# Synchronization and Transient Stability in Power Networks and Non-Uniform Kuramoto Oscillators

Florian Dörfler and Francesco Bullo

*Abstract*— **Motivated by recent interest for multi-agent systems and smart grid architectures, we discuss the synchronization problem for the network-reduced model of a power system with non-trivial transfer conductances. Our key insight is to exploit the relationship between the power network model and a first-order model of coupled oscillators. Assuming overdamped generators (possibly due to local excitation controllers), a singular perturbation analysis shows the equivalence between the classic swing equations and a non-uniform Kuramoto model characterized by multiple time constants, non-homogeneous coupling, and non-uniform phase shifts. By extending methods from synchronization theory and consensus protocols, we establish sufficient conditions for synchronization of non-uniform Kuramoto oscillators. These conditions reduce to and improve upon previously-available tests for the classic Kuramoto model. By combining our singular perturbation and Kuramoto analyses, we derive concise and purely algebraic conditions that relate synchronization and transient stability of a power network to the underlying network parameters and initial conditions.**

## I. Introduction

The vast North American interconnected power grid is often referred to as the largest and most complex machine engineered by humankind. The envisioned future power grid is expected to be even more complex than the current one and will rely increasingly on renewable energy sources, such as wind and solar power, which cause stochastic disturbances. Thus, the future power grid is more prone to instabilities, which can ultimately lead to power blackouts. The detection and rejection of such instability mechanisms will be one of the major challenges faced by the future "smart grid".

One form of power network stability is the so-called *transient stability*, which is the ability of a power system to remain in synchronism when subjected to large transient disturbances such as faults on transmission elements or loss of load, generation, or system components. For example, a recent major blackout in Italy in 2003 was caused by tripping of a tie-line and resulted in the loss of synchronism of the Italian power grid with the rest of Europe. In a classic setting the transient stability problem is posed as a special case of the more general *synchronization problem*, which considers a possibly longer time horizon, possibly drifting generator rotor angles, and local excitation controllers aiming to restore synchronism. In order to analyze the stability of a synchronous operating point of a power grid and to estimate its region of attraction, various sophisticated methods have been developed [2], [3], [4], [5], [6]. Surveys on transient stability analysis can be found in [7], [8], [9]. Unfortunately, the existing methods can cope only with simplified models and do not result in simple conditions to check if a power system synchronizes for a given network state and parameters. In fact, it is an outstanding problem to relate synchronization and transient stability of a power network to the underlying network parameters, state, and topology [10].

The recent years have witnessed a burgeoning interest of the control community in cooperative control of multi-agent systems. One of the basic tasks in a multi-agent system is a consensus of the agents' states to a common value [11]. This consensus problem finds applications in robotic coordination, distributed sensing and computation, and various other fields including synchronization. In most articles treating consensus problems the agents obey single integrator dynamics, but the synchronization of interconnected power systems has often been envisioned as possible future application [12]. However, we are aware of only one article [13] that indeed applies consensus methods to a power network model.

Another set of literature relevant to our investigation is the synchronization in the coupled oscillator model introduced by Kuramoto [14]. The synchronization of coupled Kuramoto oscillators has been widely studied by the physics [15], [16] and the dynamical systems communities [17], [18]. This vast literature with numerous theoretical results and rich applications to various scientific areas is elegantly reviewed in [19], [20]. Recent works in the control community [21], [22], [23], [24] investigate the close relationship between Kuramoto oscillators and consensus networks.

The three areas of power network synchronization, Kuramoto oscillators, and consensus protocols are apparently closely related. Indeed, the similarity between the Kuramoto model and the power network models used in transient stability analysis is striking. Even though power networks have often been referred to as coupled-oscillators systems, the similarity to a second-order Kuramoto-type model has been mentioned only recently in the power systems community in simulation studies for simplified models [25], [26], [27]. In the coupled-oscillators literature, second-order Kuramoto models have often been analyzed [20], but we know of only one article mentioning power networks [28]. In short, neither the Kuramoto nor the power systems literature has recognized and thoroughly analyzed this apparent connection.

There are three main contributions in the present paper. First, we present a coupled-oscillators approach to the problem of synchronization and transient stability in power networks. Via a singular perturbation analysis, we show that the transient stability analysis for the classic swing equations with overdamped generators reduces, on a long time-scale, to the problem of synchronizing non-uniform Kuramoto oscillators with multiple time constants, non-homogeneous

This work was supported in part by NSF grants CMS-0626457 and CNS-0834446. A full length version of this document is available at [1].

The authors gratefully acknowledge Prof. Y. Susuki and Prof. P. Kokotović for their comments that improved the presentation of this article.

Florian Dörfler and Francesco Bullo are with the Center for Control, Dynamical Systems and Computation, University of California at Santa Barbara, Santa Barbara, CA 93106, {dorfler, bullo}@engineering.ucsb.edu

coupling, and non-uniform phase shifts. This reduction to a non-uniform Kuramoto model is arguably the missing link connecting transient stability analysis and networked control, a link that was hinted at in [10], [12], [25], [26], [27], [28].

Second, we give novel and purely algebraic conditions that suffice for synchronization and transient stability of a power network. To the best of our knowledge these conditions are the first ones to relate synchronization and performance of a power network directly to the underlying network parameters and initial state. Our conditions are based on different and possibly less restrictive assumptions than those obtained by classic analysis methods [2], [3], [4], [5], [6]. We consider a network-reduction model of a power system and do not make any of the following common or classic assumptions: we do not require uniform mechanical damping, we do not require the swing equations to be formulated in relative coordinates or the existence of an infinite bus, and we do not require the transfer conductances to be "sufficiently small" or even negligible. On the other hand, our results are based on the assumption that each generator is strongly overdamped, possibly due to internal excitation control. This assumption allows us to perform a singular perturbation analysis and study a dimension-reduced system. In simulations, our synchronization conditions appear to hold even if generators are not overdamped, and in the application to real power networks the approximation via the reduced system has been used successfully in academia and industrial practice.

Our synchronization conditions are based on an analytic approach whereas classic analysis methods [2], [3], [4], [5], [6] rely on numerical procedures to approximate the region of attraction of an equilibrium by level sets of energy functions and stable manifolds. Compared to classic analysis methods, we do not aim at providing best estimates of the region of attraction or the critical clearing time. Rather, we approach the outstanding problem [10] of relating synchronization and transient stability to the underlying network structure. For this problem, we derive sufficient and purely algebraic conditions that can be interpreted as follows: "the network connectivity has to dominate the network's non-uniformity, the network's losses, and the lack of phase locking"

Third and final, we perform a synchronization analysis of non-uniform Kuramoto oscillators, as an interesting mathematical problem in its own right. Our analysis combines and extends methods from consensus protocols and synchronization theory. As an outcome, purely algebraic conditions on the network parameters and the system state establish the phase locking, frequency entrainment, and phase synchronization of the non-uniform Kuramoto oscillators. We emphasize that our results do not hold only for non-uniform network parameters but also in the case when the underlying coupling topology is not a complete graph. When our results are specialized to classic (uniform) Kuramoto oscillators, they reduce to and even improve upon various well-known conditions in the literature on the Kuramoto model [15], [17], [22], [23], [24], [29]. In the end, these conditions guaranteeing synchronization of non-uniform Kuramoto oscillators also suffice for the transient stability of the power network.

*Paper Organization:* The remainder of this section introduces some notation and recalls the consensus protocol and the Kuramoto model. Section II reviews the problem of transient stability analysis and synchronization in power networks. Section III introduces the non-uniform Kuramoto model and presents the main result of this article. Section IV applies a singular perturbation analysis to the power network model resulting in the non-uniform Kuramoto model, which is analyzed in Section V. Section VI illustrates the analytical results via simulation studies. Finally, some conclusions are drawn in Section VII. All proofs and further references can be found in the full-length version of this article [1].

*Preliminaries and Notation:* Given an $n$-tuple $(x_1, \dots, x_n)$, $\operatorname{diag}(x_i) \in \mathbb{R}^{n \times n}$ is the associated diagonal matrix, $x \in \mathbb{R}^n$ is the associated vector, $x_{\max}$ and $x_{\min}$ are the maximum and minimum elements, and $\|x\|_2$ is the 2–norm. Let $\mathbf{1}$ and $\mathbf{0}$ be the vectors with unit and zero entries of appropriate dimension. Given an array $\{A_{ij}\}$ with $i,j \in \{1,\dots,n\}$, we let $A \in \mathbb{R}^{n\times n}$ denote the associated matrix and we define $A_{\max} = \max_{i,j}\{A_{ij}\}$ $A_{\min} = \min_{i,j}\{A_{ij}\}$. Define the sinc function $\operatorname{sinc} : \mathbb{R} \to \mathbb{R}$ by $\operatorname{sinc}(x) = \sin(x)/x$. The set $\mathbb{T}^1 = (-\pi, \pi]$ is the torus and the product set $\mathbb{T}^n$ is the $n$-torus. Given two angles $\theta_1 \in \mathbb{T}^1$ and $\theta_2 \in \mathbb{T}^1$ we define their distance $|\theta_1 - \theta_2|$, with slight abuse of notation, to be the *geodesic distance* on $\mathbb{T}^1$.

A *weighted directed graph* is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, \dots, n\}$ is the set of nodes, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of directed edges, and $A \in \mathbb{R}^{n\times n}$ is the *adjacency matrix* inducing the graph. The *Laplacian* is the matrix $L(a_{ij}) := \operatorname{diag}(\sum_{j=1}^{n} a_{ij}) - A$. For an undirected graph, i.e., $A = A^T$, let $H \in \mathbb{R}^{|\mathcal{E}|\times n}$ be the incidence matrix inducing, for $x \in \mathbb{R}^n$, the vector of difference variables $Hx = (x_2 - x_1, \dots)$. If $\mathcal{G}$ is connected, then $\ker(H) = \ker(L(a_{ij})) = \operatorname{span}(\mathbf{1})$, all $n-1$ remaining eigenvalues of $L(a_{ij})$ are strictly positive, and the second-smallest eigenvalue $\lambda_2(L(a_{ij}))$ is referred to as the *algebraic connectivity* of $\mathcal{G}$. Finally, the set $\{i,j\}$ refers to the pair of nodes connected by either $(i,j)$ or $(j,i)$.

*Review of the Consensus Protocol and the Kuramoto Model:* In a system of $n$ *autonomous agents*, each characterized by a state variable $x_i \in \mathbb{R}$, a basic task is to achieve a consensus on a common state value, that is, $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ for all agent pairs $\{i,j\}$. Given a graph with adjacency matrix $A$ describing the interaction between agents, a linear continuous time algorithm to achieve consensus on the agents' state is the *consensus protocol*

$$\dot{x}_i = -\sum\nolimits_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}. \qquad (1)$$

In vector notation the consensus protocol (1) takes the form $\dot{x} = -L(a_{ij})x$ and directly reveals the underlying graph $\mathcal{G}$.

A well-known and widely used model for the synchronization among coupled oscillators is the *Kuramoto model*, which considers $n$ coupled oscillators with state $\theta_i \in \mathbb{T}^1$ and natural frequency $\omega_i \in \mathbb{R}$, and with the dynamics

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum\nolimits_{j=1}^{n} \sin(\theta_i - \theta_j), \quad i \in \{1, \dots, n\}, \quad (2)$$

where $K$ is the coupling strength among the oscillators.

## II. Problem Setup in Synchronization and Transient Stability Analysis

### A. The Mathematical Model of a Power Network

In a power network with $n$ generators we associate with each generator its internal voltage $E_i > 0$, its active power output $P_{e,i}$, its mechanical power input $P_{m,i} > 0$, its inertia $M_i > 0$, its damping constant $D_i > 0$, and its rotor angle $\theta_i$ measured with respect to a rotating frame with frequency $f_0$. All parameters are given in per unit system, except for $M_i$ and $D_i$ which are given in seconds, and $f_0$ is typically given as 50 Hz or 60 Hz. The rotor dynamics of generator $i$ are then given by the classic constant-voltage behind reactance model of interconnected *swing equations* [7], [30]

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + P_{m,i} - P_{e,i}, \quad i \in \{1, \ldots, n\} .$$

Under the common assumption that the loads are modeled as passive admittances, all passive nodes of a power network can be eliminated resulting in the *reduced admittance matrix* with components $Y_{ij} = G_{ij} + \sqrt{-1}\, B_{ij}$, where $G_{ij} = G_{ji} \geq 0$ and $B_{ij} = B_{ji} > 0$ are the conductance and susceptance between generator $i$ and $j$ in per unit values. With the *power-angle relationship*, the electrical output power $P_{e,i}$ is then

$$P_{e,i} = \sum\nolimits_{j=1}^{n} E_i E_j \left(G_{ij}\cos(\theta_i - \theta_j) + B_{ij}\sin(\theta_i - \theta_j)\right) .$$

Given the admittance $Y_{ij}$ between generator $i$ and $j$, define the magnitude $|Y_{ij}| = (G_{ij}^2 + B_{ij}^2)^{1/2} > 0$ and the *phase shift* $\varphi_{ij} = \arctan(G_{ij}/B_{ij}) \in [0, \pi/2)$ depicting the energy loss due to the transfer conductance $G_{ij}$. Recall that a lossless network is characterized by zero phase shifts. Furthermore, we define the *natural frequency* $\omega_i := P_{m,i} - E_i^2 G_{ii}$ (effective power input to generator $i$) and the *coupling weights* $P_{ij} := E_i E_j |Y_{ij}|$ (maximum power transferred between generators $i$ and $j$) with $P_{ii} := 0$ for $i \in \{1, \ldots, n\}$. The power network model can then be formulated compactly as

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j=1}^{n} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij}) . \quad (3)$$

Note that higher order electrical and flux dynamics can be reduced into an augmented damping constant $D_i$ in (3) [31]. The generator's internal excitation control essentially increases the *damping torque* towards the net frequency and can also be reduced into the damping constant $D_i$ [30], [31]. It is commonly agreed that the classical model (3) captures the power system dynamics sufficiently well during the first swing. Thus we omit higher order dynamics and control effects and assume they are incorporated into the model (3).

### B. Synchronization and Equilibrium in Power Networks

A *frequency equilibrium* of the power network model (3) is characterized by $\dot{\theta} = \mathbf{0}$ and by the *power flow equations*

$$P_i(\theta) = P_{m,i} - E_i^2 G_{ii} - P_{e,i} \equiv 0, \quad i \in \{1, \ldots, n\} , \quad (4)$$

depicting the power balance. The generators are said to be in a *synchronous equilibrium*, if the phase differences $\theta_i - \theta_j$ are constant, respectively the frequency differences $\dot{\theta}_i - \dot{\theta}_j$ are zero. We say the power network *synchronizes* (exponentially) if the phase differences $\theta_i(t) - \theta_j(t)$ become bounded and the frequency differences $\dot{\theta}_i(t) - \dot{\theta}_j(t)$ converge to zero (with exponential decay rate) as $t \to \infty$. In the literature on coupled oscillators this is also referred to as *phase locking* and *frequency entrainment*, and the case $\theta_i = \theta_j$ for all $i, j \in \{1, \ldots, n\}$ is referred to as *phase synchronization*.

In order to reformulate the synchronization problem as a stability problem, system (3) is usually formulated in relative coordinates and, to render the resulting dynamics self-contained, uniform damping is assumed, i.e., $D_i/M_i$ is constant. Alternatively, sometimes the existence of an infinite bus (a stationary generator without dynamics) as reference is postulated. We remark that both of these assumptions are not physically justified but are mathematical simplifications to reduce the synchronization problem to a stability analysis.

### C. Review of Classic Transient Stability Analysis

Classically, transient stability analysis deals with a special case of the synchronization problem, namely the stability of a post-fault equilibrium, that is, a new equilibrium of (3) arising after a change in the network parameters or topology. To analyze the stability of a post-fault equilibrium and to estimate its region of attraction various sophisticated methods have been developed [7], [8], [9], which typically employ the Hamiltonian structure of system (3). Since in general a Hamiltonian for model (3) with non-zero conductances does not exist, early analysis approaches neglect the phase shifts [2], [3]. In this case, the power network model (3) takes form

$$(M/\pi f_0)\, \ddot{\theta} = -D\dot{\theta} - \nabla U(\theta) , \quad (5)$$

where $\nabla$ is the gradient operator and $U : (-\pi, \pi]^n \to \mathbb{R}$ is the potential energy given up to an additive constant by

$$U(\theta) = -\sum\nolimits_{i=1}^{n} \omega_i \theta_i + \sum\nolimits_{j=1}^{n} P_{ij}\left(1 - \cos(\theta_i - \theta_j)\right) . \quad (6)$$

When system (5) is formulated in relative or reference coordinates (that feature equilibria), the *energy function* $(\theta, \dot{\theta}) \mapsto (1/2)\, \dot{\theta}^T (M/\pi f_0)\dot{\theta} + U(\theta)$ serves semi-globally as a Lyapunov function and clearly implies convergence of the dynamics (5) to $\dot{\theta} = \mathbf{0}$ and the largest invariant zero level set of $\nabla U(\theta)$. In order to estimate the region of attraction of a stable equilibrium, algorithms such as *PEBS* [3] or *BCU* [5] consider the associated dimension-reduced gradient flow

$$\dot{\theta} = -\nabla U(\theta) . \quad (7)$$

Then $(\theta^*, \mathbf{0})$ is a hyperbolic type-$k$ equilibrium of (5) (i.e., the Jacobian has $k$ stable eigenvalues) if and only if $(\theta^*)$ is a hyperbolic type$-k$ equilibrium of (7). Moreover, the regions of attractions of both equilibria are bounded by the stable manifolds of the same unstable equilibria [5, Theorem 5.7].

For fixed and "sufficiently small" transfer conductances the lossy power system (3) can be analyzed locally as a perturbation of the lossless system (5) [5]. Other approaches to lossy power networks compute numerical energy functions [4] or make use of an extended invariance principle [6]. Based on these results numerical methods were developed to approximate the stability boundaries of (5) by level sets of energy functions or stable manifolds of unstable equilibria.

To summarize the shortcomings of the classical transient stability analysis methods, they consider simplified models formulated in relative or reference coordinates and mostly result in numerical procedures rather than in concise and simple conditions. For lossy power networks the cited articles consider either special benchmark problems or networks with "sufficiently small" transfer conductances. To the best of our knowledge there are no results quantifying this smallness for arbitrary networks. Moreover, from a network perspective the existing methods do not result in conditions relating synchronization to the network's parameters, state, and topology.

## III. THE NON-UNIFORM KURAMOTO MODEL AND MAIN SYNCHRONIZATION RESULT

### A. The Non-Uniform Kuramoto Model

The similarity between the power network model (3) and the Kuramoto model (2) is striking. To emphasize this similarity, we define the *non-uniform Kuramoto model* by

$$D_i\,\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})\,, \qquad (8)$$

where $i \in \{1,\dots,n\}$ and the parameters satisfy the following ranges: $D_i > 0$, $\omega_i \in \mathbb{R}$, $P_{ij} > 0$, and $\varphi_{ij} \in [0, \pi/2)$, for all $i, j \in \{1,\dots,n\}$, $i \neq j$; by convention, $P_{ii}$ and $\varphi_{ii}$ are set to zero. System (8) may be regarded as a generalization of the classic Kuramoto model (2) with multiple time-constants $D_i$ and non-homogeneous but symmetric coupling terms $P_{ij}$ and phase-shifts $\varphi_{ij}$. The non-uniform Kuramoto model (8) will serve as a link between the power network model (3), the Kuramoto model (2), and the consensus protocol (1).

**Remark III.1 (Second-order mechanical systems and their first-order approximations:)** The non-uniform Kuramoto model (8) can be seen as a long-time approximation of the second order system (3) for a small "inertia over damping ratio" $M_i/D_i$ or, more specifically, for a ratio $2M_i/D_i$ much smaller than the net frequency $2\pi f_0$. Spoken differently, system (8) can be obtained by a singular perturbation analysis of the second-order system (3). Note the analogy between the non-uniform Kuramoto model (8) and the dimension-reduced gradient system (7), which is often studied in classic transient stability analysis to approximate the stability properties of the second-order system (5) [3], [5], [9]. Both models are of first order, have the same right-hand side, and thus also the same equilibria with the same stability properties. Strictly speaking, both models differ only in the time constants $D_i$. The dimension-reduced system (7) is formulated as a gradient-system and is used to study the stability of the equilibria of (7) (possibly formulated in relative coordinates). The non-uniform Kuramoto model (8), on the other hand, can be directly used to study synchronization and clearly reveals the underlying network structure. □

### B. Main Synchronization Result

We are now ready to state our main result on the power network model (3) and the non-uniform Kuramoto model (8).

**Theorem III.1 (Main synchronization result)** *Consider the power network model* (3) *and the non-uniform Kuramoto model* (8). *Assume that the minimal coupling weight is larger than a critical value, i.e., for every* $i, j \in \{1,\dots,n\}$

$$P_{\min} > P_{\text{critical}} := \frac{D_{\max}}{n\cos(\varphi_{\max})}\times \left(\max_{\{i,j\}}\Big(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}\Big) + \max_i \sum\nolimits_{j=1}^{n} \frac{P_{ij}}{D_i}\sin(\varphi_{ij})\right)\,. \qquad (9)$$

*Accordingly, define* $\gamma_{\min} = \arcsin(\cos(\varphi_{\max})P_{\text{critical}}/P_{\min})$ *taking value in* $(0, \pi/2 - \varphi_{\max})$. *For* $\gamma \in [\gamma_{\min}, \pi/2 - \varphi_{\max})$, *define the (non-empty) set of bounded phase differences* $\Delta(\gamma) := \{\theta \in \mathbb{T}^n : \max_{\{i,j\}} |\theta_i - \theta_j| \leq \gamma\}$.

*For the* **non-uniform Kuramoto model**,

1) **phase locking**: *for every* $\gamma \in [\gamma_{\min}, \pi/2 - \varphi_{\max})$ *the set* $\Delta(\gamma)$ *is positively invariant; and*
2) **frequency entrainment:** *if* $\theta(0) \in \Delta(\gamma)$, *then the frequencies* $\dot{\theta}_i(t)$ *synchronize exponentially to some frequency* $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$.

*For the* **power network model** *with initial phases satisfying* $\theta(0) \in \Delta(\gamma)$ *and any initial frequencies* $\dot{\theta}(0)$,

1) **approximation error:** *there exists a constant* $\epsilon^* > 0$ *such that, if* $\epsilon := (M_{\max})/(\pi f_0 D_{\min}) < \epsilon^*$, *then the solution* $(\theta(t), \dot{\theta}(t))$ *of* (3) *exists for all* $t \geq 0$ *and it holds uniformly in* $t$ *that*
$$\begin{aligned} \theta(t) - \bar{\theta}(t) &= \mathcal{O}(\epsilon), \quad \forall t \geq 0\,, \\ \dot{\theta}(t) - D^{-1}P(\bar{\theta}(t)) &= \mathcal{O}(\epsilon), \quad \forall t > 0\,, \end{aligned} \qquad (10)$$
*where* $\bar{\theta}(t)$ *is the solution to the non-uniform Kuramoto model* (8) *with initial condition* $\bar{\theta}(0) = \theta(0)$ *and* $D^{-1}P(\bar{\theta})$ *is the power flow* (4) *scaled by* $D^{-1}$*; and*
2) **asymptotic approximation error:** *there exists* $\epsilon$ *and* $\varphi_{\max}$ *sufficiently small, such that the* $\mathcal{O}(\epsilon)$ *approximation errors in equation* (10) *converge to zero as* $t \to \infty$.

We discuss the assumption that the *perturbation parameter* $\epsilon$ needs to be small separately and in detail in the next subsection and state the following remarks to Theorem III.1.

**Remark III.2 (Physical interpretation of Theorem III.1:)** The condition (9) on the network parameters has a direct physical interpretation when it is rewritten as

$$n\frac{P_{\min}}{D_{\max}}\cos(\varphi_{\max}) > \max_{\{i,j\}}\Big(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}\Big) + \max_i \sum_{j=1}^{n} \frac{P_{ij}}{D_i}\sin(\varphi_{ij})\,. \qquad (11)$$

The right-hand side of (11) states the worst-case non-uniformity in natural frequencies and the worst-case lossy coupling of a node to the network ($P_{ij}\sin(\varphi_{ij}) = E_iE_jG_{ij}$ reflects the transfer conductance), both of which are scaled with the rates $D_i$. These negative effects have to be dominated by the left-hand side of (11), which is a lower bound on $\min_i\{\sum_{j=1}^{n} P_{ij}\cos(\varphi_{ij})/D_i\}$, the worst-case lossless coupling of a node $i$ to the network. The gap between the left- and the right-hand side in (11) determines the ultimate lack of phase locking in $\Delta(\gamma)$. In summary, the conditions of Theorem III.1 read as "the network connectivity has to dominate the network's non-uniformity, the network's losses, and the lack of phase locking." The minimal coupling weight $P_{\min}$ in condition (9) is not only the weakest power flow but reflects for uniform voltages $E_i$ and phase shifts $\varphi_{ij}$ also the maximum pairwise *effective resistance* of the original non-reduced power network. The effective resistance is a well studied graphical property and is related to the algebraic connectivity and the graph-topological distance. □

**Remark III.3 (Refinement of Theorem III.1:)** Theorem III.1 can also be stated for a non-complete but connected coupling graph and two-nom-like bounds on the parameters and initial conditions involving the algebraic connectivity (see Theorem V.3). In the case of a lossless network, explicit values for the synchronization frequency and the exponential synchronization rate can be derived, and conditions for phase synchronization can be given (see Section V). □

**Remark III.4 (Reduction of Theorem III.1 to classic Kuramoto oscillators:)** When specialized to classic (uniform) Kuramoto oscillators (2), the presented condition (9) improves the results obtained by [15], [17], [23], [24], [29]. We refer the reader to the detailed remarks in Section V. □

### C. Discussion of the Perturbation Assumption

The assumption that each generator is strongly overdamped is captured by the smallness of the perturbation parameter $\epsilon = (M_{\text{max}})/(\pi f_0 D_{\text{min}})$. This choice of the perturbation parameter $\epsilon$ and the subsequent singular perturbation analysis is similar to the analysis of Josephson arrays [16], coupled overdamped mechanical pendula [32], and also classic transient stability analysis [3, Theorem 5.2], [27].

In the linear case, this analysis resembles the well-known overdamped harmonic oscillator, which features one slow and one fast eigenvalue. The harmonic oscillator thus exhibits two separate time-scales and the fast eigenvalue corresponding to the frequency damping can be neglected in the long-term phase dynamics. In the non-linear case these two distinct time-scales are captured by a singular perturbation analysis. In short, this dimension-reduction of a coupled-pendula system corresponds to the physical assumption that damping and synchronization happen on separate time scales.

In the application to realistic generator models one has to be careful under which operating conditions $\epsilon$ is indeed a small physical quantity. Typically, $M_i \in [2\text{s}, 8\text{s}]$ depending on the type of generator and the mechanical damping (including damper winding torques) is poor: $D_i \in [1, 2]/(2\pi f_0)$. However, for the synchronization problem also the generator's internal excitation control have to be considered which increase the *damping torque* to $D_i \in [10, 35]/(2\pi f_0)$ depending on the load [30], [31]. In this case, $\epsilon \in \mathcal{O}(0.1)$ is small and a singular perturbation approximation is accurate.

We note that the simulation studies in Section VI show an accurate approximation of the power network by the non-uniform Kuramoto model also for larger values of $\epsilon$.

The assumption that $\epsilon$ is small seems to be crucial for the approximation of the power network model by the non-uniform Kuramoto model. However, similar results can also be obtained independently of the magnitude of $\epsilon$. In Remark III.1 we discussed the similarity between the non-uniform Kuramoto model (8) and the dimension-reduced system (7) considered in classic transient stability analysis. The transient stability literature derived various static and dynamic analogies between the power network model (3) and the reduced first-order model (7) [3], [5]. Among other things, both models have the same equilibria with the same local stability properties and comparable regions of attractions, as mentioned earlier. These results hold independently of the magnitude of $\epsilon$, have been successfully applied in academia and in industry [9], and support the approximation of the power network model by the non-uniform Kuramoto model.

## IV. Singular Perturbation Analysis

This section puts the approximation of the power network model (3) by the non-uniform Kuramoto model (8) on solid mathematical ground. With the perturbation parameter $\epsilon = M_{\text{max}}/(\pi f_0 D_{\text{min}})$ the power network model (3) can be reformulated as the singular perturbation problem

$$\epsilon\ddot{\theta}_i = -F_i\dot{\theta}_i + \frac{F_i}{D_i}\Big(\omega_i - \sum\nolimits_{j=1}^{n} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})\Big)\,, \quad (12)$$

where $F_i := (D_i/D_{\text{min}})/(M_i/M_{\text{max}})$ for $i \in \{1, \ldots, n\}$. For $\epsilon = 0$, system (12) reduces to the non-uniform Kuramoto model (8) or, after freezing time, it reduces to the set of algebraic equations $\dot{\theta}_i = P_i(\theta)/D_i$, where $P_i(\theta)$ is the power flow (4). For $\epsilon$ sufficiently small, the synchronization dynamics of (12) can be approximated by the non-uniform Kuramoto model (8) and the power flow (4), where the terms $F_i$ will determine the speed of convergence of the initial approximation error.

**Theorem IV.1 (Singular Perturbation Approximation)** *Consider the power network model* (3) *written as the singular perturbation problem* (12) *with initial conditions* $(\theta(0), \dot{\theta}(0))$ *and solution* $(\theta(t,\epsilon), \dot{\theta}(t,\epsilon))$. *Consider furthermore the non-uniform Kuramoto model* (8) *as the reduced model with initial condition* $\theta(0)$ *and solution* $\bar{\theta}(t)$*, the quasi-steady state* $h(\theta)$ *defined component-wise as* $h_i(\theta) := P_i(\theta)/D_i$*, and the boundary layer error* $y_i(t/\epsilon) := (\dot{\theta}_i(0) - h_i(\theta(0)))\, e^{-F_i t/\epsilon}$ *for* $i \in \{1, \ldots, n\}$. *Let* $T > 0$ *be arbitrary but finite and assume that the initial frequencies* $\dot{\theta}_i(0)$ *are bounded.*

*Then, there exists* $\epsilon_* > 0$ *such that for all* $\epsilon < \epsilon_*$*, the singular perturbation problem* (12) *has a unique solution on* $[0, T]$*, and for all* $t \in [0, T]$ *it holds uniformly in* $t$ *that*

$$\theta(t,\epsilon) - \bar{\theta}(t) = \mathcal{O}(\epsilon)\,, \; \dot{\theta}(t,\epsilon) - h(\bar{\theta}(t)) - y(t/\epsilon) = \mathcal{O}(\epsilon). \quad (13)$$

*Moreover, given any* $T_b \in (0, T)$*, there exists* $\epsilon^* \leq \epsilon_*$ *such that for all* $t \in [T_b, T]$ *and whenever* $\epsilon < \epsilon^*$ *it holds that*

$$\dot{\theta}(t,\epsilon) - h(\bar{\theta}(t)) = \mathcal{O}(\epsilon)\,. \quad (14)$$

Theorem IV.1 holds on a finite time interval $[0, T]$. In order to render the approximation (13)-(14) valid on an infinite time interval, additionally exponential stability of the reduced system is required. Among other things, the following section will show that the non-uniform Kuramoto model synchronizes exponentially for certain initial conditions. In this case, we can state the following corollary of Theorem IV.1.

**Corollary IV.1** *Under the assumption that the non-uniform Kuramoto model* (8) *synchronizes exponentially for some initial condition* $\theta(0)$*, the singular perturbation approximation* (13)-(14) *in Theorem IV.1 is valid for any* $T > 0$.
*Moreover, there exist* $\epsilon$ *and* $\varphi_{\text{max}}$ *sufficiently small such that the approximation errors* (13)-(14) *converge to zero.*

## V. Synchronization Analysis of Non-uniform Kuramoto Oscillators

This section combines and extends methods from the consensus and Kuramoto literature to analyze the non-uniform Kuramoto model (8). The role of the time constants $D_i$ and the phase shifts $\varphi_{ij}$ is immediately revealed when dividing by $D_i$ both hand sides and expanding the right-hand side as

$$\frac{\omega_i}{D_i} - \sum_{j=1}^{n}\Big(\frac{P_{ij}}{D_i}\cos(\varphi_{ij})\sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i}\sin(\varphi_{ij})\cos(\theta_i - \theta_j)\Big).$$

The difficulties in the analysis of system (8) are the lossy (anti-synchronizing) coupling via $(P_{ij}/D_i)\sin(\varphi_{ij})\cos(\theta_i - \theta_j)$

and the non-symmetric coupling between an oscillator pair $\{i, j\}$ via $P_{ij}/D_i$ on the one hand and $P_{ij}/D_j$ on the other.

Since the non-uniform Kuramoto model (8) is derived from the power network model (3), the underlying graph induced by $P$ is *complete* and *symmetric*, i.e., except for the diagonal entries, the matrix $P$ is fully populated and symmetric. For the sake of generality, this section considers the non-uniform Kuramoto model (8) under the assumption that the graph induced by $P$ is neither complete nor symmetric, that is, some coupling terms $P_{ij}$ are zero and $P \neq P^T$.

### A. Frequency Entrainment

Under the assumption of bounded phase differences, the classic Kuramoto oscillators (2) achieve frequency entrainment. An analogous result guarantees synchronization of the non-uniform Kuramoto oscillators (8) whenever the graph induced by $P$ has a globally reachable node.

**Theorem V.1 (Frequency entrainment)** *Consider the non-uniform Kuramoto model* (8) *where the graph induced by $P$ has a globally reachable node. Assume that there exists $\gamma \in (0, \pi/2 - \varphi_{\max})$ such that the (non-empty) set of bounded phase differences $\Delta(\gamma) = \{\theta \in \mathbb{T}^n : \max_{\{i,j\}} |\theta_i - \theta_j| \leq \gamma\}$ is positively invariant. Then for every $\theta(0) \in \Delta(\gamma)$,*

1) *the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$; and*
2) *if $\varphi_{\max} = 0$ and $P = P^T$, then $\dot{\theta}_\infty = \Omega := \sum_i \omega_i / \sum_i D_i$ and the exponential synchronization rate is no worse than*

$$\lambda_{\text{fe}} = -\lambda_2(L(P_{ij})) \cos(\gamma) \cos(\angle(D\mathbf{1}, \mathbf{1}))^2 / D_{\max}. \quad (15)$$

In the convergence rate $\lambda_{\text{fe}}$ given in (15), the factor $\lambda_2(L(P_{ij}))$ is the algebraic connectivity of the graph induced by $P = P^T$, the factor $1/D_{\max}$ is the slowest time constant of the non-uniform Kuramoto model (8), the proportionality $\lambda_{\text{fe}} \sim \cos(\gamma)$ reflects the phase locking, and the proportionality $\lambda_{\text{fe}} \sim \cos(\angle(D\mathbf{1}, \mathbf{1}))^2 = (\mathbf{1}^T D\mathbf{1})^2 / (\|\mathbf{1}\|_2 \, \|D\mathbf{1}\|_2)^2$ reflects the fact that the error coordinate $\dot{\theta} - \Omega\mathbf{1}$ is for non-uniform time constants not orthogonal to the *agreement vector* $\Omega\mathbf{1}$.

In essence, the proof of Theorem V.1 is based on the insight that the frequency dynamics of the non-uniform Kuramoto oscillators can be written as the consensus protocol

$$\frac{d}{dt} \dot{\theta}_i = -\sum\nolimits_{j=1}^{n} a_{ij}(\theta(t))(\dot{\theta}_i - \dot{\theta}_j), \quad i \in \{1, \dots, n\},$$

where the weight $a_{ij}(\theta(t)) = (P_{ij}/D_i) \cos(\theta_i(t) - \theta_j(t) + \varphi_{ij})$ is strictly positive for $P_{ij} > 0$ and $\theta(t) \in \Delta(\gamma)$ for all $t \geq 0$.

**Remark V.1 (Reduction of Theorem V.1 to classic Kuramoto oscillators:)** For classic Kuramoto oscillators (2), Theorem V.1 can be reduced to [23, Theorem 3.1]. □

### B. Phase Locking

The key assumption in Theorem V.1 is that phase differences are bounded in the set $\Delta(\gamma)$. To show this phase locking assumption, the Kuramoto literature provides various methods such as quadratic Lyapunov functions [23], contraction mapping [24], geometric [17], or Hamiltonian arguments [15], [18] based on an order parameter similar to the potential energy $U(\theta)$ defined in (6). Due to the asymmetric coupling and the phase shifts none of the mentioned methods appears to be easily extendable to the non-uniform Kuramoto model.

A different approach from the literature on consensus protocols [21], [22] is based on convexity and contraction and aims to show that the arc containing all phases is of non-increasing length. This approach turns out to be applicable to completely-coupled non-uniform Kuramoto oscillators.

**Theorem V.2 (Phase locking I)** *Consider the non-uniform Kuramoto-model* (8)*, where the graph induced by $P = P^T$ is complete. Assume that the minimal coupling is larger than a critical value, i.e., for every $i, j \in \{1, \dots, n\}$*

$$P_{\min} > P_{\text{critical}} := \frac{D_{\max}}{n \cos(\varphi_{\max})} \times \left( \max_{\{i,j\}} \left( \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right) + \max_i \sum\nolimits_{j=1}^{n} \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \right). \quad (16)$$

*Accordingly, define $\gamma_{\min} = \arcsin(\cos(\varphi_{\max}) P_{\text{critical}} / P_{\min})$ taking value in $(0, \pi/2 - \varphi_{\max})$. For $\gamma \in [\gamma_{\min}, \pi/2 - \varphi_{\max})$, define the (non-empty) set of bounded phase differences $\Delta(\gamma) = \{\theta \in \mathbb{T}^n : \max_{\{i,j\}} |\theta_i - \theta_j| \leq \gamma\}$. Then*

1) **phase locking***: for every $\gamma \in [\gamma_{\min}, \pi/2 - \varphi_{\max})$ the set $\Delta(\gamma)$ is positively invariant; and*
2) **frequency entrainment:** *for every $\theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ of the non-uniform Kuramoto oscillators* (8) *synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$. Moreover, if $\varphi_{\max} = 0$, then $\dot{\theta}_\infty = \Omega$ and the exponential synchronization rate is no worse than $\lambda_{\text{fe}}$ stated in equation* (15).

Theorem V.2 relies on the *contraction property*: the positive invariance of $\Delta(\gamma)$ means geometrically that all $\theta_i \in \mathbb{T}$ are contained in a rotating arc of non-increasing maximal length $\gamma$. Thus, the non-smooth function $V : \mathbb{T}^n \to [0, \pi]$,

$$V(\theta) = \max\{|\theta_i - \theta_j| \mid i, j \in \{1, \dots, n\}\}$$

has to be non-increasing at the boundary of $\Delta(\gamma)$, which is true under condition (16) interpreted in Remark III.2. Frequency entrainment follows directly from Theorem V.1.

**Remark V.2 (Reduction of Theorem V.2 to classic Kuramoto oscillators:)** For the classic Kuramoto oscillators (2) the sufficient condition (16) of Theorem V.2 specializes to

$$K > K_{\text{critical}} := \omega_{\max} - \omega_{\min}. \quad (17)$$

In other words, if $K > K_{\text{critical}}$, then there exists a positive-measure set of initial phase differences $\Delta(\gamma)$ with $\gamma \in [\arcsin(K_{\text{critical}}/K), \pi/2)$, such that the oscillators synchronize. To the best of our knowledge, the condition (17) on the coupling gain $K$ is the tightest bound sufficient for synchronization that has been presented in the Kuramoto literature so far. In fact, the bound (17) is close to the necessary condition for synchronization $K > K_{\text{critical}} \, n/(2(n-1))$ derived in [15], [23], [24]. Thus, in the case of two oscillators, condition (17) is necessary and sufficient for the onset of synchronization.

Other sufficient bounds given in the Kuramoto literature scale asymptotically with $n$, e.g., [24, Theorem 2] or [23, proof of Theorem 4.1]. To compare condition (17) with the bounds derived in [17], [23], [29], we note that our condition can be equivalently stated as follows. The set of bounded phase differences $\Delta(\pi/2-\gamma)$, for $\gamma \in (0,\pi/2)$, is positively invariant if $K \geq K(\gamma) := K_{\text{critical}}/\cos(\gamma)$. Our bound improves the bound $K > K(\gamma)n/2$ derived in [23, proof of Theorem 4.1] via a quadratic Lyapunov function, the bound $K > K(\gamma)n/(n-2)$ derived in [29, Lemma 9] via contraction arguments similar to ours, and the bound derived geometrically in [17, proof of Proposition 1] that, after some manipulations, reads in our notation as $K \geq K(\gamma)\cos((\pi/2-\gamma)/2)/\cos(\pi/2-\gamma)$. In summary, the bound (16) in Theorem V.2 improves the known sufficient conditions for synchronization of classic Kuramoto oscillators [15], [17], [23], [24], [29], and is necessary and sufficient condition in the case of two oscillators. □

Theorem V.2 presents an infinity bound for the phase locking and is based on the infinity bound (16) on the parameters and a complete coupling graph. In the remainder of this section, we consider a different approach based on an ultimate boundedness argument requiring only two-norm bounds and connectivity of the graph induced by $P = P^T$.

With slight abuse of notation, we denote the two-norm of the vector of pairwise geodesic distances by $\|H\theta\|_2 = (\sum_{\{i,j\}} |\theta_i - \theta_j|^2)^{1/2}$, and aim at ultimately bounding the evolution of $\|H\theta(t)\|_2$. In the recent literature [23], [24], a Lyapunov function considered for the uniform Kuramoto model (2) is simply $\|H\theta\|_2^2$. Unfortunately, in the case of non-uniform rates $D_i$ this function's Lie derivative is sign-indefinite. Inspired by [23], [24], let $D_{\neq\{i,j\}} := \prod_{k\neq i,j}^{n} D_k$, and consider the function $\mathcal{W} : \mathbb{T}^n \to \mathbb{R}$ defined by

$$\mathcal{W}(\theta) = \frac{1}{2}\sum\nolimits_{\{i,j\}} \frac{1}{D_{\neq\{i,j\}}} |\theta_i - \theta_j|^2 .$$

A Lyapunov analysis of the non-uniform Kuramoto model via the Lyapunov function $\mathcal{W}$ leads to the following theorem.

**Theorem V.3 (Phase locking II)** *Consider the non-uniform Kuramoto model* (8)*, where the graph induced by $P = P^T$ is connected with incidence matrix $H$ and unweighted Laplacian $L = H^T H$. Assume that the algebraic connectivity of the lossless coupling is larger than a critical value, i.e.,*

$$\lambda_2(L(P_{ij}\cos(\varphi_{ij}))) > \lambda_{\text{critical}} := \frac{\left\|HD^{-1}\omega\right\|_2 + \sqrt{\lambda_{\max}(L)}\left\|\left[\dots,\sum_{j=1}^{n}\frac{P_{ij}}{D_i}\sin(\varphi_{ij}),\dots\right]\right\|_2}{\cos(\varphi_{\max})(\kappa/n)\mu\min_{\{i,j\}}\{D_{\neq\{i,j\}}\}}, \tag{18}$$

*where $\mu := (\min_{i\neq j}\{D_iD_j\}/\max_{i\neq j}\{D_iD_j\})^{1/2}$ and $\kappa := \sum_{k=1}^{n}(1/D_{\neq k})$. Accordingly, define $\rho_{\max} \in (\pi/2-\varphi_{\max},\pi)$ as unique solution to the equation $(\pi/2-\varphi_{\max})\,\text{sinc}(\rho_{\max}) = \cos(\varphi_{\max})\lambda_{\text{critical}}/\lambda_2(L(P_{ij}\cos(\varphi_{ij})))$. Then*

1) **phase locking:** *for every $\rho \in (\pi/2-\varphi_{\max},\rho_{\max})$ and for $\|H\theta(0)\|_2 \leq \mu\rho$ there is $T \geq 0$ such that $\|H\theta(t)\|_2 < \pi/2 - \varphi_{\max}$ for all $t > T$; and*
2) **frequency entrainment:** *if $\|H\theta(0)\|_2 \leq \mu\rho$, then the frequencies $\dot\theta_i(t)$ of the non-uniform Kuramoto oscillators* (8) *synchronize exponentially to some frequency $\dot\theta_\infty \in [\dot\theta_{\min}(0),\dot\theta_{\max}(0)]$. Moreover, if $\varphi_{\max} = 0$, then $\dot\theta_\infty = \Omega$ and the exponential synchronization rate is no worse than $\lambda_{\text{fe}}$ stated in equation* (15).

**Remark V.3 (Physical interpretation of Theorem V.3:)** In condition (18) the term $(\kappa/n)\mu\min_{\{i,j\}}\{D_{\neq\{i,j\}}\}$ weights the non-uniformity in the time constants $D_i$, $\left\|\left[\dots,\sum_{j=1}^{n} P_{ij}\sin(\varphi_{ij})/D_i,\dots\right]\right\|_2$ is the two-norm of the vector with entry $i$ reflecting the lossy coupling of a node $i$ to the network, $\left\|HD^{-1}\omega\right\|_2 = \|(\omega_2/D_2 - \omega_1/D_1,\dots)\|_2$ corresponds to the non-uniformity in the natural frequencies, $\cos(\varphi_{\max}) = \sin(\pi/2-\varphi_{\max})$ reflects the ultimate phase locking, $\lambda_{\max}(L)$ is the largest eigenvalue of the Laplacian of the unweighted coupling graph (related to the maximum degree of a node), and $\lambda_2(L(P_{ij}\cos(\varphi_{ij})))$ is the algebraic connectivity induced by the lossless coupling. □

**Remark V.4 (Reduction of Theorem V.3 to classic Kuramoto oscillators:)** For classic Kuramoto oscillators (2), condition (18) relaxes to $K > K^*_{\text{critical}} := \|H\omega\|_2$ resembling the bound $K > \omega_{\max} - \omega_{\min} = \|H\omega\|_\infty$ presented in (17). It follows that the oscillators synchronize for $\|H\theta(0)\|_2 < \rho_{\max}$, where $\rho_{\max} \in (\pi/2,\pi)$ is the solution to the equation $(\pi/2)\,\text{sinc}(\rho_{\max}) = K^*_{\text{critical}}/K$. Note also that the Lyapunov function $\mathcal{W}(\theta)$ reduces to the one used in [23], [24] and can be used to prove [23, Theorem 4.2] and [24, Theorem 1].□

### C. Phase Synchronization

For uniform natural frequencies and zero phase shifts, Theorem V.2 and Theorem V.3 imply phase synchronization.

**Theorem V.4 (Phase synchronization)** *Consider the non-uniform Kuramoto-model* (8)*, where the graph induced by $P$ has a globally reachable node, $\varphi_{\max} = 0$, and $\omega_i/D_i = \bar\omega$ for all $i \in \{1,\dots,n\}$. Then*

1) *for $\theta(0) \in \{\theta \in \mathbb{T}^n : \max_{\{i,j\}}|\theta_i - \theta_j| < \pi\}$ the phases $\theta_i(t)$ synchronize exponentially to $\theta_\infty(t) \in [\theta_{\min}(0),\theta_{\max}(0)] + \bar\omega t$; and*
2) *if $P = P^T$ and $\|H\theta(0)\|_2 \leq \mu\rho$ with $\rho \in [0,\pi)$, then $\theta_\infty(t) = \sum_i D_i\theta_i(0)/\sum_i D_i + \bar\omega t$ and the exponential synchronization rate is no worse than*

$$\lambda_{\text{ps}} = -\frac{\kappa}{n}\min_{\{i,j\}}\{D_{\neq\{i,j\}}\}\,\text{sinc}(\rho)\lambda_2(L(P_{ij})). \tag{19}$$

The worst-case phase synchronization rate $\lambda_{\text{ps}}$ can be interpreted similarly as the terms in condition (18), where $\text{sinc}(\rho)$ corresponds to the phase locking in $\|H\theta(0)\|_2 \leq \mu\rho$.

**Remark V.5 (Reduction of Theorem V.4 to classic Kuramoto oscillators:)** Statements 1) and 2) can be reduced to the Kuramoto result found in [22] and Theorem 1 in [24].□

## VI. Simulation Results

The conditions given in Theorem III.1 and Theorem V.3 are only sufficient for synchronization, and simulations show that the bounds on the network parameters and its initial state are overly conservative. For a fixed value of $\epsilon$ the accuracy of the singular perturbation approximation of the power network model (3) by the non-uniform Kuramoto model (8) is independent of the network size but becomes worse if the initial state is near the stability margin – a property that is obvious in Theorem IV.1, where $\epsilon$ is dependent on $(\theta(0),\dot\theta(0))$. Conversely, if the sufficient conditions for synchronization are satisfied, the singular perturbation approximation is expected to hold also for large values of $\epsilon$.

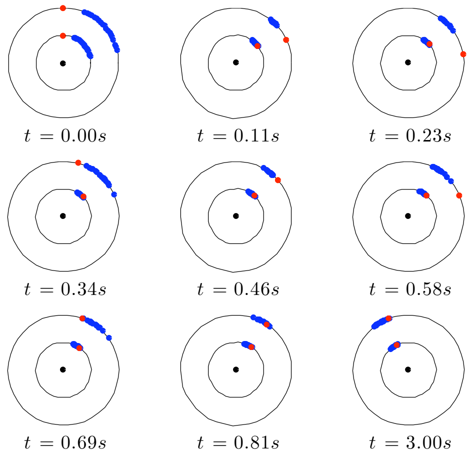


Fig. 1. Simulation of the power network model (3) (outer circle) and the non-uniform Kuramoto model (8) (inner circle)

Figure 1 shows a simulation of such a case, where $\epsilon = 0.6$s is large and all initial angles are clustered with exception of the red one. The simulation parameters can be found in [1]. The conditions of Theorem V.2 are satisfied and synchronization can be observed. Since $\epsilon$ is large, the damping of the generators in the power network is poor and their synchronization dynamics are oscillatory, whereas the non-uniform Kuramoto oscillators synchronize with the dynamics of overdamped pendula. Nevertheless, after this initial transient in the boundary layer the singular perturbation approximation is accurate, and all oscillators synchronize.

## VII. Conclusions

This paper studied the synchronization and transient stability problem for a network-reduction model of a power system. Our technical approach is based on the assumption that each generator is highly overdamped due to local excitation control. The subsequent singular perturbation analysis shows that the transient stability analysis in power networks reduces to the synchronization problem for non-uniform Kuramoto oscillators. The latter is an interesting mathematical problem in its own right and was tackled by combining and extending techniques from synchronization theory and consensus protocols. In the end, purely algebraic conditions depending on network parameters and initial phase differences suffice for the synchronization of non-uniform Kuramoto oscillators as well as the transient stability of the power network model.

## References

[1] F. Dörfler and F. Bullo, "Synchronization and transient stability in power networks and non-uniform Kuramoto oscillators," *IEEE Transactions on Automatic Control*, Jan. 2010, submitted.

[2] P. Varaiya, F. F. Wu, and R. L. Chen, "Direct methods for transient stability analysis of power systems: Recent results," *Proceedings of the IEEE*, vol. 73, no. 12, pp. 1703–1715, 1985.

[3] H.-D. Chiang, F. F. Wu, and P. P. Varaiya, "Foundations of the potential energy boundary surface method for power system transient stability analysis," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 6, pp. 712–728, 1988.

[4] T. Athay, R. Podmore, and S. Virmani, "A practical method for the direct analysis of transient stability," *IEEE Transactions on Power Apparatus and Systems*, vol. 98, no. 2, pp. 573–584, 1979.

[5] H.-D. Chiang and C. C. Chu, "Theoretical foundation of the BCU method for direct stability analysis of network-reduction power system models with small transfer conductances," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 42, no. 5, pp. 252–265, 1995.

[6] F. H. J. R. Silva, L. F. C. Alberto, J. B. A. London Jr, and N. G. Bretas, "Smooth perturbation on a classical energy function for lossy power system stability analysis," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 52, no. 1, pp. 222–229, 2005.

[7] M. A. Pai, *Energy Function Analysis for Power System Stability*. Kluwer Academic Publishers, 1989.

[8] L. F. C. Alberto, F. H. J. R. Silva, and N. G. Bretas, "Direct methods for transient stability analysis in power systems: state of art and future perspectives," in *IEEE Power Tech Proceedings*, Porto, Portugal, Sept. 2001.

[9] H.-D. Chiang, *Direct Methods for Stability Analysis of Electric Power Systems*. Wiley, 2010, ISBN: 0-470-48440-3.

[10] D. J. Hill and G. Chen, "Power systems as dynamic networks," in *IEEE Int. Symposium on Circuits and Systems*, Kos, Greece, May 2006, pp. 722–725.

[11] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[12] D. J. Hill and J. Zhao, "Global synchronization of complex dynamical networks with non-identical nodes," in *IEEE Conf. on Decision and Control*, Cancún, México, Dec. 2008, pp. 817–822.

[13] M. Arcak, "Passivity as a design tool for group coordination," *IEEE Transactions on Automatic Control*, vol.52, no.8, pp.1380–1390, 2007.

[14] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*. Dover Publications, 2003.

[15] J. L. van Hemmen and W. F. Wreszinski, "Lyapunov function for the Kuramoto model of nonlinearly coupled oscillators," *Journal of Statistical Physics*, vol. 72, no. 1, pp. 145–166, 1993.

[16] K. Wiesenfeld, P. Colet, and S. H. Strogatz, "Frequency locking in Josephson arrays: Connection with the Kuramoto model," *Physical Review E*, vol. 57, no. 2, pp. 1563–1569, 1998.

[17] F. De Smet and D. Aeyels, "Partial entrainment in the finite Kuramoto–Sakaguchi model," *Physica D: Nonlinear Phenomena*, vol. 234, no. 2, pp. 81–89, 2007.

[18] R. E. Mirollo and S. H. Strogatz, "The spectrum of the locked state for the Kuramoto model of coupled oscillators," *Physica D: Nonlinear Phenomena*, vol. 205, no. 1-4, pp. 249–266, 2005.

[19] S. H. Strogatz, "From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators," *Physica D: Nonlinear Phenomena*, vol. 143, no. 1, pp. 1–20, 2000.

[20] J. A. Acebron, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, "The Kuramoto model: A simple paradigm for synchronization phenomena," *Reviews of Modern Physics*, vol.77, no.1, pp.137–185, 2005.

[21] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.

[22] Z. Lin, B. Francis, and M. Maggiore, "State agreement for continuous-time coupled nonlinear systems," *SIAM Journal on Control and Optimization*, vol. 46, no. 1, pp. 288–307, 2007.

[23] N. Chopra and M. W. Spong, "On exponential synchronization of Kuramoto oscillators," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 353–357, 2009.

[24] A. Jadbabaie, N. Motee, and M. Barahona, "On the stability of the Kuramoto model of coupled nonlinear oscillators," in *American Control Conference*, Boston, MA, June 2004, pp. 4296–4301.

[25] G. Filatrella, A. H. Nielsen, and N. F. Pedersen, "Analysis of a power grid using a Kuramoto-like model," *The European Physical Journal B*, vol. 61, no. 4, pp. 485–491, 2008.

[26] V. Fioriti, S. Ruzzante, E. Castorini, E. Marchei, and V. Rosato, "Stability of a distributed generation network using the Kuramoto models," in *Critical Information Infrastructure Security*, ser. Lecture Notes in Computer Science. Springer, 2009, pp. 14–23.

[27] D. Subbarao, R. Uma, B. Saha, and M. Phanendra, "Self-Organization on a Power System," *IEEE Power Engineering Review*, vol. 21, no. 12, pp. 59–61, 2001.

[28] H. A. Tanaka, A. J. Lichtenberg, and S. Oishi, "Self-synchronization of coupled oscillators with hysteretic responses," *Physica D: Nonlinear Phenomena*, vol. 100, no. 3-4, pp. 279–300, 1997.

[29] G. S. Schmidt, U. Münz, and F. Allgöwer, "Multi-agent speed consensus via delayed position feedback with application to Kuramoto oscillators," in *European Control Conference*, Budapest, Hungary, Aug. 2009, pp. 2464–2469.

[30] P. M. Anderson and A. A. Fouad, *Power System Control and Stability*. Iowa State University Press, 1977.

[31] P. W. Sauer and M. A. Pai, *Power System Dynamics and Stability*. Prentice Hall, 1998.

[32] R. De Luca, "Strongly coupled overdamped pendulums," *Revista Brasileira de Ensino de Física*, vol. 30, pp. 4304–4304, 2008.