_1^2)^T$. Analogously, we let S^i denote the point set S as expressed in the reference frame Σ_i . Finally, if three reference frames Σ_i , $i \in \{1, 2, 3\}$, are considered, then simple bookkeeping arguments lead to

$$\mathbf{R}_2^1 \mathbf{R}_3^2 \mathbf{R}_1^3 = I_3, \quad \text{and} \quad \mathbf{R}_2^1 = \mathbf{R}_3^1 \mathbf{R}_2^3. \quad (3)$$

Next, we present a planar case version of these notions. In the planar case, p_1 and p_2 take values in \mathbb{R}^2 , the reference frames consist of only two orthonormal vectors, and the rotation matrices take values in $SO(2)$. It is convenient to identify \mathbb{R}^2 with the set of complex numbers \mathbb{C} and to denote the unit imaginary number by $\sqrt{-1} \in \mathbb{C}$. If we describe the planar rotation matrix $\mathbf{R}_2^1 \in SO(2)$ by its unit-length complex number representation $\exp(\theta_2^1 \sqrt{-1})$, with angle $\theta_2^1 \in [-\pi, \pi[$, then the second part of eq. (2) reads $\mathbf{v}^1 = \exp(\theta_2^1 \sqrt{-1}) \mathbf{v}^2$.

Finally, we review the exponential representation of rotations. For the unit vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$, we use Rodrigues' rotation formula (Murray et al., 1994) to define the rotation matrix about axis $\boldsymbol{\omega}$ of an angle γ as

$$\exp(\gamma \widehat{\boldsymbol{\omega}}) = I_3 + \sin \gamma \widehat{\boldsymbol{\omega}} + (1 - \cos \gamma) \widehat{\boldsymbol{\omega}}^2, \quad (4)$$

where $\widehat{\boldsymbol{\omega}} \in \mathbb{R}^{3 \times 3}$ is defined by $\widehat{\boldsymbol{\omega}} \mathbf{z} = \boldsymbol{\omega} \times \mathbf{z}$, for all $\mathbf{z} \in \mathbb{R}^3$. We also recall that for any $\mathbf{R} \in SO(d)$ and $\mathbf{v} \in \mathbb{R}^d$,

$$\mathbf{R} \exp(\widehat{\mathbf{R}\mathbf{v}}) = \exp(\widehat{\mathbf{v}}) \mathbf{R}. \quad (5)$$

2.2. Elements of graph theory

We review a few notions and a theorem from graph theory, e.g., see (Diestel, 2005; Foulds, 1995). We let $G = (V, E)$ represent an undirected graph G , with vertex set $V \triangleq \{v_i\}_{i=1}^n$ and edge set E with cardinality m . $G_D = (V, E_D)$ defines a directed graph associated to G , where E_D is an orientation of E . We denote a directed edge from v_i to v_j by $e_{ij} = (i, j)$. If the graph is undirected, (i, j) is equivalent to (j, i) .

Definition 1 (Path and cycle) *Let G be either a directed or undirected non-empty graph. A path is a non-empty graph $P = (V_P, E_P) \subseteq G$ of the form $V_P \triangleq \{v_i\}_{i=1}^k$ and*

$E_P \triangleq \{(j_i, j_{i+1})\}_{i=1}^{k-1}$, where $\{j_1, \dots, j_k\}$ is a permutation of v_1, \dots, v_k . Every sequence of edges that form a closed path in G and do not visit the same node twice, except the start/end node, is called a cycle and it is denoted by ℓ .

The direction of a cycle is the order in which the nodes are visited. We let $\mathcal{L}(G)$ denote the set of all cycles ℓ of G , and $|\ell|$ the number of edges in ℓ . It should be noted that, in a digraph G_D , the cycle directions are independent of the direction of the individual edges composing the cycles.

Definition 2 (Cycle vector) For $\ell \in \mathcal{L}(G_D)$, the cycle vector is the vector $\mathbf{1}_\ell \in \{-1, 0, +1\}^m \subset \mathbb{R}^m$ whose i th entry is $+1$ if the i th edge belongs to ℓ and its orientation is consistent with the orientation of ℓ , -1 if the i th edge belongs to ℓ and its orientation is opposite to the orientation of ℓ , and is 0 otherwise.

Definition 3 (Set of cycle and fundamental cycle vectors) The set of cycle vectors is $L = \{\mathbf{1}_\ell \mid \ell \in \mathcal{L}(G_D)\}$. A set of fundamental cycle vectors $L_f \subseteq L$ is a subset of L that constitute a basis for the linear space generated by L . The elements of L_f are called fundamental cycle vectors.

Given a set of fundamental cycle vectors L_f , we let $\mathcal{L}_f(G_D)$ denote the associated fundamental cycles $\mathcal{L}_f(G) = \{\ell \in \mathcal{L}(G_D) \mid \mathbf{1}_\ell \in L_f\}$.

Definition 4 (Cycle and fundamental cycle matrix) The cycle matrix C of a directed graph G_D is the $k \times m$ matrix $C = [\mathbf{1}_{\ell_1}, \dots, \mathbf{1}_{\ell_k}]^T$ where k is the cardinality of L , and m is the number of edges of G_D . An $r \times m$ matrix $C_f \subseteq C$, with $r = \dim(L_f)$, such that each row represents a fundamental cycle vector in L_f , is called a fundamental cycle matrix:

$$C_f = [\mathbf{1}_{\ell_1}, \dots, \mathbf{1}_{\ell_r}]^T, \quad \text{for all } \mathbf{1}_{\ell_i} \in L_f. \quad (6)$$

Note that C_f is not unique since it depends on the choice of the fundamental cycle vectors, and it is a full rank matrix.

Theorem 5 (Number of independent cycles) If G_D has n vertices and m edges, then the dimension of the fundamental cycle space L_f is $m-n+1$, i.e., there are $m-n+1$ independent cycles.

3. Network model and localization problems

In what follows we describe our notion of a *network equipped with relative sensors*. We consider a group of n distinct nodes $\{p_1, \dots, p_n\}$ in \mathbb{R}^d , for $d \in \{2, 3\}$, and corresponding reference frames $\{\Sigma_1, \dots, \Sigma_n\}$ with the property that p_i is the origin of Σ_i for all $i \in \{1, \dots, n\}$.

3.1. Relative sensing model

Each node i activates a sensor that detects the presence and returns a measurement of the relative position of any node inside a given sensor footprint. The principal sensing modality that we will use throughout the paper is the

angle-of-arrival sensing: node i measures $\text{vers}(p_j^i) \in \mathbb{R}^d$ for all nodes j within a fixed sensing range r from i , i.e., for all nodes j whose distance from i is not greater than r .

Given the nodes p_1, \dots, p_n , the directed *sensing graph*, $G_D = (V_S, E_D)$ is the directed graph where vertex v_i corresponds to node i and the directed edge $(i, j) \in E_D$ if node j is inside the sensor footprint of node i . In what follows, we assume that the sensing graph is the so-called unit-disk geometric graph. With this assumption, if node i senses node j , then node j senses node i as well. Therefore, $(i, j) \in E_D$ iff $(j, i) \in E_D$. To simplify notations we use an undirected graph $G_S = (V_S, E_S)$ with vertex set V_S and undirected edge set E_S satisfying $(i, j) \in E_S \iff (i, j) \in E_D \iff (j, i) \in E_D$. We call G_S the undirected sensing graph or simply the sensing graph. We further assume that a pair of nodes i and j communicate with each other if and only if they can sense each other, i.e., $(i, j) \in E_S$. In summary, the physical components of a *relative sensing network* consist of n nodes with identifiers in $\{1, \dots, n\}$, with configurations in $\mathbb{R}^d \times SO(d)$, and with angle-of-arrival sensors.

3.2. The frame localization problem

We call *frame localization* the problem of computing the location and orientation of each node of a relative sensing network. Additionally, we call *orientation localization* the problem of computing the orientation of each node of a relative sensing network. We begin with questions about the uniqueness of these localization problems.

Problem 6 (Frame and orientation localizability) Given a relative sensing network with reference node 1, provide graph theoretical conditions under which:

(frame localizability): the reference frame transformations $\{\mathbf{R}_i^1, p_i^1\}$, for all $i \in \{2, \dots, n\}$, are uniquely determined by the relative measurements;

(orientation localizability): the orientations \mathbf{R}_i^1 , for all $i \in \{2, \dots, n\}$, are uniquely determined by the relative measurements. \square

Problem 7 (Centralized and distributed localization) Given a frame (respectively, orientation) localizable network, give a centralized or distributed algorithm to compute the reference frames transformation $\{\mathbf{R}_i^1, p_i^1\}$ (respectively, the orientations \mathbf{R}_i^1), for all $i \in \{2, \dots, n\}$. Give algorithms for both noise-less and noisy sensor measurements. \square

Remark 8 (Data referencing motivation) It is worth remarking that the frame localization problem needs to be solved in relative sensing networks if measurements taken by arbitrary sensors in their respective reference frames need to be expressed (and possibly fused) in a common unique reference frame. Measurements might include positions of targets, environment boundaries, etc. \square

3.3. Preliminary relationships

In three dimensions, for any sensing and communication undirected edge (i, j) , the relationship between the relative positions p_j^i and p_i^j and the change of frame rotation matrix \mathbf{R}_j^i can be computed from (2) to be $p_j^i = -\mathbf{R}_j^i p_i^j$. It is possible to write a normalized version of this equation that applies to angle-of-arrival measurements:

$$\text{vers}(p_j^i) = -\mathbf{R}_j^i \text{vers}(p_i^j), \quad (7)$$

The planar version, where relative positions are complex numbers and rotations matrices are unit-length complex numbers, is (recall eq. (1))

$$\theta_j^i = \text{proj}(\angle p_j^i - \angle p_i^j + \pi). \quad (8)$$

Recall that nodes i and j measure each other's relative positions p_j^i and p_i^j , respectively. The unknown variable in eq. (7) is the rotation matrix \mathbf{R}_j^i with d degrees of freedom.

4. Two-dimensional frame localization

4.1. Orientation localizability with angle-of-arrival sensors

Theorem 9 (Orientation localizability) *For a planar relative sensing network with noiseless angle-of-arrival sensing, the following statements are equivalent:*

- (i) *the sensing graph is connected, and*
- (ii) *the network is orientation localizable.*

Proof: For every undirected edge (i, j) of the sensing graph, the angles $\angle p_j^i$ and $\angle p_i^j$ are measured. Therefore, eq. (8) implies that the relative angle θ_j^i is uniquely determined from the measurements. Now, let us prove (i) \implies (ii). If the network is connected, there exists a path from a reference node, e.g., node 1, to each $i \neq 1$. From eq. (3), the angle θ_i^1 is uniquely determined as the sum of the relative angles along the path connecting i to the reference node. Let us now prove (ii) \implies (i). Assume that there exists no path from node i to the reference node 1. Therefore, i and 1 belong to distinct connected components with the network. No measurement is available about the relative orientation of each node in the component containing i with respect to any node in the component containing 1. Therefore, it is not possible that only a single orientation θ_i^1 is compatible with the measurements. ■

Proposition 10 (Sufficient conditions for localizability)

A planar relative sensing network with noiseless angle-of-arrival sensing is both frame localizable and orientation localizable if the sensing graph is rigid and at least one of the edge lengths is known.

To prove Proposition 10, we introduce the following definitions. Consider a reference frame with nodes $\{p_1, \dots, p_n\}$, where $p_i \in \mathbb{R}^2$, $i = \{1, \dots, n\}$. We can identify the reference frame with a point $p \in \mathbb{R}^{2n}$. A *length constraint* Υ between two points p_i and p_j in \mathbb{R}^2 is an equality of the

form $\|p_i - p_j\|^2 = \Upsilon$, for any fixed Υ . A *direction constraint* Ω between two points p_i and p_j in \mathbb{R}^2 is an equality of the form $\angle(p_i - p_j) = \Omega$, for any fixed Ω , where the angle is computed with respect to a fixed reference axis. It is then possible to introduce the direction graph $G_\Omega = (V, \Omega)$ and the length graph $G_\Upsilon = (V, \Upsilon)$ and consider the double graph $G_{\Omega, \Upsilon} = (V; \Omega, \Upsilon)$. We can measure the distance among points through the *rigidity function* $\zeta: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n(n+1)/2}$ defined by $\zeta(p)_{i,j} = \|p_i - p_j\|^2$, for $i < j \leq n$. The *rigidity matrix* for p is defined by $D(p) = \frac{1}{2}\zeta'(p)$. The *constraint matrix* of the graph, denoted by $D(G_{\Omega, \Upsilon}, p)$, consists of the rows of $D(p)$ that correspond to the edges in Υ and Ω . A set of constraints are said to be *independent* if the corresponding rows of the constraint matrix are independent. The reference frame p is said to be *rigid* if $\text{rank} D(p) = 2n - 3$. The following result is taken from Servatius and Whiteley (1999).

Lemma 11 (Number of independent constraints) *A graph with n nodes and $2n - 3$ independent direction constraints plus any single length constraint, has $2n - 2$ independent constraints and a 2-dimensional space of translations in the plane.*

Proof: [Proof of Proposition 10] Since the sensing graph is rigid, it is connected and it has at least $2n - 3$ independent edges. Hence, by Theorem 9, it is orientation localizable, and by Lemma 11, the corresponding framework has a 2-dimensional space of translation in the plane. Fixing the origin as any node $i \in \{1, \dots, n\}$, the pair $\{R_j^i, p_j^i\}$, for any $j \in \{1, \dots, n\}$, is uniquely determined. Hence the network is frame localizable. ■

4.2. Orientation localization with noisy sensors: model and problem statement

As in Theorem 9, we consider a network with nodes in the plane and with angle-of-arrival sensing. We assume that, for each undirected edge (i, j) of the sensing graph, nodes i and j measure, respectively, the angles $\angle p_j^i + n_j^i$ and $\angle p_i^j + n_i^j$, where we assume the noises n_j^i and n_i^j to be independent Gaussian random variables with zero mean and variance σ^2 . We assume that the angle-of-arrival sensors are relatively accurate so that $\sigma^2 \ll \pi$. Therefore, for each undirected edge (i, j) , we can measure only

$$y_j^i = \text{proj}((\angle p_j^i + n_j^i) - (\angle p_i^j + n_i^j) + \pi), \quad (9)$$

and not the true relative orientation θ_j^i as in eq. (8).

If the sensing graph is a tree, then there is no redundant measurement and we cannot reduce the effect of measurement noise on our angle estimates. However, for every cycle in the network, we can enforce a cycle constraint (see eq. (3)) and thereby partially mitigate the noise. We formalize this statement as follows.

Consider a sensing graph with n nodes and m edges, and let G_D be the associated directed graph computed by assigning a direction to each edge in the following way: the direction is from j to i if $i > j$. For each oriented edge $e = (j, i)$ of G_D , let ψ_e denote the estimate of the

true relative angle θ_i^j associated to e . Let $\psi \in \mathbb{R}^m$ denote the vector of angle estimates for every edge of the graph. Analogously, let y denote the measurement vector with components $y_e = y_i^j$, for $i > j$. For $\ell \in \mathcal{L}(G_D)$, the cycle error ϵ_ℓ at ψ is

$$\epsilon_\ell = \text{proj}(\mathbf{1}_\ell \cdot \psi), \quad (10)$$

where $\text{proj}(\mathbf{1}_\ell \cdot \psi) = \text{proj}\left(\sum_{f \in \ell} \pm \psi_f\right)$, and \pm indicates whether or not the direction of the edge f is concordant with the direction of the cycle ℓ which f belongs to.

Motivated by this setup, we consider the nonlinear least-squares estimation problem:

$$\begin{aligned} \min_{\psi \in [-\pi, \pi]^m} \quad & \|\text{proj}(\psi - y)\|^2 \\ \text{subject to} \quad & \text{proj}(\mathbf{1}_\ell \cdot \psi) = 0, \text{ for all } \ell \in \mathcal{L}(G_D). \end{aligned} \quad (11)$$

Here, cost function and constraint set are nonconvex, e.g., the constraint set is a countable set of affine subspaces.

4.3. Orientation localization with noisy sensors: an iterative projection algorithm

In this section we propose an iterative projection algorithms that computes an approximate solution to the optimization problem (11) and begin its analysis.

For a network G_D with set of cycles $\mathcal{L}(G_D)$, let \mathcal{L}_f be a set of fundamental cycles and let ψ_e denote the estimate of the angle associated to the edge e . For $0 < \kappa \ll 1$, consider the discrete-time system:

$$\begin{aligned} \psi_e(0) &= y_e, \\ \psi_e(t+1) &= \psi_e(t) - \kappa \sum_{\ell \in \mathcal{L}_f: e \in \ell} (\mathbf{1}_\ell \cdot \mathbf{e}_e) \text{proj}(\mathbf{1}_\ell \cdot \psi(t)), \end{aligned} \quad (12)$$

for all edges e . Here \mathbf{e}_i is the m -dimensional vector whose i -th entry is 1, and all the other entries are equal to zero.

In what follows we establish in what sense this algorithm is distributed, and when and how fast the algorithm converges to a feasible solution. The next subsection discusses how the algorithm reduces the effect of noise in the measurements and approximates the solution to the optimization problem (11).

The discrete-time system (12) is distributed in the following sense. First, we define an appropriate notion of neighborhood. Given \mathcal{L}_f , two edges are neighbors if there exists a cycle in \mathcal{L}_f containing them. Second, we assume that each node of the graph contains a processor and we associate to each edge $e = (j, i)$ of the graph a unique processor (e.g., the processor j if $j > i$). The estimate ψ_e is maintained and updated by the processor associated with the edge e . Third and final, we note that eq. (12) can be evaluated if communication is exchanged between processors of neighboring edges. More precisely, the value of ψ_e can be updated according to eq. (12) if the processor associated to e exchanges information with the processors associated to every edge neighbor of e .

Theorem 12 (Exponential convergence of iterative estimation algorithm) *Consider a planar relative sensing network with noisy angle-of-arrival sensing. For the*

digraph associated to the sensing graph, let L_f be a set of fundamental cycle vectors and C_f be the corresponding fundamental cycle matrix. Consider the discrete-time system (12) and assume $\kappa < 2/(1 + \lambda_{\max}(C_f C_f^T))$, where $\lambda_{\max}(C_f C_f^T)$ is the maximum eigenvalue of $C_f C_f^T$. Then, every solution of (12) converges to the set of angles with zero cycle error and does so with exponential convergence factor $(1 - \kappa)^2$.

Proof: Let G_D be the directed digraph, and let $\dim(L_f) = r$. Given the fundamental cycles ℓ_1, \dots, ℓ_r , define the cycle error vector ϵ at ψ by $\epsilon = [\epsilon_{\ell_1}, \dots, \epsilon_{\ell_r}]^T$, where ϵ_{ℓ_i} is defined by (10), for all $i \in \{1, \dots, r\}$. With this notation we have

$$\psi(t+1) = \psi(t) - \kappa \sum_{\ell \in \mathcal{L}(G_D)} \mathbf{1}_\ell \epsilon_\ell(t).$$

Then for every loop $\alpha \in \mathcal{L}(G_D)$,

$$\hat{\epsilon}_\alpha(t+1) = \hat{\epsilon}_\alpha(t) - \kappa \sum_{\ell \in \mathcal{L}(G_D)} (\mathbf{1}_\alpha \cdot \mathbf{1}_\ell) \epsilon_\ell(t),$$

where $\hat{\epsilon}_\alpha(t) = (\mathbf{1}_\alpha \cdot \psi(t))$. By choosing a base of independent loops ℓ_i , $i \in \{1, \dots, r\}$, and an associated fundamental cycle matrix C_f as defined in (6), we can write this for all the loops as vector $\hat{\epsilon}$, whose evolution is given by

$$\hat{\epsilon}(t+1) = \hat{\epsilon}(t) - \kappa C_f C_f^T \epsilon(t),$$

and

$$\epsilon(t+1) = \text{proj}((I_r - \kappa F)\epsilon(t)), \quad (13)$$

where $F = C_f C_f^T$. Note that F is symmetric positive definite. Consider now the associated linear system

$$\mathbf{x}(t+1) = (I_r - \kappa F)\mathbf{x}(t),$$

and the Lyapunov function candidate $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$, with $P = I_r$. Next, for $\kappa \in]0, 2[$, we define $Q = (2\kappa - \kappa^2)I_r > 0$. Noting that $A = I_r - \kappa F$, we find the values of κ such that the discrete-time Lyapunov inequality $A^T P A - P \leq -Q$ holds. Because F is symmetric positive definite, it can be diagonalized as $F = U \Lambda U^T$, with an orthogonal matrix U and a positive definite diagonal matrix $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_r\}$. Accordingly, the above discrete-time Lyapunov inequality reads

$$U(I_r - \kappa \Lambda)^T (I_r - \kappa \Lambda) U^T - I_r \leq -U(2\kappa - \kappa^2)U^T,$$

which is satisfied if and only if $(1 - \kappa \lambda_i)^2 - 1 + 2\kappa - \kappa^2 < 0$, for $i \in \{1, \dots, r\}$. Since $\lambda_{\min}(F) \geq 1$ (see Arioli et al. (2006)), this is in turn satisfied for $\kappa < 2/(1 + \lambda_{\max}(F)) < 1$. Additionally, one can show that $P - Q = \rho I_r$ where $\rho = (1 - \kappa)^2$.

We are now ready to study the nonlinear system (13). It is straightforward to verify that the inequality $V(\text{proj}(\mathbf{x})) \leq V(\mathbf{x})$ holds for all $\mathbf{x} \in \mathbb{R}^r$. Therefore,

$$V(\epsilon(t+1)) = V(\text{proj}(A\epsilon(t))) \leq V(A\epsilon(t)) = \epsilon(t)^T A^T A \epsilon(t).$$

From the discrete-time Lyapunov inequality and from $P - Q = \rho I_r$, we compute $\epsilon(t)^T A^T A \epsilon(t) \leq \rho V(\epsilon(t))$, so that $V(\epsilon(t)) \leq \rho^t V(\epsilon(0))$. Given $\kappa \in]0, 2/(1 + \lambda_{\max}(F))]$, we know that $\rho < 1$ and, therefore, the cycle error converges to zero exponentially fast. ■

4.4. Orientation localization: performance analysis of iterative algorithm

For general noise models, it is clear that the proposed algorithm (12) does not compute the optimal solution to the estimation problem (11). However, for measurements subject to Gaussian noise with zero mean and small variance $\sigma^2 \ll \pi$, we believe that (i) the algorithm reduces the mean-square error substantially for large numbers of redundant measurements, (ii) in terms of noise reduction, our algorithm is comparable to the classic least-square estimator for the linear problem analogous to our nonlinear estimation problem (11). We justify these two statements in what follows.

First, the motivation behind the structure of our algorithm comes from the iterative projection method (Kaczmarz, 1993) for systems of linear equations. Suppose the sensing graph is a ring with $n \geq 3$ nodes and with undirected edges $(i, (i+1) \bmod n)$, for $i \in \{1, \dots, n\}$. In what follows, we write $(i+1)$ to denote $(i+1) \bmod n$. Then, for one step, eq. (12) in vector form becomes

$$\psi = y - \mathbf{1}_\ell \frac{1}{n} \text{proj}(\mathbf{1}_\ell \cdot y), \quad (14)$$

where $y = [y_2^1, y_3^2, \dots, y_1^n]^T$ and $\psi = [\psi_2^1, \psi_3^2, \dots, \psi_1^n]^T$, and where κ has been chosen to be equal to $\frac{1}{n}$. The choice of $\kappa = \frac{1}{n}$ is consistent with Theorem 12, because $C_f C_f^T = n$, and therefore $\kappa < \frac{2}{1 + \lambda_{\max}(C_f C_f^T)}$. Now, assume that angle measurements and measurement errors are small enough that $\min_\psi \|\text{proj}(\psi - y)\|^2 = \min_\psi \|\psi - y\|^2$, and $\text{proj}(\mathbf{1}_\ell \cdot \psi) = \mathbf{1}_\ell \cdot \psi$. Hence, the estimation problem (11) becomes the standard least-square estimation problem, whose optimal solution is given by the Kaczmarz's iterative projection method, of which (14) is a particular case.

Second, let us consider the linear problem analogous to problem (11), i.e., the classic least-squares:

$$\begin{aligned} \min_{\hat{\theta} \in [-\pi, \pi]^m} \quad & \|\hat{\theta} - y\|^2 \\ \text{subject to} \quad & \mathbf{1}_\ell \cdot \hat{\theta} = 2\mathbf{k}\pi, \text{ for all } \ell \in \mathcal{L}(G_d), \end{aligned} \quad (15)$$

where $\mathbf{k} \in \mathbb{N}^z$ such that $\mathbf{1}_\ell \cdot \theta = 2\mathbf{k}\pi$, and z is the number of cycles in $\mathcal{L}(G_d)$. We now consider the set of independent cycles \mathcal{L}_I . By definition, we know that $C_f \theta = 2\mathbf{k}_f \pi$, with $\mathbf{k}_f \in \mathbb{N}^r$, and r is the number of independent cycles. The true angle vector θ satisfies $\theta = \theta_0 + U\eta$, where $C_f \theta_0 = 2\mathbf{k}_f \pi$, $U \in \mathbb{R}^{m \times (m-r)}$ spans the null space of C_f , and $\eta = \mathbb{R}^{m-r}$ is then computed accordingly. Simple calculations show that the error covariance matrix $Q \in \mathbb{R}^{m \times m}$ is

$$Q = \mathbb{E}[(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T] = \sigma^2 U(U^T U)^{-1} U^T.$$

Note that the matrix $U(U^T U)^{-1} U^T$ is an orthogonal projector onto the vector space defined by the columns of U . It is known that the trace of a projector is equal to the rank of the projection, that is, $\text{tr}(U(U^T U)^{-1} U^T) = m - r$, and the estimation mean-squared error (MSE) is then given by

$$MSE = \frac{\text{tr}(Q)}{(\# \text{of measurements})} = \sigma^2 \frac{m-r}{m}. \quad (16)$$

For a complete graph n nodes, $m = n(n-1)/2$ implies that the estimation MSE is $\sigma^2 \frac{2}{n}$, which corresponds to a multiplicative reduction of the measurements MSE equal to $1 - \frac{2}{n}$ (e.g., a reduction of 80% for a complete graph with $n = 10$ nodes, and of 98% for a complete graph with $n = 100$ nodes).

Third, we now provide some simulations to illustrate the performance of the proposed distributed algorithm (12). We consider an arbitrary network configuration with 10 nodes with fixed positions and varying sensing footprints. Figure 2 shows the percentage decrease of the MSE $\|\text{proj}(\psi - \theta)\|^2$ with respect to the measurement MSE $\|\text{proj}(y - \theta)\|^2$, when the number of edges in the graph changes, computed for different value of variance σ^2 , which refers to the noise n_j^i of each angle measurement $\angle p_j^i$, as shown in (9). The total noise on y_j^i is additive, and therefore it has variance $2\sigma^2$. Note that the number of independent cycles is proportional to the number of redundant measurements. As expected, we can see that, for small noise variance, the estimation MSE is smaller than the measurement MSE, and the percentage decrease grows with the number of redundant measurements available. In particular, for $\sigma^2 \in \{0.01, 0.1\}$ [rad], the percentage decrease goes to 80%. For larger noise variance, however, there is no significant improvement. Figure 2 also shows the mean-squared error percentage decrease of the analogous linear problem computed via (16); as we can see, for large number of redundant measurements, our algorithm reduces the mean-square error in a way that is comparable to the amount of noise reduction obtained by the classic least-squares estimator.

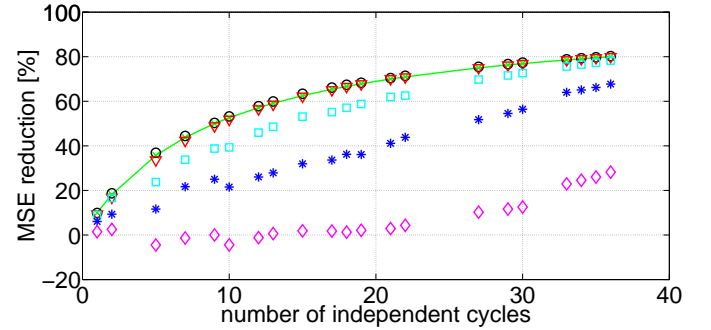


Figure 2: This plot refers to a graph with 10 nodes, using (12), and represents the percentage decrease of $\|\text{proj}(\psi - \theta)\|^2$ with respect to $\|\text{proj}(y - \theta)\|^2$ as the number of independent cycles increases. Note this is an average over 2000 iterations of (12) with Gaussian noise with error variance σ^2 set to 0.01 [rad] (black circle), 0.1 [rad] (red triangle), 0.2 [rad] (cyan square), 0.3 [rad] (blue star), and 0.5 [rad] (magenta diamond). Additionally, the green line with crosses represents the MSE percentage decrease computed with (16) for the classic least-squares problem.

5. Three-dimensional frame localization

Here, we consider first a network composed by three nodes in 3-dimensional space with a complete sensing graph.

The setup is illustrated in the left image in Figure 3.

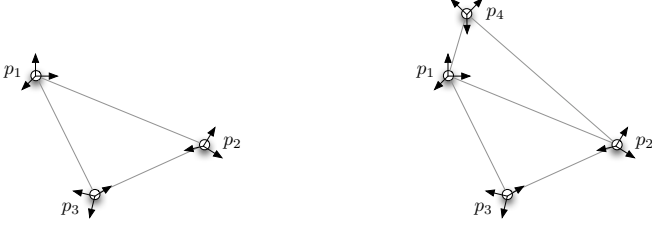


Figure 3: Complete sensing graphs in \mathbb{R}^3

Lemma 13 (Feasible orientations) *Given unit-length measurements $u_j^i = \text{vers}(p_j^i)$ and $u_i^j = \text{vers}(p_i^j)$, compute $\mathbf{H}_j^i \in SO(3)$ by $\mathbf{H}_j^i = \exp(\alpha_j^i \widehat{e}_j^i)$, where $e_j^i \in \mathbb{R}^3$, $\alpha_j^i \in [0, \pi]$ are defined¹ by*

$$e_j^i = \begin{cases} \text{vers}(u_j^i \times u_i^j), & \text{if } u_j^i \times u_i^j \neq \mathbf{0}, \\ \text{any unit-length vector } \perp u_j^i, & \text{otherwise,} \end{cases}$$

$$\alpha_j^i = \text{atan}_2(\|u_j^i \times u_i^j\|, -u_j^i \cdot u_i^j).$$

Then, for each solution \mathbf{R}_j^i to eq. (7), there exists a unique angle $\beta \in [-\pi, \pi[$ such that $\mathbf{R}_j^i = \exp(\beta \widehat{u}_j^i) \mathbf{H}_j^i$.

Proof: First, let us show that \mathbf{H}_j^i is solution of (7):

$$\begin{aligned} \mathbf{H}_j^i u_i^j &= \exp(\alpha_j^i \widehat{e}_j^i) u_i^j, \\ &= u_i^j + \sin \alpha_j^i \widehat{e}_j^i u_i^j + (1 - \cos \alpha_j^i) \widehat{e}_j^i{}^2 u_i^j. \end{aligned}$$

Because e_j^i and u_i^j are mutually orthogonal unit vectors,

$$\mathbf{H}_j^i u_i^j = u_i^j \cos \alpha_j^i + \mathbf{n} \sin \alpha_j^i, \quad (17)$$

where \mathbf{n} is a unit vector perpendicular to the plane containing e_j^i and u_i^j whose direction is given by their cross product. Let us consider the orthonormal base $\{u_i^j, \mathbf{n}, e_j^i\}$. Then, eq. (17) represents the rotation of axis u_i^j around axis e_j^i of an angle α_j^i , where α_j^i is, by definition, the angle between u_i^j and $-u_i^j$. Therefore $\mathbf{H}_j^i u_i^j = -u_i^j$, that is, \mathbf{H}_j^i is solution of (7). Now, for an arbitrary angle $\gamma \in [-\pi, \pi[$,

$$\begin{aligned} \exp(\gamma \widehat{u}_j^i) u_i^j &= \\ &= u_i^j \cos \gamma + (u_j^i \times u_i^j) \sin \gamma + (1 - \cos \gamma)(u_j^i \cdot u_i^j) u_i^j \quad (18) \\ &= u_i^j \cos \gamma + u_j^i - u_j^i \cos \gamma = u_j^i. \end{aligned}$$

Then $u_j^i = -\exp(\beta \widehat{u}_j^i) \mathbf{H}_j^i u_i^j$, for $\beta = -\gamma \in [-\pi, \pi[$, i.e., $\exp(\beta \widehat{u}_j^i) \mathbf{H}_j^i$ is solution of (7) for all $\beta \in]-\pi, \pi]$.

¹For any point (x, y) in the plane except for the origin, let $\text{atan}_2(y, x)$ be the angle between the horizontal positive axis and the point (x, y) measured counterclockwise.

Now, we want to show that any solution of (7) takes such a form. Consider the following matrix:

$$\tilde{\mathbf{R}} = \exp(\varphi \widehat{v}) \exp(\beta \widehat{u}_j^i) \mathbf{H}_j^i,$$

where v is a unit vector such that $v \parallel u_j^i$, and $\varphi \in [-\pi, \pi[$, $\varphi \neq 0$. By contradiction, let us assume $\tilde{\mathbf{R}}$ is solution of (7). Therefore,

$$u_j^i = -\exp(\varphi \widehat{v}) \exp(\beta \widehat{u}_j^i) \mathbf{H}_j^i u_i^j = \exp(\varphi \widehat{v}) u_i^j.$$

It is known that any rotation of a fixed vector that yields the same vector is equivalent to a rotation of the vector about itself. This implies $v \parallel u_j^i$ or $\varphi = 0$. Therefore, by contradiction, any solution of (7) can be written as $\tilde{\mathbf{R}} = \exp(\beta \widehat{u}_j^i) \mathbf{H}_j^i$, for an arbitrary $\beta \in [-\pi, \pi[$. ■

Lemma 14 *Consider a network composed by three nodes in 3-dimensional space with angle of arrival sensing. Pick any one of the three nodes as reference. If the sensing graph is the complete graph and the nodes are in generic positions with generic orientations, then there are precisely two feasible configurations for the three nodes and, therefore, the network is not orientation localizable.*

Proof: The frame localizability problem is described as follows. First, the unknown variables are the three matrices \mathbf{R}_{i+1}^i , for $i \in \{1, 2, 3\}$, where we write $(i+1)$ to denote $(i+1) \bmod 3$. These matrices have each three degrees of freedom, for a total of 9 degrees of freedom. Second, assuming unit-length angle of arrival measurements u_j^i , for $i \neq j \in \{1, \dots, 3\}$, the constraint equations arising from the measurements and from the closed kinematics chain relationships (3) are:

$$\begin{aligned} u_2^1 &= -\mathbf{R}_2^1 u_1^2, & u_3^2 &= -\mathbf{R}_3^2 u_2^3, \\ u_1^3 &= -\mathbf{R}_1^3 u_3^1, & I_3 &= \mathbf{R}_2^1 \mathbf{R}_3^2 \mathbf{R}_1^3. \end{aligned} \quad (19)$$

Given these measurements and according to Lemma 13, we compute the three rotation matrices \mathbf{H}_{i+1}^i and we know that there exist three angles $\beta_{i+1}^i \in [-\pi, \pi[$ such that

$$\mathbf{R}_{i+1}^i = \exp(\beta_{i+1}^i \widehat{u}_{i+1}^i) \mathbf{H}_{i+1}^i.$$

Thus, eq. (19) admits a unique solution $\{\mathbf{R}_2^1, \mathbf{R}_3^2, \mathbf{R}_1^3\}$ precisely when there exist unique $\beta_{i+1}^i \in [-\pi, \pi[$ such that

$$I_3 = \exp(\beta_2^1 \widehat{u}_2^1) \mathbf{H}_2^1 \exp(\beta_3^2 \widehat{u}_3^2) \mathbf{H}_3^2 \exp(\beta_1^3 \widehat{u}_1^3) \mathbf{H}_1^3.$$

Applying eq. (5) repeatedly, we compute

$$\begin{aligned} &(\mathbf{H}_1^3)^T (\mathbf{H}_3^2)^T (\mathbf{H}_2^1)^T \\ &= \exp(\beta_2^1 \widehat{u}_2^1) \exp(\beta_3^2 \widehat{\mathbf{H}_2^1 u_3^2}) \exp(\beta_1^3 \widehat{\mathbf{H}_2^1 \mathbf{H}_3^2 u_1^3}). \end{aligned}$$

We now rely on the assumption of generic positions and orientations to infer that $u_2^1 \not\parallel \mathbf{H}_2^1 u_3^2$ and that $u_3^2 \not\parallel \mathbf{H}_3^2 u_1^3$. Also the left hand side term $(\mathbf{H}_1^3)^T (\mathbf{H}_3^2)^T (\mathbf{H}_2^1)^T$ is generic. From the formulation of the problem, we know that at least a real solution for (19) exists. In particular, as shown by Shuster and Markley (2003) and by Wittenburg and Lilov (2003), such equations admit two solutions. Therefore, the network is not orientation localizable. ■

Now, let us consider a network composed by four nodes, whose connected sensing graph consists of two 3-nodes loops, with an edge in common. For example, consider the setup in the right image in Figure 3.

Lemma 15 *Consider a network composed by four nodes in the 3-dimensional space with angle of arrival sensing. If the sensing graph is connected and there are at least two independent loops, then the network is orientation localizable.*

Proof: As in the three-nodes case, the set of eqs. (19) is extended by

$$\begin{aligned} u_2^1 &= -\mathbf{R}_2^1 u_1^2, & u_4^2 &= -\mathbf{R}_4^2 u_2^4, \\ u_1^4 &= -\mathbf{R}_1^4 u_4^1, & I_3 &= \mathbf{R}_2^1 \mathbf{R}_4^2 \mathbf{R}_1^4. \end{aligned} \quad (20)$$

As eqs. (19), also eqs. (20) admit two solutions, i.e. two different sets of values for β_i^j . It is straightforward to show that only one of the two solutions for β_2^1 of (20) matches with one of the solutions for β_2^1 in (19) (see Shuster and Markley (2003)). Therefore, all angles β_i^j are uniquely determined, and the network is orientation localizable. ■

Lemma 16 *A necessary condition for a network in the 3-dimensional space with angle of arrival sensing to be orientation localizable is to have at least 4 nodes.*

Proof: If the network has less than 3 nodes, there are no loops, so it is not orientation localizable. Assume now the network has 3 nodes. If the sensing graph is not complete, the network has no loops and therefore is not orientation localizable. If the sensing graph is complete, according to Lemma 14, the network is not orientation localizable. ■

Lemma 17 *Any network in 3-dimensional space with a complete sensing graph is orientation localizable if it has at least 4 nodes.*

Proof: In a complete network every loop belongs to a three-edges loop. Therefore, by what has been shown before, the network is orientation localizable. ■

Definition 18 (3-dimensional triangulation) *Consider a connected network composed of nodes in 3-dimensional space. We call such network a 3-dimensional triangulation if there exists a basis for the cycle space such that each cycle in the basis has 3 nodes and it shares at least one edge with another cycle of the basis.*

Lemma 15 directly implies the following final result.

Lemma 19 *Consider a network with $n \geq 4$ nodes in 3-dimensional space, and assume its angle of arrival sensing is a 3-dimensional triangulation. Then, the network is orientation localizable.*

6. Conclusions

This paper has introduced the frame localization problem in a connected network. For the planar orientation localization problem with angle-of-arrival (bearing) sensors, we developed an exponentially fast algorithm that reduces the effect of noise. For the three-dimensional case, we have explored necessary and sufficient conditions for a noiseless network to be orientation localizable.

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